Академик Градимир В. Миловановић

НАУЧНЕ АКТИВНОСТИ
У 2017. ГОДИНИ

ПРОЈЕКАТ Ф-96
НАУЧНЕ КОНФЕРЕНЦИЈЕ И ПРЕДАВАЊА

Београд, 2018

СРПСКА АКАДЕМИЈА НАУКА И УМЕТНОСТИ
Одељење за математику, физику и гео-науке

Академик Градимир В. Миловановић

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НАУЧНЕ КОНФЕРЕНЦИЈЕ И ПРЕДАВАЊА

Библиотека
Математика и примене

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Тираж
300

Предговор

Научне активности које сам имао током 2017. године, а које су пре свега везане за рад на индивидуалном академијском пројекту Ф-96 (*Интерполациони и квадратурни процеси засновани на теорији ортогоналности*), описане су у овој публикацији, као и учешће на међународним научним конференцијама током ове године.

У првом делу описују се истраживања обављена у 2017. години и даје се списак објављених радова у том периоду. Сва неопходна документа која су поднета Одељењу за математику, физику и гео-науке САНУ, а односе се на научну активност, су садржај другог дела ове публикације. Активности на научним конференцијама током 2017. године су приказане у трећем делу. Најзад, комплетни радови се дају у четвртом делу публикације.

Мада се овом проблематиком бавим више од три деценије, са радом на пројекту Ф-96 формално сам почео након мог избора у САНУ 2006. године. Паралелно сам радио и на пројекту Министарства науке Републике Србије (сада Министарство просвете, науке и технолошког развоја) све до 2015. године, када због пензионисања престаје финансирање мог научног рада по том основу.

Резултати мог досадашњег научног рада саопштавани су на међународним скуповима и објављивани у познатим светским часописима или монографијама (детаљи се могу наћи на сајту: http://www.mi.sanu.ac.rs/~gvm/). Као један од значајних резултата помињем монографију *Interpolation Processes – Basic Theory and Applications*, Springer Monographs in Mathematics, Springer – Verlag, Berlin – Heidelberg, 2008 (XIV+444 pp.), коју сам објавио заједно са колегом Ђузепе Мастројанијем (Giuseppe Mastroianni) из Италије.

У Београду, 22. децембра 2017. Градимир В. Миловановић

Садржај

1. ИЗВЕШТАЈ О РАДУ НА ПРОЈЕКТУ Ф-96 У 2017. ГОДИНИ

1.1. Опис истраживања

У складу са планом настављена су истраживања на пројекту у 2017. години у области квадратурних и сумационих формула, ортогоналних полинома и специјалних функција.

Опште тежинске квадратурне формуле Биркоф-Јунговог типа са максималним степеном тачности за интеграцију аналитичких функција у комплексној равни, укључујући карактеризацију и јединственост таквих формула, као и нумеричку конструкцију чворова и тежинских коефицијената разматране су у раду [1]. Експлицитни облик "чворног полинома" је добијен за генералисану Гегенбауерову тежинску функцију. Најзад, проучаван је низ генералисаних квадратурних формула и њихови "чворни полиноми" су интерпретирани у терминима тзв. мултипл-ортогоналних полинома.

Нумеричко израчунавање интеграла са брзо осцилаторним функцијама и примене у израчунавању Фуријеових и Беселових трансформација, као и развој формула Гаусовог типа са модификованом Хермитовом тежином дато је у [2]. Методи су засновани на идеји метода комплексне интеграције (видети: [G.V. Milovanović, *Comput. Math. Appl*. **36** (1998), 19–39]).

Симболичко-нумеричка израчунавања нових класа ортогоналних полинома и одговарајућих Гаусових квадратурних формула у односу на кардиналне B-сплајнове (као тежине функције) разматрана су у [3], док је у [4] дат метод конструкције симетричних квадратура Гаусовог типа за парне тежинске функције, који је примењен на квадратуре са Полачековом тежином на (–1,1), као и на симетричне квадратурне формуле које се појављују у Абел-Плана сумационим формулама. Детаљна студија таквих сумационих формула, као и одговарајућих формула Ојлер-Маклореновог типа, укључујући класичне и нове резултате, изложена је као посебно поглавље у Springer-овој монографији [5] (*Progress in Approximation Theory and Applicable Complex Analysis*), која је посвећена недавно преминулом професору Рахману [Q.I. Rahman (1934-2013)].

Квадратурне формуле Гаусовог типа за експоненцијалне тежинске функције на реалној полуоси, које омогућавају интеграцију функција са сингуларитетима у нули и/или бесконачности (видети: [G. Mastroianni, I. Notarangelo, G.V. Milovanović, *IMA J. Numer. Anal.* **34** (2014), 1654–1685]), примењене су на конструкцију метода Нистремовог типа за решавање одговарајуће класе Фредхолмових интегралних једначина друге врсте [6]. Студија стабилности и конвергенције добијеног нумеричког метода базирана је на претходним резултатима о тежинској полиномијалној апроксимацији и тзв. "одсеченим" Гаусовим квадратурним формулама.

Конструкција оптималних квадратурних формула у Сард-овом смислу за израчунавање Фуријеових интеграла у Хилберовом простору не-периодичних функција $W_2^{(m,m-1)}$ је дата у раду [7], који је објављен у току 2017. године. Иначе, ови резултати су добијени у претходној календарској години и приказани су у извештају за 2016. годину, тако да се у одговарајућим извештајима о раду на пројекту у 2017. години за Билтен Фонда овај чланак не помиње (видети одељке 2.3 и 2.4 у овој публикацији).

Генераторске функције и особине нових класа специјалних полинома и њихове везе са добро познатим класама полинома Бернулијевог, Ојлеровог, Апостол-Бернулијевог, Апостол-Ојлеровог, Геночијевог и Фибоначијевог типа су разматране у раду [9]. Такође су уведене и проучаване класе полинома Фибоначијевог типа са две променљиве, као оне добијене модификацијом генераторске функције Хумбертових полинома.

Потребни и довољни услови за постизање глобалног минимума функције са слабим субдиференцијалом дати су у раду [10], као и веза између субдиференцијала и Фрешеовог диференцијала са слабим субдиференцијалом.

На основу извесних специјалних особина гама функције, у раду [11] се дефинише екстензија Похамеровог симбола, која омогућава увођење екстензије Гаусове и конфлуентне хипергеометријске функције, као и добијање њихових општих особина.

У раду [12] се даје побољшана верзија реверзне Хелдерове неједнакости помоћу $(k,s)-$Риман-Лиувиловог фракционог интеграла, као и одговарајуће примене.

Нова класа тежинских Адамс-Башфортових формула за решавање Кошијевих проблема за сингуларне диференцијалне једначине предложена је у раду [13], при чему је тежинска функција она која се појављује код класичних ортогоналних полинома.

Најзад у раду [8], који припада области бежичних телекомуникација, анализира се пренос информација кроз слободни простор коришћењем оптичких сигнала. У раду се утврђује утицај хардверских несавршености пријемника у комбинацији са несавршеним позиционирањем предајника у односу на пријемник и атмосферске турбуленције, на вероватноћу грешке када се користи диференцијална фазна модулација. Израз за вероватноћу грешке је одређен у облику конвергентног функционалног реда за који је нађена прецизна горња граница грешке одсецања као интеграл модификоване Беселове функције прве врсте $I_v(b)$ ро параметру v на неограниченом интервалу за фиксно b. Овим је омогућено израчунавање вероватноће грешке са произвољном тачношћу за различите параметре система.

1.2. Списак објављених радова у 2017. години

1. **G.V. Milovanović**: *Generalized weighted Birkhoff-Young quadratures with the maximal degree of exactness,* Appl. Numer. Math. **116** (2017), 238 – 255. [RJ234]*
2. **G.V. Milovanović**: *Computing integrals of highly oscillatory special functions using complex integration methods and Gaussian quadratures,* Dolomites Res. Notes Approx. **10** (2017), Special Issue, 79 – 96. [RJ240]
3. **G.V. Milovanović**: *Symbolic-numeric computation of orthogonal polynomials and Gaussian quadratures with respect to the cardinal B-spline,* Numer. Algorithms **76** (2017), 333 – 347. [RJ237]
4. M. Masjed-Jamei, **G.V. Milovanović**: *Construction of Gaussian quadrature formulas for even weight functions,* Appl. Anal. Discrete Math. **11** (2017), 177 – 198. [RJ233]
5. **G.V. Milovanović**: *Summation formulas of Euler-Maclaurin and Abel-Plana: old and new results and applications*, In: Progress in Approximation Theory and Applicable Complex Analysis – In the Memory of Q.I. Rahman (N.K. Govil, R.N. Mohapatra, M.A. Qazi, G. Schmeisser, eds.), pp. 429 – 461, Springer, 2017, ISBN 978-3-3199-49240-1. [BC33]
6. G. Mastroianni, **G.V. Milovanović**, I. Notarangelo: *A Nyström method for a class of Fredholm integral equations on the real semiaxis,* Calcolo **54** (2017), 567 – 585. [RJ235]
7. N.D. Boltaev, A.R. Hayotov, **G.V. Milovanović**, Kh.M. Shadimetov: *Optimal quadrature formulas for Fourier coefficients in* $W_2^{(m,m-1)}$ *space,* J. Appl. Anal. Comput. **7** (2017), 1233 – 1266. [RJ236]
8. M.I. Petković, G.T. Djordjević, G.K. Karagiannidis, **G.V. Milovanović**: *Performance of SIM-MDPSK FSO systems with hardware imperfections,* IEEE Trans. Wireless Commun. **16** (2017), 5442 – 5451. [RJ241]
9. G. Ozdemir, Y. Simsek, **G.V. Milovanović**: *Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials,* Mediterr. J. Math. **14** (2017), no. 3, Art. 117, 17 pp. [RJ238]
10. P. Cheraghi, A.P. Farajzadeh, **G.V. Milovanović**: *Some notes on weak subdifferential,* FILOMAT **31** (2017), 3407 – 3420. [RJ239]
11. M. Masjed-Jamei, **G.V. Milovanović**: *An extension of Pochhammer's symbol and its application to hypergeometric functions,* FILOMAT **31** (2017), 207 – 215. [RJ232]
12. M. Tomar, P. Agarwal, S. Jain, **G.V. Milovanović**: *Some reverse Hölder type inequalities involving* (*k,s*)*–Riemann-Liouville fractional integrals*, In: Functional Analysis in Interdisciplinary Applications. FAIA 2017 (T. Kalmenov, E. Nursultanov, M. Ruzhansky, M. Sadybekov, eds.), Springer Proceedings in Mathematics & Statistics, Vol. 216, pp. 302 – 311, Springer, Cham, 2017, ISBN 978-3-319-67052-2. [BC34]
13. M. Masjed-Jamei, **G.V. Milovanović**, A.H. Salehi Shayegan: *On weighted Adams-Bashforth rules,* Math. Commun. **23** (2018), 127 – 144. [RJ242]

*) Ознака на крају референце указује на позицију те референце на сајту http://www.mi.sanu.ac.rs/~gvm/publ0.html ; на пример [RJ234] је 234. референца у списку "Refereed Journals".

2. ДОКУМЕНТА ПОДНЕТА ОДЕЉЕЊУ ЗА МАТЕМАТИКУ, ФИЗИКУ И ГЕО-НАУКЕ САНУ

Одељење zа математику, физику и гео-науке САНУ, Београд

Лични извештај САНУ академика Градимира В. Миловановића за 2017. годину

Редовни професор Електронског факултета Универзитета у Нишу, у пензији. Предаје на докторским студијама на Електротехничком факултету Универзитета у Београду. Члан је Научног већа Математичког института. Од марта 2016. године у САНУ обавља функцију секретара Одељења за математику, физику и гео-науке и члан је Председништва САНУ.

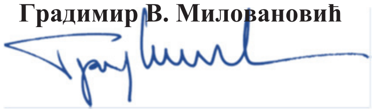
Главни уредник је у часописима "**Journal of Inequalities and Applications**" (*Springer*), "**Bulletin, Classe des Sciences Mathématiques et Naturelles, Sciences mathématiques**" (*САНУ, Београд*) и "***Publications de l'Institut Mathématique***" (*Математички институт САНУ, Београд*), уредник (Associate Editor) у часописима са SCI листе "***Applied Mathematics and Computation***" (*Elsevier*), "***Optimization Letters***" (*Springer*), "***Applicable Analysis and Discrete Mathematics***" (*Електротехнички факултет у Београду*) и "***FILOMAT***" (*Природно-математички факултет у Нишу*), као и члан редакција више часописа у Србији, Бугарској, Румунији, Јерменији, Индији, Ирану, Хрватској и Турској.

У САНУ ради на индивидуалном пројекту Ф-96 под насловом ***Интерполациони и квадратурни процеси засновани на теорији ортогоналности***. У оквиру Министарства просвете, науке и технолошког развоја Републике Србије без финансирања учествује и руководи пројектом „***Апроксимација интегралних и диференцијалних оператора и примене***" у области основних истраживања. Учествовао је у комисијама за одбрану више докторских дисертација у земљи и иностранству. У САНУ је организовао међународну научну конференцију у САНУ, **ACTA 2017: Approximation and Computation – Theory and Applications** (30.11. – 02.12), посвећену светски познатом научнику Валтеру Гаучију (W. Gautschi) поводом његовог 90-тог рођендана.

У току 2017. године објавио је 13 радова (10 у часописима са SCI листе) и одржао три пленарна предавања на међународним научним скуповима [ICRAPAM 2017 (May 11-15, Kusadasi – Aydin, Turkey); EnuMDeS-17 (September 11-12, UH1-EST Berrechid, Morocco); ACTA 2017 (November 30-December 2, Belgrade)].

Редовни члан САНУ

Градимир В. Миловановић



Gradimir V. Milovanović, redovni član SANU

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8. M.I. Petković, G.T. Djordjević, G.K. Karagiannidis**, G.V. Milovanović**: *Performance of SIM-MDPSK FSO systems with hardware imperfections*, IEEE Transactions on Wireless Communications **16** (2017), 5442 – 5451.
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13. M. Masjed-Jamei, **G.V. Milovanović**, A.H. Salehi Shayegan: *On weighted Adams-Bashforth rules*, Mathematical Communications **23** (2018), 127 – 144.

Пројекат Ф-96: **Интерполациони и квадратурни процеси засновани на теорији ортогоналности**

Руководилац: **Градимир В. Миловановић**

Извештај за 2017. годину

У складу са планом настављена су истраживања у области квадратурних и сумационих формула, ортогоналних полинома и специјалних функција. Опште тежинске квадратурне формуле Биркоф-Јунговог типа са максималним степеном тачности за интеграцију аналитичких функција у комплексној равни, укључујући карактеризацију и јединственост формула, као и нумеричку конструкцију чворова и тежинских коефицијената, разматране су у раду [1]. Експлицитни облик "чворног полинома" је добијен за генералисану Гегенбауерову тежинску функцију, а проучаван је и низ генералисаних квадратурних формула и њихови "чворни полиноми" су интерпретирани у терминима тзв. мултипл-ортогоналних полинома. Нумеричко израчунавање интеграла са брзо осцилаторним функцијама и примене у израчунавању Фуријеових и Беселових трансформација, као и развој формула Гаусовог типа са модификованом Хермитовом тежином дато је у [2]. Методи су засновани на идеји метода комплексне интеграције (видети: [G.V. Milovanović, Comput. Math. Appl. **36** (1998), 19–39]). Симболичко-нумеричка израчунавања нових класа ортогоналних полинома и одговарајућих Гаусових квадратурних формула у односу на кардиналне B-сплајнове разматрана су у [3], док је у [4] дат метод конструкције симетричних квадратура Гаусовог типа за парне тежинске функције, који је примењен на квадратуре са Полачековом тежином на (–1,1), као и на симетричне квадратурне формуле које се појављују у Абел-Плана сумационим формулама. Детаљна студија таквих сумационих формула, као и одговарајућих формула Ојлер-Маклореновог типа, укључујући класичне и нове резултате, изложена је као посебно поглавље у Springer-овој монографији [5]. Квадратурне формуле Гаусовог типа за експоненцијалне тежинске функције на реалној полуоси, које омогућавају интеграцију функција са сингуларитетима у нули и/или бесконачности (видети: [G. Mastroianni, I. Notarangelo, G.V. Milovanović, IMA J. Numer. Anal. **34** (2014), 1654–1685]), примењене су на конструкцију метода Нистремовог типа за решавање одговарајуће класе Фредхолмових интегралних једначина друге врсте [6]. Нова класа тежинских Адамс-Башфортових формула за решавање Кошијевих проблема за сингуларне диференцијалне једначине предложена је у [12], док су у [8] изведене генераторске функције за специјалне полиноме и бројеве, укључујући полиноме Апостоловог и Хумбертовог типа. У радовима [9] и [10] излажу се извесни теоријски концепти слабог субдиференцијала у теорији оптимизације, као и екстензија Поххамеровог симбола и његова примена на хипергеометријске функције. У раду [11] се даје побољшана верзија реверзне Хелдерове неједнакости помоћу $(k,s)-$Риман-Лиувиловог фракционог интеграла и одговарајуће примене. Најзад, у раду [7] третира се проблем у телекомуникацијама.

1. **G.V. Milovanović**: *Generalized weighted Birkhoff-Young quadratures with the maximal degree of exactness,* Applied Numerical Mathematics **116** (2017), 238 – 255.

2. **G.V. Milovanović**: *Computing integrals of highly oscillatory special functions using complex integration methods and Gaussian quadratures*, Dolomites Research Notes on Approximation **10** (2017), Special Issue, 79 – 96.

3. **G.V. Milovanović**: *Symbolic-numeric computation of orthogonal polynomials and Gaus-sian quadratures with respect to the cardinal B-spline*, Numerical Algorithms **76** (2017), 333 – 347.

4. M. Masjed-Jamei, **G.V. Milovanović**: *Construction of Gaussian quadrature formulas for even weight functions,* Applicable Analysis and Discrete Mathematics **11** (2017), 177 – 198.

5. **G.V. Milovanović**: *Summation formulas of Euler-Maclaurin and Abel-Plana: old and new results and applications*, In: Progress in Approximation Theory and Applicable Complex Analysis – In the Memory of Q.I. Rahman (N.K. Govil, R.N. Mohapatra, M.A. Qazi, G. Schmeisser, eds.), pp. 429 – 461, Springer, 2017, ISBN 978-3-3199-49240-1.

6. G. Mastroianni, **G.V. Milovanović**, I. Notarangelo: A Nyström method for a class of Fredholm integral equations on the real semiaxis, Calcolo **54** (2017), 567 – 585.

7. M.I. Petković, G.T. Djordjević, G.K. Karagiannidis, **G.V. Milovanović**: Performance of SIM-MDPSK FSO systems with hardware imperfections, IEEE Transaction on Wireless Communications **16** (2017), 5442 – 5451.

8. G. Ozdemir, Y. Simsek, **G.V. Milovanović**: *Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials*, Mediterranean Journal of Mathematics 14 (2017), no. 3, Art. 117, 17 pp.

9. P. Cheraghi, A.P. Farajzadeh, **G.V. Milovanović**: *Some notes on weak subdifferen-tial*, FILOMAT **31** (2017), 3407 – 3420.

10. M. Masjed-Jamei, **G.V. Milovanović**: An extension of Pochhammer's symbol and its application to hypergeometric functions, FILOMAT **31** (2017), 207 – 215.

11. M. Tomar, P. Agarwal, S. Jain, **G.V. Milovanović**: *Some reverse Hölder type inequalities involving* (*k,s*)*–Riemann-Liouville fractional integrals*, In: Functional Analysis in Interdisciplinary Applications. FAIA 2017 (T. Kalmenov, E. Nursultanov, M. Ruzhansky, M. Sadybekov, eds.), Springer Proceedings in Mathematics & Statistics, Vol. 216, pp. 302 – 311, Springer, Cham, 2017, ISBN 978-3-319-67052-2.

12. M. Masjed-Jamei, **G.V. Milovanović**, A.H. Salehi Shayegan: *On weighted Adams-Bashforth rules*, Mathematical Communications **23** (2018), 127 – 144.

Редовни члан САНУ
Градимир В. Миловановић

Project Φ-96: **Interpolation and quadrature processes based on the theory of orthogonality**

Leader: **Gradimir V. Milovanović**

Report for 2016

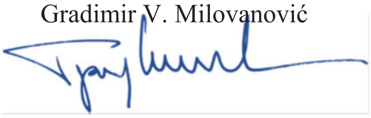
According to the plan of this project, the work on the development and applications of quadrature and summation processes, orthogonal polynomials and special functions is continued. General weighted quadrature formulas of Birkhoff–Young type with the maximal degree of exactness are given in [1]. It includes a characterization and uniqueness of such rules, as well as numerical construction of nodes and weight coefficients. An explicit form of the node polynomial of such kind of quadratures with respect to the generalized Gegenbauer weight function is obtained. Also, a sequence of generalized quadrature formulas is studied and their node polynomials are interpreted in terms of multiple orthogonal polynomials. An account on computation of integrals of highly oscillatory functions based on the so-called *complex integration methods* is presented in [2]. Beside the basic idea of this an approach established by Milovanović [Comput. Math. Appl. **36** (1998), 19–39] some applications in computation of Fourier and Bessel transformations are given. Also, Gaussian quadrature formulas with a modified Hermite weight are considered. Symbolic–numeric computation of orthogonal polynomials and the corresponding Gaussian quadratures with respect to the cardinal B-spline are considered in [3]. A method for constructing symmetric Gaussian formulas with respect to an even weight function is given in [4], and it is applied to quadratures related to the Pollaczek-type weight functions on $(-1, 1)$, as well as to symmetric Gaussian quadrature rules on the real line, which appear in the Abel-Plana summation formulas. A detailed study of such summation formulas, as well as ones of the Euler–Maclaurin type, including classical and new results, is presented as a chapter in the Springer monograph [5]. Quadrature rules of Gaussian type with exponential weight functions on the real semiaxis (see [G. Mastroianni, I. Notarangelo, G.V. Milovanović, IMA J. Numer. Anal. **34** (2014), 1654–1685]) are applied for getting a Nyström method for a class of Fredholm integral equations of the second kind on the real semiaxis [6]. A new class of Adams-Bashforth rules for solving the Cauchy problem for singular differential equations is proposed in [12], and in [8] new generating functions for some special classes of polynomials and numbers are derived, including polynomials of Apostol and Humbert type. Certain theoretical concepts of weak subdifferential in the optimization theory, as well as an extension of the Pochhammer symbol and its application to hypergeometric functions are given in [9] and [10]. An improved version of the reverse Hölder type inequalities by taking $(k,s)-$Riemann-Liouville fractional integrals, as well as some applications are presented in [11]. Finally, a problem in telecommunication is treated in [7].

1. **G.V. Milovanović**: *Generalized weighted Birkhoff-Young quadratures with the maximal degree of exactness,* Applied Numerical Mathematics **116** (2017), 238 – 255.
2. **G.V. Milovanović**: *Computing integrals of highly oscillatory special functions using*
3. *complex integration methods and Gaussian quadratures*, Dolomites Research Notes on Approximation **10** (2017), Special Issue, 79 – 96.

4. **G.V. Milovanović**: *Symbolic-numeric computation of orthogonal polynomials and Gaussian quadratures with respect to the cardinal B-spline*, Numerical Algorithms **76** (2017), 333 – 347.
5. M. Masjed-Jamei, **G.V. Milovanović**: *Construction of Gaussian quadrature formulas for even weight functions,* Applicable Analysis and Discrete Mathematics **11** (2017), 177 – 198.
6. **G.V. Milovanović**: *Summation formulas of Euler-Maclaurin and Abel-Plana: old and new results and applications*, In: Progress in Approximation Theory and Applicable Complex Analysis – In the Memory of Q.I. Rahman (N.K. Govil, R.N. Mohapatra, M.A. Qazi, G. Schmeisser, eds.), pp. 429 – 461, Springer, 2017, ISBN 978-3-3199-49240-1.
7. G. Mastroianni, **G.V. Milovanović**, I. Notarangelo: A Nyström method for a class of Fredholm integral equations on the real semiaxis, Calcolo **54** (2017), 567 – 585.
8. M.I. Petković, G.T. Djordjević, G.K. Karagiannidis, **G.V. Milovanović**: Performance of SIM-MDPSK FSO systems with hardware imperfections, IEEE Transaction on Wireless Communications **16** (2017), 5442 – 5451.
9. G. Ozdemir, Y. Simsek, **G.V. Milovanović**: *Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials*, Mediterranean Journal of Mathematics 14 (2017), no. 3, Art. 117, 17 pp.
10. P. Cheraghi, A.P. Farajzadeh, **G.V. Milovanović**: *Some notes on weak subdifferential*, FILOMAT **31** (2017), 3407 – 3420.
11. M. Masjed-Jamei, **G.V. Milovanović**: An extension of Pochhammer's symbol and its application to hypergeometric functions, FILOMAT **31** (2017), 207 – 215.
12. M. Tomar, P. Agarwal, S. Jain, **G.V. Milovanović**: *Some reverse Hölder type inequalities involving* (*k,s*)*–Riemann-Liouville fractional integrals*, In: Functional Analysis in Interdisciplinary Applications. FAIA 2017 (T. Kalmenov, E. Nursultanov, M. Ruzhansky, M. Sadybekov, eds.), Springer Proceedings in Mathematics & Statistics, Vol. 216, pp. 302 – 311, Springer, Cham, 2017, ISBN 978-3-319-67052-2.
13. M. Masjed-Jamei, **G.V. Milovanović**, A.H. Salehi Shayegan: *On weighted Adams-Bashforth rules*, Mathematical Communications **23** (2018), 127 – 144.

Full member of SASA

Gradimir V. Milovanović



Пројекат Ф-96: Интерполациони и квадратурни процеси засновани на теорији ортогоналности

Руководилац: Градимир В. Миловановић

План рада за 2018.

Наставак истраживања у области специјалних функција, теорије апроксимација, ортогоналности и нумеричке интеграције: (1) развој генералисаних линеарних интегралних оператора Канторовичевог типа и (*p*,*q*)-функција и примене у теорији апроксимација; (2) екстензија Pochhammer-овог симбола и примене на хипергеометријске функције; (3) студија генералисаних сумационих теорема за хипергеометријске функције $_2F_1$ и примена на Лапласову трансформацију конволуционих интеграла са Kummer-овим функцијама $_1F_1$; (4) проучавање ортогоналних полинома са модификованом Чебишевљевом мером и развој квадратура са вишеструким чворовима за израчунавање Фурије-Чебишевљевих коефицијената; (5) примене квадратурних и сумационих метода у телекомуникацијама и електромагнетици.

Такође се наставља рад на монографијама:

1. **Квадратурни процеси у сумирању споро конвергентних редова (Quadrature Processes in Summation of Slowly Convergent Series)** и

2. **Квадартурне формуле са вишеструким чворовима (Quadrature Formulae with Multiple Nodes)**.

Редовни члан САНУ

Градимир В. Миловановић

3. НАУЧНЕ КОНФЕРЕНЦИЈЕ И ПРЕДАВАЊА У 2017. ГОДИНИ

3.1. Конференција ICRAPAM 2017 у Турској

У периоду од 11. до 15. маја одржана је међународна конференција: **ICRAPAM 2017 (International Conference on Recent Advances in Pure and Applied Mathematics)** у Турској (Palm Wings Ephesus Resort Hotel, Kusadasi – Aydin). Била је то четврта по реду конференција са истим називом, у организацији **Istanbul Commerce University**, **Turkish Cooperation and Coordination Agency** и **Турске академије наука** (**TÜBA**), а главни организатор је био проф. Ekrem SAVAS, редовни члан Турске академије наука. Поред седам пленарних предавања на конференцији је, кроз више секција, излажено и око 200 краћих предавања.

Током конференције организована је и једнодневна екскурзија са обиласком древног грчког града Ефеса (Ephesos), посети Кући Блажене Богородице Марије (The House of the Virgin Mary) и селу Sirince.

У даљем тексту биће дати неки детаљи са конференције, апстракт пленарног предавања, као и интересантне фотографије са екскурзије.

Одељење za математику, физику и гео-науке САНУ, Београд

Извештај о боравку у Турској

У периоду од 11. до 16. маја боравио сам у Турској, где сам учествовао на **International Conference on Recent Advances in Pure and Applied Mathematics – ICRAPAM 2017** (May 11 – 15, 2017, Palm Wings Ephesus Resort Hotel, Kusadasi – Aydin, Turkey), као пленарни предавач. Том приликом сам одржао предавање под насловом "*Summation and Quadrature Processes for Slowly Convergent Series*". Конференцију су организовали **Istanbul Commerce University**, **Turkish Cooperation and Coordination Agency** и **Турска академија наука** (**TÜBA**).

Трошкови борвка су покривени од стране домаћина.

23. мај 2017.

Редовни члан САНУ

Градимир В. Миловановић

INTERNATIONAL CONFERENCE on RECENT ADVANCES in PURE AND APPLIED MATHEMATICS

(ICRAPAM 2017)

May 11-15, 2017, Palm Wings Ephesus Resort Hotel, Kusadasi - Aydin, TURKEY

www.icrapam.org

Conference Program Booklet

ICRAPAM

International Conference on Recent Advances in Pure and Applied Mathematics

Conference Program Overview

11th of May	
9:30-12:00	Registration
12:00-14:00	Lunch Break
14:00-14:50	Plenary Talk by ***Robin Harte***, Chair: **Okay Çelebi**
15:00-16:40	Sessions 1
16:40-18:00	Sessions 2
18:00-18:20	Coffee Break
18:20-19:10	Plenary Talk by ***Mujahid Abbas***, ***Chair: Mohammed al-Gwaiz***
12th of May	
9:00-10:00	Plenary Talk by ***Taras Banakh***, ***Chair:*** E***krem SAVAS***
10:00-10:50	Opening Ceremony
10:50-11:00	Coffee Break
11:00-12:40	Sessions 3
13:00-14:30	Lunch Break
14:30-15:20	*Plenary Talk by* ***Mohammed al-Gwaiz , Chair: Taras Banakh***
15:20-16:40	Sessions 4
16:40- 17:00	Coffee Break
17:00- 19:30	Sessions 5
19:10-19:40	Poster Session
20:00-23:00	Gala Dinner
13rd of May	
10:00-19:00	**EXCURSION**
14th of May	
9:00-9:50	*Plenary Talk by* ***A. Okay Çelebi, Chair: Robin Harte***
10:00-12:00	Sessions 6
12:00-14:00	Lunch Break
14:00-14:50	*Plenary Talk by* ***Gradimir V. Milovanovic, Chair: Reza Langari***
15:00-16:40	Sessions 7
16:40-17:00	Coffee Break
17:00-18:20	Session 8
18:20-18:30	Coffee Break
18:30-19:20	Plenary Talk by ***Brahim Mezerdi***, Chair: ***Mujahid Abbas***

9:00-10:40	Sessions 9
10:40-11:00	Coffee Break
11:00-12:00	Closing Ceremony

Plenary Talk Titles

11st May, 14:00-14:50	Robin Harte	Gelfand Theory Unplugged
11st May, 18:20-19:10	Mujahid Abbas	Solution of an implicit complementarity problem on isotone projection cones
12th May, 9:00-10:00	Taras Banakh	Topological spaces with an $omega^omega$-base
12th May, 14:30-15:20	Mohammed Al-Gwaiz	The Sturm-Liouville Theory And Fourier Analysis
14th May, 9:00-9:50	Okay Çelebi	Schwarz problem for higher order equations in a polydisc
14th May, 14:00-14:50	Gradimir V. Milovanovic	Summation and Quadrature Processes for Slowly Convergent Series
14th May, 18:30-19:20	Brahim Mezerdi	On optimal control of stochastic mean field systems

Summation and Quadrature Processes for Slowly Convergent Series

Gradimir V. Milovanović

Serbian Academy of Sciences and Arts, Belgrade, Serbia
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Abstract: An account on summation/integration methods for computation of slowly convergent series and finite sums, as well as some new results on this subject and new applications, are presented. Methods are based on Gaussian quadrature formulas with respect to some non-classical weight functions over the real line or the halfline. For constructing such quadrature rules we use recent progress in symbolic compuation and variableprecision arithmetic, implemented through our **Mathematica** package "*OrthogonalPolynomials*" [1], [2]. Some details on these methods can be found in [3], [4], [5].

Keywords: Summation, Gaussian quadrature rules, weight function, convergence, orthogonal polynomials.

References:

[1] A. S. Cvetković and G. V. Milovanović, "The Mathematica Package OrthogonalPolynomials", Facta Univ. Ser. Math. Inform. 19 (2004), 17-36.
[2] G. V. Milovanović and A. S. Cvetković, "Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type", Math. Balkanica 26 (2012), 169-184.
[3] G. V. Milovanović, "Summation of series and Gaussian quadratures", In: Approximation and Computation (R.V.M. Zahar, ed.), ISNM Vol. 119, pp. 459-475, Birkhäuser Verlag, Basel-Boston-Berlin, 1994.
[4] G. Mastroianni and G. V. Milovanović, Interpolation Processes - Basic Theory and Applications, Springer Monographs in Mathematics, Springer Verlag, Berlin - Heidelberg - New York, 2008.
[5] G. V. Milovanović, "Summation formulas of Euler-Maclaurin and Abel-Plana: old and new results and applications", In: Progress in Approximation Theory and Applicable Complex Analysis – In the Memory of Q,I. Rahman (N.K. Govil, R.N. Mohapatra, M.A. Qazi, G. Schmeisser, eds.), Springer, 2017 (to appear).

ICRAPAM

International Conference on Recent Advances in Pure and Applied Mathematics

Организатор конференције Prof. Dr. Ekrem Savas (лево) додељује признање академику Градимиру В. Миловановићу (десно) за одржано пленарно предавање *Summation and Quadrature Processes for Slowly Convergent Series.*

Посета Ефесу (Ephesus): Г. Миловановић, Екрем Савас, Љубиша Кочинац

3.2. Конференција ENuMDeS-17 у Мароку

Поводом 60-тог рођендана Алал Гесаба (Allal Guessab), професора Université de Pau из Француске, у Мароку је на Université Hassan 1st Settat, Ecole Superieure de Technologie, Berrechid, одржан научни скуп под насловом "**Effective Numerical Methods for Decision Support**" (Berrechid, September 11-12, 2017), као и одбрана докторске дисертације кандидата **Yassine ZAIM** под насловом: *Approximation by enriched conforming and nonconforming finite elements.*

Професор Гесаб је иначе пореклом из Марока. Пре 30 година учествовао сам као инострани члан Комисије за одбрану његове докторске дисертације на Université de Pau, на југу Француске. Након тога развили смо успешну вишегодишњу сарадњу у области екстремалних проблема и квадратурних процеса, из које је проистекао већи број заједничких радова. То је био главни разлог да будем позван од стране оба универзитета да одржим пленарно предавање на овом скупу, као и да учествујем у Комисији за одбрану докторске дисерације кандидата Y. Zaim, у оквиру заједничких докторских студија поменутих универзитета из Француске и Марока. Иначе, Berrechid је место у региону Казабланке (Casablanca).

У даљем тексту дају се детаљи са скупа, насловна страна докторске дисертације, извештај о боравку у Мароку, као и више фотографија из Марока.

UNIVERSITE HASSAN 1er
ECOLE SUPERIEURE DE TECHNOLOGIE
BERRECHID

The Higher School of Technology of Berrechid organizes a closing workshop of the UH1 Thematic Project in honor of Professor Allal Guessab on the occasion of his 60th birthday

"Effective Numerical Methods for Decision Support :

ENuMDeS-17"

September, 11-12, 2017

Invited Speakers:
Abellatif Agouzal : University Lyon 1- France
Roland Becker : Université de Pau et des Pays de l'Adour-France
Domingo Barrera : University of Granada- Espagne
Abdellah Lamnii : University Hassan 1er Settat – Marocco
Gradimir V. Milovanovic, Serbian Academy of Sciences and Arts, Serbia
Othman Nouisser, University Ibnou Tofail Kénitra-Marocco
Gerhald Scheisser, University of Erlangen–Nuremberg- Germany
Ahmed Taik : University Hassan II – Casablanca- Marocco

This event is an opportunity to pay tribute to Professor Allal Guessab of the University of Pau and Pays de l'Adour, on the occasion of his 60th anniversary, for his brilliant scientific career and for the services rendered to the Moroccan University in general, University Hassan 1st in particular Allal Guessab holds a first doctoral thesis (1983) and a Ph-D thesis in Mathematical Sciences at the University of Pau and the Pays de l'Adour (1987). He has been Professor of French Universities at UPPA since 1993. He has been a member of UPPA since 1983 and has since acquired recognized experience in teaching, research and academia: former Director of the Research Department (1998-2006), Former Director of the Education Department (1992-1998) and former Director of the Institute of French Studies for Foreign Students (2008-2011), Director of the Approximation Approximation Research Team for the period 1992-2000. For thirty-three years, Professor A. Guessab's teachings have focused on mathematics and their applications. They are taught at different levels, mainly in France and French-speaking Africa (Continuing Education, Post-Doctoral Training Seminars, Doctoral School, Master, etc.). His main field of research concerns the theory of approximation and its various applications. In addition to his scientific activities and as a trainer, Professor A. Guessab is also a consultant to major multinational companies and public and private sector decision-makers. He has created and chaired Ifed, his sector of activity being publishing Of scientific software. As part of its consulting and training activities for large multinational companies or public and private institutions, it has developed expertise in these areas. Professor A. Guessab is a confirmed Teacher-Researcher: more than 60 articles published in international journals. He is an associate editor of eight international peer-reviewed journals.

The day also aims to present the state of progress of the research carried out in the framework of the UH1 project "Simulations and high-performance numerical methods for decision support: problems of the environment" which deals with the mathematical and numerical study Complex systems arising from problems related to natural resource management, particularly water management and pollution problems in the Casablanca-Settat region.

To this end, we will focus on the modeling and simulation of flows in porous media modeling aquifers and the risks of pollution. Emphasis will be placed on the development of high-performing and reliable numerical methods, making available to decision-makers and managers a scientific simulation tool for decision-making support in natural resource management and for the fight against pollution, which has reached alarming levels in Morocco.

Scientific Committee

Boujemâa Achchab : UH1- Morocco
Abdellatif Agouzal : UCB-Lyon1-France
Ahmed Nafidi : UH1-Morocco
Allal Guessab : UPPA-France
Domingo Barrera : UGR- Espagne
Gerhald Scheisser : FAU- Germany
Ali Souissi : UMV-Rabat-Morocco
Gradimir V. Milovanovic : Serbian Academy of Sciences and Arts, Serbia

Organisation Committee

Boujemâa Achchab : UH1-ESTB- Morocco
Domingo Barrera UGR- Espagne
Khalid Bouihat UH1- ESTB- Morocco
Mohamed Lamnii UH1- ESTB Morocco
Ahmed Nafidi UH1- ESTB-Morocco
Mohamed Naimi UH1- ESTB-Morocco
Abdelmjid Qadi Idrissi UH1 – ESTB- Morocco

Programme du Workshop : ENuMDeS-17
11 et 12 septembre 2017,
UH1- EST Berrechid

- Lundi 11 septembre :

14h-18h : Soutenance de la thèse de doctorat : Yassine Zaim :
« Approximation by enriched conforming and nonconforming finite elements »
Devant le jury composé des professeurs, R. Aboulaich, B. Achchab, R. Becker, A. Guessab, G. Milovanovic, G. Schmeisser, A. Souissi

- Mardi 12 septembre :

Matin 9h-13h

1. **Gradimir Milovanovic**, Académies des Sciences et des Arts de la Serbie.
« Summation and Quadrature Processes for Slowly Convergent Series. » (45 mn)

2. **Roland Becker**, Université de Pau et des Pays de l'Adour, France
« Convergence of adaptive finite element methods » (45 mn)

3. **Abdellatif Agouzal**, Université Claude Bernard Lyon 1, France
« Discretization of BVP on unstructured meshes. » (45 mn)

4. **Gerhard Schmeisser**, Université Erlangen–Nuremberg, Allemagne
« Extension of basic relations valid for bandlimited functions to larger spaces via a unified distance concept » (45 mn)

Après-midi : 14h30-18h30

5. **Ahmed Taik**, FST Mohammadia, Université Hassan 2, Casablanca, Maroc
« Analysis and numerical simulation of some combustion problems with critical parameters » (45 mn)

6. **Abdellah LAMNII**, Université Hassan 1er, FST Settat, Maroc
« Control curves and wavelets for Uniform Hyperbolic Trigonometric spline » (30 mn)

7. **Domingo Barrera**, Université de Grenade, Espagne
« On trivariate near-best blending quasi-interpolation operators. » (30 mn)

8. **Othmen Nouisser**, Université Ibnou Tofail Kénitra-Maroc

« Approximation des données à plusieurs variables et analyse de l’erreur » (30 mn)

Извршни Одбор САНУ
Одељење za математику, физику и гео-науке САНУ

Предмет: **Боравак у Мароку**

На позив Université de Pau et des Pays de l'Adour (PAU, France) **и** Université Hassan 1st Settat (Ecole Superieure de Technologie, Berrechid, Morocco) **боравио сам у Мароку од 9. до 13. септембра.**

1) Том приликом одржао сам пленарно предавање на Workshop-u “**Effective Numerical Methods for Decision Support**” под насловом: *Summation and Quadrature Processes for Slowly Convergent Series*. Скуп је одржан поводом 60-тог рођендана др **Алал Гесаба** (**Allal Guessab**), професора са Université de Pau из Француске, који је пореклом из Марока. Иначе, давне 1988. године учествовао сам у Комисији за одбрану његове докторске дисертације (државни докторат) на поменутом универзитету у Француској.

2) Учествовао у Комисији за одбрану докторске дисертације кандидата **Yassine ZAIM** под насловом: *Approximation by enriched conforming and nonconforming finite elements,* у оквиру заједничких докторских студија поменутих универзитета из Француске и Марока.

Трошкови боравка су покривени од стране домаћина, а повратне авио карте на релацији Београд-Франкфурт-Казаблака обезбедио је француски универзитет.

22. септембар 2017. **Редовни члан САНУ**

Градимир В. Миловановић

Учесници скупа “**Effective Numerical Methods for Decision Support**”

Предавање Г. В. Миловановића

THÈSE

présentée pour obtenir le grade de

docteur de l'université hassan premier de settat
et
de l'université de pau et des pays de l'adour

Spécialité : Mathématiques appliquées

par

Yassine ZAIM

Approximation par éléments finis conformes et non conformes enrichis
Approximation by enriched conforming and nonconforming finite elements

soutenue publiquement le 11 Septembre 2017

Devant la commission d'examen composée de :

M.	ACHCHAB Boujemâa	Professeur à l'Université Hassan Premier de Settat	Directeur
M.	GUESSAB Allal	Professeur à l'Université de Pau et des Pays de l'Adour	Directeur
Mme	LÓPEZ DE SILANES María Cruz	Professeur à l'Universidad de Zaragoza	Rapporteur
M.	SCHMEISSER Gerhard	Professeur à Friedrich-Alexander-University of Erlangen-Nürnberg	Rapporteur
M.	TOUHAMI Ahmed	Maitre de conférences HDR à l'Université Hassan Premier de Settat	Rapporteur
Mme	ABOULAICH Rajae	Professeur à l'Université Mohammed V de Rabat	Présidente
M.	BECKER Roland	Professeur à l'Université de Pau et des Pays de l'Adour	Examinateur
M.	CARDENAS-MORALES Daniel	Professeur à l'Université de Jaen, Espagne	Examinateur
M.	SOUISSI Ali	Professeur à l'Université Mohammed V de Rabat	Examinateur
M.	V. MILOVANOVIĆ Gradimir	Professeur à Serbian Academy of Sciences and Arts, Serbia	Invité

laboratoire d'analyse et modélisation des systèmes pour l'aide à la décision (LAMSAD)
et le laboratoire de mathématiques et de leurs applications de pau (LMAP)

Чланови Комисије за одбрану докторске дисертације (слева на десно): R. Aboulaich, G. V. Milovanović, G. Schmeisser, A. Souissi, R. Becker, B. Achchab, A. Guessab

Са колегом Герхардом (G. Schmeisser) из Немачке приликом посете највећој џамији у Африци (Grande Mosquee Hassan II, Casablanca)

3.3. Конференција ACTA 2017 у Србији

Тродневна конференција **ACTA 2017: АПРОКСИМАЦИЈЕ И ИЗРАЧУНА-ВАЊА – ТЕОРИЈА И ПРИМЕНЕ** (**APPROXIMATION AND COMPUTATION – THEORY AND APPLICATIONS**), посвећена Волетру Гаучију (Walter Gautschi), професору емеритусу на **Purdue** универзитету (Индијана, САД), поводом његовог 90-тог рођендана, одржана је од 30. новембра до 2. децембра 2017. године у САНУ и Машинском факултетету Универзитета у Београду са око шездесет учесника из 12 земаља.

У даљем тексту наводе се сви детаљи почев од Предлога за организацију конференције ACTA 2017, списка суорганизатора конференције, састава Научног и Организационог одбора, списка пленарних предавача, Извештаја о одржаној конференцији, сајта и постера конференције, програма конференције, као и низ фотографија са отварања конференције у САНУ и током наредна два дана на Машинском факултету. Такође, дата је и презентација пленарног предавања "**Walter Gautschi – A Master in Approximation and Computation**", које је одржао академик Градимир В. Миловановић, након отварања конференције у САНУ.

СРПСКА АКАДЕМИЈА НАУКА И УМЕТНОСТИ
Одељење за математику, физику и гео-науке

Предмет: Предлог за организацију међународне конференције

За 2017. годину планира се организација међународне научне конференције под насловом **APPROXIMATION AND COMPUTATION – THEORY AND APPLICATIONS**, која ће бити посвећена Волетру Гаучију (Walter Gautschi), професору емеритусу на **Purdue** универзитету (Индијана, САД), поводом његовог 90-тог рођендана.

Место одржавања конференције:
САНУ и Машински факултет Универзитета у Београду, Београд

Време одржавања конференције:
30. новембар – 2. децембар, 2017.

Организатори конференције:

САНУ (Одељење за математику, физику и гео-науке),
Машински факултет Универзитета у Београду,
Електротехнички факултет Универзитета у Београду,
Природно-математички факултет Универзитета у Нишу,
Природно-математички факултет Универзитета у Крагујевцу,
Природно-математички факултет Универзитета у Новом Саду,
Математички факултет у Београду,
Математички институт САНУ

Научни одбор конференције:

академик Градимир В. Миловановић, председник Одбора,
академик Стеван Пилиповић,
академик Александар Ивић,
проф. Ендре Шили, инострани члан САНУ, Универзитет у Оксфорду,
проф. Миодраг Спалевић, Машински факултет у Београду,
проф. Бошко Јовановић, Математички факултет у Београду,
проф. Клод Брезински (Claude Brezinski, France),
проф. Лотар Рајчел (Lothar Reichel, USA),
проф. Волтер Ван Асе (Walter Van Assche, Belgium),
проф. Франциско Марцелан (Francisco Marcellán, Spain),
проф. Ђовани Монегато (Giovanni Monegato, Italy),
проф. Ђузепе Мастројани (Giuseppe Mastroianni, Italy),
проф. Сотирис Нотарис (Sotirios E. Notaris, Greece).

Организациони одбор конференције:

проф. Миодраг Спалевић, председник Одбора,
проф. Александар Цветковић,
проф. Марија Станић,
проф. Ненад Цакић,
проф. Драган Ђорђевић,
проф. Наташа Крејић,
проф. Зоран Огњановић,
проф. Зорица Станимировић,
проф. Миодраг Матељевић, дописни члан САНУ,
проф. Владимир Ракочевић, дописни члан САНУ,
доцент Даворка Јаснрлић.

Пленарни предавачи (у овом моменту):

Claude Brezinski, France

Martin J. Gander, University of Geneva, Geneva, Switzerland

Francisco Marcellán, Universidad Carlos III de Madrid, Spain

Sotirios Notaris, University of Athens, Greece

Lothar Reichel, Kent State University, Ohio, United States

Walter Van Assche, University of Leuven, Leuven, Belgium

Образложење:

Волтер Гаучи (рођен у Базелу, Швајцарска, 11. децембра 1927) спада у ред водећих светских научника у области нумеричке анализе, специјалних функција и теорије апроксимација и дугогодишњи је професор на **Purdue** универзитету у САД (сада емеритус професор). У већини водећих међународних часописа био је дуго година главни уредник или члан редакције (*Mathematics of Computation* [American Mathematical Society], *Numerische Mathematik* [Springer], *SIAM* часописима, итд.). Члан је редакције нашег најстаријег часописа *Publications de l'Institut Mathématique.* На конгресу српских математичара 2008. године у Новом Саду био је пленарни предавач. Сарадњу са Волтером Гаучијем је отпочео проф. Миловановић 1983. године, управо у време када је Волтер Гаучи почињао са развојем тзв. *конструктивне теорије ортогоналних полинома* и развојем *нумеричких квадратура*. Од свих сарадника са којима је објављивао радове, Гаучи има далеко највећи број радова са Миловановићем. Данас у овим областима *нумеричке анализе* и *теорије апроксимација* успешно ради десетак математичара у Србији, на факултетима Универзитета у Београду, Крагујевцу и Нишу, што је и разлог да се као суорганизатори ове конференције појављују и ти факултети.

Тема конференције обухвата претходно поменуте области, али је и знатно шира и представља тренутно актуелну проблематику у истраживањима у свету, а која су заступљена и у текућим пројектима Министарства просвете, науке и технолошког развоја Републике Србије.

На конференцији се очекује велики број учесника из земље и иностранства. Предвиђено је да се први дан конфренције (30. новембар) одвија у Свечаној сали на другом спрату Палате САНУ, са ручком у Клубу САНУ. Друга два дана конференције би се одвијала у просторијама Машинског факултета у Београду. Остали детаљи биће дати накнадно.

14. децембар 2016. Академик Градимир В. Миловановић

Извршни Одбор САНУ
Одељење zа математику, физику и гео-науке САНУ

Предмет: **Извештај о одржаној конференцији АСТА 2017**

У организацији САНУ од 30. новембра до 2. децембра 2017. године одржана је међународна конференција под насловом **АПРОКСИМАЦИЈЕ И ИЗРАЧУНАВАЊА – ТЕОРИЈА И ПРИМЕНЕ (APPROXIMATION AND COMPUTATION – THEORY AND APPROXIMATION**), која је била посвећена проф. **Валтер Гаучију** (**Walter Gautschi**), једном од оснивача модерне нумеричке анализе, поводом његовог 90-тог рођендана. Волтер Гаучи је рођен у Базелу у Шварцајској 11. децембра 1927. године, дугогодишњи је професор, а сада емеритус професор, на Парду универзитету (Purdue University) у САД. Један је од водећих научника у области нумеричке анализе, специјалних функција и теорије апроксимација. Више од 30 година активно сарађује са математичарима у Србији.

Коорганизатори конференције су били Математички институт САНУ, природно-математички факултети у Нишу, Новом Саду и Крагујевцу, као и Математички факултет, Машински и Електротехнички факултет Универзитета у Београду.

У раду конференције било је 5 пленарних предавача и 60 учесника из 12 земаља (Белгија, Босна и Херцеговина, Грчка, Италија, Иран, Израел, Иран, Пољска, Шпанија, Швајцарска, САД , Србија, Турска).

22. децембар 2017. **Редовни члан САНУ**

Градимир В. Миловановић

ACTA 2017: APPROXIMATION AND COMPUTATION – THEORY AND APPLICATIONS

Overview

Dedicated to Professor Walter Gautschi on the Occasion of his 90th Anniversary

Belgrade, November 30 – December 2, 2017

The international conference APPROXIMATION AND COMPUTATION – THEORY AND APPLICATIONS (ACTA 2017) will be held in **Belgrade, Serbia** on **November 30 - December 2, 2017**. The event will be held over three days, with presentations delivered by researchers from the international community, including presentations from keynote speakers and state-of-the-art lectures. The aim of the conference is to bring together leading scientists of the international Numerical and Applied Mathematics community and young researchers from all over the world working in mathematics and its applications to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other.

The conference is dedicated to the renowned mathematician Walter Gautschi, one of the founders of modern numerical analysis. Walter Gautschi, born in Basel, Switzerland, 11 December 1927, is one of the the world leading scientists in the field of numerical analysis, special functions and approximation theory and a longtime professor at Purdue University (now emeritus professor).

The topics to be covered include (but are not limited to): All the research areas of Numerical Analysis and Computational Mathematics and all the research areas of Applied Mathematics.

Topics

- Polynomials and Orthogonal Systems
- Numerical Integration (Quadrature and Cubature formulae)
- Approximation Theory
- Scientific Computing
- Applied Mathematics

The session's organizer is responsible for the selection of papers and the submission of full papers for the conference proceedings.

ACTA 2017: APPROXIMATION AND COMPUTATION – THEORY AND APPLICATIONS

Conference poster

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ACTA 2017: APPROXIMATION AND COMPUTATION – THEORY AND APPLICATIONS

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PROGRAM

Days: Thursday, November 30th Friday, December 1st Saturday, December 2nd

Thursday, November 30th, 2017

View this program: with abstracts session overview talk overview

09:00-10:00 Session : Registration

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

10:00-10:30 Session : Opening Ceremony

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

10:30-11:30 Session A

CHAIRS: *Gradimir Milovanović*, *Lothar Reichel* and *Miodrag Spalević*

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

10:30 *Gradimir Milovanović*
WALTER GAUTSCHI - A Master in Approximation and Computation (abstract)

11:00 *Walter Gautschi*
Progress by Accident: Some Reflections on my Career (abstract)

11:30-12:00 Coffee Break

12:00-12:30 Session B

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

12:00 *Miodrag Spalević*
Walter Gautschi and Serbian School of Numerical Integration (abstract)

12:30-13:15 Session C

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

12:30 *Walter Van Assche*
Multiple Hermite polynomials and simultaneous quadrature (abstract)

13:30-15:00 Lunch Break

15:00-16:30 Session D

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

15:00 *Martin Gander*
Five Decades of Time Parallel Time Integration: Best Current Methods for Parabolic and Hyperbolic Problems (abstract)

15:45 *Francisco Marcellan*, *Cleonice Bracciali* and *Serhan Varma*
Orthogonal polynomials, Geronimus transformations and quadrature rules (abstract)

16:30-17:00 Coffee Break

17:00-18:30 Session E

LOCATION: Main hall 2nd floor (Serbian Academy of Sciences and Arts)

17:00 *Sotirios Notaris*
Gauss-Kronrod quadrature: Recent advances and open questions (abstract)

17:45 *Lothar Reichel*, *Hessah Alqahtani* and *Miroslav Pranic*
Generalized Anti-Gauss-Type Quadrature Rules (abstract)

Friday, December 1st, 2017

View this program: with abstracts session overview talk overview

09:00-11:00 Session A1

CHAIR: *Lothar Reichel*
LOCATION: 514 (Faculty of Mechanical Engineering)

09:00 *Maria Carmela De Bonis* and *Donatella Occorsio*
A product integration rule for hypersingular integrals on the positive semi-axis (abstract **)**

09:30 *Miroslav Pranic*, *Stefano Pozza* and *Zdenek Strakos*
Gauss quadrature and incurable breakdown in the Lanczos algorithm (abstract **)**

10:00 *Giuseppe Mastroianni* and *Incoronata Notarangelo*
Polynomial approximation of functions with exponential monotonicity (abstract **)**

10:30 *Giuseppe Mastroianni*, *Gradimir V. Milovanovic* and *Incoronata Notarangelo*
A Nyström method for Fredholm integral equations with exponential weights on (0,+∞) (abstract **)**

11:00-11:30 Coffee Break

11:30-13:30 Session A2

CHAIR: *Sotirios Notaris*
LOCATION: 514 (Faculty of Mechanical Engineering)

11:30 *Walter Gautschi* and *Gradimir Milovanovic*
Binet-type polynomials and their zeros (abstract **)**

12:00 *Yilmaz Simsek*
Generating functions for some special polynomials including Poisson-Charlier, Hermite type, Milne-Thomson type and the other polynomials (abstract **)**

12:30 *Maria Carmela De Bonis* and *Donatella Occorsio*
On a quadrature method for Prandtl's integro-differential equations in weighted Zygmund spaces with uniform norm (abstract **)**

13:00 *Maria Carmela De Bonis* and *Concetta Laurita*
On the stability of a modified Nyström method for Mellin convolution equations (abstract **)**

13:30-15:00 Lunch Break

15:00-16:30 Session A3

CHAIR: *Francisco Marcellan*
LOCATION: 514 (Faculty of Mechanical Engineering)

15:00 *Mahmoud Behroozifar*
Pseudospectral method for time-fractional differential equation with boundary conditions (abstract **)**

15:30 *Ramon Orive*
Minimax Approximation and Probability. Estimating the parameter of a biased coin (abstract **)**

16:00 *Dora Selesi*
Approximation of generalized stochastic processes (abstract **)**

15:00-16:30 Session B3

CHAIR: *Katica R Stevanovic Hedrih*
LOCATION: 513 (Faculty of Mechanical Engineering)

15:00 *Irem Kucukoglu* and *Yilmaz Simsek*
Numerical evaluations on power series including the numbers of Lyndon words and interpolation functions for the Apostol-type polynomials (abstract **)**

15:30 *Svetislav Savovic*, *Branko Drljaca* and *Alexandar Djordjevich*
Unconditionally positive finite difference and standard finite difference schemes for advection–diffusion reaction equations (abstract **)**

16:00 *Svetislav Savovic*, *Branko Drljaca* and *Alexandar Djordjevich*
Numerical solution of one-dimensional advection-diffusion equation with constant and periodic boundary conditions (abstract **)**

16:30-17:00 Coffee Break

17:00-19:00 Session A4

CHAIR: *Donatella Occorsio*
LOCATION: 514 (Faculty of Mechanical Engineering)

17:00 *Agnieszka Prusinska* and *Alexey Tretyakov*

P-regular nonlinear optimization -- calculus and methods (abstract)

17:30 *Predrag Stanimirovic*, *Marko Petkovic* and *Miroslav Ciric*
RNN solution of linear matrix equation and its applications (abstract)

18:00 *Tuğba Bostancı* and *Gülen Başcanbaz-Tunca*
An Extension of Stancu Operator (abstract)

18:30 *Snezana S. Djordjevic*
Analysis of a class of conjugate gradient methods (abstract)

17:00-19:00 Session B4

CHAIR: *Bilge Peker*
LOCATION: 513 (Faculty of Mechanical Engineering)

17:00 *Zorica Milovanović Jeknić*
Parabolic-Hyperbolic Transmission Problem in Disjoint Domains (abstract)

17:30 *Bratislav Sredojević* and *Dejan Bojović*
Finite difference method for the 2D heat equation with concentrated capacity (abstract)

18:00 *Katica R Stevanovic Hedrih*
Approximations in an investigation of the vibro-impact dynamics of rolling bodies in successive central collisions on curvilinear trace (abstract)

18:30 *Haldun Alpaslan Peker*
A Semi-Analytical Approach to Solve a Flow Model (abstract)

Saturday, December 2nd, 2017

View this program: with abstracts session overview talk overview

09:00-11:00 Session AA1

CHAIR: *Miroslav Pranic*
LOCATION: 514 (Faculty of Mechanical Engineering)

09:00 *Aleksandar Jovanović*, *Marija Stanić* and *Tatjana Tomović*
Construction of the optimal set of quadrature rules in the sense of Borges (abstract)

09:30 *Dušan Đukić*, *Lothar Reichel* and *Miodrag Spalević*
Internality of truncated generalized averaged Gaussian quadratures (abstract)

10:00 *Davorka Jandrlić*, *Miodrag Spalević* and *Jelena Tomanović*
Error Estimates for Certain Cubature Formulae (abstract)

10:30 *Rada Mutavdžić* and *Aleksandar Pejčev*
Error bounds for Kronrod extension of generalizations of Micchelli-Rivlin quadrature formula for analytic functions (abstract)

09:00-11:00 Session BB1

CHAIR: *Haldun Alpaslan Peker*
LOCATION: 513 (Faculty of Mechanical Engineering)

09:00 *Bilge Peker*
On the Use of Continued Fractions to Solve Binary Quadratic Diophantine Equations (abstract)

09:30 *Nenad Cakić* and *Ivana Jovović*
On generalized Whitney numbers (abstract)

10:00 *Rabia Aktas* and *Fatma Tasdelen*
Miscellaneous Properties for a Class of Analytic Functions Defined by Rodrigues Type Formula (abstract)

10:30 *Dragan Pavlović*, *Gradimir Milovanović* and *Jovan Cvetić*
Calculation of the channel discharge function for the generalized lightning traveling current source return stroke model (abstract)

11:00-11:30 Coffee Break

11:30-13:30 Session AA2

CHAIR: *Maria Carmela De Bonis*
LOCATION: 514 (Faculty of Mechanical Engineering)

11:30 *Miodrag Mateljević*
Interior estimate for elliptic PDE and distortion of quasiconformal harmonic mappings (abstract)

12:00 *Zoran Ovcin*, *Nataša Krejić* and *Nataša Krklec Jerinkić*
Stochastic Approximation Method with Second Order Search Directions (abstract **)**

12:30 *Ljubica Mihić*, *Aleksandar Pejčev* and *Miodrag Spalević*
Error estimations of Turan formulas with Gori-Micchelli and generalized Chebyshev weight functions (abstract **)**

13:00 *Zoran Vidović*
Bayesian prediction of order statistics based on record values from generalized exponential distribution (abstract **)**

11:30-13:30 Session BB2

CHAIR: *Dejan Bojović*
LOCATION: 513 (Faculty of Mechanical Engineering)

11:30 *Milan Dotlić*, *Boris Pokorni*, *Milenko Pušić* and *Milan Dimkić*
Non-linear multi-point flux approximation in the near-well region (abstract **)**

12:00 *Bojan Banjac*, *Tatjana Lutovac* and *Branko Malešević*
One method for proving some classes of analytical inequalities (abstract **)**

12:30 *Ljubica Vujovic* and *Zeljko Djurovic*
Application of machine learning algorithms to high frequency trading (abstract **)**

13:00 *Zoran Pucanović*
Note on right zero divisors in the ring of infinite upper triangular matrices over a field (abstract **)**

13:30-15:00 Lunch Break

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WALTER GAUTSCHI – A Master in Approximation and Computation

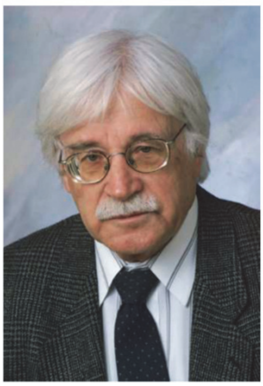
Gradimir V. Milovanović

Serbian Academy of Sciences and Arts, Beograd, Serbia

International Conference ACTA 2017:
APPROXIMATION AND COMPUTATION
Theory and Applications

Belgrade (November 30 – December 2, 2017)

This Conference is Dedicated to Professor Walter GAUTSCHI



On the Occasion of His 90th Birthday

- **Walter Gautschi** is world-renowned scientist in the field in numerical analysis and approximation theory.
- He is one of the founders of modern numerical analysis.
- His research cover a wide range of topics including
  - ordinary differential equations,
  - linear difference equations,
  - interpolation and approximation,
  - special functions,
  - orthogonal polynomials,
  - quadrature processes,
  - history of mathematics.

- **Walter Gautschi:**

  Born: December 11, 1927 (Basel, Switzerland)

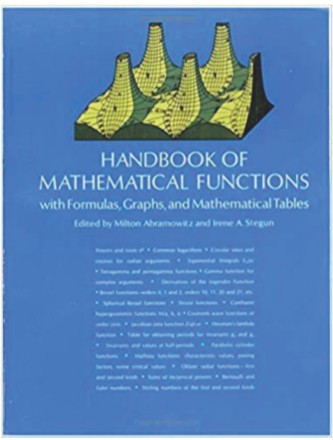
  Schools: Primary and secondary in Basel graduating in 1947

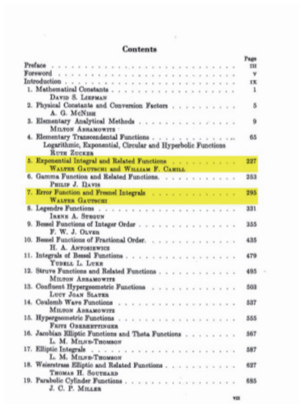
  University of Basel: – *Primary subject:* mathematics; – *Secondary subjects:* physics, physical chemistry, and actuarial mathematics
- **University of Basel** has a long tradition going back to 1460, with the world-class mathematicians (brothers **Jacob** and **Johann Bernoulli**, Johann's sons **Daniel** and **Johann II**, etc.)
- Full professors: **Andreas Speiser** and **Alexander Ostrowski**
- **A. Ostrowski** (b. 1893, Kiev) came to Basel in 1927
- During Walter's studies he improved a graphical method for solving ordinary differential equations, due to **Richard Grammel**

Professor Alexander Markowich Ostrowski (1893–1986)

- By suggestion of **A. Ostrowski**, Gautschi developed methods for analyzing the error of Grammel's method and expanded it into a Ph.D. thesis (1953)
- Beside his thesis work, Walter looked also at numerical methods!
- Walter studied the book **Numerische Behandlung von Differentialgleichungen** (by **Lothar Collatz**) from cover to cover.
- Using **Bieberbach's techniques to Runge-Kutta-Zurmühl methods**, Walter obtained local error bounds for all derivatives of order $< n$.
- These results were published in 1955 in the journal **Zeitschrift für Angewandte Mathematik und Physik (ZAMP)**

- After his Ph.D. exam, Walter received a two-year fellowship for study abroad from a private Swiss foundation in St. Gallen.
- In 1954, he went to Roma for a year as a Research Fellow at **the National Institute for Application and Computation**, founded and directed by **Mauro Picone**.
- He then came to the United States for appointments in three laboratories:
- **Harvard Computation Lab** in 1955,
- **National Bureau of Standards** in 1956,
- **Oak Ridge National Lab** in 1959.
- Experience with electronic computers, programming (in machine code) on (Professor Aiken's) **MARK III** computer.
- Two chapters of the **Handbook of Mathematical Functions**, edited by **Milton Abramowitz** and **Irene A. Stegun**.





Contents

VII

- Abramowitz introduced Walter to the work of **J.C.P. Miller** on his backward recurrence algorithm, which became one of the early areas of emphasis in Walter's research.
- Walter's goal was to find **a stable algorithm for computing a minimal solution of a three-term recurrence relation**.
- Walter worked on this for several years, applying these ideas to **the recursive computation of many special functions** and then published a comprehensive account of this work in 1967.
- As a byproduct of his work on special functions, Walter published inequalities involving ratios of gamma functions.

  This work has a very high citations!
- The work on the Handbook required knowledge of methods for **(numerical) calculating special functions**.
- Seminar on computing special functions at the **Mathematics Center of the University of Wisconsin** (**Dick Askey**).

- During the 1960s, in addition to theoretical work in several domains of special functions, Walter developed a number of computer algorithms for evaluating special functions:

  – the gamma function and incomplete beta function ratios,

  – Bessel functions of the first kind, Legendre functions,

  – regular Coulomb wave functions, the complex error function,

  – repeated integrals of the coerror function, and incomplete gamma functions.

  Oak Ridge National Laboratory (1959-1963)
- Alston Householder's Mathematics Panel.
- During this period he was twice invited to lecture at the **Michigan University Engineering Summer Conferences** then organized by **Robert C. F. Bartels**.
- In 1963, Walter started his permanent academic career at **Purdue University** as Professor of Mathematics and Computer Science.

- Through contacts with chemists during his stay in the **Oak Ridge National Laboratory**, he became interested in the **numerical aspects of Gaussian quadrature and orthogonal polynomials**.
- **Later, this become one of the principal areas of Walter's research!**
- In about two dozen papers, Walter Gautschi developed the so-called **constructive theory of orthogonal polynomials on the real line**

  – effective algorithms for numerically generating orthogonal polynomials with respect to an arbitrary measure,

  – a rigorous and detailed stability analysis of algorithms,

  – new applications of orthogonal polynomials.
- **Methods for constructing OP**:

  – Method of (modified) moments,

  – Discretized Stieltjes-Gautschi procedure,

  – Lanczos algorithm.

- Walter's work and his contributions in the **constructive theory of orthogonal polynomials** allow the construction of many **new classes of polynomials** and **their application** in diverse areas of applied and numerical analysis, e.g.,

  – numerical integration,

  – interpolation processes,

  – integral equations,

  – probability,

  – moment-preserving spline approximation,

  – summation of slowly convergent series,

  – approximation theory, e.t.c.
- General algorithms for **modifications of the measures** by linear and quadratic factors and divisors.
- **Software:**

  – ORTHPOL (1994) in FORTRAN;

  – OPQ and SOPQ (Matlab routines);

Teaching Activities:

- Walter regularly taught the beginning graduate course on **Numerical Analysis**, an advanced course on the **Numerical solution of ordinary differential equations**, and occasionally courses on **Numerical linear algebra** and **Optimization**.
- He had 8 Ph.D. students.
- Walter officially retired from Purdue University in 2000 with the title of **Professor Emeritus**, but both his research and lecturing activities continued ever since!

  European Academies: In 2001, he was elected

  – Foreign Member of the **Bavarian Academy of Sciences** in Munich

  – Corresponding Member of the **Turin Academy of Sciences**
- He was also named a **SIAM Fellow** in 2012.

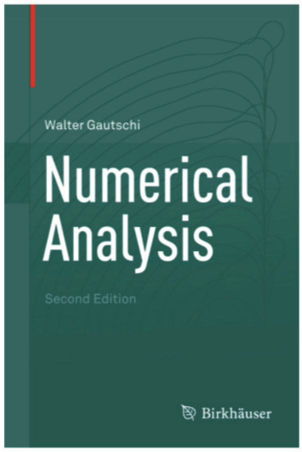
Publications:

- Walter has published 4 books, 34 book chapters, 170 refereed journal papers, 7 refereed papers in conference proceedings, translated 3 books, and edited 5 conference proceedings.
- He wrote 279 reviews for **Mathematical Reviews**.
- His books have set a high standard for graduate textbooks in their respective subjects.

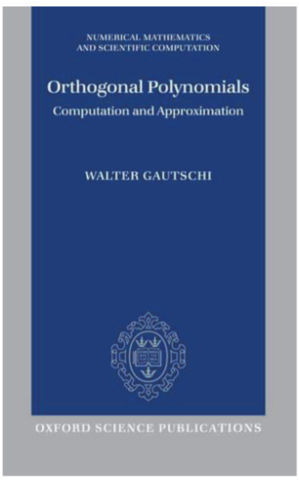
  Numerical analysis – an introduction, published by Birkhäuser (1997; 2012)

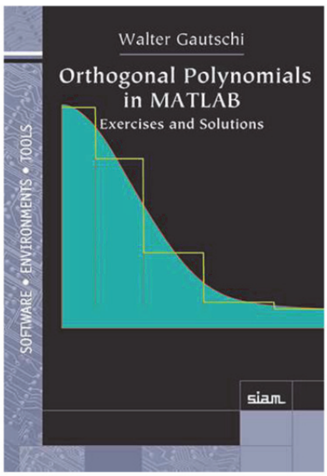
  Orthogonal polynomials – computation and approximation, published by Oxford University Press (2004);

  Orthogonal Polynomials in MATLAB – Exercises and Solutions, published by SIAM (2016).



Gradimir V. Milovanović, gvm@mi.sanu.ac.rs





Gradimir V. Milovanović, gvm@mi.sanu.ac.rs

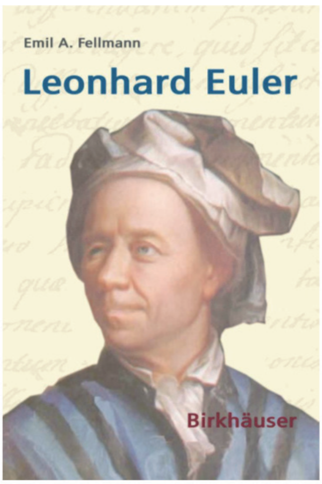
- Walter was also active as a translator (from German), translating

  – (jointly with R. Bartels and C. Witzgall) the text **Einführung in die Numerische Mathematik** by J. Stoer and R. Bulirsch,

  – preparing an annotated translation of H. Rutishauser's **Vorlesungen über numerische Mathematik**, and

  – (jointly with his wife Erika) an English translation of E. A. Fellmann's **Leonhard Euler**.

  Emil Alfred Fellmann (1927–2012) was a Swiss historian of science, which was particularly known for his collaboration in the publication of the works of Leonhard Euler.



Leonhard Euler

Emil A. Fellmann
Translated by Erika Gautschi and Walter Gautschi

Birkhäuser Verlag
Basel · Boston · Berlin

- Walter's contributions have had a significant impact on the field, and his papers are widely cited.
- They are characterized by their clarity of exposition and will remain excellent resources for researchers in the field.
- Throughout his academic career, Walter participated and lectured at numerous national and international meetings.
- He was also a frequent visitor at other academic institutions (Polytechnics of Milan and Turin, University of Padua, ETH in Zurich, University of Basel, etc.)
- Plenary lecture about **Leonhard Euler** (1707–1783) at the **International Congress on Industrial and Applied Mathematics** in Zürich, 2007.

  2007 was the Euler year!

- Walter Gautschi visited Serbia 3 times (1984, 1987, and 2008).
- In 1987 at the international conference **Numerical Methods and Approximation Theory** in Niš, he gave a plenary talk;
- In 2008 also as a plenary lecturer he attended

  – the international conference **Approximation and Computation** at University of Niš,

  – the **12th Serbian Mathematical Congress** in Novi Sad.

During the Conference "Numer. Methods & Approx. Theory"
(Niš, August 18–21, 1987)

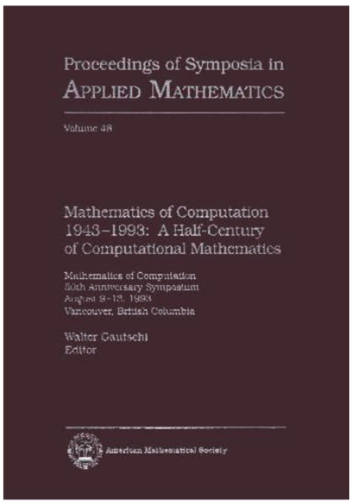
Archaeological site "Iustiniana Prima" (6th century)
(Lebane, South Serbia)

AFTER EXCURSION: They are waiting for dinner
(Lebane, South Serbia)

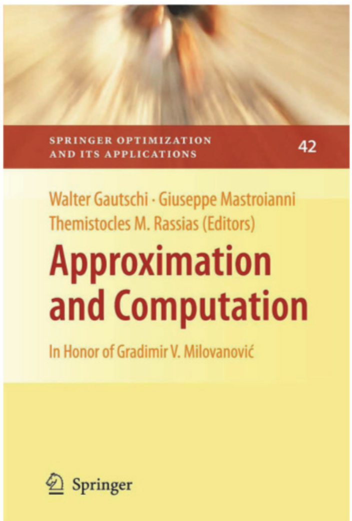
Temple E (Selinunte), Sicily, Italy
During an excursion at OPSFA3, Erice, 1990

Editorial Activities:

- From 1966 to 1999, Walter was a member of the Editorial Committee of **Mathematics of Computation** and its **Managing Editor** from 1984 to 1995.
- On the 50th anniversary of **Mathematics of Computation**, Walter edited an AMS proceedings volume entitled **A half-century of computational mathematics**.
- Other journals for which he served as an **Associate Editor** are
  – **Numerische Mathematik**, 1971 to the present (**Honorary Editor** since 1991),
  – **SIAM Journal on Mathematical Analysis**, 1970–1973,
  – **Calcolo**, 1975–1987.
- In addition, in 1981–1983, Walter served as a **Special Editor** of **Linear Algebra and its Applications**.
- Gautschi was **co-editor** of a number of other proceedings volumes.



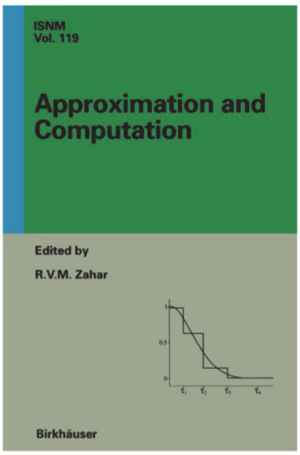
Gradimir V. Milovanović, gvm@mi.sanu.ac.rs



- **Walter's 65th birthday** was celebrated by a conference held in his honor in December 1993 at Purdue University
- The proceedings volume of this conference under title **Approximation and Computation** was published by Birkhaäuser in 1994.

  Editor of this volume was **Ramsay V.M. Zahar** (the first Walter's Ph.D. student)

Purdue Conference (December 3, 1993)



Approximation and Computation:
A Festschrift in Honor of Walter Gautschi
Proceedings of the Purdue Conference, December 2-5, 1993

Edited by
R. V. M. Zahar

Birkhäuser
Boston · Basel · Berlin

In Walter's house, December 1993
(Ramsay V.M. Zahar, Louis de Branges and GVM)

Birkhäuser/Springer Project in 2014
Series: **Contemporary Mathematicians**

Walter Gautschi – Selected Works with Commentaries
(Edited by Claude Brezinski & Ahmed Sameh)

- This set of three volumes includes reprints of many of Gautschi's papers and commentaries on his work by some of his colleagues.
- **List of Contributors:**

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- **Volume 1** (xii+694 pp.) includes a **biography and summary** of Gautschi's work, as well as **commentaries** on his work in

  – numerical conditioning (by Nicholas Higham),
  – special functions (by Javier Segura), and
  – interpolation and approximation (by Miodrag Spalević).

  The main part of this volume are **reprints** of **44 papers** by Walter Gautschi from these three areas.
- **Volume 2** (xiv+914 pp.) includes **commentaries** on his work in

  – orthogonal polynomials on the real line (by Gradimir Milovanović),
  – polynomials orthogonal on the semicircle (by Lothar Reichel),
  – Chebyshev quadrature (by Jacob Korevaar),
  – Kronrod and other quadratures (by Giovanni Monegato), and
  – Gauss-type quadratures (by Walter van Asche).

  The main part are **reprints** of **54 Walter's papers** from these several areas.

- **Volume 3** (xii+767 pp.) includes **commentaries** on Walter's work on
  – linear recurrence relations (by Lisa Lorentzen),
  – ordinary differential equations (by John Butcher),
  – computer algorithms and software packages (by G. Milovanović),
  – history and biography (by Gerhard Wanner), and
  – miscellaneous topics (by Martin J. Gander).

  Again main part of this volume are **reprints** of **38 papers** by Walter Gautschi from these several areas.

  This is an impressive project on more than 2400 pages with 136 reprints on his papers in 13 subjects!

- **Walter Gautschi** is one of the leading numerical analysts of the second half of the 20th and the beginning of the 21st century!
- His scientific work is very wide, ranging from special functions, quadrature and orthogonal polynomials to difference and differential equations, software implementations, and the history of mathematics!
- He has contributed to almost all areas of numerical analysis, and many of his results have proven to be highly significant and timeless!

Отварање конференције (САНУ, 30. новембар 2017. године)

Поздравни говор академика Зорана Поповића, потпредседника САНУ

У првом реду у САНУ: проф. Виктор Недовић, представник Министарства просвете, науке и технолошког развоја, академик Зоран Поповић, потпредседник САНУ, проф. Волтер Гаучи (Walter Gautschi) и проф. Франциско Марцелан (Francisco Marcellán)

Радно Председништо: Л. Рајчел (L. Reichel), Г.В. Миловановић и М.М. Спалевић


На Машинском факултету (1. децембар): Y. Simsek, W. Gautschi, GVM i G. Mastroianni

Групна фотографија учесника конференције

Волтер и Ерика Гаучи на Машинском факултету са ГВМ

2. КОПИЈЕ ОБЈАВЉЕНИХ РАДОВА У 2017. ГОДИНИ

G.V. Milovanović: *Generalized weighted Birkhoff-Young quadratures with the maximal degree of exactness,* Appl. Numer. Math. **116** (2017), 238 – 255. [RJ234]

Applied Numerical Mathematics 116 (2017) 238–255

Contents lists available at ScienceDirect

Applied Numerical Mathematics

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Generalized weighted Birkhoff–Young quadratures with the maximal degree of exactness [☆]

Gradimir V. Milovanović [a,b,*]

[a] Serbian Academy of Sciences and Arts, Beograd, Serbia
[b] State University of Novi Pazar, Serbia

ARTICLE INFO

Article history:
Available online 12 July 2016

Dedicated to Professor Francesco A. Costabile on his 70th birthday

Keywords:
Quadrature formula
Weight function
Error term
Orthogonality
Analytic function
Nodes
Weight coefficients
Multiple orthogonal polynomial

ABSTRACT

Several types of quadratures of Birkhoff–Young type, as well as a sequence of the weighted generalized quadrature rules and their connection with multiple orthogonal polynomials, are considered. Beside a short account on a recent result on the generalized $(4n+1)$-point Birkhoff–Young quadrature, general weighted quadrature formulas of Birkhoff–Young type with the maximal degree of exactness are given. It includes a characterization and uniqueness of such rules, as well as numerical construction of nodes and weight coefficients. An explicit form of the node polynomial of such kind of quadratures with respect to the generalized Gegenbauer weight function is obtained. Finally, a sequence of generalized quadrature formulas is studied and their node polynomials are interpreted in terms of multiple orthogonal polynomials.

1. Introduction and preliminaries

The well-known quadrature formula for numerical integration over the line segment $[z_0-h, z_0+h]$ of analytic functions in the complex domain $\Omega = \{z : |z-z_0| \le r\}$, $|h| \le r$,

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z = \frac{h}{15}\Big\{24 f(z_0) + 4\big[f(z_0+h)+f(z_0-h)\big] - \big[f(z_0+\mathrm{i}h)+f(z_0-\mathrm{i}h)\big]\Big\} + R_5^{BY}(f)$$

was obtained by Birkhoff and Young [5], and it is exact for all algebraic polynomials of degree at most five. Young [32] proved that its error term can be estimated by

$$|R_5^{BY}(f)| \le \frac{|h|^7}{1890} \max_{z\in S} |f^{(6)}(z)|,$$

where S denotes the square with vertices $z_0 + \mathrm{i}^k h$, $k = 0, 1, 2, 3$ (see also the monograph [6, p. 136]). Birkhoff–Young rule can be compared with the so-called extended Simpson rule (cf. [28, p. 124]) with the nodes z_0, $z_0 \pm h$, $z_0 \pm 2h$, and

☆ This paper was supported by the Serbian Ministry of Education, Science and Technological Development (No. #OI 174015).
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http://dx.doi.org/10.1016/j.apnum.2016.06.012
0168-9274/

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z \approx \frac{h}{90}\Big\{114 f(z_0)+34\Big[f(z_0+h)+f(z_0-h)\Big]-\Big[f(z_0+2h)+f(z_0-2h)\Big]\Big\}+R_5^{ES}(f),$$

where

$$|R_5^{ES}(f)| \sim \frac{|h|^7}{756}|f^{(6)}(\zeta)|, \qquad 0<\frac{\zeta-(z_0-2h)}{4h}<1.$$

Both formulas use $N=5$ points and have the same algebraic degree of exactness $d=5$, but $|R_5^{BY}(f)| \approx 0.4\,|R_5^{ES}(f)|$.

In 1976 Lether [10] transformed Birkhoff–Young formula from $[z_0-h, z_0+h]$ to $[-1,1]$ (of course, without loss of generality),

$$I(f)=\int_{-1}^{1} f(z)\,\mathrm{d}z=\frac{8}{5}f(0)+\frac{4}{15}\big[f(1)+f(-1)\big]-\frac{1}{15}\big[f(\mathrm{i})+f(-\mathrm{i})\big]+R_5(f), \tag{1.1}$$

and pointed out that the three point Gauss–Legendre quadrature which is also exact for all polynomials of degree at most five, is more precise than (1.1) and he recommended it for numerical integration. However, Tošić [29] improved the quadrature (1.1) in a simple way taking its nodes at the points $\pm r$ and $\pm \mathrm{i}r$, with $r\in(0,1)$, instead of ± 1 and $\pm\mathrm{i}$, respectively, and derived an one-parametric family of quadrature rules in the form

$$\begin{aligned} I(f) &= 2\left(1-\frac{1}{5r^4}\right)f(0)+\left(\frac{1}{6r^2}+\frac{1}{10r^4}\right)\big[f(r)+f(-r)\big] \\ &\quad +\left(-\frac{1}{6r^2}+\frac{1}{10r^4}\right)\big[f(\mathrm{i}r)+f(-\mathrm{i}r)\big]+R_5^T(f;r). \end{aligned} \tag{1.2}$$

It is clear that for $r=1$ it reduces to (1.1). However, for $r=\sqrt{3/5}$, the coefficient of $f(\mathrm{i}r)+f(-\mathrm{i}r)$ vanishes, and it reduces to the three point Gauss–Legendre formula,

$$I(f)=\frac{8}{9}f(0)+\frac{5}{9}\left[f\left(\sqrt{\frac{3}{5}}\right)+f\left(-\sqrt{\frac{3}{5}}\right)\right]+R_3^G(f), \tag{1.3}$$

where $R_3^G(f)=R_5^T(f;\sqrt{3/5})$.

Expanding the error-term $R_5^T(f;r)$ in (1.2) in the form

$$R_5^T(f;r)=\left(-\frac{2}{3\cdot 6!}r^4+\frac{2}{7!}\right)f^{(6)}(0)+\left(-\frac{2}{5\cdot 8!}r^4+\frac{2}{9!}\right)f^{(8)}(0)+\cdots, \tag{1.4}$$

and putting $r=\sqrt[4]{3/7}$ in order to vanish the first term in (1.4), Tošić [29] obtained a five-point formula of algebraic degree of exactness seven,

$$\begin{aligned} I(f) &= \frac{16}{15}f(0)+\frac{1}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right)\left[f\left(\sqrt[4]{\frac{3}{7}}\right)+f\left(-\sqrt[4]{\frac{3}{7}}\right)\right] \\ &\quad +\frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\left[f\left(\mathrm{i}\sqrt[4]{\frac{3}{7}}\right)+f\left(-\mathrm{i}\sqrt[4]{\frac{3}{7}}\right)\right]+R_5^{MF}(f), \end{aligned} \tag{1.5}$$

with the error-term

$$R_5^{MF}(f)=R_5^T(f;\sqrt[4]{3/7})=\frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots \approx 1.26\cdot 10^{-6}f^{(8)}(0).$$

We note that the error term in the Gaussian formula (1.3) is given by

$$R_3^G(f)=R_5^T(f;\sqrt{3/5})=\frac{1}{15750}f^{(6)}(0)-\frac{1}{226800}f^{(8)}(0)+\cdots \approx 6.35\cdot 10^{-5}f^{(6)}(0).$$

Quadrature formulae of Birkhoff–Young type for analytic functions have been investigated in several papers in different directions (cf. [1,15,20,22]). These formulas can also be used to integrate real harmonic functions (see [5]). In addition, we mention also that Lyness and Delves [12] and Lyness and Moler [13], and later Lyness [11], developed formulae for numerical integration and numerical differentiation of complex functions.

An extra motivation for a development of this kind of quadratures lies in the possible application of such rules in a construction of orthogonal polynomials on the radial rays in the complex plane (cf. [16,24,25,17,18]). In order to explain this fact we consider a case of these polynomials on the four rays, with the inner product defined by

$$(f,g)=\int_0^1\Big[f(z)\overline{g(z)}+f(\mathrm{i}z)\overline{g(\mathrm{i}z)}+f(-z)\overline{g(-z)}+f(-\mathrm{i}z)\overline{g(-\mathrm{i}z)}\Big]w(z)\,\mathrm{d}z,$$

where w is a given weight function. Extending w to an even function on $(-1,1)$ (again denoted as w), this inner product can be expressed in the form

$$(f,g)=\int_{-1}^1\Big[f(z)\overline{g(z)}+f(\mathrm{i}z)\overline{g(\mathrm{i}z)}\Big]\,w(z)\,\mathrm{d}z.$$

For the numerical construction of recursive coefficients for these orthogonal polynomials on the radial rays, by using the discretized Stieltjes–Gautschi procedure (see [18]), we need a quadrature rule for exactly computing integrals of the form $(f,1)$, when f is an algebraic polynomial (except for rounding errors). Weighted quadratures of Birkhoff–Young type (of sufficiently large degree of exactness) would be very appropriate for this kind of integration, since their nodes are on the real and the imaginary axis. As a simple illustration of this fact we can see that the rule (1.5), with $w(x)=1$, gives the following formula

$$\int_{-1}^1\big[f(z)+f(\mathrm{i}z)\big]\,\mathrm{d}z\approx\frac{32}{15}f(0)+\frac{7}{15}\left[f\left(\sqrt[4]{\frac{3}{7}}\right)+f\left(-\sqrt[4]{\frac{3}{7}}\right)+f\left(\mathrm{i}\sqrt[4]{\frac{3}{7}}\right)+f\left(-\mathrm{i}\sqrt[4]{\frac{3}{7}}\right)\right]$$

of algebraic degree of exactness seven. On the other hand, the corresponding formula of the same complexity, obtained by the Gaussian rule (1.3),

$$\int_{-1}^1\big[f(z)+f(\mathrm{i}z)\big]\,\mathrm{d}z\approx\frac{16}{9}f(0)+\frac{5}{9}\left[f\left(\sqrt{\frac{3}{5}}\right)+f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\mathrm{i}\sqrt{\frac{3}{5}}\right)+f\left(-\mathrm{i}\sqrt{\frac{3}{5}}\right)\right],$$

has the algebraic degree of exactness five.

In this paper we consider several types of quadrature rules of Birkhoff–Young type, as well as a sequence of the weighted generalized quadratures and their connection with multiple orthogonal polynomials. The paper is organized as follows. In Section 2 we gave a short account of a recent result on a generalized $(4n+1)$-point Birkhoff–Young quadrature. Section 3 is devoted to general weighted quadrature formulas of Birkhoff–Young type with the maximal degree of exactness, including a characterization and the uniqueness of such rules as well as the numerical construction of the nodes and the weights of the rule. In the case of the generalized Gegenbauer weight function, explicit form of the node polynomials is derived. Finally, a sequence of generalized quadrature formulas and their connection to multiple orthogonal polynomials are presented in Section 4.

2. Generalized $(4n+1)$-point Birkhoff–Young quadrature

In 1982 Milovanović and Đorđević [23] extended Tošić's formula (1.2) to the following nine-point quadrature rule of interpolatory type

$$\begin{aligned}I(f)&=Af(0)+B\big[f(x_1)+f(-x_1)\big]+C\big[f(\mathrm{i}x_1)+f(-\mathrm{i}x_1)\big]\\&\quad+D\big[f(x_2)+f(-x_2)\big]+E\big[f(\mathrm{i}x_2)+f(-\mathrm{i}x_2)\big]+R_9(f;x_1,x_2),\end{aligned}\tag{2.1}$$

with $0<x_1<x_2<1$. Taking

$$x_1=x_1^*=\sqrt[4]{\frac{63-4\sqrt{114}}{143}}\quad\text{and}\quad x_2=x_2^*=\sqrt[4]{\frac{63+4\sqrt{114}}{143}},\tag{2.2}$$

this formula has the algebraic degree of exactness $d=13$, with the error-term

$$R_9(f;x_1^*,x_2^*)\approx 3.56\cdot 10^{-14}f^{(14)}(0).$$

Recently, this result has been extended to the $(4n+1)$-point interpolatory quadrature formula of the form [21]

$$I(f):=\int_{-1}^1 f(z)\,\mathrm{d}z=Q_{4n+1}(f)+R_{4n+1}(f),\tag{2.3}$$

where

$$Q_{4n+1}(f) = A_0 f(0) + \sum_{k=1}^{n} \Big\{ A_k\big[f(x_k) + f(-x_k)\big] + B_k\big[f(\mathrm{i}x_k) + f(-\mathrm{i}x_k)\big] \Big\} \tag{2.4}$$

and $R_{4n+1}(f)$ is the corresponding remainder term. The nodes in (2.4) are connected with the zeros of a monic polynomial of degree $4n+1$,

$$\omega_{4n+1}(z) = z \sum_{j=0}^{n} a_j z^{4j} = z \prod_{k=1}^{n} (z^4 - r_k), \quad 0 < r_1 < \cdots < r_n < 1, \tag{2.5}$$

i.e., $x_k = \sqrt[4]{r_k}$, $k = 1, \ldots, n$. In [21] it has been proved that there exists a unique interpolatory quadrature formula of the form (2.4) with the maximal degree of exactness $d_{\max} = 6n+1$, and that the respective coefficients a_j in (2.5) are given by

$$a_j = (-1)^{n-j} \binom{n}{j} \frac{\left(2j + \frac{3}{2}\right)_{2n-2j}}{\left(n + 2j + \frac{3}{2}\right)_{2n-2j}}, \quad j = 0, 1, \ldots, n, \tag{2.6}$$

where $(s)_j$ is the standard notation for Pochhammer's symbol

$$(s)_j = s(s+1)\cdots(s+j-1) = \frac{\Gamma(s+j)}{\Gamma(s)} \quad (\Gamma \text{ is the gamma function}).$$

The weight coefficients A_0 and A_k, B_k, $k = 1, \ldots, n$, in the interpolatory quadrature formula (2.4), can be expressed in the form

$$A_0 = \frac{1}{\widehat{p}_n(0)} \int_{-1}^{1} \widehat{p}_n(z^4)\,\mathrm{d}z,$$

$$A_k = \frac{1}{4r_k \widehat{p}\,'_n(r_k)} \int_{-1}^{1} \frac{z^2 \widehat{p}_n(z^4)}{z^2 - \sqrt{r_k}}\,\mathrm{d}z, \quad B_k = \frac{1}{4r_k \widehat{p}\,'_n(r_k)} \int_{-1}^{1} \frac{z^2 \widehat{p}_n(z^4)}{z^2 + \sqrt{r_k}}\,\mathrm{d}z, \quad k = 1, \ldots, n.$$

The corresponding node polynomials $\widehat{p}_n(z)$ are (see [21]):

$$\widehat{p}_1(z) = z - \frac{3}{7}, \quad \widehat{p}_2(z) = z^2 - \frac{126z}{143} + \frac{15}{143}, \quad \widehat{p}_3(z) = z^3 - \frac{429z^2}{323} + \frac{693z}{1615} - \frac{7}{323},$$

$$\widehat{p}_4(z) = z^4 - \frac{204z^3}{115} + \frac{14586z^2}{15295} - \frac{1716z}{10925} + \frac{9}{2185},$$

$$\widehat{p}_5(z) = z^5 - \frac{1995z^4}{899} + \frac{4522z^3}{2697} - \frac{92378z^2}{186093} + \frac{1001z}{20677} - \frac{77}{103385},$$

$$\widehat{p}_6(z) = z^6 - \frac{690z^5}{259} + \frac{32775z^4}{12617} - \frac{3714500z^3}{3293037} + \frac{20995z^2}{99789} - \frac{442z}{33263} + \frac{13}{99789}.$$

For $n = 1$ and $n = 2$, the previous result reduces to (1.5) and (2.1), with parameters (2.2), respectively.

3. A general weighted quadrature rule of Birkhoff–Young type

We consider now a generalized weighted N-point quadrature formula of interpolatory type for the numerical integration of analytic function,

$$I(w; f) := \int_{-1}^{1} f(z) w(z)\,\mathrm{d}z = Q_N(w; f) + R_N(w; f), \tag{3.1}$$

with respect to an arbitrary even positive function $w : (-1, 1) \to \mathbb{R}^+$, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z)\,\mathrm{d}z$, $k = 0, 1, \ldots,$ exist. Notice that $\mu_{2k+1} = 0$ and $\mu_{2k} > 0$ for each $k \in \mathbb{N}_0$. The quadrature sum $Q_N(w; f)$ has the form

$$Q_N(w; f) = \sum_{j=0}^{\nu-1} C_j^{(\nu)} f^{(j)}(0) + \sum_{k=1}^{n} \Big\{ A_k^{(\nu)}\big[f(x_k) + f(-x_k)\big] + B_k^{(\nu)}\big[f(\mathrm{i}x_k) + f(-\mathrm{i}x_k)\big] \Big\}, \tag{3.2}$$

with nodes at the zeros of a monic polynomial with real coefficients of degree N,

$$\omega_N(z) = z^\nu p_{n,\nu}(z^4) = z^\nu \prod_{k=1}^{n}(z^4 - r_k), \quad 0 < r_1 < \cdots < r_n < 1, \tag{3.3}$$

i.e., $x_k = \sqrt[4]{r_k}$, $k = 1, \ldots, n$, where $N = 4n + \nu$, with

$$n = \left[\frac{N}{4}\right], \quad \nu = N - 4\left[\frac{N}{4}\right] \in \{0, 1, 2, 3\},$$

and $R_N(w; f)$ is the corresponding remainder term. Notice that r_k's in the node polynomial (3.3) are also dependent on ν, but we write only r_k instead of $r_k^{(\nu)}$.

The N-point quadrature formula (3.1)–(3.2) of interpolatory type has degree of exactness at least $N - 1 = 4n + \nu - 1$ for an arbitrary distribution of nodes r_k, $k = 1, \ldots, n$, in (3.3).

If $\nu = 0$, the first sum in $Q_N(w; f)$ is empty. Also, in order to have $Q_N(w; f) = I(w; f) = 0$ for $f(z) = z$, it must be $C_1^{(\nu)} = 0$, so that $Q_{4n+1}(w; f) \equiv Q_{4n+2}(w; f)$.

In the simplest case when $1 \le N \le 3$ ($n = 0$, $N = \nu$), the quadrature sum reduces only to $Q_N(w; f) = Q_\nu(w; f) = \sum_{j=0}^{\nu-1} C_j^{(\nu)} f^{(j)}(0)$, i.e.,

$$Q_1(w; f) = Q_2(w; f) = \mu_0 f(0) \quad \text{and} \quad Q_3(w; f) = \mu_0 f(0) + \frac{\mu_2}{2} f''(0). \tag{3.4}$$

Here, we are interested in weighted quadrature formulae of type (3.2), with a maximal degree of exactness for an arbitrary $N \in \mathbb{N}$. In that case, the corresponding quadrature sum will be denoted by $\widehat{Q}_N(w; f)$.

3.1. Characterization of $\widehat{Q}_N(w; f)$ and its numerical construction

Let $\mathcal{P}$ be the set of all algebraic polynomials with real coefficients (real polynomials) and let $\mathcal{P}_n$ be its subset of degree at most n.

Throughout this paper we assume that $w : (-1, 1) \to \mathbb{R}^+$ is a given even nonnegative function, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z)\,dz$, $k = 0, 1, \ldots,$ exist and $\mu_0 > 0$. Then, the inner product, defined with this weight function as

$$(p, q) = \int_{-1}^{1} p(z)q(z)w(z)\,dz \quad (p, q \in \mathcal{P}), \tag{3.5}$$

gives rise to a unique system of monic real orthogonal polynomials $\pi_k(\cdot) = \pi_k(w; \cdot)$, such that

$$\pi_k(z) \equiv \pi_k(w; z) = z^k + \text{terms of lower degree}, \quad k = 0, 1, \ldots,$$

and

$$(\pi_k, \pi_n) = ||\pi_n||^2 \delta_{kn} = \begin{cases} 0, & n \ne k, \\ ||\pi_n||^2, & n = k, \end{cases}$$

where $||\pi_n||^2 = \int_{-1}^{1} \pi_n(z)^2 w(z)\,dz$. Such monic polynomials (orthogonal with respect to an even weight function) satisfy the following three-term recurrence relation (cf. [14, p. 102])

$$\pi_{k+1}(z) = z\pi_k(z) - \beta_k \pi_{k-1}(z), \quad k = 0, 1, \ldots, \tag{3.6}$$

with $\pi_0(z) = 1$ and $\pi_{-1}(z) = 0$, where $\beta_k > 0$, $k = 1, 2, \ldots$. It is convenient to put $\beta_0 = \mu_0$.

The following theorem gives a characterization of the quadrature formula (3.1)–(3.2) with a maximal degree of exactness.

Theorem 3.1. *For a given weight function $w : (-1, 1) \to \mathbb{R}^+$ and each $N \in \mathbb{N}$ there exists a unique interpolatory quadrature $\widehat{Q}_N(w; f)$, with a maximal degree of exactness $d_{\max} = 6n + s$, where $N = 4n + \nu$, with $n = [N/4]$, $\nu = N - 4[N/4] \in \{0, 1, 2, 3\}$, and*

$$s = \begin{cases} \nu - 1, & \nu = 0, 2, \\ \nu, & \nu = 1, 3. \end{cases} \tag{3.7}$$

The nodes of such a quadrature rule are zeros of the monic real polynomial $\widehat{\omega}_N(z) = z^\nu \widehat{p}_{n,\nu}(z^4)$ which is characterized by the following orthogonality relation

$$(h(z^2), z^{s+1}\widehat{p}_{n,\nu}(z^4)) = \int_{-1}^{1} h(z^2) z^{s+1}\widehat{p}_{n,\nu}(z^4) w(z)\,dz = 0, \quad h \in \mathcal{P}_{n-1}, \tag{3.8}$$

or

$$(\pi_{2k}, z^{s+1}\widehat{p}_{n,\nu}(z^4)) = \int_{-1}^{1} \pi_{2k}(z) z^{s+1}\widehat{p}_{n,\nu}(z^4) w(z)\,dz = 0, \quad 0 \le k < n, \tag{3.9}$$

where $\{\pi_k\}$ is the sequence of monic polynomials orthogonal with respect to the inner product (3.5).

Proof. The case $N \le 3$ is solved by (3.4). Therefore, we suppose that $N \ge 4$ and that the node polynomial for the quadrature formula $Q_N(w; f)$ is given by (3.3).

Suppose that $f \in \mathcal{P}_d$, where $d \ge N = 4n + \nu$ ($n = [N/4]$, $\nu = N - 4[N/4]$). Then, this polynomial can be expressed in the form

$$f(z) = u(z)\omega_N(z) + v(z) = u(z) z^\nu p_{n,\nu}(z^4) + v(z), \tag{3.10}$$

where $u \in \mathcal{P}_{d-N}$ and $v \in \mathcal{P}_{N-1}$. By integrating (3.10) we obtain

$$I(w; f) = \int_{-1}^{1} u(z)\omega_N(z) w(z)\,dz + \int_{-1}^{1} v(z) w(z)\,dz. \tag{3.11}$$

Since $v \in \mathcal{P}_{N-1}$, it is clear that the last integral on the right hand side in (3.11), i.e., $I(w; v)$ can be exactly calculated by using the interpolatory quadrature formula (3.1) (of degree of exactness $N - 1$). Thus, $I(w; v) = Q_N(w; v)$. However, since

$$\omega_N(\mathrm{i}^\mu x_k) = \mathrm{i}^{\nu\mu} x_k^\nu p_{n,\nu}(x_k^4) = \mathrm{i}^{\nu\mu} x_k^\nu p_{n,\nu}(r_k) = 0, \quad 1 \le k \le n; \quad 0 \le \mu \le 3,$$

where $N = 4n + \nu$ and $\nu = N - 4[N/4] \in \{0, 1, 2, 3\}$, and

$$\omega_{4n+\nu}^{(j)}(0) = 0, \quad 0 \le j \le \nu - 1, \quad \nu = 1, 2, 3,$$

according to (3.10), we conclude that

$$f(\mathrm{i}^\mu x_k) = v(\mathrm{i}^\mu x_k), \quad 1 \le k \le n, \quad 0 \le \mu \le 3,$$

and

$$f^{(j)}(0) = v^{(j)}(0), \quad 0 \le j \le \nu - 1, \quad \nu = 1, 2, 3,$$

and therefore, $I(w; v) = Q_N(w; v) = Q_N(w; f)$. Thus, (3.11) reduces to $I(w; f) = (u, \omega_N) + Q_N(w; f)$, where the inner product is given by (3.5).

Now, we can see that the quadrature formula $Q_N(w; f)$ has a maximal degree of exactness *if and only if* $(u, \omega_N) = 0$ for a maximal degree of the polynomial $u \in \mathcal{P}_{d-N}$. Such $Q_N(w; f)$ and $\omega_N(z)$ we denote by $\widehat{Q}_N(w; f)$ and $\widehat{\omega}_N(z)$ $(= z^\nu \widehat{p}_{n,\nu}(z^4))$, respectively, and the previous "orthogonality conditions" can be considered with respect to the values of ν, i.e.,

$$(h(z^2), z^\nu \widehat{p}_{n,\nu}(z^4)) = 0 \quad \text{and} \quad (zh(z^2), z^\nu \widehat{p}_{n,\nu}(z^4)) = 0 \tag{3.12}$$

for $\nu = 0, 2$ and $\nu = 1, 3$, respectively, where $h \in \mathcal{P}_{n-1}$. These relations can be represented in a compact form (3.8), where s is defined by (3.7). Notice that $s + 1 \in \{0, 2, 4\}$. Alternatively, (3.8) can be expressed in the form (3.9).

The maximal degree of exactness of the quadrature $\widehat{Q}_N(w; f)$ can be found as the maximal degree of a polynomial $u \in \mathcal{P}_{d-N}$ for which $(u, \omega_N) = 0$, i.e., (3.12). Thus,

$$d_{\max} - N = \begin{cases} 2n - 1, & \nu = 0, 2, \\ 2n, & \nu = 1, 3, \end{cases}$$

from which we conclude that $d_{\max} = 6n + s$. $\square$

In the sequel we use the "orthogonality conditions" (3.9) in order to construct our quadratures $\widehat{Q}_N(w; f)$.

According to (3.3) and using the elementary symmetric functions, defined by

$$\begin{aligned}\sigma_1^{(\nu)} &= r_1 + r_2 + \cdots + r_n,\\ \sigma_2^{(\nu)} &= r_1 r_2 + \cdots + r_{n-1} r_n,\\ &\vdots\\ \sigma_n^{(\nu)} &= r_1 r_2 \cdots r_n,\end{aligned}$$

we can express the polynomial $z^{s+1}\widehat{p}_{n,\nu}(z^4)$ in the form

$$z^{s+1}\widehat{p}_{n,\nu}(z^4) = \sum_{j=0}^{n}(-1)^j \sigma_j^{(\nu)} z^{4(n-j)+s+1}, \tag{3.13}$$

where, for the convenience, we put $\sigma_0^{(\nu)} = 1$. Then, using (3.13), the "orthogonality conditions" (3.9) reduce to the following system of linear equations

$$\sum_{j=0}^{n}(-1)^j \sigma_j^{(\nu)} (\pi_{2k}, z^{4(n-j)+s+1}) = 0, \quad k = 0, 1, \ldots, n-1,$$

i.e.,

$$\sum_{j=1}^{n}(-1)^{j-1} s_{k,2n-2j+\eta}\sigma_j^{(\nu)} = s_{k,2n+\eta}, \quad k = 0, 1, \ldots, n-1, \tag{3.14}$$

where $s_{k,j}$ are the inner products given by

$$s_{k,j} = (\pi_{2k}, z^{2j}) = \int_{-1}^{1} \pi_{2k}(z) z^{2j} w(z)\,\mathrm{d}z, \quad 0 \le k \le j, \tag{3.15}$$

and $2\eta = s + 1$. Notice that $s + 1$ is an even number (according to (3.7)) and $\eta \in \{0, 1, 2\}$.

The system of linear equations (3.14) can be done in the matrix form

$$\mathbf{A}^{(\nu)}\boldsymbol{\sigma}^{(\nu)} = \mathbf{b}^{(\nu)}, \tag{3.16}$$

where

$$\mathbf{A}^{(\nu)} = \begin{bmatrix} s_{0,2n-2+\eta} & -s_{0,2n-4+\eta} & \cdots & (-1)^{n-1}s_{0,\eta} \\ s_{1,2n-2+\eta} & -s_{1,2n-4+\eta} & & (-1)^{n-1}s_{1,\eta} \\ \vdots & & & \\ s_{n-1,2n-2+\eta} & -s_{n-1,2n-4+\eta} & & (-1)^{n-1}s_{n-1,\eta} \end{bmatrix} \tag{3.17}$$

and

$$\mathbf{b}^{(\nu)} = \begin{bmatrix} s_{0,2n+\eta} \\ s_{1,2n+\eta} \\ \vdots \\ s_{n-1,2n+\eta} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma}^{(\nu)} = \begin{bmatrix} \sigma_1^{(\nu)} \\ \sigma_2^{(\nu)} \\ \vdots \\ \sigma_n^{(\nu)} \end{bmatrix}. \tag{3.18}$$

The inner products $s_{k,2n-2j+\eta}$, which appear as elements of $\mathbf{A}^{(\nu)}$ and $\mathbf{b}^{(\nu)}$, should be calculated for $0 \le k \le n-1$ and $0 \le 2n - 2j + \eta \le 2n + \eta$. In other words, for generating the system of equations (3.14), i.e., (3.16), we use only entries from the following matrix of the type $n \times (2n + \eta)$,

$$\mathbf{S}^{(\nu)} = \begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,n-1} & \cdots & s_{0,2n+\eta} \\ & s_{1,1} & \cdots & s_{1,n-1} & \cdots & s_{1,2n+\eta} \\ & & \ddots & & & \\ & & & s_{n-1,n-1} & \cdots & s_{n-1,2n+\eta} \\ & & & s_{n,n-1} & \cdots & s_{n,2n+\eta} \end{bmatrix}, \tag{3.19}$$

where $\eta = \eta(\nu) = 0$, 1, and 2, depending on $\nu = 0$, $\nu = 1$ or $\nu = 2$, and $\nu = 3$, respectively.

It is clear that $s_{0,j}$ are moments of the weight function,

$$s_{0,j} = (1, z^{2j}) = \int_{-1}^{1} z^{2j} w(z)\,\mathrm{d}z = \mu_{2j}, \quad j = 0, 1, \ldots, \tag{3.20}$$

as well as that $s_{k,j} = 0$ for $k > j$. Using the recurrence relation (3.6) for orthogonal polynomials with respect to the weight function $w : (-1, 1) \to \mathbb{R}^+$, it can be proved that the following two-dimensional recurrence relation

$$s_{k,j+1} = s_{k+1,j} + (\beta_{2k} + \beta_{2k+1}) s_{k,j} + \beta_{2k}\beta_{2k-1} s_{k-1,j} \tag{3.21}$$

holds. The proof of this relation and an algorithm for calculating a matrix of the form (3.19) have been recently done in [20]. Notice that $s_{k,k} = \beta_0\beta_1 \cdots \beta_{2k}$, $k \geq 0$.

Remark 1. Coefficients in the two-dimensional relation (3.21) appear also in the three-term recurrence relation for polynomials $\{\pi_{2k}(\sqrt{t})\}_{k\in\mathbb{N}_0}$ orthogonal with respect to the weight function $w(\sqrt{t})/\sqrt{t}$ on $(0, 1)$ (cf. [14, pp. 101–103]). Otherwise,

$$\pi_{2k+2}(z) = (z^2 - \beta_{2k} - \beta_{2k+1})\pi_{2k}(z) - \beta_{2k}\beta_{2k-1}\pi_{2k-2}(z), \quad k = 1, 2, \ldots .$$

Thus, using Theorem 3.1 and the previous facts we have the following result:

Theorem 3.2. *Let $N = 4n + \nu$ (≥ 4), with*

$$n = \left[\frac{N}{4}\right], \quad \nu = N - 4\left[\frac{N}{4}\right] \in \{0, 1, 2, 3\},$$

the inner products $s_{k,j}$ be given by (3.15), and $\boldsymbol{\sigma}^{(\nu)} = [\sigma_1^{(\nu)}\ \sigma_2^{(\nu)}\ \ldots\ \sigma_n^{(\nu)}]^T$ be the unique solution of the system of linear equations (3.16), where the matrix $\mathbf{A}^{(\nu)}$ and the vector $\mathbf{b}^{(\nu)}$ are given by (3.17) and (3.18), respectively.

Then, all zeros of the polynomial

$$\widehat{p}_{n,\nu}(z) = \sum_{j=0}^{n} (-1)^j \sigma_j^{(\nu)} z^{n-j} = \prod_{k=1}^{n} (z - r_k),$$

are real, simple and contained in $(0, 1)$, and they determine the (nonzero) nodes of the quadrature formula $\widehat{Q}_{4n+\nu}(w; f)$, with maximal degree of exactness $d_{\max} = 6n + s$, where s is given by (3.7).

Proof. According to (3.9) for $k = 0$, the equality

$$\int_{-1}^{1} z^{s+1} \widehat{p}_{n,\nu}(z^4) w(z)\,\mathrm{d}z = 0$$

holds. This is exactly the first equation in the system (3.16). Since $s + 1$ is an even number $(= 2\eta \in \{0, 2, 4\})$ and $\widehat{p}_{n,\nu}(z^4)$ is an even polynomial (of degree $4n$), we conclude that $\widehat{p}_{n,\nu}(z)$ must change its sign at least at one point in $(0, 1)$. Suppose that this polynomial changes its sign at m points in $(0, 1)$, e.g. at $r_1, \ldots, r_m$. It means that $\widehat{p}_{n,\nu}(z^4)$ changes its sign at the points $\pm\sqrt[4]{r_k}$, $k = 1, \ldots, n$.

Define now an even polynomial of degree $2m$ such that

$$\Phi(z) = (z^2 - \sqrt{r_1}) \cdots (z^2 - \sqrt{r_m}).$$

Then, we can conclude that the polynomial $\widehat{p}_{n,\nu}(z^4)\Phi(z)$ does not change its sign on $(-1, 1)$ and therefore

$$(\Phi, z^{s+1}\widehat{p}_{n,\nu}(z^4)) = \int_{-1}^{1} \Phi(z) z^{s+1} \widehat{p}_{n,\nu}(z^4) w(z)\,\mathrm{d}z \neq 0. \tag{3.22}$$

Since $\Phi(z)$ can be expressed as linear combination of even orthogonal polynomials $\{\pi_{2k}\}_{k=0}^{m}$ (w.r.t. the weight function w on $(-1, 1)$), i.e.,

$$\Phi(z) = \sum_{k=0}^{m} \gamma_k \pi_{2k}(z),$$

we get

$$(\Phi, z^{s+1}\widehat{p}_{n,\nu}(z^4)) = \sum_{k=0}^{m} \gamma_k(\pi_{2k}, z^{s+1}\widehat{p}_{n,\nu}(z^4)).$$

Using the "orthogonality conditions" (3.9) we conclude that

$$(\Phi, z^{s+1}\widehat{p}_{n,\nu}(z^4)) = \begin{cases} 0, & m < n, \\ \gamma_n(\pi_{2n}, z^{s+1}\widehat{p}_{n,\nu}(z^4)), & m = n. \end{cases}$$

Because of (3.22), it means that the case $m < n$ is not possible, so it must be $m = n$. $\square$

3.2. Weight coefficients in the quadrature formula $\widehat{Q}_N(w; f)$

Let $Q_N(w; f)$ be an N-point quadrature of interpolation type with simple (in general, complex) nodes $z_j \in Z$,

$$Q_N(w; f) = \sum_{z_j \in Z} W_j f(z_j),$$

and with the corresponding weight coefficients W_j. They can be obtained by an integration of the Lagrange polynomial constructed on the set Z, i.e.,

$$L_N(f; z) = \sum_{z_j \in Z} \frac{\omega_N(z)}{(z - z_j)\omega_N'(z_j)} f(z_j),$$

where $\omega_N(z)$ is the node polynomial. Then, the weight coefficients can be expressed in the form

$$W_j = \frac{1}{\omega_N'(z_j)} \int_{-1}^{1} \frac{\omega_N(z)}{z - z_j} w(z)\,\mathrm{d}z. \tag{3.23}$$

We separately consider cases $\nu = 0$, $\nu = 1$, and $\nu = 3$.

3.2.1. Quadrature formula $\widehat{Q}_{4n}(w; f)$

In this case ($\nu = 0$), the quadrature nodes belong to $Z = \{\pm x_k, \pm \mathrm{i}x_k, k = 1, \ldots, n\}$, so that

$$\omega_{4n}(z) = p_{n,0}(z^4), \quad \omega_{4n}'(z) = 4z^3 p_{n,0}'(z^4).$$

Using (3.23) and notations for coefficients as in (3.2), we can formulate and prove the following result:

Theorem 3.3. *The weight coefficients in the quadrature formula $\widehat{Q}_{4n}(w; f)$ with the maximal degree of exactness $d = 6n - 1$ are given by*

$$A_k^{(0)} = \frac{1}{4\sqrt{r_k}\,\widehat{p}\,'_{n,0}(r_k)} \int_{-1}^{1} \frac{\widehat{p}_{n,0}(z^4)}{z^2 - \sqrt{r_k}} w(z)\,\mathrm{d}z, \quad k = 1, \ldots, n,$$

$$B_k^{(0)} = \frac{-1}{4\sqrt{r_k}\,\widehat{p}\,'_{n,0}(r_k)} \int_{-1}^{1} \frac{\widehat{p}_{n,0}(z^4)}{z^2 + \sqrt{r_k}} w(z)\,\mathrm{d}z, \quad k = 1, \ldots, n,$$

where

$$\widehat{p}_{n,0}(z) = \prod_{k=1}^{n}(z - r_k) = \sum_{j=0}^{n}(-1)^j \sigma_j^{(0)} z^{n-j}.$$

3.2.2. Quadrature formula $\widehat{Q}_{4n+1}(w; f)$

In this case ($\nu = 1$), the quadrature nodes belong to $Z = \{0, \pm x_k, \pm \mathrm{i}x_k, k = 1, \ldots, n\}$, so that

$$\omega_{4n+1}(z) = z p_{n,1}(z^4), \quad \omega_{4n+1}'(z) = p_{n,1}(z^4) + 4z^4 p_{n,1}'(z^4).$$

Theorem 3.4. *The weight coefficients in the quadrature formula $\widehat{Q}_{4n+1}(w;f)$ with the maximal degree of exactness $d=6n+1$ are given by*

$$C_0^{(1)}=\frac{1}{p_{n,1}(0)}\int_{-1}^{1}p_{n,1}(z^4)w(z)\,dz,$$

$$A_k^{(1)}=\frac{1}{4r_kp'_{n,1}(r_k)}\int_{-1}^{1}\frac{z^2p_{n,1}(z^4)}{z^2-\sqrt{r_k}}w(z)\,dz,\quad k=1,\ldots,n,$$

$$B_k^{(1)}=\frac{1}{4r_kp'_{n,1}(r_k)}\int_{-1}^{1}\frac{z^2p_{n,1}(z^4)}{z^2+\sqrt{r_k}}w(z)\,dz,\quad k=1,\ldots,n,$$

where

$$\widehat{p}_{n,1}(z)=\prod_{j=1}^{n}(z-r_j)=\sum_{j=0}^{n}(-1)^j\sigma_j^{(1)}z^{n-j}.$$

3.2.3. Quadrature formula $\widehat{Q}_{4n+3}(w;f)$

In the case $\nu=3$ we have the following result:

Theorem 3.5. *Let $\widetilde{C}_0^{(1)}$, $\widetilde{A}_k^{(1)}$, $\widetilde{B}_k^{(1)}$, $k=1,\ldots,n$, be the weight coefficients of the quadrature formula $\widehat{Q}_{4n+1}(\widetilde{w};f)$, with the maximal degree of exactness $d=6n+1$, where $\widetilde{w}(z)=z^2w(z)$. Then, the corresponding weight coefficients in the quadrature formula $\widehat{Q}_{4n+3}(w;f)$, with the maximal degree of exactness $d=6n+3$, are given by*

$$C_0^{(3)}=\mu_0-2\sum_{k=1}^{n}\frac{\widetilde{A}_k^{(1)}-\widetilde{B}_k^{(1)}}{x_k^2},\quad C_1^{(3)}=\mu_1=0,\quad C_2^{(3)}=\frac{1}{2}\widetilde{C}_0^{(1)},$$

$$A_k^{(3)}=\frac{\widetilde{A}_k^{(1)}}{x_k^2},\quad B_k^{(3)}=\frac{\widetilde{B}_k^{(1)}}{x_k^2},\quad k=1,\ldots,n,$$

where μ_k, $k=0,1,\ldots$, are moments of the weight function w, and $x_k^4=r_k$, $k=1,\ldots,n$, are zeros of the polynomial

$$\widehat{p}_{n,3}(z)=\prod_{j=1}^{n}(z-r_j)=\sum_{j=0}^{n}(-1)^j\sigma_j^{(3)}z^{n-j}.$$

Proof. Let $x_0=\sqrt{r_0}$ and $x_k=\sqrt[4]{r_k}$, $k=1,\ldots,n$. According to

$$\omega'_{4n+3}(z)=(3z^2-r_0)\prod_{\nu=1}^{n}(z^4-r_\nu)+4z^4(z^2-r_0)\sum_{j=1}^{n}\prod_{\nu\neq j}(z^4-r_\nu),$$

we have

$$\omega'_{4n+3}(0)=-r_0\prod_{\nu=1}^{n}(-r_\nu)=-r_0p_n(0),$$

$$\omega'_{4n+3}(\pm x_0)=2r_0\prod_{\nu=1}^{n}(r_0^2-r_\nu)=2r_0p_n(r_0^2),$$

$$\omega'_{4n+3}(\pm x_k)=4r_k(\sqrt{r_k}+r_0)\prod_{\nu\neq k}(r_k-r_\nu)=4r_k(\sqrt{r_k}-r_0)p'_n(r_k),$$

$$\omega'_{4n+3}(\pm \mathrm{i}x_k)=4r_k(-\sqrt{r_k}-r_0)\prod_{\nu\neq k}(r_k-r_\nu)=-4r_k(\sqrt{r_k}+r_0)p'_n(r_k),$$

where $k=1,\ldots,n$. Now, applying (3.23) and using notations for coefficients as in (3.2), we get the desired results. □

3.3. Explicit form of the node polynomial in $\widehat{Q}_N(w; f)$ for the generalized Gegenbauer weight

In this part we determined the expressions in a closed form of $s_{k,j}$ for the generalized Gegenbauer weight defined by

$$w(z) = |z|^\gamma (1-z^2)^\alpha \qquad (\alpha, \gamma > -1), \tag{3.24}$$

on $(-1, 1)$. The monic polynomials $W_\nu^{(\alpha,\beta)}(z)$, $\nu = 0, 1, \dots$, orthogonal with respect to this weight function, where $\beta = (\gamma-1)/2$, were introduced by Laščenov [9] (cf. [14, pp. 147–148]). These polynomials can be expressed in terms of the Jacobi polynomials $P_\nu^{(\alpha,\beta)}(z)$, $\nu = 0, 1, \dots$, which are orthogonal on $(-1, 1)$ with respect to the weight function $w^{(\alpha,\beta}(z) = (1-z)^\alpha (1+z)^\beta$, $\alpha, \beta > -1$. Namely,

$$W_{2k}^{(\alpha,\beta)}(z) = \frac{k!}{(k+\alpha+\beta+1)_k} P_k^{(\alpha,\beta)}(2z^2-1), \tag{3.25}$$

$$W_{2k+1}^{(\alpha,\beta)}(z) = \frac{k!}{(k+\alpha+\beta+2)_k} z P_k^{(\alpha,\beta+1)}(2z^2-1).$$

Notice that $W_{2k+1}^{(\alpha,\beta)}(z) = zW_{2k}^{(\alpha,\beta+1)}(z)$. The polynomials $W_\nu^{(\alpha,\beta)}(z)$ satisfy the following three-term recurrence relation

$$W_{\nu+1}^{(\alpha,\beta)}(z) = zW_\nu^{(\alpha,\beta)}(z) - \beta_\nu W_{\nu-1}^{(\alpha,\beta)}(z), \quad \nu = 0, 1, \dots,$$

$$W_{-1}^{(\alpha,\beta)}(z) = 0, \quad W_0^{(\alpha,\beta)}(z) = 1,$$

where

$$\beta_{2k} = \frac{k(k+\alpha)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}, \quad \beta_{2k-1} = \frac{(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)},$$

for $k = 1, 2, \dots$, except when $\alpha + \beta = -1$; then $\beta_1 = (\beta+1)/(\alpha+\beta+2)$.

First, in our case, we need explicit expressions for the products

$$s_{k,j} = (W_{2k}^{(\alpha,\beta)}, z^{2j}) = \int_{-1}^{1} W_{2k}^{(\alpha,\beta)}(z) z^{2j} |z|^{2\beta+1} (1-z^2)^\alpha \, dz, \quad 0 \le k \le j. \tag{3.26}$$

Let $\alpha, \gamma > -1$ (i.e., $\beta > -1$). Then the products defined in (3.26) are (cf. [20, Lemma 5.1])

$$s_{k,j} = \frac{k!}{(k+\alpha+\beta+1)_k} \binom{j}{k} \frac{\Gamma(k+\alpha+1)\Gamma(j+\beta+1)}{\Gamma(k+j+\alpha+\beta+2)}, \quad 0 \le k \le j. \tag{3.27}$$

Otherwise, because of orthogonality, $s_{k,j} = 0$ for $k > j$.

In the following theorem we give the explicit form of the node polynomial (3.3):

Theorem 3.6. *For the generalized Gegenbauer weight function (3.24), the coefficients of the node polynomial*

$$\widehat{p}_{n,\nu}(z) = \sum_{j=0}^{n} (-1)^j \sigma_j^{(\nu)} z^{n-j}$$

in the quadrature formula $\widehat{Q}_N(w; f)$ are given by

$$\sigma_j^{(\nu)} = \binom{n}{j} \frac{(\beta+\eta+2(n-j)+1)_{2j}}{(\alpha+\beta+\eta+2(n-j)+n+1)_{2j}}, \qquad j = 0, 1, \dots, n, \tag{3.28}$$

where $\beta = (\gamma-1)/2$ and $\eta = (s+1)/2$.

Proof. Regarding Theorem 3.1 the node polynomial $\widehat{p}_{n,\nu}(z)$ exists uniquely. Therefore, it is enough to prove that the coefficients (3.28) satisfy the system of equations (3.14), i.e.,

$$\sum_{j=0}^{n} (-1)^j s_{k,2n-2j+\eta} \sigma_j^{(\nu)} = \sum_{j=0}^{n} (-1)^{n-j} s_{k,2j+\eta} \sigma_{n-j}^{(\nu)} = 0,$$

for each $k = 0, 1, \dots, n-1$.

Using (3.27) we see that

$$s_{k,2j+\eta} = \frac{k!}{(k+\alpha+\beta+1)_k}\binom{2j+\eta}{k}\frac{\Gamma(k+\alpha+1)\Gamma(2j+\beta+\eta+1)}{\Gamma(k+2j+\alpha+\beta+\eta+2)}, \quad j \ge j_k^{(\nu)},$$

where

$$j_k^{(\nu)} = \left\lceil \frac{k-\eta}{2} \right\rceil \quad \text{and} \quad \eta = \eta(\nu) = \begin{cases} 0, & \nu = 0, \\ 1, & \nu = 1 \text{ or } \nu = 2, \\ 2, & \nu = 3. \end{cases}$$

For $j < j_k^{(\nu)}$ the inner product $s_{k,2j+\eta} = 0$. Also, from (3.28) we conclude that

$$\sigma_{n-j}^{(\nu)} = \binom{n}{j}\frac{(\beta+\eta+2j+1)_{2n-2j}}{(\alpha+\beta+\eta+2j+n+1)_{2n-2j}} = \binom{n}{j}\frac{\Gamma(\alpha+\beta+\eta+2j+n+1)}{\Gamma(\alpha+\beta+\eta+3n+1)}\cdot\frac{\Gamma(\beta+\eta+2n+1)}{\Gamma(\beta+\eta+2j+1)}.$$

Therefore,

$$s_{k,2j+\eta}\sigma_{n-j}^{(\nu)} = C_k^{(\nu)}\binom{n}{j}(2j+\eta-k+1)_k(k+2j+\alpha+\beta+\eta+2)_{n-k-1},$$

where $C_k^{(\nu)}$ is a coefficient which does not depend on j, i.e.,

$$C_k^{(\nu)} = \frac{1}{(k+\alpha+\beta+1)_k}\cdot\frac{\Gamma(k+\alpha+1)\Gamma(\beta+\eta+2n+1)}{\Gamma(\alpha+\beta+\eta+3n+1)},$$

so that we should prove that the sums

$$S_k^{(\nu)} = \sum_{j=j_k^{(\nu)}}^{n}(-1)^j\binom{n}{j}(2j+\eta-k+1)_k(k+2j+\alpha+\beta+\eta+2)_{n-k-1} \tag{3.29}$$

are equal to zero for each $k = 0, 1, \dots, n-1$.

Since $(2j+\eta-k+1)_k = 0$ for $j < j_k^{(\nu)}$, the summation in (3.29) is equivalent to

$$S_k^{(\nu)} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}(2j+\eta-k+1)_k(k+2j+\alpha+\beta+\eta+2)_{n-k-1}. \tag{3.30}$$

Expanding $(2j+\eta-k+1)_k(k+2j+\alpha+\beta+\eta+2)_{n-k-1}$ in powers of j, we conclude that it is a polynomial in j of degree $n-1$ for each $k = 0, 1, \dots, n-1$, i.e.,

$$(2j+\eta-k+1)_k(k+2j+\alpha+\beta+\eta+2)_{n-k-1} = \sum_{i=0}^{n-1}\gamma_i j^i \quad (0 \le k \le n-1),$$

where the coefficients γ_i depend, in general, on n, k, ν, α, and β. In this way, we see that (3.30) becomes

$$S_k^{(\nu)} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{i=0}^{n-1}\gamma_i j^i = \sum_{i=0}^{n-1}\gamma_i\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^i \quad (0 \le k \le n-1).$$

Finally, using the identity (see Gould [8, p. 2, Formula (1.13)])

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^i = \begin{cases} 0, & 0 \le i \le n-1, \\ (-1)^n n!, & i = n, \end{cases}$$

we conclude that $S_k^{(\nu)} = 0$ for each $0 \le k \le n-1$. □

4. A sequence of generalized quadrature formulas

In this section we consider a sequence of quadrature rules $\{Q_N^{[m]}(f)\}_m$,

$$Q_N^{[m]}(f)=\sum_{j=0}^{\nu-1}C_j^{(\nu)}f^{(j)}(0)+\sum_{k=1}^{n}\sum_{j=1}^{m}A_{k,j}^{(\nu)}\left[f\left(x_k\mathrm{e}^{\mathrm{i}\theta_j}\right)+f\left(-x_k\mathrm{e}^{\mathrm{i}\theta_j}\right)\right],\quad m=1,2,\ldots, \tag{4.1}$$

for weighted numerical integration of an analytic function,

$$I(w;f):=\int_{-1}^{1}f(z)w(z)\,\mathrm{d}z=Q_N^{[m]}(f)+R_N^{[m]}(w;f),$$

where

$$x_k=\sqrt[2m]{r_k},\quad k=1,\ldots,n;\quad \theta_j=\frac{(j-1)\pi}{m},\quad j=1,\ldots,m,$$

and the node polynomial is defined by

$$\omega_N(z)=z^\nu p_{n,\nu}^{[m]}(z^{2m})=z^\nu\prod_{k=1}^{n}(z^{2m}-r_k),\quad 0<r_1<\cdots<r_n<1, \tag{4.2}$$

with $N=2mn+\nu$, $n=[N/2m]$, and $\nu\in\{0,1,\ldots,2m-1\}$.

For $\nu=0$, the first sum in $Q_N^{[m]}(f)$ is empty. $R_N^{[m]}(w;f)$ in (3.1) is the corresponding remainder.

As before in Section 3, the coefficients $C_j^{(\nu)}$ for odd j must be zero, so we can conclude that $Q_{2mn+\nu}^{[m]}(f)\equiv Q_{2mn+\nu-1}^{[m]}(f)$ for $\nu=2,4,\ldots,2m-2$. In this set of quadrature rules there is a unique interpolatory quadrature $\widehat{Q}_N^{[m]}(f)$ with a maximal degree of exactness, and we can prove the following result (see [19]):

Theorem 4.1. *Let m be a fixed positive integer and w be a nonnegative even weight function on $(-1,1)$, for which all moments $\mu_k=\int_{-1}^{1}z^kw(z)\,\mathrm{d}z$, $k\ge 0$, exist and $\mu_0>0$. For any $N\in\mathbb{N}$ there exists a unique interpolatory quadrature $\widehat{Q}_N^{[m]}(f)$ with the maximal degree of exactness $d_{\max}=2(m+1)n+s$, where*

$$n=\left[\frac{N}{2m}\right],\quad \nu=N-2mn\in\{0,1,\ldots,2m-1\},\quad s=\begin{cases}\nu-1, & \nu\ \text{even},\\ \nu, & \nu\ \text{odd}.\end{cases}$$

The node polynomial (4.2) is characterized by the following orthogonality relations

$$\int_0^1 t^k\widehat{p}_{n,\nu}^{[m]}(t^m)t^{s/2}w(\sqrt{t})\,\mathrm{d}t=0,\quad k=0,1\ldots,n-1. \tag{4.3}$$

In the case $m=1$ (see Fig. 1 (left)), the node polynomial $\widehat{\omega}_{2n+\nu}(z)=z^\nu\widehat{p}_{n,\nu}^{[1]}(z^2)$, with $\nu=0$ or $\nu=1$, is a monic polynomial of degree $2n+\nu$, which is orthogonal to $\mathcal{P}_{2n+\nu-1}$ with respect to the even weight function w on $(-1,1)$, so that the rule $\widehat{Q}_N^{[1]}(f)$ is a standard Gaussian formula. Since $s=-1$ for $\nu=0$ and $s=1$ for $\nu=1$, according to the orthogonality relations (4.3), the sequences of polynomials

$$\widehat{p}_{n,0}^{[1]}(t)=\widehat{\omega}_{2n}(\sqrt{t})\quad\text{and}\quad \widehat{p}_{n,1}^{[1]}(t)=\frac{\widehat{\omega}_{2n+1}(\sqrt{t})}{\sqrt{t}}$$

are orthogonal on $(0,1)$ with respect to the weight functions $w(\sqrt{t})/\sqrt{t}$ and $w(\sqrt{t})\sqrt{t}$, respectively (see also [14, Theorem 2.2.11]). Notice that the origin appears as a quadrature node only when $\nu=1$.

In the case $m=2$, the nodes are symmetrically distributed on the real and imaginary axes (see Fig. 1 (right)), and the quadrature rules (4.1) reduce to the generalized quadrature of Birkhoff–Young type considered in Section 3. The orthogonality conditions (3.8) which characterize the interpolatory quadrature $\widehat{Q}_N(w;f)$, with a maximal degree of exactness $d_{\max}=6n+s$, can be expressed in the equivalent form (4.3) (for $m=2$ and $\widehat{p}_{n,\nu}=\widehat{p}_{n,\nu}^{[2]}$). Indeed, (3.8), i.e.,

$$(z^{2k},z^{s+1}\widehat{p}_{n,\nu}(z^4))=\int_{-1}^{1}z^{2k}z^{s+1}\widehat{p}_{n,\nu}(z^4)w(z)\,\mathrm{d}z=0,\quad k=0,1,\ldots,n-1,$$

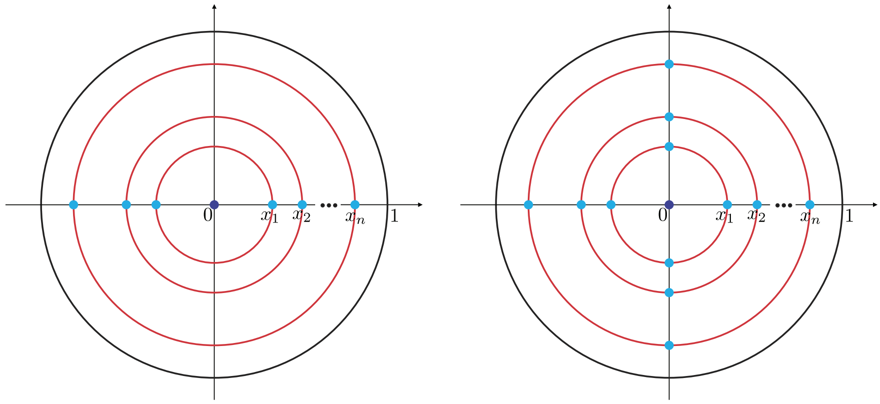

Fig. 1. Distribution of nodes for $m = 1$ (left) and $m = 2$ (right).

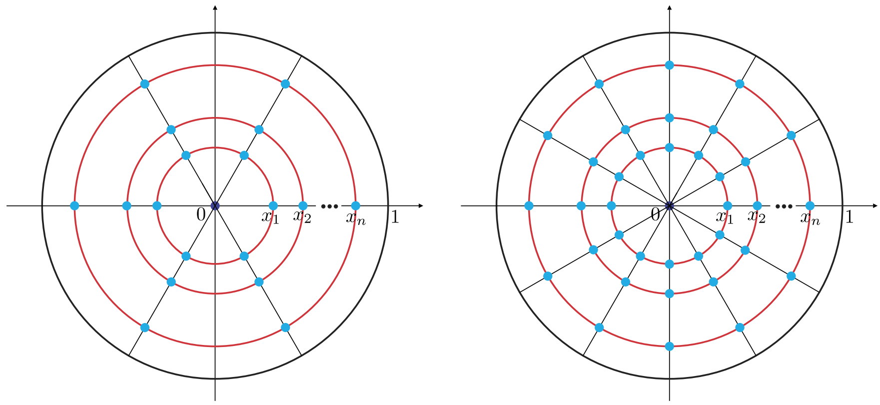

Fig. 2. Distribution of nodes for $m = 3$ (left) and $m = 6$ (right).

can be written as

$$(z^{2k}, z^{s+1}\widehat{p}_{n,\nu}(z^4)) = 2\int_0^1 z^{2k}z^{s+1}\widehat{p}_{n,\nu}(z^4)w(z)\,\mathrm{d}z = \int_0^1 t^k\widehat{p}^{[2]}_{n,\nu}(t^2)t^{s/2}w(\sqrt{t})\,\mathrm{d}t = 0 \quad (0 \le k < n),$$

after changing variables $z = \sqrt{t}$.

The characterization (4.3) shows that $\widehat{p}^{[m]}_{n,\nu}(t^m)$ must be orthogonal to $\mathcal{P}_{n-1}$ with respect to the weight function $w_\nu(t) = t^{s/2}w(\sqrt{t})$ on $(0, 1)$. Precisely, these polynomials $\widehat{p}^{[m]}_{n,0}, \widehat{p}^{[m]}_{n,1}, \dots, \widehat{p}^{[m]}_{n,2m-1}$ (each of degree n) are orthogonal to $\mathcal{P}_{n-1}$ with respect to the weight functions $w_0(t) = t^{-1/2}w(\sqrt{t})$, $w_1(t) = t^{1/2}w(\sqrt{t})$, ..., $w_{2m-1}(t) = t^{m-1/2}w(\sqrt{t})$, respectively. The distribution of the nodes for $m = 3$ and $m = 6$ and some $\nu \in \{0, 1, \dots, 2m - 1\}$ are presented in Fig. 2.

All results obtained in Section 3 for the rules $\widehat{Q}_N(w; f)$ can be directly extended to the generalized quadrature formulas $\widehat{Q}^{[m]}_N(f)$. Here, we mention only the extension of Theorem 3.6.

Theorem 4.2. *For the weight function (3.24), the coefficients of the node polynomial in (4.2),*

$$p^{[m]}_{n,\nu}(z) = \sum_{j=0}^{n}(-1)^j\sigma_j^{(\nu)}z^{n-j},$$

are given in explicit form by

$$\sigma_j^{(\nu)} = \binom{n}{j}\frac{(\beta + \eta + m(n - j) + 1)_{mj}}{(\alpha + \eta + \beta + m(n - j) + n + 1)_{mj}}, \qquad j = 1, \dots, n,$$

where $\beta = (\gamma - 1)/2$ and $\eta = (s + 1)/2$.

In this section, however, we give another approach to this kind of quadrature rules. Namely, the orthogonality conditions (4.3) can be interpreted in terms of the so-called multiple orthogonal polynomials. Because of that, in the sequel we give some basic facts on this kind of orthogonality.

4.1. Multiple orthogonal polynomials

Multiple orthogonal polynomials are intimately related to Hermite–Padé approximants and, because of that, they are known as Hermite–Padé polynomials (for a nice survey see Aptekarev [2]). Multiple orthogonal polynomials are a generalization of the standard orthogonal polynomials in the sense that they satisfy m orthogonality conditions.

Let $m \ge 1$ be an integer and let w_j, $j = 1, \dots, m$, be weight functions on the real line so that the support of each w_j is a subset of an interval E_j. Let $\mathbf{n} = (n_1, n_2, \dots, n_m)$ be a vector of m nonnegative integers, which is called a *multi-index* with the length $|\mathbf{n}| = n_1 + n_2 + \cdots + n_m$. There are two types of multiple orthogonal polynomials, but here we consider only the so-called *type II multiple orthogonal polynomials* $\pi_{\mathbf{n}}(t)$ of degree $|\mathbf{n}|$. Such monic polynomials are defined by the m orthogonality relations

$$\left.\begin{array}{ll}
\displaystyle\int_{E_1} \pi_{\mathbf{n}}(t)\, t^{\ell} w_1(t)\,\mathrm{d}t = 0, & \ell = 0, 1, \dots, n_1 - 1, \\
\displaystyle\int_{E_2} \pi_{\mathbf{n}}(t)\, t^{\ell} w_2(t)\,\mathrm{d}t = 0, & \ell = 0, 1, \dots, n_2 - 1, \\
\vdots & \\
\displaystyle\int_{E_m} \pi_{\mathbf{n}}(t)\, t^{\ell} w_m(t)\,\mathrm{d}t = 0, & \ell = 0, 1, \dots, n_m - 1.
\end{array}\right\} \tag{4.4}$$

Evidently, for $m = 1$ they reduce to the ordinary orthogonal polynomials.

The conditions (4.4) give $|\mathbf{n}|$ linear equations for the $|\mathbf{n}|$ unknown coefficients $a_{k,\mathbf{n}}$ of the polynomial $\pi_{\mathbf{n}}(t) = \sum_{k=0}^{|\mathbf{n}|} a_{k,\mathbf{n}} t^k$, where $a_{|\mathbf{n}|,\mathbf{n}} = 1$. However, the matrix of coefficients of this system of equations can be singular and we need some additional conditions on the m weight functions to provide the uniqueness of the multiple orthogonal polynomials. If the polynomial $\pi_{\mathbf{n}}(t)$ is unique, then we say that $\mathbf{n}$ is a *normal multi-index* and if all multi-indices are normal then we have a *perfect system* of weight functions.

One important perfect system is the AT system, in which all weight functions are supported on the same interval E $(= E_1 = E_2 = \cdots = E_m)$ and the $|\mathbf{n}|$ functions:

$$w_1(t),\ t w_1(t),\ \dots,\ t^{n_1-1} w_1(t),\ w_2(t),\ t w_2(t),\ \dots, t^{n_2-1} w_2(t),\ \dots,\ w_m(t),\ t w_m(t),\ \dots,\ t^{n_m-1} w_m(t)$$

form a *Chebyshev system* on E for each multi-index $\mathbf{n}$. This means that every linear combination

$$\sum_{j=1}^{m} Q_{n_j-1}(t) w_j(t),$$

where Q_{n_j-1} is a polynomial of degree at most $n_j - 1$, has at most $|\mathbf{n}| - 1$ zeros on E.

In 2001 Van Assche and Coussement [31] proved the following result:

Theorem 4.3. *For an* AT *system the type II multiple orthogonal polynomial $\pi_{\mathbf{n}}(x)$ has exactly $|\mathbf{n}|$ zeros on E.*

In the last decade there is a growing interest in the study of multiple orthogonal polynomials, including recurrence relations, numerical constructions, special weight functions, etc. (cf. [3,4,7,26,27,30]).

4.2. Generalized quadrature formulae in terms of multiple orthogonal polynomials

Starting from Theorem 4.1 we can prove the following statement:

Theorem 4.4. *Let m be a fixed positive integer and w be a nonnegative even weight function on $(-1, 1)$, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z)\,\mathrm{d}z$, $k \ge 0$, exist and $\mu_0 > 0$. For any $N \in \mathbb{N}$ there exists a unique interpolatory quadrature rule $\widehat{Q}_N^{[m]}(f)$, with the maximal degree of exactness $d_{\max} = 2(m+1)n + s$, if and only if the polynomial $\widehat{p}_{n,\nu}^{[m]}(t)$ is the type II multiple orthogonal polynomial $\pi_{\mathbf{n}}(t)$, with respect to the weight functions*

$$w_j(t) = t^{(s+2j)/(2m)-1} w(t^{1/(2m)}), \quad j = 1, \ldots, m,$$

on $(0, 1)$, *with* $n_j = 1 + \left[\frac{n-j}{m}\right]$, $j = 1, \ldots, m$.

Proof. At first, if we put $t^{1/m}$ instead of t in (4.3), it is easy to see that the orthogonality relations (4.3) are equivalent to

$$\int_0^1 t^{k/m} \widehat{p}_{n,\nu}^{[m]}(t) t^{(s+2)/(2m)-1} w(t^{1/(2m)})\, dt = 0, \quad k = 0, 1, \ldots, n-1.$$

Now, putting $k = m\ell + j - 1$, where $\ell = [k/m]$ and $j = 1, \ldots, m$, and defining

$$w_j(t) := t^{(s+2j)/(2m)-1} w(t^{1/(2m)}) \quad \text{and} \quad n_j := 1 + \left[\frac{n-j}{m}\right], \quad j = 1, \ldots, m,$$

we conclude that the polynomial $\widehat{p}_{n,\nu}^{[m]}(t)$ satisfies m orthogonality relations like (4.4), i.e.,

$$\int_0^1 t^\ell \widehat{p}_{n,\nu}^{[m]}(t) w_1(t)\, dt = 0, \quad \ell = 0, 1 \ldots, n_1 - 1,$$

$$\int_0^1 t^\ell \widehat{p}_{n,\nu}^{[m]}(t) w_2(t)\, dt = 0, \quad \ell = 0, 1 \ldots, n_2 - 1,$$

$$\vdots$$

$$\int_0^1 t^\ell \widehat{p}_{n,\nu}^{[m]}(t) w_m(t)\, dt = 0, \quad \ell = 0, 1 \ldots, n_m - 1.$$

Notice that these weight functions w_j, $j = 1, 2, \ldots, m$, defined on the same interval $E_1 = E_2 = \cdots = E_m = E = (0, 1)$, can be expressed in the form $w_j(t) = t^{(j-1)/m} w_1(t)$, $j = 1, \ldots, m$, where $w_1(t) = t^{(s+2)/(2m)-1} w(t^{1/(2m)})$. Since the Müntz system $\{t^{k+(j-1)/m}\}$, $k = 0, 1, \ldots, n_j - 1$; $j = 1, \ldots, m$, is a Chebyshev system on $[0, \infty)$, and also on $E = (0, 1)$, and $w_1(t) > 0$ on E, we conclude that $\{w_j,\ j = 1, \ldots, m\}$ is an AT system on E.

Therefore, according to Theorem 4.3, the unique type II multiple orthogonal polynomial $\widehat{p}_{n,\nu}^{[m]}(t) = \pi_{\mathbf{n}}(t)$ has exactly

$$|\mathbf{n}| := \sum_{j=1}^m n_j = \sum_{j=1}^m \left(1 + \left[\frac{n-j}{m}\right]\right) = n$$

zeros in $(0, 1)$. $\square$

Example 1. Let $w(z) = 1/\sqrt{1-z^2}$ (Chebyshev weight of the first kind), $n = 8$, and $m = 3$. According to Theorem 4.2, the node polynomials for $\nu = 0, 1, \ldots, 5$ are

$$\widehat{p}_{5,0}^{[3]}(t) = t^5 - \frac{10875t^4}{5168} + \frac{50025t^3}{34816} - \frac{250125t^2}{720896} + \frac{350175t}{16777216} - \frac{10005}{268435456},$$

$$\widehat{p}_{5,1}^{[3]}(t) = t^5 - \frac{2697t^4}{1216} + \frac{2171085t^3}{1323008} - \frac{1550775t^2}{3407872} + \frac{310155t}{8388608} - \frac{310155}{1073741824},$$

$$\widehat{p}_{5,3}^{[3]}(t) = t^5 - \frac{9889t^4}{4256} + \frac{17058525t^3}{9261056} - \frac{3411705t^2}{5963776} + \frac{3411705t}{58720256} - \frac{3411705}{3758096384},$$

$$\widehat{p}_{5,5}^{[3]}(t) = t^5 - \frac{155t^4}{64} + \frac{337125t^3}{165376} - \frac{1550775t^2}{2228224} + \frac{7753875t}{92274688} - \frac{2171085}{1073741824},$$

as well as $\widehat{p}_{5,2}^{[3]}(t) = \widehat{p}_{5,1}^{[3]}(t)$ and $\widehat{p}_{5,4}^{[3]}(t) = \widehat{p}_{5,3}^{[3]}(t)$.

On the other side, regarding Theorem 4.4, we have that $\mathbf{n} = (2, 2, 1)$, $|\mathbf{n}| = 5$, and

$$
\begin{aligned}
\{w_1(t), w_2(t), w_3(t)\} &= \left\{ \frac{1}{\sqrt[6]{t^5}\sqrt{1-\sqrt[3]{t}}}, \frac{1}{\sqrt{t}\sqrt{1-\sqrt[3]{t}}}, \frac{1}{\sqrt[6]{t}\sqrt{1-\sqrt[3]{t}}} \right\} \quad (\text{for } \nu = 0), \\
&= \left\{ \frac{1}{\sqrt{t}\sqrt{1-\sqrt[3]{t}}}, \frac{1}{\sqrt[6]{t}\sqrt{1-\sqrt[3]{t}}}, \frac{\sqrt[6]{t}}{\sqrt{1-\sqrt[3]{t}}} \right\} \quad (\text{for } \nu = 1 \text{ and } \nu = 2), \\
&= \left\{ \frac{1}{\sqrt[6]{t}\sqrt{1-\sqrt[3]{t}}}, \frac{\sqrt[6]{t}}{\sqrt{1-\sqrt[3]{t}}}, \frac{\sqrt{t}}{\sqrt{1-\sqrt[3]{t}}} \right\} \quad (\text{for } \nu = 3 \text{ and } \nu = 4), \\
&= \left\{ \frac{\sqrt[6]{t}}{\sqrt{1-\sqrt[3]{t}}}, \frac{\sqrt{t}}{\sqrt{1-\sqrt[3]{t}}}, \frac{\sqrt[6]{t^5}}{\sqrt{1-\sqrt[3]{t}}} \right\} \quad (\text{for } \nu = 5),
\end{aligned}
$$

so that it is easy to check the equalities

$$
\begin{aligned}
&\int_0^1 t^\ell \widehat{p}_{5,\nu}^{[3]}(t) w_1(t)\,\mathrm{d}t = 0, \quad \ell = 0, 1, \\
&\int_0^1 t^\ell \widehat{p}_{5,\nu}^{[3]}(t) w_2(t)\,\mathrm{d}t = 0, \quad \ell = 0, 1, \\
&\int_0^1 t^\ell \widehat{p}_{5,\nu}^{[3]}(t) w_3(t)\,\mathrm{d}t = 0, \quad \ell = 0,
\end{aligned}
$$

for each $\nu = 0, 1, \ldots, 5$.

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Computing Integrals of Highly Oscillatory Special Functions Using Complex Integration Methods and Gaussian Quadratures

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Abstract

An account on computation of integrals of highly oscillatory functions based on the so-called *complex integration methods* is presented. Beside the basic idea of this approach some applications in computation of Fourier and Bessel transformations are given. Also, Gaussian quadrature formulas with a modified Hermite weight are considered, including some numerical examples.

1 Introduction and Preliminaries

In this paper we give an account on computing integrals of highly oscillatory functions based on the so-called *complex integration methods* and using quadrature processes in general, as well as some new results and numerical examples. Some of these results have been recently presented during author's lecture at the *4th Dolomites Workshop on Constructive Approximation and Applications*, Session: *Numerical integration, integral equations and transforms* (September 8–13, 2016, Alba di Canazei, Italy).

We deal here with integration of functions of the form

$$I(f,K)=I(f(\cdot),K(\cdot;x))=\int_a^b w(t)f(t)K(t;x)\,\mathrm{d}t, \tag{1}$$

where (a,b) is an interval on the real line, which may be finite or infinite, $w(t)$ is a given weight function, and the kernel $K(t;x)$ is a function depending on a parameter x and such that it is highly oscillatory or/and has singularities on the interval (a,b) or in its nearness. Typical examples of such kernels are:

(a) Oscillatory kernel $K(t;x)=\mathrm{e}^{\mathrm{i}xt}$, where $x=\omega$ is a large positive parameter. Then we have Fourier integrals over $(0,+\infty)$ (*Fourier transforms*)

$$\mathcal{F}(f;\omega)=\int_0^{+\infty} t^{\mu}f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \qquad (\mu>-1)$$

or *Fourier coefficients* (on a finite interval)

$$c_k(f)=a_k(f)+\mathrm{i}b_k(f)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t, \tag{2}$$

where $\omega=k\in\mathbb{N}$.

(b) Oscillatory kernels $K(t;x)=H_\nu^{(m)}(xt)$, where $x=\omega$ is also a large positive parameter. These integral transforms are known as *Hankel* (or *Bessel*) *transforms* (see Wong [51]),

$$\mathcal{H}_m(x)=\int_0^{+\infty} t^{\mu}f(t)H_\nu^{(m)}(\omega t)\,\mathrm{d}t \qquad (m=1,2), \tag{3}$$

where $H_\nu^{(m)}(t)$, $m=1,2$, are the Hankel functions of first and second type and order ν,

$$H_\nu^{(1)}(z)=J_\nu(z)+\mathrm{i}Y_\nu(z) \quad \text{and} \quad H_\nu^{(2)}(z)=J_\nu(z)-\mathrm{i}Y_\nu(z),$$

where J_ν is the *Bessel* function of the first kind and order (index) ν, defined by

$$J_\nu(z)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{z}{2}\Big)^{2k+\nu}, \qquad J_{-n}(z)=(-1)^nJ_n(z).$$

Otherwise, J_ν is a particular solution of the so-called *Bessel* differential equation

$$z^2y''+zy'+(z^2-\nu^2)y=0.$$

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The second linearly independent solution of this equation is the *Bessel* function of the second kind Y_ν (sometimes known as *Weber* or *Neumann* function),

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

(c) Logarithmic singular kernel $K(t;x) = \log|t-x|$, where $a \le x \le b$.

(d) Algebraic singular kernel $K(t;x) = |t-x|^\alpha$, where $\alpha > -1$ and $a < x < b$.

Also, we mention here an important case when $K(t;x) = 1/(t-x)$, where $a < x < b$ and the integral (1) is taken to be a Cauchy principal value integral.

Integrals of rapidly oscillating functions appear mainly in the theory of special functions and Fourier analysis, but also in other applied and computational sciences and engineering, e.g., in theoretical physics (in particular, theory of scattering), acoustic scattering, quantum chemistry, theory of transport processes, electromagnetics, telecommunication, fluid mechanics, etc. For example, in the last time, a very attractive problem is the numerical solution of *Volterra* integral equation of the second (or first) kind with highly oscillatory kernel

$$y(x) + \int_0^x \frac{J_\nu(\omega(x-t))}{(x-t)^\alpha}\, y(t)\,\mathrm{d}t = \varphi(x),$$

or

$$\lambda y(x) + \int_0^x \frac{\mathrm{e}^{\mathrm{i}\omega g(x-t)}}{(x-t)^\alpha}\, y(t)\,\mathrm{d}t = \varphi(x),$$

where $x \in [0,1]$, $0 \le \alpha < 1$, $\omega \gg 1$, $\varphi(x)$ and $g(x)$ are given functions, and $y(x)$ is unknown function.

We mention also a type of integrals involving Bessel functions

$$I_\nu(f;\omega) = \int_0^{+\infty} \mathrm{e}^{-t^2} J_\nu(\omega t) f(t^2) t^{\nu+1}\,\mathrm{d}t, \quad \nu > -1,$$

with a large positive parameter ω. Such integrals appear in some problems of high energy nuclear physics (cf. [14]).

In Fig. 1 we present the graphics of $J_3(\omega x)$ and $Y_3(\omega x)$ on $[1,10]$ for some values of the parameter ω

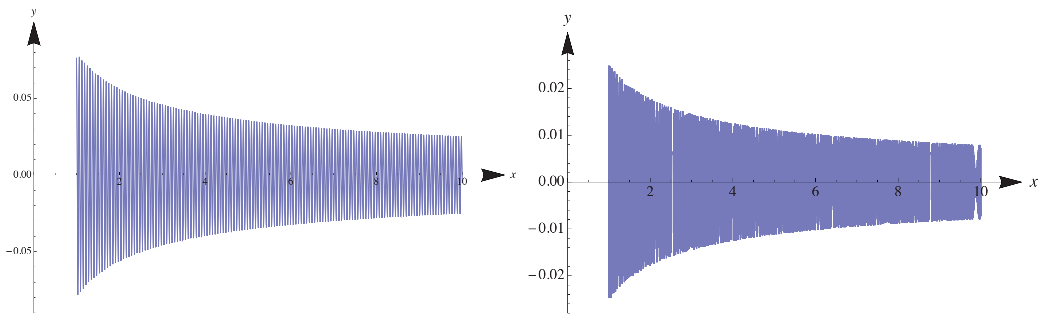

Figure 1: The graphics of $J_3(100x)$ *(left)* and $Y_3(1000x)$ *(right)* on $[1,10]$

Conventional techniques for computing values of special functions are power series, Chebyshev expansions, asymptotic expansions, recurrence relations, sequence transformations, continued fractions and best rational approximations, differential and difference equations, quadrature methods, etc. A nice survey on these methods, including a list of recent software for special functions as well as a list of new publications on computational aspects of special functions is given recently by Gil, Segura and Temme [18]. An application of standard quadrature formulas to $I(f;K)$ usually requires a large number of nodes and too much computation work in order to achieve a modest degree of accuracy. In a recent joint survey paper with M. Stanić [40] we discussed some specific nonstandard methods for numerical integration of highly oscillating functions, mainly based on some contour integration methods and applications of some kinds of Gaussian quadratures, including complex oscillatory weights. In particular, Filon-type quadratures for weighted Fourier integrals, exponential-fitting quadrature rules, Gaussian-type quadratures with respect to some complex oscillatory weights, methods for irregular oscillators, as well as two methods for integrals involving highly oscillating Bessel functions have been considered, including some numerical examples. In addition, we mention also the so-called integrals with *irregular oscillators*

$$I[f;g] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x, \tag{4}$$

where $-\infty < a < b < +\infty$, $|\omega|$ is large, and both f and g are sufficiently smooth functions. In a special case when $g(x) = x$, we have the so-called regular oscillators. Numerical calculation of the integrals 4 has been treated in a large number of papers (cf. [10, 11, 12], [22], [24], [26, 27, 28, 29], [31], [43, 44, 45, 46], etc.). The most important are asymptotic methods, Filon–type methods, and Levin–type methods. Asymptotic method was presented by Iserles and Nørsett [29].

Using suitable *integral representations of special functions*, in this paper, we show how existing or specially developed quadrature rules can be successfully applied to effectively calculation of highly oscillatory integrals (Fourier type integrals, oscillatory Bessel transformation, Bessel-Hilbert transformation, etc.). The procedure is based on an idea from our paper [35] from 1998, where, beside an account on some special – fast and efficient – quadrature methods for weighted integrals of strongly oscillatory functions, we introduced the so-called *Complex Integration Methods* for some classes of oscillatory integrals (1).

This paper is organized as follows. In Section 2 we give some basic facts on the *Complex Integration Methods*. Applications of these methods to integrals of highly oscillatory special functions are treated in Section 3. Finally, in Section 4 we consider Gaussian quadrature formulas with respect to a modified Hermite weight on $\mathbb{R}$.

2 Complex Integration Methods – Basic Idea

The basic idea of the *Complex Integration Methods* is to transform the integral of an oscillatory function to a weighed integral with respect to the exponentially decreasing weight function on $(0,+\infty)$.

First we illustrate this idea to calculation of the Fourier integrals on the finite interval $[-1,1]$,

$$I(f;\omega)=\int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x, \tag{5}$$

assuming that f is an analytic real-valued function in the half-strip of the complex plane, $-1\le \operatorname{Re} z\le 1$, $\operatorname{Im} z\ge 0$, with possible singularities at the points z_ν $(\nu=1,\ldots,m)$ inside the region

$$G_\delta=\Big\{z\in\mathbb{C}\ \Big|\ -1\le \operatorname{Re} z\le 1,\ 0\le \operatorname{Im} z\le\delta\Big\},$$

where δ is sufficiently large.

Now we suppose that the corresponding residues of these singularities give

$$2\pi\mathrm{i}\sum_{\nu=1}^{m}\operatorname*{Res}_{z=z_\nu}\Big\{f(z)\mathrm{e}^{\mathrm{i}\omega z}\Big\}=P+iQ, \tag{6}$$

as well as that there exist the constants $M>0$, $\delta_0>0$ and $\xi<\omega$ such that

$$\int_{-1}^{1}|f(x+\mathrm{i}\delta)|\,\mathrm{d}x\le M\mathrm{e}^{\xi\delta}\quad(\delta>\delta_0>0). \tag{7}$$

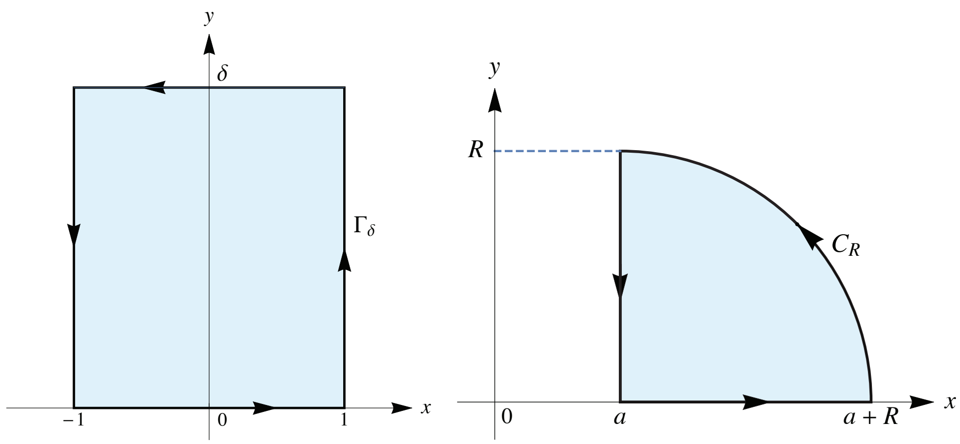

Figure 2: The contours of integration Γ_δ *(left)* and C_R *(right)*

By integrating the function $z\mapsto f(z)\mathrm{e}^{\mathrm{i}\omega z}$ over the contour $\Gamma_\delta=\partial G_\delta$ (see Fig. 2 (left)), we have

$$\begin{aligned}\oint_{\Gamma_\delta} f(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z &= \int_0^\delta f(1+\mathrm{i}y)\mathrm{e}^{\mathrm{i}\omega(1+\mathrm{i}y)}\mathrm{i}\,\mathrm{d}y+\int_1^{-1} f(x+\mathrm{i}\delta)\mathrm{e}^{\mathrm{i}\omega(x+\mathrm{i}\delta)}\,\mathrm{d}x+\int_\delta^0 f(-1+\mathrm{i}y)\mathrm{e}^{\mathrm{i}\omega(-1+\mathrm{i}y)}\mathrm{i}\,\mathrm{d}y+I(f;\omega)\\ &= 2\pi\mathrm{i}\sum_{\nu=1}^{m}\operatorname*{Res}_{z=z_\nu}\Big\{f(z)\mathrm{e}^{\mathrm{i}\omega z}\Big\}=P+\mathrm{i}Q,\end{aligned}$$

i.e.,

$$I(f;\omega)=P+\mathrm{i}Q+\mathrm{i}\int_0^\delta\Big[\mathrm{e}^{-\mathrm{i}\omega}f(-1+\mathrm{i}y)-\mathrm{e}^{\mathrm{i}\omega}f(1+\mathrm{i}y)\Big]\mathrm{e}^{-\omega y}\,\mathrm{d}y+\int_{-1}^{1}(x+\mathrm{i}\delta)\mathrm{e}^{\mathrm{i}\omega}f(x+\mathrm{i}\delta)\,\mathrm{d}x.$$

Because of (7) we conclude that

$$
\begin{aligned}
|I_\delta| &= \left|\int_{-1}^{1} f(x+\mathrm{i}\delta)\mathrm{e}^{\mathrm{i}\omega(x+\mathrm{i}\delta)}\,\mathrm{d}x\right| = \mathrm{e}^{-\omega\delta}\left|\int_{-1}^{1} f(x+\mathrm{i}\delta)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x\right| \\
&\leq \mathrm{e}^{-\omega\delta}\int_{-1}^{1} |f(x+\mathrm{i}\delta)|\,\mathrm{d}x \leq M\mathrm{e}^{(\xi-\omega)\delta}.
\end{aligned}
$$

Thus, $I_\delta \to 0$ when $\delta \to +\infty$, and

$$
I(f;\omega) = P + \mathrm{i}Q + \frac{1}{\mathrm{i}\omega}\int_0^{+\infty}\left[\mathrm{e}^{\mathrm{i}\omega} f\Big(1+\mathrm{i}\frac{t}{\omega}\Big) - \mathrm{e}^{-\mathrm{i}\omega} f\Big(-1+\mathrm{i}\frac{t}{\omega}\Big)\right]\mathrm{e}^{-t}\,\mathrm{d}t. \tag{8}
$$

In this way we proved the following result:

Theorem 2.1 ([35]). *Let f be an analytic real-valued function in the half-strip of the complex plane, $-1 \leq \mathrm{Re}\, z \leq 1$, $\mathrm{Im}\, z \geq 0$, with possible singularities z_ν ($\nu = 1,\ldots,m$) in the region $G_\delta = \mathrm{int}\,\Gamma_\delta$, such that (6) holds. Supposing that there exist the constants $M > 0$ and $\xi < \omega$ such that the condition (7) holds for sufficiently large δ, we have (8).*

The obtained integral (8) in Theorem 2.1 can be solved by using the Gauss-Laguerre rule.

In order to illustrate the efficiency of this method we consider a simple example – Fourier coefficients (2), with $f(t) = 1/(t^2+\varepsilon^2)^m$ ($m \in \mathbb{N}$, $\varepsilon > 0$). Thus, we are interested in the integrals

$$
c_k(f) = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}k\pi x}\,\mathrm{d}x, \qquad \omega = k\pi.
$$

According to (8), for $1 \leq m \leq 3$, we have

$$
c_k(f) = P + \mathrm{i}Q + \frac{(-1)^k}{\mathrm{i}k\pi}\int_0^{+\infty}\left[f\Big(1+\mathrm{i}\frac{t}{k\pi}\Big) - f\Big(-1+\mathrm{i}\frac{t}{k\pi}\Big)\right]\mathrm{e}^{-t}\,\mathrm{d}t,
$$

where, in our case, we have

$$
f(z) = \frac{1}{(z^2+\varepsilon^2)^m}, \quad P + \mathrm{i}Q = 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}\varepsilon}\left\{f(z)\mathrm{e}^{\mathrm{i}k\pi z}\right\} = \begin{cases} \dfrac{\pi}{\varepsilon}\mathrm{e}^{-k\pi\varepsilon}, & m=1, \\ \dfrac{\pi(1+k\pi\varepsilon)}{2\varepsilon^3}\mathrm{e}^{-k\pi\varepsilon}, & m=2, \\ \dfrac{\pi(3+3k\pi\varepsilon+k^2\pi^2\varepsilon^2)}{8\varepsilon^5}\mathrm{e}^{-k\pi\varepsilon}, & m=3. \end{cases}
$$

For calculating $c_5(f)$, $c_{10}(f)$ and $c_{40}(f)$, when $\varepsilon = 1$ and $\varepsilon = 10^{-2}$, we apply the n-point Gauss-Laguerre rule for $n = 1,\ldots,7$ nodes. The corresponding relative errors in quadrature approximations are given in Table 1. Numbers in parentheses indicate decimal exponents. As we can see the convergence is faster for larger k (and smaler ε).

Table 1: Relative errors in n-point Gauss-Laguerre approximations of $c_k(f)$ for $k = 5, 10, 40$ and $\varepsilon = 1$ and 10^{-2}

	$k=5$		$k=10$		$k=40$	
n	$\varepsilon=1$	$\varepsilon=10^{-2}$	$\varepsilon=1$	$\varepsilon=10^{-2}$	$\varepsilon=1$	$\varepsilon=10^{-2}$
1	1.11(−2)	1.69(−9)	2.60(−3)	1.28(−10)	1.59(−4)	7.91(−13)
2	3.48(−4)	1.38(−10)	2.56(−5)	3.40(−12)	1.04(−7)	1.45(−15)
3	2.12(−5)	8.83(−12)	2.71(−7)	1.02(−13)	5.78(−11)	3.35(−18)
4	3.84(−7)	1.03(−13)	3.25(−9)	3.21(−15)	5.45(−14)	9.92(−21)
5	3.49(−8)	7.80(−14)	1.29(−10)	8.69(−17)	8.20(−13)	4.48(−22)
6	8.46(−9)	9.35(−15)	4.06(−12)	2.94(−19)	4.77(−12)	2.39(−21)
7	1.61(−9)	6.62(−16)	1.65(−13)	2.21(−19)	5.40(−14)	2.75(−23)

Table 2: Gaussian approximation of the integral $c_k(f)$

k	$\varepsilon=1$	$\varepsilon=10^{-2}$
5	$4.00392583\underline{46130827412}\,(-3)$	$1.55333209782728\underline{2899812027}\,(+6)$
10	$-1.010071027052\underline{0897087}\,(-3)$	$1.507753137017524820\underline{873537}\,(+6)$
40	$-6.331369411209\underline{4129150}\,(-5)$	$1.0088603450377737040756\underline{38}\,(+6)$

Approximative values obtained by 7-point Gauss-Laguerre rule are presented in Table 2. Digits in error are underlined.

Now we consider the Fourier integral on $(0,+\infty)$,

$$F(f;\omega)=\int_0^{+\infty} f(x)e^{i\omega x}\,dx,$$

which can be transformed to

$$F(f;\omega)=\frac{1}{\omega}\int_0^{+\infty} f\left(\frac{x}{\omega}\right)e^{ix}\,dx=\frac{1}{\omega}F\left(f\left(\frac{\cdot}{\omega}\right);1\right),$$

which means that is enough to consider only the case $\omega=1$.

In order to calculate $F(f;1)$ we select a positive number a and divide the integral over $(0,+\infty)$ into two integrals,

$$F(f;1)=\int_0^{a} f(x)e^{ix}\,dx+\int_a^{+\infty} f(x)e^{ix}\,dx=L_1(f)+L_2(f),$$

where

$$L_1(f)=a\int_0^1 f(at)e^{iat}\,dt \quad \text{and} \quad L_2(f)=\int_a^{+\infty} f(x)e^{ix}\,dx.$$

For calculating the second integral $L_2(f)$ we use the complex integration method over the closed circular contour C_R presented in Fig. 2 (right).

Theorem 2.2 ([35]). *Suppose that the function $z\mapsto f(z)$ is defined and holomorphic in the region $D=\{z\in\mathbb{C}\,|\,\mathrm{Re}\,z\ge a>0,\ \mathrm{Im}\,z\ge 0\}$, and such that*

$$|f(z)|\le \frac{A}{|z|},\quad \textit{when } |z|\to+\infty, \tag{9}$$

for some positive constant A. Then

$$L_2(f)=ie^{ia}\int_0^{+\infty} f(a+iy)e^{-y}\,dy \qquad (a>0). \tag{10}$$

In this case, by Cauchy's residue theorem, we have

$$\int_a^{a+R} f(x)e^{ix}\,dx+\int_0^{\pi/2}\left[f(z)e^{iz}\right]_{z=a+Re^{i\theta}}Rie^{i\theta}\,d\theta+\int_R^0 f(a+iy)e^{i(a+iy)}i\,dy=0. \tag{11}$$

Let $z=a+Re^{i\theta}$, $0\le\theta\le\pi/2$. Because of (9), we have that

$$|f(z)|\le\frac{A}{|a+R\cos\theta+i\sin\theta|}=\frac{A}{\sqrt{a^2+2aR\cos\theta+R^2}}\le\frac{A}{\sqrt{a^2+R^2}}\qquad(0\le\theta\le\pi/2).$$

Using Jordan's inequality $\sin\theta\ge 2\theta/\pi$, when $0\le\theta\le\pi/2$, we obtain the following estimate for the integral over the arc

$$\left|\int_0^{\pi/2}\left[f(z)e^{iz}\right]_{z=a+Re^{i\theta}}Rie^{i\theta}\,d\theta\right|\le\int_0^{\pi/2}\left|f(a+Re^{i\theta})\right|e^{-R\sin\theta}R\,d\theta\le\frac{\pi}{2}\cdot\frac{A}{\sqrt{a^2+R^2}}\cdot\frac{\pi}{2}\left(1-e^{-R}\right)\to 0,$$

when $R\to+\infty$, and then (10) follows directly from (11).

In the numerical implementation we use the Gauss-Legendre rule on $(0,1)$ and Gauss-Laguerre rule for calculating $L_1(f)$ and $L_2(f)$, respectively.

3 Computing Integrals of Highly Oscillatory Special Functions

The idea on complex integration methods has been exploited in many papers, which are dealing with integrals of special functions, in particular with a highly oscillatory Bessel kernels (cf. Chen [4, 5, 6, 7, 8], Kang and Xiang [30], Xu, Milovanović and Xiang [53], Xu and Milovanović [52], Xu and Xiang [54], etc.). For example, Chen [4] considered the numerical evaluation of the integrals on (a,b), $0<a<b$, involving highly oscillatory Bessel kernel $J_\nu(\omega x)$, where $J_\nu(\omega x)$ is the Bessel function of the first kind and of order $\nu(>0)$ and ω is a large positive parameter. Using the integral form of Bessel function and its analytic continuation, he applied the complex integration methods to transform these integrals into the forms on $[0,+\infty)$ that the integrand does not oscillate and decays exponentially fast, and which can be efficiently computed by using Gauss-Laguerre quadrature rule.

Evaluation of Cauchy principal value integrals of oscillatory functions was also considered in such a way by Wang and Xiang [50], as well as applications to the computation of highly oscillatory Bessel Hilbert transforms [52]. We mention also the corresponding applications in solving Volterra and Fredholm integral equations with highly oscillatory kernels (cf. [13], [23], [32]).

Recently, Xu, Milovanović and Xiang [53] developed a method for efficient computation of highly oscillatory integrals with Hankel kernel,

$$I_1[f]=\int_a^b f(x)H_\nu^{(1)}(\omega x)\,dx \quad\text{and}\quad I_2[f]=\int_a^{+\infty} f(x)H_\nu^{(1)}(\omega x)\,dx, \tag{12}$$

for $\omega \gg 1$ and $b > a > 0$. Using the integral form of the Hankel function for $x > 0$ (see [20, p. 915])

$$H_\nu^{(1)}(\omega x) = \sqrt{\frac{2}{\pi\omega x}}\,\frac{\mathrm{e}^{\mathrm{i}(\omega x - \frac{\pi}{2}\nu - \frac{\pi}{4})}}{\Gamma(\nu+\frac{1}{2})}\int_0^{+\infty}\Big(1+\frac{\mathrm{i}t}{2\omega x}\Big)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}}\mathrm{e}^{-t}\,\mathrm{d}t,$$

they obtained the following integral representations for the previous integrals:

$$I_1[f] = \sqrt{\frac{2}{\pi\omega}}\,\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma(\nu+\frac{1}{2})}\int_a^b f(x)x^{-1/2}g(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x \quad \text{and} \quad I_2[f] = \sqrt{\frac{2}{\pi\omega}}\,\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma(\nu+\frac{1}{2})}\int_a^{+\infty} f(x)x^{-1/2}g(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x,$$

where

$$g(x) = \int_0^{+\infty}\Big(1+\frac{\mathrm{i}t}{2\omega x}\Big)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{13}$$

Supposing that f be a holomorphic function in the half-strip of the complex plane, $a \le \mathrm{Re}(z) \le b$, $\mathrm{Im}(z) \ge 0$, as well as that there exist two constants C and ω_0, such that $|f(x+\mathrm{i}R)| \le C\mathrm{e}^{\omega_0 R}$, $a \le x \le b$, with $0 < \omega_0 < \omega$, the integral $I_1[f]$ can be reduced to (see [53])

$$I_1[f] = \frac{\mathrm{i}}{\omega}\sqrt{\frac{2}{\pi\omega}}\,\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma\big(\nu+\frac{1}{2}\big)}(G(a)-G(b)), \tag{14}$$

where

$$G(c) = \mathrm{e}^{\mathrm{i}\omega c}\int_0^{+\infty} F\Big(c+\frac{\mathrm{i}}{\omega}t\Big)\mathrm{e}^{-t}\,\mathrm{d}t. \tag{15}$$

Really, (14) follows after an application of the complex integration method over the contour $\Gamma = \partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ (see Fig. 3 (left)), where D is the region

$$D = \Big\{z \in \mathbb{C} \;\Big|\; a \le \mathrm{Re}(z) \le b,\ 0 \le \mathrm{Im}(z) \le R\Big\}.$$

In this case, the integrand $F(z) = f(z)z^{-1/2}g(z)$ is a holomorphic function in D, such that $\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4} F(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z = 0$.

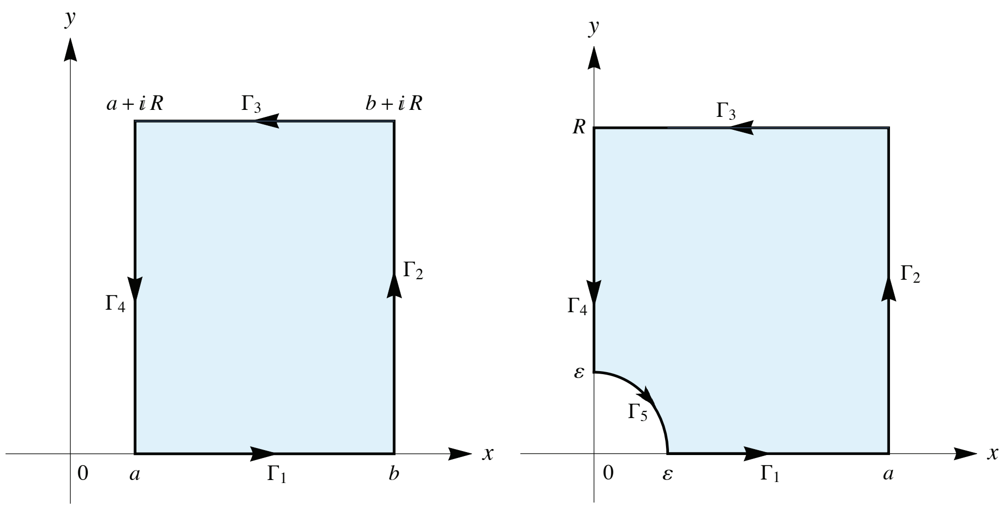

Figure 3: The contours of integration $\Gamma = \partial D$ (*left*) and $\Gamma = \partial(G \setminus G')$ (*right*)

Regarding the assumptions we can see that

$$\Big|\int_{\Gamma_3} F(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z\Big| \le \int_{\Gamma_3} |F(z)\mathrm{e}^{\mathrm{i}\omega z}||\,\mathrm{d}z| \le CM\mathrm{e}^{-(\omega-\omega_0)R}(b-a) \to 0 \quad \text{as} \quad R \to +\infty,$$

i.e., $\int_{\Gamma_3} F(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z \to 0$ as $R \to +\infty$, so that

$$\begin{aligned}
\int_{\Gamma_1} F(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z &= -\lim_{R\to+\infty}\int_{\Gamma_2\cup\Gamma_3\cup\Gamma_4} F(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z \\
&= \lim_{R\to+\infty}\left\{\mathrm{i}\int_0^R F(a+\mathrm{i}y)\mathrm{e}^{\mathrm{i}\omega(a+\mathrm{i}y)}\,\mathrm{d}y - \mathrm{i}\int_0^R F(b+\mathrm{i}y)\mathrm{e}^{\mathrm{i}\omega(b+\mathrm{i}y)}\,\mathrm{d}y\right\} \\
&= \frac{\mathrm{i}}{\omega}\int_0^{+\infty} \mathrm{e}^{\mathrm{i}\omega a}F\Big(a+\mathrm{i}\frac{t}{\omega}\Big)\mathrm{e}^{-t}\,\mathrm{d}t - \frac{\mathrm{i}}{\omega}\int_0^{+\infty} \mathrm{e}^{\mathrm{i}\omega b}F\Big(b+\mathrm{i}\frac{t}{\omega}\Big)\mathrm{e}^{-t}\,\mathrm{d}t \\
&= \frac{\mathrm{i}}{\omega}(G(a)-G(b)),
\end{aligned}$$

where

$$G(c) = \mathrm{e}^{\mathrm{i}\omega c}\int_0^{+\infty} F\Big(c+\frac{\mathrm{i}}{\omega}t\Big)\mathrm{e}^{-t}\,\mathrm{d}t.$$

Thus, we have

$$I_1[f] = \int_a^b f(x)H_\nu^{(1)}(\omega x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi\omega}}\,\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma\big(\nu+\frac{1}{2}\big)}\int_{\Gamma_1} F(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z,$$

i.e., (14).

Similarly, using a circular contour like one in Fig. 2 (right), the second integral in (12) can be reduced to

$$I_2[f] = \int_a^{+\infty} f(x)H_\nu^{(1)}(\omega x)\,\mathrm{d}x = \frac{\mathrm{i}}{\omega}\sqrt{\frac{2}{\pi\omega}}\,\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma\big(\nu+\frac{1}{2}\big)}G(a).$$

Since $F(z) = f(z)z^{-1/2}g(z)$ and $g(x)$ defined in (13), after certain transformations, $G(c)$ can be transformed to (see [53])

$$G(c) = \mathrm{e}^{\mathrm{i}\omega c}\int_0^{+\infty}\!\!\int_0^{+\infty} \frac{f\Big(c+\frac{\mathrm{i}}{\omega}t\Big)}{\Big(c+\frac{\mathrm{i}}{\omega}t\Big)^{\nu}}\Big(c+\frac{\mathrm{i}}{\omega}t+\frac{\mathrm{i}}{2\omega}s\Big)^{\nu-1/2}\mathrm{e}^{-t}s^{\nu-1/2}\mathrm{e}^{-s}\,\mathrm{d}t\,\mathrm{d}s.$$

For computing this double integral, in [53] we used two classical Gaussian quadrature rules

$$\int_0^{+\infty} h(x)w_\ell(x)\,\mathrm{d}x = \sum_{k=1}^{n} A_{n,k}^{(\ell)}h(x_{n,k}^{(\ell)}) + R_n^{(\ell)}[h], \quad \ell = 1,2; \tag{16}$$

one with respect to the Laguerre weight $w_1(t) = \mathrm{e}^{-t}$ and the second one to the generalized Laguerre weight $w_2(s) = s^{\nu-1/2}\mathrm{e}^{-s}$. The coefficients in the three-term recurrence relations for the corresponding orthogonal polynomials,

$$\pi_{k+1}^{(\ell)}(x) = (x-\alpha_k^{(\ell)})\pi_k^{(\ell)}(x) - \beta_k^{(\ell)}\pi_{k-1}^{(\ell)}(x), \quad k = 0,1,\ldots,$$

with $\pi_0^{(\ell)}(x) = 1$, $\pi_{-1}^{(\ell)}(x) = 0$, are given by

$$\begin{aligned}
&\alpha_k^{(1)} = 2k+1, && \beta_0^{(1)} = 1,\ \ \beta_k^{(1)} = k^2; \\
&\alpha_k^{(2)} = 2k+\nu+\frac{1}{2}, && \beta_0^{(2)} = \Gamma\Big(\nu+\frac{1}{2}\Big),\ \ \beta_k^{(2)} = k\Big(k+\nu-\frac{1}{2}\Big),
\end{aligned}$$

respectively. With these recursive coefficients, it is easy to compute quadrature parameters in (16), the nodes $x_{n,k}^{(\ell)}$ and the weights (Christoffel numbers) $A_{n,k}^{(\ell)}$, using the well-known Golub-Welsch algorithm [19] (see also [33, p. 100]), with the Jacobi matrices

$$J_n(w_\ell) = \begin{bmatrix}
\alpha_0^{(\ell)} & \sqrt{\beta_1^{(\ell)}} & & & \mathbf{O} \\
\sqrt{\beta_1^{(\ell)}} & \alpha_1^{(\ell)} & \sqrt{\beta_2^{(\ell)}} & & \\
 & \sqrt{\beta_2^{(\ell)}} & \alpha_2^{(\ell)} & \ddots & \\
 & & \ddots & \ddots & \sqrt{\beta_{n-1}^{(\ell)}} \\
\mathbf{O} & & & \sqrt{\beta_{n-1}^{(\ell)}} & \alpha_{n-1}^{(\ell)}
\end{bmatrix} \qquad (\ell = 1,2).$$

This algorithm is implemented in our MATHEMATICA package `OrthogonalPolynomials` (see [9], [38]), which is freely downloadable from the web site: `http://www.mi.sanu.ac.rs/~gvm/`.

Now, an application of quadrature formulas (16) to (14) gives

$$I_1[f] = Q_{n_1,n_2}[f] + R_{n_1,n_2}[f],$$

where the cubature sum $Q_{n_1,n_2}[f]$ (with n_1 nodes in the first quadrature and n_2 nodes in the second one) is given by

$$Q_{n_1,n_2}[f] = \frac{\mathrm{i}}{\omega}\sqrt{\frac{2}{\pi\omega}}\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{k=1}^{n_1}\sum_{j=1}^{n_2} A_{n_1,k}^{(1)}A_{n_2,j}^{(2)}\left[\varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; a) - \varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; b)\right],$$

where

$$\varphi(t,s;c) = \mathrm{e}^{\mathrm{i}\omega c}\frac{f\left(c+\frac{\mathrm{i}}{\omega}t\right)}{\left(c+\frac{\mathrm{i}}{\omega}t\right)^{\nu}}\left(c+\frac{\mathrm{i}}{\omega}t+\frac{\mathrm{i}}{2\omega}s\right)^{\nu-1/2}.$$

Theorem 3.1 ([53]). *Suppose that f is a holomorphic function in the half-strip of the complex plane, $a \le \mathrm{Re}(z) \le b$, $\mathrm{Im}(z) \ge 0$, and there exist two constants C and ω_0, such that $|f(x+iR)| \le Ce^{\omega_0 R}$, $a \le x \le b$, with $0 < \omega_0 < \omega$. Then the error bound of the method for the integral $I_1[f]$ is given by*

$$I_1[f] - Q_{n_1,n_2}[f] = O\left(\omega^{-\frac{3}{2}-2\tau}\right), \quad \omega \gg 1,$$

where $\tau = \min\{n_1, n_2\}$.

A similar result has been proved for the quadrature method

$$\overline{Q}_{n_1,n_2}[f] = \frac{\mathrm{i}}{\omega}\sqrt{\frac{2}{\pi\omega}}\frac{\mathrm{e}^{-\mathrm{i}\pi(2\nu+1)/4}}{\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{k=1}^{n_1}\sum_{j=1}^{n_2} A_{n_1,k}^{(1)}A_{n_2,j}^{(2)}\varphi\left(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; a\right)$$

for calculating $I_2[f]$.

Theorem 3.2 ([53]). *Suppose that f is a holomorphic function in the complex plane $\{0 \le \arg(z) \le \pi/2\}$, and there exists some constant C_1, such that $|f(z)| \le C_1$ as $|z| \to +\infty$. Then the error bound of the method for the integral $I_2[f]$ is given by*

$$I_2[f] - \overline{Q}_{n_1,n_2}[f] = O\left(\omega^{-\frac{3}{2}-2\tau}\right), \quad \omega \gg 1,$$

where $\tau = \min\{n_1, n_2\}$.

As we can see the convergence of quadrature sums $Q_{n_1,n_2}[f]$ and $\overline{Q}_{n_1,n_2}[f]$ to $I_1[f]$ and $I_2[f]$, respectively, is very fast, especially for larger ω.

In the sequel we mention another approach for computing the Bessel transformations

$$I_1[f] = \int_0^a f(x)J_\nu(\omega x)\,\mathrm{d}x \quad \text{and} \quad I_2[f] = \int_0^{+\infty} f(x)J_\nu(\omega x)\,\mathrm{d}x,$$

where $a > 0$ and ν is an arbitrary nonnegative number. The method has been recently developed in a joint paper by Xu [52] and it is based on the use of the following important identity

$$J_\nu(z) = \frac{1}{(2\pi z)^{1/2}}\left\{\mathrm{e}^{\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}W_{0,\nu}(2\mathrm{i}z) + \mathrm{e}^{-\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}W_{0,\nu}(-2\mathrm{i}z)\right\}, \tag{17}$$

where $W_{\kappa,\mu}(z)$ is the *Whittaker W* function, as well as its asymptotic property as $z \to 0$,

$$W_{0,\nu}(z) \sim \begin{cases} z^{1/2}\log z, & \nu = 0, \\ z^{1/2-\nu}, & \nu > 0. \end{cases} \tag{18}$$

Based on an idea of Chen [8], we rewrite the integral $I_1[f]$ as a sum $I_1[f] = I_1'[f] + I_1''[f]$, where

$$I_1'[f] = \int_0^a F(x)J_\nu(\omega x)\,\mathrm{d}x \quad \text{and} \quad I_1''[f] = \sum_{k=0}^{2n-1+n_1}\frac{f^{(k)}(0)}{k!}\int_0^a x^k J_\nu(\omega x)\,\mathrm{d}x, \tag{19}$$

where $n_1 = \lceil \nu \rceil$ is the smallest integer not less than ν, and

$$F(x) = f(x) - \sum_{k=0}^{2n-1+n_1}\frac{f^{(k)}(0)}{k!}x^k. \tag{20}$$

The integral in $I_1''[f]$ can be expressed in the explicit form [20, p. 676]

$$\int_0^a x^k J_\nu(\omega x)\,\mathrm{d}x = \frac{2^k\Gamma(\frac{k+\nu+1}{2})}{\omega^{k+1}\Gamma(\frac{\nu-k+1}{2})} + \frac{a}{\omega^k}\left\{(k+\nu-1)J_\nu(\omega a)s_{k-1,\nu-1}^{(2)}(\omega a) - J_{\nu-1}(\omega a)s_{k,\nu}^{(2)}(\omega a)\right\},$$

where $s_{k,\nu}^{(2)}(z)$ denotes the second kind of Lommel function.

For the integral $I_1'[f]$ we put

$$F_1(x) = F(x)x^{-1/2}\mathrm{e}^{-\mathrm{i}\omega x}W_{0,\nu}(-2\mathrm{i}\omega x) \quad \text{and} \quad F_2(x) = F(x)x^{-1/2}\mathrm{e}^{\mathrm{i}\omega x}W_{0,\nu}(2\mathrm{i}\omega x), \tag{21}$$

where F is defined in (20). Now, according to the identity (17), we can see that

$$\begin{aligned} F(z)J_\nu(\omega z) &= \frac{1}{\sqrt{2\pi\omega}}\Big\{\mathrm{e}^{\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}F(z)z^{-1/2}W_{0,\nu}(2\mathrm{i}\omega z) + \mathrm{e}^{-\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}F(z)z^{-1/2}W_{0,\nu}(-2\mathrm{i}\omega z)\Big\} \\ &= \frac{1}{\sqrt{2\pi\omega}}\Big\{\mathrm{e}^{-\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}F_1(z)\mathrm{e}^{\mathrm{i}\omega z} + \mathrm{e}^{\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}F_2(z)\mathrm{e}^{-\mathrm{i}\omega z}\Big\}. \end{aligned}$$

In order to calculate the integral $I_1'[f]$ defined in (19) we suppose that f is a holomorphic function in the half-strip of the complex plane $0 \le \mathrm{Re}(z) \le a$ and define we define the regions $G = \Big\{z \in \mathbb{C} \;\Big|\; 0 \le \mathrm{Re}(z) \le a,\ 0 \le \mathrm{Im}(z) \le R\Big\}$ and $G' = \Big\{z \in \mathbb{C} \;\Big|\; |z| \le \varepsilon,\ 0 \le \arg(z) \le \pi/2\Big\}$, such that G contains G', i.e., $0 < \varepsilon < \min\{a, R\}$ (see Fig. 3 (right)). Then, we note that $z \mapsto F_1(z)\mathrm{e}^{\mathrm{i}\omega z}$ is holomorphic in $G \setminus G'$ (see (18) for behaviour at $z = 0$), as well as the function $z \mapsto F_2(z)\mathrm{e}^{-\mathrm{i}\omega z}$ in a symmetric region with respect to the real axis. Therefore, by the Cauchy Residue Theorem, $\int_\Gamma F_1(z)\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z = 0$, where $\Gamma = \partial(G \setminus G') = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ (displayed in Fig. 3 (right)), as well as $\int_{\Gamma^*} F_2(z)\mathrm{e}^{-\mathrm{i}\omega z}\,\mathrm{d}z = 0$ over the symmetric contour Γ^* (w.r.t. the real axis).

Applying the complex integration method Xu and Milovanović proved the following result:

Theorem 3.3 ([53]). *Assume that f is a holomorphic function in the half-strip of the complex plane, $0 \le \mathrm{Re}(z) \le a$, and there exist two constants C and ω_0, such that for $0 < \omega_0 < \omega$, the inequalities*

$$\int_0^a |F_1(x + \mathrm{i}R)|\,\mathrm{d}x \le C\mathrm{e}^{\omega_0 R} \quad \textit{and} \quad \int_0^a |F_2(x + \mathrm{i}R)|\,\mathrm{d}x \le C\mathrm{e}^{\omega_0 R}$$

hold, where F_1 and F_2 are defined in (21). Then the integral $I_1'[f]$ can be rewritten in the following form

$$\int_0^a F(x)J_\nu(\omega x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi\omega}}\Big\{\mathrm{e}^{\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}\Big[I[F_2, a] - I[F_2, 0]\Big] + \mathrm{e}^{-\frac{1}{2}(\nu+\frac{1}{2})\pi\mathrm{i}}\Big[I[F_1, 0] - I[F_1, a]\Big]\Big\},$$

where

$$I[F_1, y] = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\omega y}}{\omega}\int_0^{+\infty} F_1\Big(y + \frac{\mathrm{i}p}{\omega}\Big)\mathrm{e}^{-p}\,\mathrm{d}p \quad \textit{and} \quad I[F_2, y] = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega y}}{\omega}\int_0^{+\infty} F_2\Big(y - \frac{\mathrm{i}p}{\omega}\Big)\mathrm{e}^{-p}\,\mathrm{d}p, \tag{22}$$

and F_1 and F_2 are defined in (21).

A similar result has been obtained for the integral $I_2[f]$ over $(0, +\infty)$ [53]. Also, numerical quadrature rules of Gaussian type for computing the line integrals $I[F_j, a]$ and $I[F_j, 0]$ $(j = 1, 2)$ have been analyzed in detail in [53].

In the case $a > 0$ these integrals can be evaluated by the n-point Gauss-Laguerre quadrature rule as

$$I[F_1, a] \approx Q^n_{I[F_1,a]} = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\omega a}}{\omega}\sum_{k=1}^n w_k F_1\Big(a + \frac{\mathrm{i}x_k}{\omega}\Big) \quad \text{and} \quad I[F_2, a] \approx Q^n_{I[F_2,a]} = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega a}}{\omega}\sum_{k=1}^n w_k F_2\Big(a - \frac{\mathrm{i}x_k}{\omega}\Big).$$

However, when $a = 0$ the behavior of the functions F_1 and F_2 at $z = 0$ should be taken into account. According to (18) we have introduced the functions

$$L_j(x) = \begin{cases} \dfrac{F_j(x)}{\log x}, & \nu = 0, \\[2mm] \dfrac{F_j(x)}{x^\alpha}, & \nu > 0, \end{cases}$$

for $j = 1, 2$, where $\alpha = \lceil \nu \rceil - \nu$, and then we concluded that for $\nu > 0$ the previous integrals can be evaluated by the generalized Gauss-Laguerre quadrature rule (with the parameter α), e.g.,

$$I[F_1, 0] = \frac{\mathrm{i}}{\omega}\int_0^{+\infty} F_1\Big(\frac{\mathrm{i}p}{\omega}\Big)\mathrm{e}^{-p}\,\mathrm{d}p = \Big(\frac{\mathrm{i}}{\omega}\Big)^{1+\alpha}\int_0^{+\infty} L_1\Big(\frac{\mathrm{i}p}{\omega}\Big)p^\alpha\mathrm{e}^{-p}\,\mathrm{d}p \approx Q^n_{I[F_1,0]} = \Big(\frac{\mathrm{i}}{\omega}\Big)^{1+\alpha}\sum_{k=1}^n w_k^\alpha L_1\Big(\frac{\mathrm{i}x_k^\alpha}{\omega}\Big).$$

Finally, the most complicated case is when $a = 0$ and $\nu = 0$. Then for the integral $I[F_1, 0]$ we have

$$I[F_1, 0] = \frac{\mathrm{i}}{\omega}\int_0^{+\infty} F_1\Big(\frac{\mathrm{i}p}{\omega}\Big)\mathrm{e}^{-p}\,\mathrm{d}p = \frac{\mathrm{i}}{\omega}\int_0^{+\infty} L_1\Big(\frac{\mathrm{i}p}{\omega}\Big)\log\Big(\frac{\mathrm{i}p}{\omega}\Big)\mathrm{e}^{-p}\,\mathrm{d}p. \tag{23}$$

Evidently, the Gauss-Laguerre (GL) quadrature rule is not feasible, because of logarithmic singularity. However, if we rewrite the integral $I[F_1, 0]$ as a linear combination of two integrals,

$$I[F_1, 0] = \frac{\mathrm{i}}{\omega}\Big\{\int_0^{+\infty} L_1\Big(\frac{\mathrm{i}p}{\omega}\Big)\Big[\log\Big(\frac{\mathrm{i}}{\omega}\Big) - 1 + p\Big]\mathrm{e}^{-p}\,\mathrm{d}p - \int_0^{+\infty} L_1\Big(\frac{\mathrm{i}p}{\omega}\Big)(p - 1 - \log p)\mathrm{e}^{-p}\,\mathrm{d}p\Big\},$$

then, we can apply the ordinary Gauss-Laguerre rule to the first integral and the so-called *logarithmic Gauss-Laguerre* (logGL) rule to the second one. Thus, the application of such two n-point rules leads to the following approximate formula

$$I[F_1,0]\approx Q^n_{I[F_1,0]}=\frac{\mathrm{i}}{\omega}\Big\{\sum_{k=1}^{n}w_kL_1\Big(\frac{\mathrm{i}x_k}{\omega}\Big)\Big[\log\Big(\frac{\mathrm{i}}{\omega}\Big)-1+x_k\Big]-\sum_{k=1}^{n}w_k^G\Big(\frac{\mathrm{i}x_k^G}{\omega}\Big)\Big\},$$

where x_k^G and w_k^G, $k=1,\dots,n$, are the nodes and weights of the n-point logGL-rule. A similar formula can be done for $I[F_2,0]$ (see [53]).

The last quadrature rule on $(0,+\infty)$ with respect to the weight function

$$w_\alpha^G(x)=x^\alpha(x-1-\log x)\mathrm{e}^{-x}\quad\text{on}\;\;(0,+\infty),$$

has been constructed recently by Gautschi [16], using his MATLAB package `SOPQ` for symbolic/variable-precision calculations (see Appendix B in [17]). Graphics of this weight for $\alpha=-1/2,0,1/2$ are presented in Fig. 4. Following Gautschi [16], the

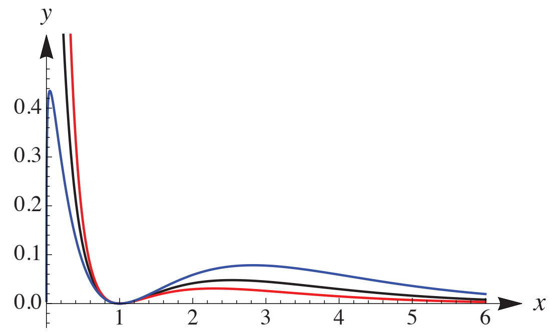

Figure 4: Gautschi's logGL weight function for $\alpha=-1/2$ (*red line*), $\alpha=0$ (*black line*), and $\alpha=1/2$ (*blue line*)

moments with of the weight function $x\mapsto w_\alpha^G(x)$ on $\mathbb{R}^+$ are

$$\mu_k=\int_0^{+\infty}x^{k+\alpha}(x-1-\log x)\mathrm{e}^{-x}\,\mathrm{d}x=\Gamma(\alpha+k+1)[\alpha+k-\psi(\alpha+k+1)],\quad k\ge 0,$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function, as well as the modified moments relative to the system of monic generalized monic Laguerre polynomials $\widehat{L}_k^{(\alpha)}(x)$,

$$m_k=\int_0^{+\infty}x^\alpha(x-1-\log x)\widehat{L}_k^{(\alpha)}(x)\mathrm{e}^{-x}\,\mathrm{d}x\begin{cases}[\alpha-\psi(\alpha+1)]\Gamma(\alpha+1), & k=0,\\ \alpha\,\Gamma(\alpha+1), & k=1,\\ (-1)^k(k-1)!\Gamma(\alpha+1), & k\ge 2.\end{cases}$$

Using these moments and the previous mentioned Mathematica package `OrthogonalPolynomials` we can obtain the recursive coefficients α_k^G and β_k^G. For example for $\alpha=0$, we have

$$\alpha_0^G=1,\quad \alpha_1^G=\frac{3\gamma+5}{\gamma+1},\quad \alpha_2^G=\frac{20\gamma^4+106\gamma^3+111\gamma^2+32\gamma-1}{(\gamma+1)(4\gamma^3+14\gamma^2+5\gamma-1)},$$

$$\alpha_3^G=\frac{4032\gamma^7+48480\gamma^6+176768\gamma^5+237320\gamma^4+72624\gamma^3-31006\gamma^2-8839\gamma+2489}{(4\gamma^3+14\gamma^2+5\gamma-1)(144\gamma^4+1104\gamma^3+1652\gamma^2+184\gamma-237)};$$

$$\beta_0^G=\gamma,\quad \beta_1=\frac{\gamma+1}{\gamma},\quad \beta_2^G=\frac{4\gamma^3+14\gamma^2+5\gamma-1}{\gamma(\gamma+1)^2},\quad \beta_3^G=\frac{\gamma(\gamma+1)\big(144\gamma^4+1104\gamma^3+1652\gamma^2+184\gamma-237\big)}{(4\gamma^3+14\gamma^2+5\gamma-1)^2},\quad\text{etc.},$$

where γ is the well-known Euler's constant (see [53]).

Theorem 3.4 ([53]). *If the functions $F_1(x)$ and $F_2(x)$ defined by (21) satisfy the condition of Theorem 3.3, the error bound of the method for the integral $I_1[f]$ can be estimated as*

$$\big|Q^n_{I_1[f]}-I_1[f]\big|=\begin{cases}O\big(\omega^{-2n-3/2}(1+\log\omega)\big), & \nu=0,\\ O\big(\omega^{-2n-3/2}\big), & \nu>0.\end{cases}$$

An alternative approach for computing the integral (23) has been also developed in [53]. Namely, we constructed the so-called *universal* (*direct*) quadrature formulas of Gaussian type

$$\int_0^{+\infty} g(t)\mathrm{e}^{-t}\,\mathrm{d}t = \sum_{k=1}^{n} A_k g(\tau_k) + R_n(g), \tag{24}$$

which are exact for each $g(t) = p(t) + q(t)\log t$, where $p(t)$ and $q(t)$ are algebraic polynomials of degree at most $n-1$. These quadrature rules can calculate integrals with a sufficient accuracy, regardless of whether their integrands contain a logarithmic singularity, or they do not. Thus, an application of such rules avoids the separation into singular and non-singular parts in integrands, as well as an additional integration of such a singular part using some special logarithmically weighted quadrature formula like one w.r.t. the weight function $w_\alpha^G(t)$. Thus, with the universal quadrature formula (24) we can directly calculate the integrals $I[F_1, y]$ and $I[F_2, y]$ given by (22) in Theorem 3.3; for example,

$$I[F_1, y] \approx \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\omega y}}{\omega} \sum_{k=1}^{n} A_k F_1\Big(y + \frac{\mathrm{i}\tau_k}{\omega}\Big).$$

Unfortunately, the construction of such universal quadrature formulas is not simple. Namely, there are not elegant tools for their construction like Golub-Welsch procedure in the case of construction quadrature rules with a polynomial degree of precision. In this non-polynomial case, in order to construct the quadrature formula (24), we must solve the following system of $2n$ nonlinear equations

$$\sum_{k=1}^{n} A_k \varphi_j(\tau_k) = \int_0^{+\infty} \varphi_j(t)\mathrm{e}^{-t}\,\mathrm{d}t, \quad j = 1, 2, \ldots, 2n, \tag{25}$$

in τ_k and A_k, $k = 1, \ldots, n$, taking an orthonormal system $\{\varphi_1, \varphi_2, \ldots, \varphi_{2n}\}$ obtained from the system of $2n$ linearly independent functions $U = \{1, t, \ldots, t^{n-1}, \log t, t\log t, \ldots, t^{n-1}\log t\}$ by an orthogonalization process (cf. [33, pp. 75–77]). Since $\varphi_1(t) = 1$, the right-hand side in the previous system of Eqs. becomes

$$\int_0^{+\infty} \varphi_j(t)\varphi_1(t)\mathrm{e}^{-t}\,\mathrm{d}t = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0. \end{cases}$$

Otherwise, a direct use of the non-orthogonal system of the basis functions U leads to a very ill-conditioned iterative process.

The orthonormal system of functions $\{\varphi_1, \varphi_2, \ldots, \varphi_{2n}\}$ can be considered as a Müntz system $\{t^{\lambda_0}, t^{\lambda_1}, \ldots, t^{\lambda_{2n-1}}\}$ on $(0, +\infty)$, with $\lambda_j = \lambda_{n+j} = j$, $j = 0, 1, \ldots, n-1$. Then, we can see that $\varphi_j(t) = \overline{L}_{j-1}(t)$, $j = 1, \ldots, n$, are normalized classical Laguerre polynomials. So, for different $n \in \mathbb{N}$, we obtain the following orthogonal functions:

1° $n = 1$:

$$\varphi_1(t) = 1, \quad \varphi_2(t) = \frac{\sqrt{6}}{\pi}(\gamma + \log t);$$

2° $n = 2$:

$$\varphi_1(t) = 1, \quad \varphi_2(t) = t - 1, \quad \varphi_3(t) = \sqrt{\frac{6}{\pi^2 - 6}}\big(\gamma + 1 - t + \log t\big),$$

$$\varphi_4(t) = \sqrt{\frac{6}{216 - 12\pi^4 + \pi^6}}\Big\{6 - \gamma(\pi^2 - 12) - \big[\pi^2 + \gamma(6 - \pi^2)\big]t + \big[12 - \pi^2 + (\pi^2 - 6)t\big]\log t\Big\};$$

3° $n = 3$:

$$\varphi_1(t) = 1, \quad \varphi_2(t) = t - 1, \quad \varphi_3(t) = \frac{1}{2}\big(t^2 - 4t + 2\big), \quad \varphi_4(t) = \frac{1}{2}\sqrt{\frac{3}{2\pi^2 - 15}}\big(6 + 4\gamma - 8t + t^2 + 4\log t\big),$$

$$\varphi_5(t) = C_5\Big\{24 - 2\pi^2 + \gamma(21 - 2\pi^2) + \big[2\pi^2 - 27 + \gamma(2\pi^2 - 15)\big]t + (9 - \pi^2)t^2 + \big[(2\pi^2 - 15)t - 2\pi^2 + 21\big]\log t\Big\},$$

$$\begin{aligned}\varphi_6(t) = C_6\Big\{&504 - 51\pi^2 + 2\gamma\big(279 - 48\pi^2 + 2\pi^4\big) + 2\big[4\pi^4 - 24\pi^2 - 153 - \gamma(4\pi^4 - 66\pi^2 + 261)\big]t \\ &+ \big[54 + 24\pi^2 - 3\pi^4 + \gamma(72 - 27\pi^2 + 2\pi^4)\big]t^2 \\ &+ \big[\big(72 - 27\pi^2 + 2\pi^4\big)t^2 - 2\big(261 - 66\pi^2 + 4\pi^4\big)t + 2(2\pi^4 - 48\pi^2 + 279)\big]\log t\Big\},\end{aligned}$$

where

$$C_5 = \sqrt{\frac{6}{-1080 + 549\pi^2 - 84\pi^4 + 4\pi^6}} \quad \text{and} \quad C_6 = \sqrt{\frac{3}{159408 - 65610\pi^2 + 2727\pi^4 + 1584\pi^6 - 216\pi^8 + 8\pi^{10}}},$$

etc.

For solving the system of equations (25) we use the well-known Newton-Kantorovich method, with quadratic convergence, but the main problem which then arises is how to provide sufficiently good starting values. Our strategy in the construction is based on the method of continuation, starting from the corresponding standard Gauss-Laguerre formula (with a polynomial degree of exactness). Numerical values of parameters τ_k and A_k, $k=1,\dots,n$, for $1\le n\le 6$ was presented in [53]. For some additional details on the generalized Gaussian quadratures on a finite interval and for Müntz systems of functions see [36], [39] and [37].

4 Gaussian Quadrature Formulas with a Modified Hermite Weight

I this section we consider the Gaussian quadrature formula on $\mathbb{R}$ with respect to a modified Hermite weight $x\mapsto \mathrm{e}^{-x^2}$ by the square root term $x\mapsto\sqrt{1+\alpha x+\beta x^2}$, i.e.,

$$w^{(\alpha,\beta)}(x)=\frac{\mathrm{e}^{-x^2}}{\sqrt{1+\alpha x+\beta x^2}}, \tag{26}$$

with the real parameters α and β such that $\alpha^2<4\beta$.

Remark 1. The weight function $w^{(\alpha,\beta)}(x)$ has the quasi-singularities near to the real axis if $\alpha^2\to 4\beta$. In the limit case, $w^{(\alpha,\alpha^2/4)}(x)$ has a singularity, i.e., a pole of the first order at the point $-\alpha/(2\beta)$ on the real line.

Several methods for modified weights (measures) by the rational terms (linear and quadratic factors and divisors) can be found in [15, Subsection 2.4], as well as the corresponding MATLAB software in [17, pp. 19–27].

Thus, we are interested here in constructing Gaussian quadrature rules of the form

$$\int_{-\infty}^{+\infty}\frac{f(x)}{\sqrt{1+\alpha x+\beta x^2}}\mathrm{e}^{-x^2}\,\mathrm{d}x=\sum_{\nu=1}^{N}A_\nu f(x_\nu)+R_N(f), \tag{27}$$

where $A_\nu=A_\nu^{(\alpha,\beta)}$ are weight coefficients (Christoffel numbers), and $R_n(f)$ is the corresponding remainder term, such that $R_N(f)=0$ for each $f\in\mathcal{P}_{2N-1}$.

Remark 2. In 1997 Bandrauk [3] stated a problem how to evaluate the integral

$$I_{m,n}^{\alpha,\beta}=\int_{-\infty}^{+\infty}\frac{H_m(x)H_n(x)}{\sqrt{1+\alpha x+\beta x^2}}\mathrm{e}^{-x^2}\,\mathrm{d}x, \tag{28}$$

where $H_m(x)$ is the Hermite polynomial of degree m, defined by

$$H_n(x)=(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathrm{e}^{-x^2}\big),\quad n\ge 0.$$

Alternatively, the question was how to find computationally effective approximations for the integral (28). The function $x\mapsto H_m(x)\mathrm{e}^{-x^2/2}$ is the quantum-mechanical wave function of m photons, the quanta of the electromagnetic field. The integral (28) expresses the modification of atomic Coulomb potentials by electromagnetic fields. In the case $m=n=0$, the integral $I_{0,0}^{\alpha,\beta}$ represents the vacuum or zero-field correction (for details see [2, Chaps. 1 and 3]).

Evidently, for $\alpha=\beta=0$, the integral $I_{m,n}^{0,0}$ expresses the orthogonality of the Hermite polynomials, i.e, $I_{m,n}^{0,0}=2^m m!\sqrt{\pi}\delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta.

A solution for $I_{0,0}^{\alpha,\beta}$ was derived by Grosjean [21] in the following form

$$I_{0,0}^{\alpha,\beta}=\frac{1}{\beta}\sum_{j=0}^{+\infty}\frac{\big[(4\beta-\alpha^2)/4\beta^2\big]^j}{2^{2j}(j!)^2}\sum_{r=0}^{+\infty}(-1)^r\frac{(2r+2j)!}{(2r)!(r+j)!}\left(\frac{\alpha}{2\beta}\right)^{2r}c_{r,j},$$

where

$$c_{r,j}=-\gamma+\log 4-\log\left(\frac{4\beta-\alpha^2}{4\beta^2}\right)+2H_j+H_{r+j}-2H_{2r+2j},$$

γ $(=0.57721566490\dots)$ is Euler's constant, and H_j is the j-th harmonic number,

$$H_j=1+\frac{1}{2}+\cdots+\frac{1}{j}.$$

Also, he gave a study of $I_{m,0}^{\alpha,\beta}$, $m=1,2,\dots$, as well as a five-term recurrence relation for these integrals.

The problem from Remark 2 was also considered in [35], with the monic Hermite polynomials $\widehat{H}_k(x)=2^{-k}H_k(x)$ in (28). For constructing the coefficients α_k and β_k, $k=0,1,\dots$, in the three-term recurrence relation

$$\pi_{k+1}(x)=(x-\alpha_k)\pi_k(x)-\beta_k\pi_{k-1}(x),\quad k\ge 0\quad (\pi_0(x)=1,\ p_{-1}(x)=0) \tag{29}$$

for polynomials $\pi_k(x)$ orthogonal on $(-\infty,\infty)$ with respect to the modified Hermite weight function (26), it was used the *discretized Stieltjes-Gautschi procedure* with the discretization based on the standard Gauss-Hermite quadratures,

$$\begin{aligned}\int_{-\infty}^{+\infty}P(t)w^{(\alpha,\beta)}(x)\,\mathrm{d}x &= \int_{-\infty}^{+\infty}\frac{P(x)}{\sqrt{1+\alpha x+\beta x^2}}\mathrm{e}^{-x^2}\,\mathrm{d}x\\ &\cong \sum_{k=1}^{N}\frac{\lambda_k^H P(\tau_k^H)}{\sqrt{1+\alpha\tau_k^H+\beta(\tau_k^H)^2}},\end{aligned}$$

where P is an arbitrary algebraic polynomial, and $\tau_k^H = \tau_k$ are nodes (zeros of $H_N(x)$) and

$$\lambda_k^H = \frac{2^{N-1}(N-1)!\sqrt{\pi}}{NH_{N-1}(\tau_k)^2}$$

are the weights (Christoffel numbers) of the N-point Gauss-Hermite quadrature formula (cf. [33, p. 325]). Such a procedure is needed for each of selected pairs (α,β). The recurrence coefficients for $k<20$ and $\alpha=\beta=1$ were presented in [35]. The corresponding Gaussian approximations were tested in double precision arithmetic in two cases: $m=3$, $n=6$, and $m=10$, $n=15$.

In this section we give a simple way for constructing the coefficients in the three-term recurrence relation (29), using the modified method of moments, realized in the MATHEMATICA package `OrthogonalPolynomials` ([9], [38]) in variable-precision arithmetic in order to overcome the numerical instability. All that is required is a procedure for numerical calculation of the modified moments in variable-precision arithmetic. In the same time, we give answer to the problem stated in Remark 2.

In our case we use the first $2N$ modified moments with respect to the sequence of the monic Hermite polynomials, i.e.,

$$m_k = m_k^{(\alpha,\beta)} = \int_{-\infty}^{+\infty} \frac{\widehat{H}_k(x)}{\sqrt{1+\alpha x+\beta x^2}} \mathrm{e}^{-x^2}\,\mathrm{d}x, \quad k=0,1,\ldots,2N-1, \tag{30}$$

in order to get quadrature rules of Gaussian type (27) for each $n \le N$, using the Golub-Welsch algorithm [19]. For the sequence $\{\widehat{H}_k(x)\}_{k\in\mathbb{N}_0}$ the following recurrence relation $\widehat{H}_{k+1}(x) = x\widehat{H}_k(x) - (k/2)\widehat{H}_{k-1}(x)$ holds, with $\widehat{H}_0(x)=1$ and $\widehat{H}_1(x)=x$.

First we transform the trinomial in the integral (30) to a canonical form

$$1+\alpha x+\beta x^2 = \beta\left[(x-p)^2+q^2\right], \quad p=-\frac{\alpha}{2\beta}, \quad q=\frac{4\beta-\alpha^2}{2\beta} \quad (p^2+q^2=\beta),$$

and then we apply the so-called *double-exponential* (DE) *transformation*

$$x = u(t) = p + q\sinh\left(\frac{\pi}{2}\sinh t\right),$$

in order to reduce the modified moments (30) to

$$m_k = m_k^{(\alpha,\beta)} = \frac{\pi}{2}\sqrt{p^2+q^2}\int_{-\infty}^{+\infty} \widehat{H}_k(u(t))\mathrm{e}^{-u(t)^2}\cosh t\,\mathrm{d}t, \quad k=0,1,\ldots,2N-1. \tag{31}$$

The crucial point in this DE transformation is the decay of the integrand be at least double exponential ($\approx \exp(-C\exp|t|)$ as $|t|\to+\infty$, where C is some positive constant. For integrals of such form of an analytic function on $\mathbb{R}$, it is known that the trapezoidal formula with an equal mesh size gives an optimal formula (cf. [25, 34, 41, 42, 47, 48, 49]).

For calculating the modified moments (31) we apply the trapezoidal formula with an equal mesh size h, i.e.,

$$m_k[h] = \frac{\pi h}{2}\sqrt{p^2+q^2}\sum_{j=-\infty}^{+\infty}\widehat{H}_k(u(jh))\mathrm{e}^{-u(jh)^2}\cosh jh, \quad k=0,1,\ldots,2N-1.$$

Since the integrand decays double exponentially, in actual computation of these sums we can truncate the infinite summation at $k=-M$ and $k=M$, so that

$$m_k \approx m_k[h;M] = \frac{\pi h}{2}\sqrt{p^2+q^2}\sum_{j=-M}^{M}\widehat{H}_k(u(jh))\mathrm{e}^{-u(jh)^2}\cosh jh, \quad k=0,1,\ldots,2N-1. \tag{32}$$

Because of some symmetry in the expression for $m_k[h;M]$, (32) can be implemented in the following way. Namely, if we put

$$t_j = jh, \quad \xi_j = q\sinh\left(\frac{\pi}{2}\sinh t_j\right), \quad c_j = 2\cosh(2p\xi_j), \quad s_j = 2\sinh(2p\xi_j),$$

we have $u(t_j)=p+\xi$, $u(-t_j)=p-\xi$, $u(0)=p$, and therefore

$$m_k[h;M] = \frac{\pi h}{2}\sqrt{p^2+q^2}\,\mathrm{e}^{-p^2}\left\{\widehat{H}_k(p)+\sum_{j=1}^{M}\mathrm{e}^{-\xi_j^2}\cosh(t_j)\left[\widehat{H}_k(p+\xi_j)\mathrm{e}^{-2p\xi_j}+\widehat{H}_k(p-\xi_j)\mathrm{e}^{2p\xi_j}\right]\right\}, \quad k=0,1,\ldots,2N-1.$$

Lemma 4.1. *Let*

$$\varphi_k(p,\xi) = \widehat{H}_k(p+\xi)\mathrm{e}^{-2p\xi}+\widehat{H}_k(p-\xi)\mathrm{e}^{2p\xi}, \quad \psi_k(p,\xi) = \widehat{H}_k(p+\xi)\mathrm{e}^{-2p\xi}-\widehat{H}_k(p-\xi)\mathrm{e}^{2p\xi}, \quad k=0,1,\ldots.$$

Then, the following recurrence relations

$$\varphi_{k+1}(p,\xi) = p\varphi_k(p,\xi)-\frac{k}{2}\varphi_{k-1}(p,\xi)-\xi\psi_k(p,\xi), \quad k=0,1,\ldots, \tag{33}$$

$$\psi_{k+1}(p,\xi) = p\psi_k(p,\xi)-\frac{k}{2}\psi_{k-1}(p,\xi)-\xi\varphi_k(p,\xi), \quad k=0,1,\ldots, \tag{34}$$

hold, where $\varphi_0(p,\xi)=2\cosh(2p\xi)$, $\psi_0((p,\xi)=2\sinh(2p\xi)$, *and* $\varphi_{-1}(p,\xi)=\psi_{-1}(p,\xi)=0$.

A proof of this lemma can be done using the three-term recurrence relation of the monic Hermite polynomials.

According to Lemma 4.1 we see that

$$m_k[h;M] = \frac{\pi h}{2}\sqrt{p^2+q^2}\,\mathrm{e}^{-p^2}\left\{\widehat{H}_k(p) + \sum_{j=1}^{M}\mathrm{e}^{-\xi_j^2}\cosh(t_j)\varphi_k(p,\xi)\right\}, \quad k=0,1,\ldots,2N-1. \tag{35}$$

In the sequel, as an example, we take $\alpha=\beta=50/13$ in the weight function (26) and $N=40$. Then we have $p=-1/2$ and $q=1/10$, which means that the integrands in (30) have quasi-singularites at $p\pm \mathrm{i}q$ in the complex plane.

In order to illustrate the effect of the before mentioned double-exponential decay of integrands, we present the graphics of integrands for $k=0,1,2,3$ (left) and $k=65$ (right) in Figure 5. The values of all integrands in (31), $k=0,1,\ldots,79$, at $t=2.1$, are:

$$\{1.\times10^{-321},\ 3.\times10^{-320},\ 7.\times10^{-319},\ 2.\times10^{-317},\ 5.\times10^{-316},\ 1.\times10^{-314},\ 4.\times10^{-313},\ 1.\times10^{-311},\ 3.\times10^{-310},\ 8.\times10^{-309},$$
$$2.\times10^{-307},\ 6.\times10^{-306},\ 2.\times10^{-304},\ 4.\times10^{-303},\ 1.\times10^{-301},\ 3.\times10^{-300},\ 8.\times10^{-299},\ 2.\times10^{-297},\ 6.\times10^{-296},\ 2.\times10^{-294},$$
$$4.\times10^{-293},\ 1.\times10^{-291},\ 3.\times10^{-290},\ 8.\times10^{-289},\ 2.\times10^{-287},\ 6.\times10^{-286},\ 2.\times10^{-284},\ 4.\times10^{-283},\ 1.\times10^{-281},\ 3.\times10^{-280},$$
$$8.\times10^{-279},\ 2.\times10^{-277},\ 6.\times10^{-276},\ 1.\times10^{-274},\ 4.\times10^{-273},\ 1.\times10^{-271},\ 3.\times10^{-270},\ 7.\times10^{-269},\ 2.\times10^{-267},\ 5.\times10^{-266},$$
$$1.\times10^{-264},\ 4.\times10^{-263},\ 1.\times10^{-261},\ 3.\times10^{-260},\ 7.\times10^{-259},\ 2.\times10^{-257},\ 5.\times10^{-256},\ 1.\times10^{-254},\ 3.\times10^{-253},\ 8.\times10^{-252},$$
$$2.\times10^{-250},\ 6.\times10^{-249},\ 2.\times10^{-247},\ 4.\times10^{-246},\ 1.\times10^{-244},\ 3.\times10^{-243},\ 7.\times10^{-242},\ 2.\times10^{-240},\ 5.\times10^{-239},\ 1.\times10^{-237},$$
$$3.\times10^{-236},\ 9.\times10^{-235},\ 2.\times10^{-233},\ 6.\times10^{-232},\ 2.\times10^{-230},\ 4.\times10^{-229},\ 1.\times10^{-227},\ 3.\times10^{-226},\ 7.\times10^{-225},\ 2.\times10^{-223},$$
$$5.\times10^{-222},\ 1.\times10^{-220},\ 3.\times10^{-219},\ 8.\times10^{-218},\ 2.\times10^{-216},\ 5.\times10^{-215},\ 1.\times10^{-213},\ 4.\times10^{-212},\ 9.\times10^{-211},\ 2.\times10^{-209}\},$$

and their maximal value is $2.\times10^{-209}$. Similarly, the maximal absolute value of all values at $t=-2.1$ is $4.\times10^{-232}$.

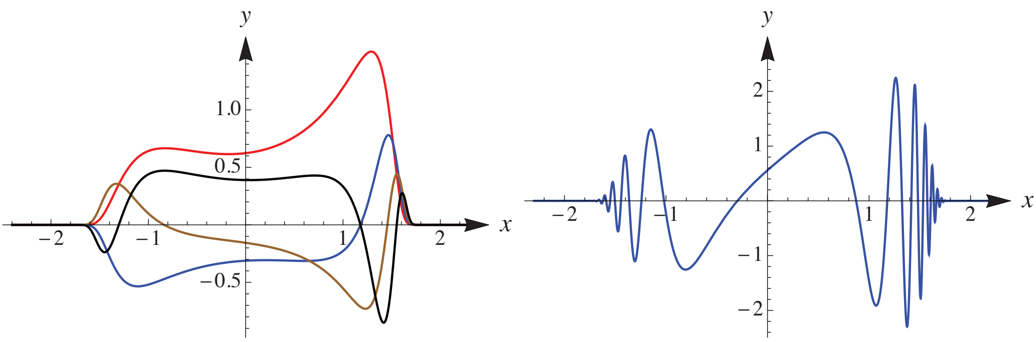

Figure 5: (*Left*) The integrands in $m_k^{(\alpha,\beta)}$ for $k=0$ (*red line*), $k=1$ (*blue line*), $k=2$ (*brown line*), and $k=3$ (*black line*) for $\alpha=\beta=50/13$; (*Right*) The integrand in $m_{65}^{(\alpha,\beta)}\times10^{-35}$ for $\alpha=\beta=50/13$

The corresponding MATHEMATICA code, which includes our package `OrthogonalPolynomials`, can be done in the following form:

```
<< orthogonalPolynomials`
(* Input of parameters alpha, beta, and Nmax *)
alpha = 50/13; beta = 50/13; Nmax = 40;
alphaH = Table[0,{k,0,2 Nmax}]; betaH = Prepend[Table[k/2,{k,1,2 Nmax}],Sqrt[Pi]];
HerM[x_] := aMakePolynomial[2 Nmax,alphaH,betaH,x,ReturnList -> True];
p=-alpha/(2 beta); q= Sqrt[4 beta-alpha^2]/(2 beta);
u[t_] := p + q Sinh[Pi/2 Sinh[t]];
fMH[t_,k_] := Pi/2 Sqrt[p^2+q^2] HerM[u[t]][[k+1]]Exp[-u[t]^2]Cosh[t];
(* Print values of integrands of all moments at t=2.1 *)
Tp = Table[N[fMH[21/10, k], 1], {k, 0, 2 Nmax-1}]; Print[Tp]; Max[Abs[Tp]]
```

The following code represents a procedure (DExpT) for calculating all moments (35), using the recurrence relations (33) and (34), as well as a command for calculating the recursive coefficients in (29), α_k and β_k, $k=0,1,\ldots,N-1$ (lists `alphaM` and `betaM`), by the Chebyshev methods of modified moments (`aChebyshevAlgorithmModified`):

```
Options[DExpT] = {WorkingPrecision -> $MachinePrecision};
DExpT[Pol_,b_,M_,p_,q_,Nmax_,Ops___] :=
Module[{wp,h,fac,momM,vt,j,xi,cvt,ec,c,s,phi0,phi1,phi2,psi0,psi1,psi2,k},
 {wp} = {WorkingPrecision} /. {Ops} /. Options[DExpT];
 Block[{$MinPrecision = wp}, h = b/M; fac = N[Pi/2 Sqrt[p^2+q^2]Exp[-p^2],wp];
 momM = N[Pol[p], wp]; vt = N[Table[j h, {j,1,M}], wp];
```

```
   xi = q Sinh[Pi/2 Sinh[vt]]; cvt = Cosh[vt]; ec = Exp[-xi^2] cvt;
   c = 2 Cosh[2 p xi]; s = 2 Sinh[2 p xi]; phi0 = c;
   phi1 = p c - xi s; psi0 = s; psi1 = p s - xi c;
   momM[[1]] = momM[[1]] + Total[ec  phi0];
   momM[[2]] = momM[[2]] + Total[ec  phi1];
   For[k = 1, k <= 2 Nmax - 2, k++,
    phi2 = p phi1 - k/2 phi0 - xi psi1;
    psi2 = p psi1 - k/2 psi0 - xi phi1;
    momM[[k + 2]] = momM[[k + 2]] + Total[ec phi2];
    phi0 = phi1; psi0 = psi1; phi1 = phi2; psi1 = psi2;]; momM = h fac  momM;
   Return[momM];];];

momM = DExpT[Function[x,HerM[x]], 21/10, 800, p, q, Nmax, WorkingPrecision -> 52];
{alphaM, betaM} = aChebyshevAlgorithmModified[momM, alphaH, betaH, WorkingPrecision -> 52];
```

As we can see, in this case, the moment integrals are calculated by the trapezoidal rule, taking $M = 800$ (positive) equidistant nodes on the finite interval $[0, b] = [0, 21/10]$. In in order to overcome the numerical instability and obtain the first $N = 40$ recursion coefficients α_k and β_k with 40 exact decimal digits, we had used the working precision of 52 decimal digits (`WorkingPrecision -> 52`). These recursion coefficients for $\alpha = \beta = 50/13$ are shown in Table 3.

Table 3: Recursion coefficients for the polynomials $\{\pi_k(\cdot\,; w^{(\alpha,\beta)})\}$, $\alpha = \beta = 50/13$

k	alpha(k)	beta(k)
0	-3.056007650989553610006858836445558057864E-01	2.372619381077149609357146735045269999374E+00
1	1.091135901172070725048795730904031636823E-01	2.245468863371941107162002605406537898164E-01
2	-6.393650751044613184510213541531714004955E-02	7.464045492492715756061806875559370152332E-01
3	2.896093622914383390462496451022656640930E-02	1.302942897121644523552877231408192507139E+00
4	-1.209308423360305534840259415165592137550E-02	1.709113932903603587555654516890103590942E+00
5	-1.718837754000391722650945735332100058077E-03	2.309675711077846031637617355708075984524E+00
6	7.750720489169099233811949642062773337494E-03	2.711933128251153801071850610620211255924E+00
7	-1.298438968330839486761114881503272529107E-02	3.295423910768401255759852684415586881992E+00
8	1.412241831101624089616484008974666521998E-02	3.727067165030535190750968123128118634984E+00
9	-1.536929634325645177722701317932091457588E-02	4.276650074098969245628776026578674761372E+00
10	1.429372116400763580863048010758003995197E-02	4.743791732695879570081833190100403204112E+00
11	-1.371654702276508069074812825473119164151E-02	5.259693049405470172104203173535811336196E+00
12	1.176798222770504655068245919798468067142E-02	5.757791419391763379586773316034897076418E+00
13	-1.047335910809081780000034574295255003176E-02	6.246792977846921565057499044107858314470E+00
14	8.340981954066187739248307612282341319947E-03	6.767670276167376192729000834886725961325E+00
15	-6.902153058283111921150338986783630810628E-03	7.238344308003691658560233194861394606057E+00
16	4.929945437819196532285480021173326789508E-03	7.773406567159667299886098793924001808522E+00
17	-3.631375626543328827099446310229842010077E-03	8.233910000676667222653908148429153190199E+00
18	1.972847873088252308562475996047991553534E-03	8.775587423340927445076726663037241731381E+00
19	-9.382315978354561145269552590737997646937E-04	9.232721978517029915230829146365560298491E+00
20	-3.590681123674328669455571404324200078543E-04	9.775014853263387106715308533241278438528E+00
21	1.095358678598460476718502577077385730842E-03	1.023393662540638845246860030635602508313E+01
22	-2.041461679283772492661795151991013174894E-03	1.077250428073515075650921973182315999394E+01
23	2.493038018253023527313131199875957019916E-03	1.123676305795445168215946077180108462115E+01
24	-3.127157761850318637784558126478989924399E-03	1.176878719309650540780247278527980428697E+01
25	3.331307697373371667460005653852087490994E-03	1.224052203413264299811344867071655763280E+01
26	-3.705066401755012230703476782449637934130E-03	1.276447094822281468910882302771173682594E+01
27	3.708472613355638464086302889208344266282E-03	1.324466634505785425765285743005522902919E+01
28	-3.876118282811593232978939167314294489251E-03	1.376002971186171368411809734326371916264E+01
29	3.726629521970398538846032669091781797121E-03	1.424878006310961035489193536070686057143E+01
30	-3.739541281204642760970291432757018722020E-03	1.475581173402593548748610073608230163091E+01
31	3.481647957368403622687805428236969037966E-03	1.525256676562352665482501057345643055499E+01
32	-3.385493240656504254144846089209709418270E-03	1.575205446688666048604769866201807323007E+01
33	3.058077360327979878387310350953018114909E-03	1.625583271515342584196983999075092016028E+01
34	-2.891602583824989303750000660963506068601E-03	1.674890265429714582600860755968340910384E+01
35	2.527072773248249037028648170239124387933E-03	1.725846851667865695915038035877920062809E+01
36	-2.321882218704978824348409214597430203417E-03	1.774642665450064142184409594060804319341E+01
37	1.946133267460015383516991475529693281566E-03	1.826043126925382043488200028053223782116E+01
38	-1.727043261257568539548073321379504740500E-03	1.874463952401863690524176781475423384354E+01
39	1.359884433345210740624392306660577854045E-03	1.926172829906529820447076427312396940827E+01

These recursive coefficients enable us to construct the Gaussian formulas (27) for each $N \le 40$.

We return now to the problem (28) given in Remark 2. Note that the integrand $x \mapsto H_m(x)H_n(x)w^{(\alpha,\beta)}(x)$ in (28) has $m+n$ zeros on $\mathbb{R}$ and very large oscillations (see graphics in Fig. 6).

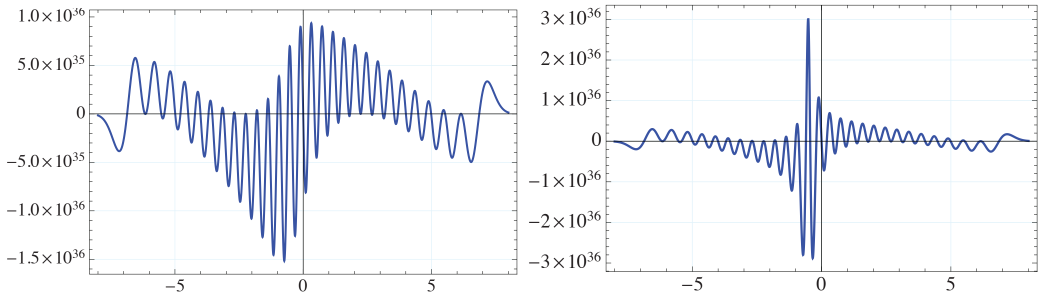

Figure 6: The integrand $x \mapsto H_{30}(x)H_{25}(x)w^{(\alpha,\beta)}(x)$ for $\alpha = \beta = 1$ (*left*) and $\alpha = \beta = 50/13$ (*right*)

Using the well-known Feldheim's linearization formula for Hermite polynomials (cf. Askey [1, p. 42])

$$H_m(x)H_n(x) = \sum_{\nu=0}^{\min(m,n)} \binom{m}{\nu}\binom{n}{\nu} 2^\nu \nu! H_{m+n-2\nu}(x),$$

we can transform (28) to

$$I_{m,n}^{\alpha,\beta} = 2^{m+n} \sum_{\nu=0}^{\min(m,n)} \binom{m}{\nu}\binom{n}{\nu} \frac{\nu!}{2^\nu} \int_{-\infty}^{+\infty} \frac{\widehat{H}_{m+n-2\nu}(x)}{\sqrt{1+\alpha x+\beta x^2}} \mathrm{e}^{-x^2}\,\mathrm{d}x = 2^{m+n} \sum_{\nu=0}^{\min(m,n)} \binom{m}{\nu}\binom{n}{\nu} \frac{\nu!}{2^\nu} m_{m+n-2\nu}^{(\alpha,\beta)},$$

i.e., $I_{m,n}^{\alpha,\beta}$ can be expressed in terms of the modified moments (30) or approximatively by $m_k[h;M]$, i.e.,

$$I_{m,n}^{\alpha,\beta} \approx 2^{m+n} \sum_{\nu=0}^{\min(m,n)} \binom{m}{\nu}\binom{n}{\nu} \frac{\nu!}{2^\nu} m_{m+n-2\nu}[h;M],$$

with some appropriate h and M.

Table 4: Gaussian approximations $Q_{30,25}^{(N)}$ of the integral $I_{30,25}^{\alpha,\beta}$ for $\alpha = \beta = 50/13$ and $N = 25(1)30$

N	$Q_{30,25}^{(N)}$
25	3.89824405255802820082386454675769487675 8(+35)
26	−1.42723752156172556525453646696194608710 1(+36)
27	−3.38570855433939840091913763148415647327 1(+35)
28	−6.86613808469115622651744579460148014601 9(+35)
29	−6.86613808469115622651744579460148014601 9(+35)
30	−6.86613808469115622651744579460148014601 9(+35)

Alternatively, $I_{m,n}^{\alpha,\beta}$ can be exactly calculated (up to rounding errors) by applying the N-point Gaussian formula (27), for a given parameters α and β, taking the number of nodes N to be such that $m+n \le 2N-1$. Thus,

$$I_{m,n}^{\alpha,\beta} \approx Q_{m,n}^{(N)} = \sum_{\nu=1}^{N} A_\nu H_m(x_\nu)H_n(x_\nu). \tag{36}$$

For example, to calculate $I_{30,25}^{\alpha,\beta}$ we need $N \ge 28$.

Taking recursion coefficients from Table 3 we can evaluate nodes and weights (x_ν and A_ν) in the quadrature formula (27) by the function `aGaussianNodesWeights` from our Mathematica package `OrthogonalPolynomials`, in this case, up to $N \le 40$. The corresponding Gaussian approximations of the integral $I_{30,25}^{50/13,50/13}$ are presented in Table 4 for $N = 25(1)30$. As we can see, the obtained results for $N \ge 28$ are exact (up to rounding errors). Results in error are displayed in red.

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ORIGINAL PAPER

Symbolic–numeric computation of orthogonal polynomials and Gaussian quadratures with respect to the cardinal B-spline

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Abstract The first 60 coefficients in the three-term recurrence relation for monic polynomials orthogonal with respect to cardinal B-splines φ_m as the weight functions on $[0, m]$ ($m \in \mathbb{N}$) are obtained in a symbolic form. They enable calculation of parameters, nodes, and weights, in the corresponding Gaussian quadrature up to 60 nodes. The efficiency of these Gaussian quadratures is shown in some numerical examples. Finally, two interesting conjectures are stated.

Keywords Orthogonal polynomial · Cardinal B-spline · Moment · Recurrence relation · Gaussian quadrature formula · Symbolic computation

Mathematics Subject Classification (2010) 42C05 · 41A55 · 41A15 · 65D30 · 65D32

1 Introduction

In this paper, we consider quadrature formulae of Gaussian type,

$$\int_{\mathbb{R}} f(x)\varphi_m(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} A_{n,\nu}^{[m]} f\left(x_{n,\nu}^{[m]}\right) + R_n^{(m)}(f), \tag{1}$$

Dedicated to Prof. Walter Gautschi on the occasion of his 90th birthday

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as well as the corresponding system of orthogonal polynomials $\{\pi_k^{[m]}\}_{k=0}^{+\infty}$ with respect to cardinal B-splines as the weight functions on $[0, m]$ ($m \in \mathbb{N}$).

Cardinal B-splines play an important role in many problems in approximation (e.g., spline interpolation, multiresolution approximation, different methods for solving initial and boundary value problems, etc.). They can be defined recursively starting from the cardinal spline of the first order $\varphi_1(\cdot)$, which is the characteristic function of the interval $[0, 1)$, i.e.,

$$\varphi_1(x) = \begin{cases} 1, & x \in [0, 1), \\ 0, & \text{otherwise} \end{cases}$$

Then, the cardinal B-spline $\varphi_m(\cdot)$ of order m is defined as the convolution

$$\varphi_m(x) = (\varphi_{m-1} * \varphi_1)(x) = \int_{\mathbb{R}} \varphi_{m-1}(x-t)\varphi_1(t)\,\mathrm{d}t = \int_0^1 \varphi_{m-1}(x-t)\,\mathrm{d}t.$$

It is supported and symmetric on $[0, m]$, i.e., for each $x \in [0, m]$, $\varphi_m(x) = \varphi_m(m-x)$. On each interval $[k, k+1]$, $0 \le k \le m-1$, the cardinal B-spline of order m is a polynomial of degree $m-1$ and $\varphi_m(\cdot) \in C^{m-2}[0, m]$.

The (monic) orthogonal polynomials $\{\pi_k^{[m]}\}_{k=0}^{+\infty}$ (with respect to the weight function $\varphi_m(\cdot)$ on $[0, m]$) satisfy a three-term recurrence relation

$$\pi_{k+1}^{[m]}(x) = (x - \alpha_k^{[m]})\pi_k^{[m]}(x) - \beta_k^{[m]}\pi_{k-1}^{[m]}(x), \quad k = 0, 1, \ldots, \tag{2}$$

with $\pi_0^{[m]}(x) = 1$, $\pi_{-1}^{[m]}(x) = 0$, where $\alpha_k^{[m]}$ and $\beta_k^{[m]}$ are real resp. positive numbers. The coefficient $\beta_0^{[m]}$ may be arbitrary, but usually, it is appropriate to take $\beta_0^{[m]} = \int_{\mathbb{R}} \varphi_m(x)\,\mathrm{d}x$.

It is known that the nodes $x_{n,k}^{[m]}$ in the Gaussian quadrature rule (1) are eigenvalues of the symmetric tridiagonal Jacobi matrix (cf. [17, pp. 325–328])

$$J_n(\varphi_m) = \begin{bmatrix} \alpha_0^{[m]} & \sqrt{\beta_1^{[m]}} & & & \mathbf{O} \\ \sqrt{\beta_1^{[m]}} & \alpha_1^{[m]} & \sqrt{\beta_2^{[m]}} & & \\ & \sqrt{\beta_2^{[m]}} & \alpha_2^{[m]} & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}^{[m]}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}^{[m]}} & \alpha_{n-1}^{[m]} \end{bmatrix}, \tag{3}$$

and the weight coefficients $A_{n,k}^{[m]}$ are given by $A_{n,k}^{[m]} = \beta_0^{[m]} v_{k,1}^2$, $k = 1, \ldots, n$, where $v_{k,1}$ is the first component of the eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,n}]^{\mathrm{T}})$ corresponding to the eigenvalue $x_{n,k}^{[m]}$ and normalized such that $\mathbf{v}_k^{\mathrm{T}}\mathbf{v}_k = 1$. The most popular method for solving this eigenvalue problem is the Golub–Welsch procedure, obtained by a simplification of the QR algorithm [15].

Therefore, if we know the recurrence coefficients $\alpha_k^{[m]}$ and $\beta_k^{[m]}$ in the fundamental three-term recurrence relation (2), the problem of constructing Gaussian rules can be easily solved by the Golub–Welsch procedure. This procedure is implemented in several packages including the best known, ORTHPOL, given by Gautschi [11]. According to (3), for constructing Gauss–Christoffel quadratures (1) for any number of nodes less than or equal to n, we need the first n recursion coefficients $\alpha_k^{[m]}$ and $\beta_k^{[m]}$, $k = 0, 1, \ldots, n-1$.

In general, the recursion coefficients are known explicitly only for some narrow classes of orthogonal polynomials, e.g., for the so-called very classical orthogonal polynomials (Jacobi, the generalized Laguerre, and Hermite polynomials). However, in the case of the so-called *strongly non-classical polynomials*, for which their weight function is given explicitly, or implicitly via moment information, the recursion coefficients must be constructed numerically. Such a problem is very sensitive with respect to small perturbations in the input data. In the eighties on the last century, Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, including effective algorithms for numerically generating recursion coefficients (the method of (modified) moments, the discretized Stieltjes–Gautschi procedure, and the Lanczos algorithm), a detailed stability analysis of such algorithms as well as several new applications of orthogonal polynomials. The basic references are [10, 12, 18]. An interesting stable recursive technique for the determination of Jacobi matrices associated with multi-fractal measures via iterated functions systems was given by Mantica [16].

Some particular cases of the Gaussian quadratures for B-splines of degree 1 or 3 were constructed numerically by Phillips and Hanson [22], who obtained the first 17 coefficients in the three-term recurrence formula for orthonormal polynomials (rounded to 14 decimal digits). In connection with these results, we mention also that Gaussian quadrature and orthogonal polynomials for refinable weight functions were considered by Gautschi, Gori, and Pitolli [9] and Laurie and de Villiers [14]. For related quadratures with assigned nodes, see a recent work by Calabrò, Manni, and Pitolli [3]. Also, Calabrò and Corbo Esposito [2] considered numerical methods for integration with respect to binomial measures and gave several numerical tests to verify the efficiency and accuracy of their methods. A connection between refinable functions, functionals, and iterated function systems has been described recently in [4].

Recent progress in symbolic computation and variable-precision arithmetic now makes it possible to generate the recursion coefficients in the three-term recurrence relation (2) directly by using the original Chebyshev method of moments. Respective symbolic/variable-precision software for orthogonal polynomials is available: Gautschi's package SOPQ in MATLAB (see Appendix B in [13]) and our MATHEMATICA package OrthogonalPolynomials (see [7] and [19]), which is downloadable from the web site http://www.mi.sanu.ac.rs/~gvm/.

In this paper, we construct the recursion coefficients $\alpha_k^{[m]}$ and $\beta_k^{[m]}$, using the Chebyshev method (implemented in the package OrthogonalPolynomials) with the moments

$$\mu_k^{(m)} = \int_0^m x^k \varphi_m(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots, \tag{4}$$

represented in an appropriate (polynomial) form for each $m \in \mathbb{N}$, which is enough to obtain these coefficients in a symbolic form. Namely, the Chebyshev method can be represented as the mapping of the sequence of moments into the coefficients of the three-term recurrence relation. The algorithm is rational and nonlinear and it can be realized using the recurrence relation which uses only two basic operations – the addition and the multiplication (cf. [7]).

The paper is organized as follows. A procedure for calculating the moments (4) as polynomials in m is given in Section 2, and the symbolic generation of recursion coefficients $\alpha_k^{[m]}$ and $\beta_k^{[m]}$ and a discussion of the corresponding Gaussian quadratures (1) are given in Section 3. Some applications of these Gaussian quadratures are illustrated in Section 4. Finally, in Section 5, two interesting conjectures are stated relating to the recurrence coefficients obtained.

2 Moments

Using the relations for the cardinal B-spline (cf. [5, p. 86], [6, p. 56])

$$\varphi_m(x) = \frac{x}{m-1}\varphi_{m-1}(x) + \frac{m-x}{m-1}\varphi_{m-1}(x-1), \quad m \geq 2, \tag{5}$$

and

$$\varphi_m'(x) = \varphi_{m-1}(x) - \varphi_{m-1}(x-1), \quad m \geq 2, \tag{6}$$

after some simplification, Milovanović and Udovičić [20] have recently obtained the following differential equation

$$(m-x)\varphi_m'(x) + (m-1)\varphi_m(x) = m\varphi_{m-1}(x), \tag{7}$$

and then proposed an effective method for calculating the coefficients of polynomials defining a cardinal B-spline.

For example,

$$\varphi_2(x) = \begin{cases} x, & 0 \leq x < 1, \\ 2-x, & 1 \leq x < 2, \\ 0, & \text{otherwise}, \end{cases} \qquad \varphi_3(x) = \begin{cases} \frac{1}{2}x^2, & 0 \leq x < 1, \\ -x^2+3x-\frac{3}{2}, & 1 \leq x < 2, \\ \frac{1}{2}x^2-3x+\frac{9}{2}, & 2 \leq x < 3, \\ 0, & \text{otherwise}, \end{cases}$$

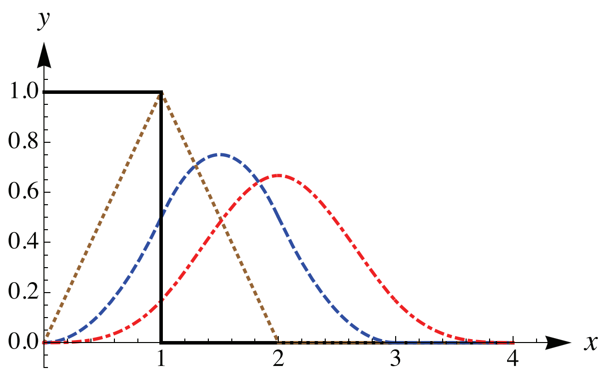

Fig. 1 Cardinal B-spline $\varphi_m(t)$ for $m = 1$ (*solid line*), $m = 2$ (*dotted*), $m = 3$ (*dashed*), and $m = 4$ (*dot-dashed*)

etc. In Fig. 1 we display $\varphi_m(t)$ for $m = 1, 2, 3, 4$.

Now, we want to calculate the moments (4) for each $m \in \mathbb{N}$.

Using (7), after multiplying by x^k and integrating over $[0, m]$, we get

$$\int_0^m (m-x)x^k\varphi_m'(x)\,\mathrm{d}x + (m-1)\int_0^m x^k\varphi_m(x)\,\mathrm{d}x = m\int_0^m x^k\varphi_{m-1}(x)\,\mathrm{d}x.$$

Since, by integration by parts,

$$\begin{aligned}\int_0^m (m-x)x^k\varphi_m'(x)\,\mathrm{d}x &= (m-x)x^k\varphi_m(x)\,\Big|_0^m \\ &\quad -\int_0^m \left[kmx^{k-1}-(k+1)x^k\right]\varphi_m(x)\,\mathrm{d}x \\ &= (k+1)\mu_k^{(m)} - km\mu_{k-1}^{(m)},\end{aligned}$$

because $\varphi_m(0) = \varphi_m(m) = 0$, we conclude that the moments (4) satisfy the following recurrence relation

$$(k+m)\mu_k^{(m)} - km\mu_{k-1}^{(m)} = m\mu_k^{(m-1)}, \quad m \geq 2. \tag{8}$$

It is not difficult to calculate directly

$$\begin{aligned}\mu_k^{(1)} &= \int_0^1 x^k\varphi_1(x)\,\mathrm{d}x = \frac{1}{k+1} \quad \text{(Legendre case shifted to } [0,1]) \\ \mu_k^{(2)} &= \int_0^2 x^k\varphi_2(x)\,\mathrm{d}x = \frac{2(2^{k+1}-1)}{(k+1)(k+2)}, \\ \mu_k^{(3)} &= \int_0^3 x^k\varphi_3(x)\,\mathrm{d}x = \frac{3(3^{k+2}-2\cdot 2^{k+2}+1)}{(k+1)(k+2)(k+3)}, \\ \mu_k^{(4)} &= \int_0^4 x^k\varphi_4(x)\,\mathrm{d}x = \frac{4(4^{k+3}-3\cdot 3^{k+3}+3\cdot 2^{k+3}-1)}{(k+1)(k+2)(k+3)(k+4)},\end{aligned}$$

etc. These formulae suggest a general formula for an arbitrary $m \in \mathbb{N}$,

$$\begin{aligned}\mu_k^{(m)} &= \int_0^m x^k\varphi_m(x)\,\mathrm{d}x \\ &= \frac{m(-1)^m}{(k+1)_m}\sum_{\nu=1}^{m}(-1)^\nu\binom{m-1}{\nu-1}\nu^{k+m-1}, \quad k = 0, 1, \ldots,\end{aligned} \tag{9}$$

where $(p)_m$ is the standard notation for Pochhammer's symbol

$$(p)_m = p(p+1)\cdots(p+m-1) = \frac{\Gamma(p+m)}{\Gamma(p)} \quad (\Gamma \text{ is the gamma function}).$$

Indeed, (9) can be proved very easily by induction, using the recurrence relation (8).

These moments can also be expressed in terms of the Stirling numbers of the second kind $S(n, m)$, which are the coefficients in the expansion

$$x^n = \sum_{m=0}^{n} S(n, m)(x)^{(m)},$$

where the so-called falling factorial is defined by $(x)^{(m)} = x(x-1)\cdots(x-m+1)$ and $(x)^{(0)} = 1$. Precisely, $S(n, m)$ gives the number of ways of partitioning a set of n elements into m non-empty subsets. These numbers obey the following recurrence relation

$$S(n+1, m) = mS(n, m) + S(n, m-1). \tag{10}$$

Note that $S(n, 1) = S(n, n) = 1$ and $S(n, m) = 0$ for $m > n$, as well as $S(n, 2) = 2^{n-1} - 1$ and $S(n, n-1) = \frac{1}{2}n(n-1)$ (cf. [23]). In general, there is the following explicit formula (cf. [8, p. 69])

$$S(n, m) = \frac{1}{m!}\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}(m-j)^n = \frac{1}{m!}\sum_{\nu=1}^{m}(-1)^{m-\nu}\binom{m}{\nu}\nu^n,$$

which can be related to (9) by Theorem 1 below. Namely, taking $n = m + k$, the above equality becomes

$$S(m+k, m) = \frac{1}{m!}\sum_{\nu=1}^{m}(-1)^{m-\nu}\frac{m}{\nu}\binom{m-1}{\nu-1}\nu^{m+k},$$

i.e.,

$$S(m+k, m) = \frac{(-1)^m m}{m!}\sum_{\nu=1}^{m}(-1)^{\nu}\binom{m-1}{\nu-1}\nu^{m+k-1}. \tag{11}$$

In the sequel, we need the following representation of the moments:

Theorem 1 *The moments of the weight function $x \mapsto \varphi_m(x)$ on $[0, m]$ can be expressed in terms of Stirling numbers of the second kind,*

$$\mu_k^{(m)} = \int_0^m x^k \varphi_m(x)\,\mathrm{d}x = \frac{S(m+k, m)}{\binom{m+k}{m}}, \quad k = 0, 1, \ldots, \tag{12}$$

and their exponential generating function is given by

$$\sum_{k=0}^{+\infty}\mu_k^{(m)}\frac{x^k}{k!} = \left(\frac{\mathrm{e}^x - 1}{x}\right)^m \quad (m \in \mathbb{N}). \tag{13}$$

Proof According to (9) and (11), we obtain

$$\mu_k^{(m)} = \frac{m!S(m+k, m)}{(k+1)_m}.$$

Since $(k+1)_m = (k+m)!/k!$, this reduces to (12).

In order to prove (13), we start with the exponential generating function for Stirling polynomials of the second kind (cf. [1, p. 30])

$$\frac{(\mathrm{e}^x-1)^m}{m!}=\sum_{n=m}^{+\infty}S(n,m)\frac{x^n}{n!}. \tag{14}$$

Since the equality (14) can be written in the form

$$\frac{(\mathrm{e}^x-1)^m}{m!}=\sum_{k=0}^{+\infty}S(m+k,m)\frac{x^{m+k}}{(m+k)!},$$

we conclude that

$$\sum_{k=0}^{+\infty}\mu_k^{(m)}\frac{x^k}{k!}=\sum_{k=0}^{+\infty}\frac{S(m+k,m)}{\binom{m+k}{m}}\frac{x^k}{k!}=\left(\frac{\mathrm{e}^x-1}{x}\right)^m,$$

i.e., the m-th power of the Bose–Einsten function $x\mapsto(\mathrm{e}^x-1)/x$ is the exponential generating function for the moments $\mu_k^{(m)}$. □

Remark 1 The generating function (13) can be found also in [21].

In order to determine the first n recurrence coefficients $\alpha_k^{[m]}$ and $\beta_k^{[m]}$, $k=0,1,\ldots,n-1$, in the recurrence relation (2), using our MATHEMATICA package `OrthogonalPolynomials` (see [7] and [19]), we only need a procedure for symbolic calculation of the first $2n$ moments.

Expanding the exponential generating function (13) in a power series in x, we can find the moments $\mu_k^{(m)}$, $k\geq 0$, for a fixed $m\in\mathbb{N}$. For example, for $n=50$ (in our case this is reasonable), the first 100 moments can be obtained in a symbolic form by simple commands in MATHEMATICA 11.0.1.0:

```
ser = Series[((Exp[x] - 1)/x)^m, {x, 0, 99}] // Simplify;
mommu = Table[k! Coefficient[ser, x, k], {k, 0, 99}];
```

In this way, we obtain $\mu_k^{(m)}$ as polynomials in m of degree k,

$$\mu_0^{(m)}=1,\quad \mu_1^{(m)}=\frac{m}{2},\quad \mu_2^{(m)}=\frac{m}{12}(3m+1),\quad \mu_3^{(m)}=\frac{m^2}{8}(m+1),$$

$$\mu_4^{(m)}=\frac{m}{240}\left(15m^3+30m^2+5m-2\right),\quad \mu_5^{(m)}=\frac{m^2}{96}\left(3m^3+10m^2+5m-2\right),$$

$$\mu_6^{(m)}=\frac{m}{4032}\left(63m^5+315m^4+315m^3-91m^2-42m+16\right),$$

$$\mu_7^{(m)}=\frac{m^2}{1152}\left(9m^5+63m^4+105m^3-7m^2-42m+16\right),$$

$$\mu_8^{(m)}=\frac{m}{34560}\left(135m^7+1260m^6+3150m^5+840m^4-2345m^3+540m^2+404m-144\right),$$

$$\mu_9^{(m)}=\frac{m^2}{7680}\left(15m^7+180m^6+630m^5+448m^4-665m^3-100m^2+404m-144\right),$$

$$\mu_{10}^{(m)} = \frac{m}{101376}\Big(99m^9 + 1485m^8 + 6930m^7 + 8778m^6 - 8085m^5 - 8195m^4 + 11792m^3 -2068m^2 - 2288m + 768\Big),$$

$$\mu_{11}^{(m)} = \frac{m^2}{18432}\Big(9m^9 + 165m^8 + 990m^7 + 1914m^6 - 847m^5 - 3179m^4 + 2904m^3 +1100m^2 - 2288m + 768\Big),$$

$$\mu_{12}^{(m)} = \frac{m}{50319360}\Big(12285m^{11} + 270270m^{10} + 2027025m^9 + 5495490m^8 + 315315m^7 -12882870m^6 + 5760755m^5 + 14444430m^4 - 15875860m^3 +2037672m^2 + 3327584m - 1061376\Big),$$

$$\mu_{13}^{(m)} = \frac{m^2}{7741440}\Big(945m^{11} + 24570m^{10} + 225225m^9 + 810810m^8 + 495495m^7 -2320890m^6 - 389389m^5 + 4978974m^4 - 3383380m^3 - 2155608m^2 +3327584m - 1061376),$$

$$\mu_{14}^{(m)} = \frac{m}{6635520}\Big(405m^{13} + 12285m^{12} + 135135m^{11} + 621621m^{10} + 765765m^9 -1898325m^8 - 2141139m^7 + 6565559m^6 - 990990m^5 - 8790964m^4 +8132904m^3 - 712672m^2 - 1810176m + 552960\Big),$$

$$\mu_{15}^{(m)} = \frac{m^2}{884736}\Big(27m^{13} + 945m^{12} + 12285m^{11} + 70161m^{10} + 135135m^9 - 190905m^8 -566137m^7 + 986843m^6 + 778778m^5 - 2802436m^4 + 1477736m^3 +1410080m^2 - 1810176m + 552960\Big),$$

etc.

Remark 2 It could be of some interest to investigate properties of the polynomials $P_k(m) := \mu_k^{(m)}$.

3 Recurrence coefficients and Gaussian quadratures

Using our MATHEMATICA package `OrthogonalPolynomials`, with the obtained moments `mommu`, and executing only the following commands:

```
<< orthogonalPolynomials`
{alpha,beta}=aChebyshevAlgorithm[mommu,Algorithm->Symbolic];
```

we obtain the first 50 recurrence coefficients for monic orthogonal polynomials $\{\pi_k^{[m]}\}_{k=0}^{+\infty}$ in (2) in a symbolic form,

$$\alpha_k^{(m)} = \frac{m}{2} \quad (k = 0, 1, 2, \ldots);$$

$$\beta_0^{(m)} = 1, \quad \beta_1^{(m)} = \frac{m}{12}, \quad \beta_2^{(m)} = \frac{5m-3}{30}, \quad \beta_3^{(m)} = \frac{175m^2 - 315m + 158}{140(5m-3)},$$

$$\beta_4^{(m)} = \frac{6125m^4 - 25725m^3 + 41965m^2 - 29547m + 7230}{21(5m-3)\big(175m^2 - 315m + 158\big)},$$

$$\beta_5^{(m)} = \frac{25(5m-3)S_6(m)}{132\big(175m^2 - 315m + 158\big)S_4(m)},$$

where

$$S_4(x) = 6125m^4 - 25725m^3 + 41965m^2 - 29547m + 7230,$$
$$S_6(x) = 471625m^6 - 3678675m^5 + 12324235m^4 - 22096305m^3 + 22009540m^2 - 11549748m + 2519472,$$

etc. It is clear that the coefficients $\alpha_k^{(m)}$ are equal to $m/2$ for each k, because of the property $\varphi_m(t) = \varphi_m(m-t)$, $0 \le t \le m$. For higher k, the corresponding $\beta_k^{(m)}$ coefficients become quite complicated. Specific values of these coefficients for a given m (e.g., $m = 10$ and $k \le 9$) can be obtained directly by the command:

$$\left\{1, \frac{5}{6}, \frac{47}{30}, \frac{3627}{1645}, \frac{3286105}{1193283}, \frac{168899331365}{52442292474}, \frac{187763661474877107}{51959406895719290}, \frac{28601715426755929240411636}{72553550395220666602115205},\right.$$

$$\left.\frac{4176933381056765717383014530658198 50}{99070749556201898603801123009129259}, \frac{354961553929825993746744991832568494429934906 27}{798495299147790584739336069068158838653075784 2}\right\}$$

```
TraditionalForm[Table[beta[[k + 1]], {k, 0, 9}] /. {m -> 10}]
```

Also, numerical values of the first thirty recursion coefficients $\beta_k^{(m)}$, $0 \le k < 30$, $m = 10$, e.g., rounded to 20 decimal digits, can be obtained by

```
N[Table[beta[[k + 1]], k, 0, 29] /. m -> 10, 20]
{1.0000000000000000000, 0.83333333333333333333,1.5666666666666666667,
2.2048632218844984802, 2.7538354271367311861, 3.2206702529020337274,
3.6136605995467230798,  3.9421524199648285636, 4.2161116169684685200,
4.4453806341585920021, 4.6387745713978611295, 4.8033973772389131640,
4.9445758344946528460, 5.0663654173861162855, 5.1720546573331317560,
5.2643250282396780468, 5.3453418145822515729, 5.4168480119278469826,
5.4802573999449810555, 5.5367328963993764315, 5.5872424476189429104,
5.6325949837650829666, 5.6734665363296058787, 5.7104246096440169106,
5.7439503253769660554, 5.7744545047515811131, 5.8022886980934010953,
5.8277547166249649692, 5.8511129345858539784, 5.8725889108639372690}
```

All computations were performed in MATHEMATICA, Ver. 11.0.1, on MacBook Pro Retina, OS X 10.11.6, using the package `OrthogonalPolynomials` (see [7] and [19]). The running time for calculating the first $2n$ moments and the corresponding first n recursive coefficients depends on n, and, expressed in minutes and seconds are given in Table 1. The running time is evaluated by the function `Timing` in MATHEMATICA and includes only CPU time spent in the MATHEMATICA kernel. This may give different results on different occasions within a session, because of the use of internal system caches. In order to generate worst-case

Table 1 Running time for calculating the first $2n$ moments and n recursive coefficients

Calculations	$n = 30$	$n = 40$	$n = 50$
Calculation of the first $2n$ moments	$4''$	$9''$	$18''$
Calculation of the first n recursive coefficients	$19''$	$2' \, 14''$	$13' \, 47''$

timing results independent of previous computations, we used the command `ClearSystemCache[]`.

Remark 3 As we can see from Table 1, the running time for calculating the first n recursive coefficients increases exponentially with respect to n. We also calculated the first $n = 60$ coefficients in a symbolic form and it lasted about 1 h and 7 min.

Finally, we can now calculate Gaussian parameters (nodes and weights) in (1) very easily for each $n \leq 60$ and each $m \in \mathbb{N}$, with an arbitrary precision (`prec`), because the Golub–Welsch algorithm is fast and well-conditioned. In the package `OrthogonalPolynomials`, this algorithm is realized by the function `aGaussianNodesWeights`, with obligatory arguments `n`, `alpha`, `beta`, and two optional arguments for `WorkingPrecision` and `Precision`. Usually, we put `WorkingPrecision -> prec+5` and `Precision -> prec`. Defining

```
PQ[n_,mm_,prec_]:=N[aGaussianNodesWeights[n,alpha/.{m->mm},
                  beta/.{m->mm},WorkingPrecision->prec+5,
                  Precision->prec],prec];
```

we can calculate `nodes` and `weights` in n-point quadrature with precision `prec` for each m (= `mm`). For example, for $n = 12$, $m = 5$, and a precision of 20 decimal digits (`prec = 20`), we obtain

```
{node,weights}=PQ[12,5,20]
```

```
{{0.28615439012499870599, 0.59173541249019933863, 0.96214478860900355154,
1.3767311734123912366, 1.8167168862233427683, 2.2707179213614517070,
2.7292820786385482930, 3.1832831137766572317, 3.6232688265876087634,
4.0378552113909964485, 4.4082645875098006614, 4.7138456098750012940},
{0.000075225476023619541079, 0.0017396925219340034584, 0.014170925759385921532,
0.062426717943609193267, 0.16216672426261319389, 0.25942071403643406831,
0.25942071403643406831, 0.16216672426261319389, 0.062426717943609193267,
0.014170925759385921532, 0.0017396925219340034584, 0.000075225476023619541079}}}
```

The first list of 12 elements represents nodes and the second one the weight coefficients (Christoffel numbers).

4 Numerical examples

In order to illustrate the efficiency of the Gaussian quadrature formulae (1), we consider the behavior of quadrature sums

$$Q_n^{[m]}(f) = \sum_{\nu=1}^{n} A_{n,\nu}^{[m]} f\big(x_{n,\nu}^{[m]}\big),$$

more precisely their relative errors

$$e_n^{[m]} \equiv e_n^{[m]}(f) := \left| \frac{Q_n^{[m]}(f) - I_m(f)}{I_m(f)} \right| \quad \left(I_m(f) = \int_0^m f(x)\varphi_m(x)\,\mathrm{d}x \right), \qquad (15)$$

for some selected values of $m \in \mathbb{N}$ and $n = 5(5)50$. In the following examples, we take $m = 3, 4, 5, 6$ and $m = 10$. For calculating true solutions $I_m(f)$, we use the piecewise polynomial representation of the B-spline functions (see [20]) and integration in sufficient precision using MATHEMATICA, Ver. 11.0.1.

Example 1 In this example, we consider two functions

$$f_1(x) = 10 \sin x \quad \text{and} \quad f_2(x) = \left(1 - 2\sin\frac{19x}{3}\right)^3 \sinh\left(1 - \frac{x}{2}\right),$$

the graphics of which are presented in Fig. 2 (left). In the same figure (right),0 we display graphics of $\varphi_5(x) f_k(x)$, $k = 1, 2$.

The relative errors (15) in Gaussian approximations for these functions f_1 and f_2 are presented in Fig. 3 for $n = 5(5)50$ and some selected values of m. As expected, the sequence of quadrature sums $\{Q_n^{[m]}(f_1)\}_n$ converges very fast! On the other hand, convergence for f_2 is slower. We see also that the rate of convergence is smaller for larger m.

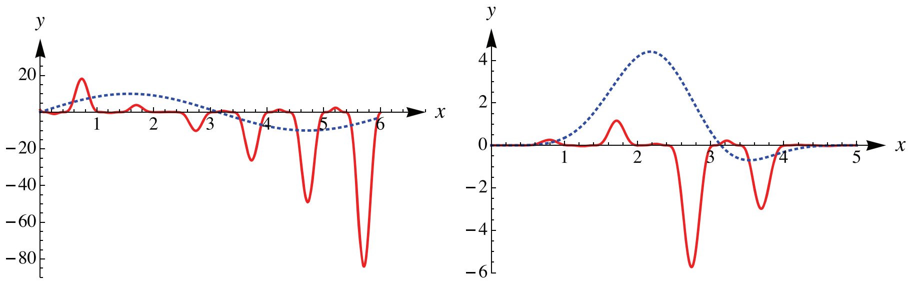

Fig. 2 (*left*) Graphics of functions $f_1(x)$ (*dotted line*) and $f_2(x)$ (*solid line*) on [0, 6]; (*right*) Graphics of functions $\varphi_5(x) f_1(x)$ (dotted line) and $\varphi_5(x) f_2(x)$ (solid line) on [0, 5]

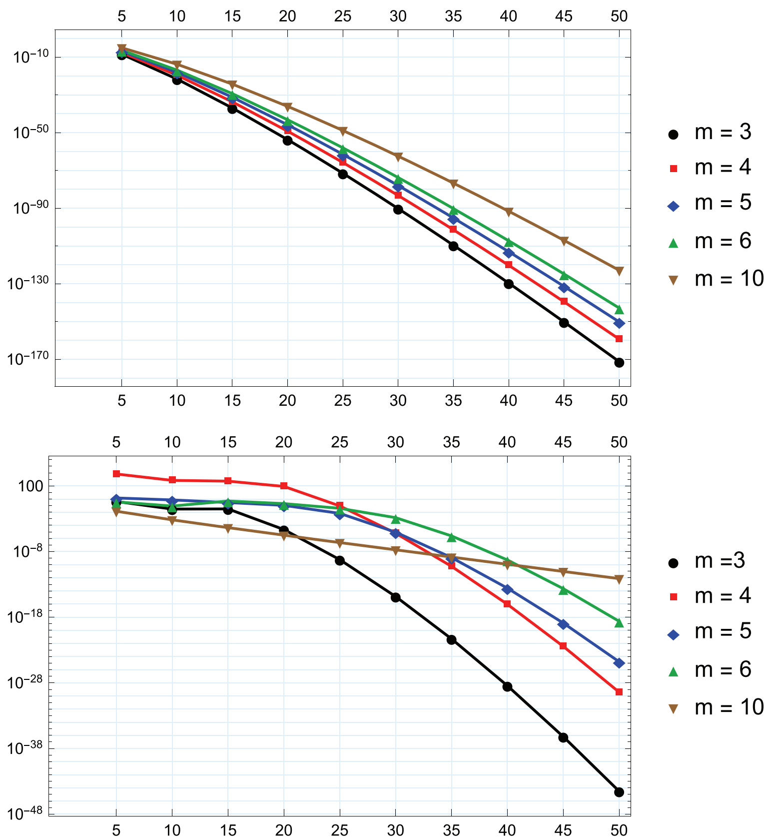

Fig. 3 Relative errors in Gaussian approximations $Q_n^{[m]}(f_1)$ (*top*) and $Q_n^{[m]}(f_2)$ (*bottom*) for $m = 3, 4, 5, 6, 10$ and $n = 5(5)50$

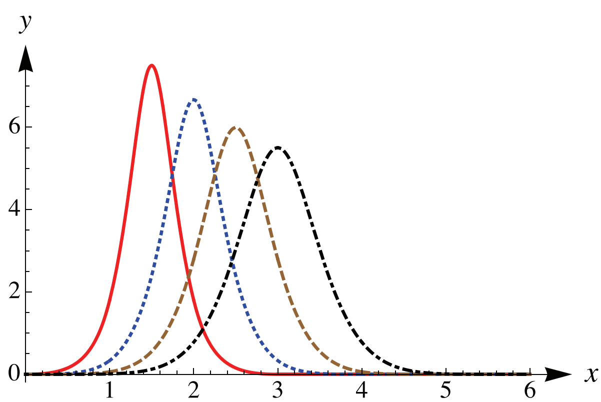

Fig. 4 Graphics of functions $\varphi_m(x)\,f_m(x)$ on $[0, m]$ for $m = 3, 4, 5, 6$

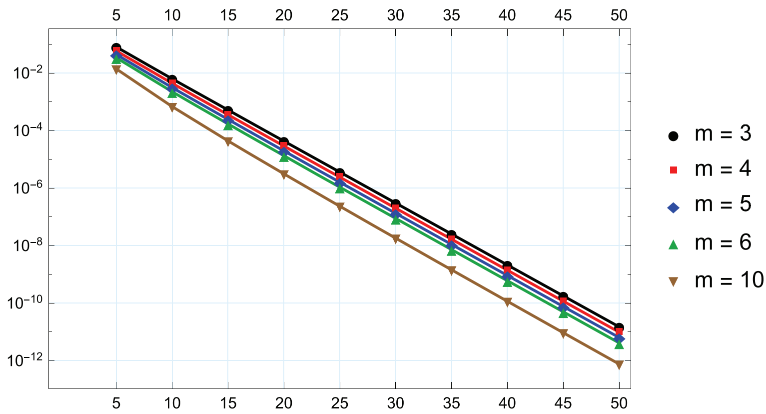

Fig. 5 (*left*) Graphics of functions $\varphi_m(x) f_m(x)$ for $m = 3, 4, 5, 6$ on $[0, m]$; (*right*) Relative errors in Gaussian approximations $Q_n^{[m]}(f_m)$, $n = 5(5)50$, for $m = 3, 4, 5, 6, 10$

Example 2 Now, we consider Runge's function (translated from $[-1, 1]$ to $[0, m]$)

$$f_m(x) = \frac{10}{1 + 16\left(\dfrac{2x}{m} - 1\right)^2},$$

with singularities at $(4 \pm \mathrm{i})m/8$. Graphics of this function multiplied by $\varphi_m(x)$, for $m = 3, 4, 5, 6$, are presented in Fig. 4. (for details on Runge's function, as well as the corresponding interpolation process see [17, p. 60].)

As we expect, the convergence of Gaussian quadrature sums $\{Q_n^{[m]}(f_m)\}_k$ in this case is relatively slow. The relative errors $e_n^{[m]}(f_m)$ in log-scale, for $m = 3, 4, 5, 6, 10$, when $n = 5(5)50$ are presented in Fig. 5.

5 Two conjectures

On the basis of the results obtained (for the first 60 recursive coefficients), we can state the following conjectures:

Conjecture 1 For $k \geq 3$, the recurrence coefficients $\beta_k^{(m)}$ can be expressed in the form

$$\beta_k^{(m)} = C_k \frac{q_{s_{k-3}}(m) q_{s_k}(m)}{q_{s_{k-2}}(m) q_{s_{k-1}}(m)}, \quad k \geq 3,$$

where $\{q_{s_k}\}_{k=0}^{+\infty}$ is a system of algebraic polynomials in m, with integer coefficients, of degrees $s_k = (2k^2 - 1 + (-1)^k)/8$, and C_k are rational constants.

In our computations, we obtained

$$
\begin{aligned}
q_{s_0}(m) &= q_{s_1}(m) = 1, \quad q_{s_2}(m) = 5m-3, \quad q_{s_3}(m) = 175m^2-315m+158,\\
q_{s_4}(m) &= 6125m^4-25725m^3+41965m^2-29547m+7230,\\
q_{s_5}(m) &= 471625m^6-3678675m^5+12324235m^4-22096305m^3\\
&\quad +22009540m^2-11549748m+2519472,
\end{aligned}
$$

$$
\begin{aligned}
q_{s_6}(m) &= 11802415625m^9-155791886250m^8+931311756375m^7\\
&\quad -3260380223100m^6+7287484078875m^5-10710163424730m^4\\
&\quad +10297166185537m^3-6207823757520m^2+2111356988868m\\
&\quad -304962099120,
\end{aligned}
$$

$$
\begin{aligned}
q_{s_7}(m) &= 8438727171875m^{12}-172150034306250m^{11}+1635521768505625m^{10}\\
&\quad -9486710999766000m^9+37170226215247125m^8\\
&\quad -103140503000384850m^7+207092542088006575m^6\\
&\quad -302311914052408260m^5+317690740984945328m^4\\
&\quad -233988179502757680m^3+114592266201395664m^2\\
&\quad -33539482979925120m+4449204584379648,
\end{aligned}
$$

$$
\begin{aligned}
q_{s_8}(m) &= 718009101418984375m^{16}-21540273042569531250m^{15}\\
&\quad +306466799031377359375m^{14}-2729829574503468937500m^{13}\\
&\quad +16961867348957379598750m^{12}-77671409630345446815900m^{11}\\
&\quad +270334750775463299675750m^{10}-727653744705777548699040m^9\\
&\quad +1527372133343317471965755m^8-2503317024698062132145586m^7\\
&\quad +3186852942000043203666779m^6-3112364522558715589144980m^5\\
&\quad +2281592249060977336852368m^4-1210995067166438020207200m^3\\
&\quad +437667158101790582119440m^2-95869567107365446403712m\\
&\quad +9538859825773941438720,
\end{aligned}
$$

etc. The corresponding coefficients C_k are

$$
\begin{aligned}
C_3 &= \frac{1}{140}, \quad C_4 = \frac{1}{21}, \quad C_5 = \frac{25}{132}, \quad C_6 = \frac{1}{1430}, \quad C_7 = \frac{49}{780}, \quad C_8 = \frac{10}{51},\\
C_9 &= \frac{3}{9044}, \quad C_{10} = \frac{1}{3990}, \quad C_{11} = \frac{121}{1380}, \quad C_{12} = \frac{7}{23}, \quad C_{13} = \frac{169}{12}, \quad C_{14} = \frac{1}{870},
\end{aligned}
$$

etc.

Conjecture 2 For the recurrence coefficients $\beta_k^{(m)}$ $(k \geq 1)$, the following asymptotic formula

$$\beta_k^{[m]} \asymp \frac{k}{12}m - \frac{k(k-1)}{20} + \frac{4k(k-1)(k-2)}{525m} + \frac{2k(k-1)(k-2)(5k-9)}{2625m^2}$$
$$+\frac{2k(k-1)(k-2)\left(552k^2-1279k+255\right)}{1010625m^3} + \cdots$$

holds as $m \to \infty$.

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CONSTRUCTION OF GAUSSIAN QUADRATURE FORMULAS FOR EVEN WEIGHT FUNCTIONS

*Mohammad Masjed-Jamei, Gradimir V. Milovanović**

Instead of a quadrature rule of Gaussian type with respect to an even weight function on $(-a, a)$ with n nodes, we construct the corresponding Gaussian formula on $(0, a^2)$ with only $[(n+1)/2]$ nodes. Especially, such a procedure is important in the cases of nonclassical weight functions, when the elements of the corresponding three-diagonal Jacobi matrix must be constructed numerically. In this manner, the influence of numerical instabilities in the process of construction can be significantly reduced, because the dimension of the Jacobi matrix is halved. We apply this approach to Pollaczek's type weight functions on $(-1, 1)$, to the weight functions on $\mathbb{R}$ which appear in the Abel-Plana summation processes, as well as to a class of weight functions with four free parameters, which covers the generalized ultraspherical and Hermite weights. Some numerical examples are also included.

1. INTRODUCTION

Let $\mathcal{P}$ be the set of all algebraic polynomials and $\mathcal{P}_n$ be its subset of degree at most n. In this paper, we consider the Gauss-Christoffel quadrature rules with respect to the even weight function $x \mapsto w(x) = w(-x)$ on a symmetric interval $(-a, a)$ for $a > 0$,

$$\int_{-a}^{a} f(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{n} w_k f(x_k) + R_n(f; w), \tag{1}$$

where $R_n(f; w) = 0$ for each $f \in \mathcal{P}_{2n-1}$ and they are automatically exact for all odd functions.

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Suppose that the moments $\mu_k = \int_{-a}^{a} x^k w(x)\,\mathrm{d}x$, exist and are finite for any $k = 0, 1, \ldots$, and also $\mu_0 = \int_{-a}^{a} w(x)\,\mathrm{d}x > 0$. Then the quadrature rules (1) exist for each $n \in \mathbb{N}$ as well as the corresponding orthogonal polynomials. It is well known that $\mu_{2k+1} = 0$ for any $k = 0, 1, \ldots$, and the monic symmetric polynomials $\pi_k(x)$ orthogonal with respect to the even weight w on $(-a, a)$ satisfy the three-term recurrence relation (cf. [**12**, p. 102])

$$\pi_{k+1}(x) = x\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, \ldots, \tag{2}$$

with $\pi_{-1}(x) = 0$, $\pi_0(x) = 1$ and $\pi_1(x) = x$.

The recurrence coefficients β_k in (2) can be computed from the moments in terms of Hankel determinants

$$\Delta_k = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{k-1} \\ \mu_1 & \mu_2 & & \mu_k \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-2} \end{vmatrix},$$

by

$$\beta_k = \frac{\Delta_{k-1}\Delta_{k+1}}{\Delta_k^2} \quad (k \geq 1) \qquad \text{with} \quad \Delta_0 = 1.$$

Although β_0 in (2) may be arbitrary, it is sometimes convenient to define it as $\beta_0 = \mu_0 = \int_{-a}^{a} w(x)\,\mathrm{d}x$. By noting the definition

$$(p, q) = \int_{-a}^{a} p(x)q(x)w(x)\,\mathrm{d}x \quad \text{and} \quad \|p\| = \sqrt{(p, p)},$$

one can prove that the norm of π_n equals to

$$\|\pi_n\| = \sqrt{\beta_0\beta_1 \cdots \beta_n} = \sqrt{\frac{\Delta_{n+1}}{\Delta_n}}.$$

For instance, the first few monic symmetric polynomials π_k in terms of moments are as follows

$$\begin{aligned} \pi_2(x) &= x^2 - \frac{\mu_2}{\mu_0}, \\ \pi_3(x) &= x^3 - \frac{\mu_4}{\mu_2}\,x, \\ \pi_4(x) &= x^4 - \frac{\mu_6\mu_0 - \mu_4\mu_2}{\mu_4\mu_0 - \mu_2^2}\,x^2 + \frac{\mu_6\mu_2 - \mu_4^2}{\mu_4\mu_0 - \mu_2^2}, \\ \pi_5(x) &= x^5 - \frac{\mu_8\mu_2 - \mu_6\mu_4}{\mu_6\mu_2 - \mu_4^2}\,x^3 + \frac{\mu_8\mu_4 - \mu_6^2}{\mu_6\mu_2 - \mu_4^2}\,x. \end{aligned}$$

A standard method for calculating the nodes x_k and the weight coefficients (Christoffel numbers) w_k in the quadrature (1) is based on their characterization via an eigenvalue problem for the Jacobi matrix of order n associated with the even weight function $x \mapsto w(x)$. Thus, the nodes x_k are the eigenvalues of the symmetric tridiagonal Jacobi matrix (cf. [**12**, pp. 325–328])

$$J_n(w) = \begin{bmatrix} 0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & 0 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & 0 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & 0 \end{bmatrix}, \tag{3}$$

and the weight coefficients w_k are given by $w_k = \beta_0 v_{k,1}^2$ $(k = 1, \ldots, n)$, where $v_{k,1}$ is the first component of the eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,n}]^{\mathrm{T}})$ corresponding to the eigenvalue x_k, normalized such that $\mathbf{v}_k^{\mathrm{T}}\mathbf{v}_k = 1$. This popular method is called the Golub-Welsch procedure [**8**].

Unfortunately, for many weight functions the coefficients β_k in (2) are not explicitly known. In such cases, the corresponding polynomials π_k are known as *strong non–classical* orthogonal polynomials, and their recursion coefficients must be constructed numerically from the moment information. Such problems are very sensitive with respect to small perturbations in the input data. Fortunately, in the eighties of the last century, Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, with effective algorithms for numerically generating the first n recursion coefficients (the method of (modified) moments, the discretized Stieltjes–Gautschi procedure, and the Lanczos algorithm), which allow us to compute all orthogonal polynomials of degree $\leq n$ by a straightforward application of the three-term recurrence relation. A detailed stability analysis of these algorithms as well as several new applications of orthogonal polynomials are also included in the previously mentioned theory. The basic references are [**6, 7, 15**].

Because of $w(-x) = w(x)$ on $(-a, a)$, the nodes in the quadrature sum

$$Q_n(f; w) := \sum_{k=1}^{n} w_k f(x_k)$$

in (1) are symmetrically distributed with respect to the origin, and their weight coefficients are mutually equal for symmetric nodes. Taking only positive nodes, denoted by $x_k^{(n)}$ and the corresponding weight coefficients by $A_k^{(n)}$ for $k = 1, \ldots, m$ $(= [n/2])$, the quadrature sum can be expressed as

$$Q_n(f; w) := \begin{cases} \displaystyle\sum_{k=1}^{m} A_k^{(n)} \big(f(x_k^{(n)}) + f(-x_k^{(n)})\big), & n = 2m, \\ \displaystyle A_0^{(n)} f(0) + \sum_{k=1}^{m} A_k^{(n)} \big(f(x_k^{(n)}) + f(-x_k^{(n)})\big), & n = 2m+1, \end{cases} \tag{4}$$

where, in the case of odd n, $A_0^{(n)}$ (> 0) is the weight coefficient for the node 0. Here,

$$0 < x_1^{(n)} < \cdots < x_m^{(n)} < a \quad \text{and} \quad A_k^{(n)} > 0, \quad k = 1, \ldots, m.$$

This paper is organized as follows. In Section 2, we shortly describe a simple transformation from $(-a, a)$ to $(0, a^2)$ and give recurrence coefficients for the corresponding orthogonal polynomials. Section 3 is devoted to the construction of two quadratures on $(0, a^2)$ and their connection with symmetric Gaussian quadratures on $(-a, a)$. These sections are introductory and record material that is essentially known (cf. [**12**], [**13**]), but needed in subsequent sections. The numerical construction of Gaussian rules related to the Pollaczek-type weight functions on $(-1, 1)$ is presented in Section 4, together with some numerical examples. Symmetric Gaussian quadrature rules on $\mathbb{R}$, which appear in the Abel-Plana summation formulas, are considered in Section 5. Finally, a class of symmetric weight functions with four free parameters that covers many well-known weights on $(-1, 1)$ and $\mathbb{R}$ are considered in Section 6.

2. TRANSFORMATION AND PRESERVATION OF ORTHOGONALITY

Suppose in (1) that $x \mapsto f(x)$ is an even function, so that

$$\int_{-a}^{a} f(x)w(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)w(x)\,\mathrm{d}x = \int_{0}^{a^2} f(\sqrt{t})\,\frac{w(\sqrt{t})}{\sqrt{t}}\,\mathrm{d}t. \tag{5}$$

On the other hand, according to (1), (4) and (5) we have

$$I(\varphi_1; w_1) = \int_{0}^{a^2} f(\sqrt{t})\,\frac{w(\sqrt{t})}{\sqrt{t}}\,\mathrm{d}t = Q_n(f; w) + R_n(f; w), \tag{6}$$

where two new functions are defined on $(0, a^2)$ as

$$w_1(t) := \frac{w(\sqrt{t})}{\sqrt{t}} \quad \text{and} \quad \varphi_1(t) := f(\sqrt{t}). \tag{7}$$

Similarly, we need to define

$$w_2(t) := \sqrt{t}\,w(\sqrt{t}) \quad \text{and} \quad \varphi_2(t) := \frac{f(\sqrt{t}) - f(0)}{t}. \tag{8}$$

The orthogonal polynomials with respect to the weight functions $w_1(t)$ and $w_2(t)$ defined on $(0, a^2)$ can be directly expressed in terms of the polynomials $\pi_k(x)$ which are orthogonal with respect to the symmetric weight w on $(-a, a)$. In fact, according to Theorem 2.2.11 of [**12**, p. 102] we have:

(i) $p_\nu(t) := \pi_{2\nu}(\sqrt{t})$ are orthogonal with respect to the weight function $w_1(t) = w(\sqrt{t})/\sqrt{t}$ on $(0, a^2)$, and

(ii) $q_n(t) := \pi_{2n+1}(\sqrt{t})/\sqrt{t}$ are orthogonal with respect to the weight function $w_2(t) = \sqrt{t}\, w(\sqrt{t})$ on $(0, a^2)$.

Also, these (monic) polynomials satisfy the three-term recurrence relations (Theorem 2.2.12 in [**12**, p. 102]),

$$p_{\nu+1}(t) = (t - a_\nu)p_\nu(t) - b_\nu p_{\nu-1}(t), \quad \nu = 0, 1, \ldots, \tag{9}$$

and

$$q_{\nu+1}(t) = (t - c_\nu)q_\nu(t) - d_\nu q_{\nu-1}(t), \quad \nu = 0, 1, \ldots, \tag{10}$$

with $p_0(t) = 1$, $p_{-1}(t) = 0$ and $q_0(t) = 1$, $q_{-1}(t) = 0$, respectively, where the coefficients in (9) and (10) are given by

$$a_0 = \beta_1, \quad a_\nu = \beta_{2\nu} + \beta_{2\nu+1}, \quad b_\nu = \beta_{2\nu-1}\beta_{2\nu},$$

and

$$c_0 = \beta_1 + \beta_2, \quad c_\nu = \beta_{2\nu+1} + \beta_{2\nu+2}, \quad d_\nu = \beta_{2\nu}\beta_{2\nu+1},$$

in which β_k are the same values as in (2). In addition, we can define

$$b_0 := \int_0^{a^2} w_1(t)\,\mathrm{d}t = \int_{-a}^{a} w(x)\,\mathrm{d}x = \mu_0$$

and

$$d_0 := \int_0^{a^2} w_2(t)\,\mathrm{d}t = \int_0^{a^2} t\, w_1(t)\,\mathrm{d}t = \int_{-a}^{a} x^2 w(x)\,\mathrm{d}x = \mu_2,$$

i.e., $b_0 := \beta_0$ and $d_0 := \beta_0\beta_1$.

In the case of strong nonclassical weights, the coefficients a_ν and b_ν in (9), as well as c_ν and d_ν in (10), must be constructed numerically (cf. [**6**], [**12**, pp. 160–166]).

The orthogonal polynomials $p_\nu(t)$ and their recurrence relation (9) are applied in constructing Gaussian quadratures with respect to the weight function $w_1(t) = w(\sqrt{t})/\sqrt{t}$ on $(0, a^2)$, while the polynomials $q_\nu(t)$ and their recurrence relation (10) are appropriate for constructing Gauss-Radau rules (cf. [**12**, p. 329]).

By noting these comments and (4), the construction of quadratures (1) will be significantly simplified. Namely, instead of constructing a quadrature formula on $(-a, a)$ with n nodes, we construct a quadrature formula on $(0, a^2)$ with only $[(n + 1)/2]$ nodes. In particular, it is very important in the cases of nonclassical weight functions, when the recurrence coefficients in the three-term relations for the corresponding orthogonal polynomials must be constructed numerically, before the procedure for constructing nodes and Christoffel numbers (by the Golub-Welsch procedure from the Jacobi matrices). In this manner, the influence of numerical instabilities in the process of construction can be significantly reduced. Also, in this way, the dimensions of the corresponding Jacobi matrices are halved.

3. CONSTRUCTION OF TWO RULES OF GAUSSIAN TYPE

We consider now two quadrature formulas for computing the integral $I(\varphi_1; w_1)$ given in (6).

3.1. Gauss-Christoffel quadrature formula with the weight $\mathbf{w_1(t)}$

The first formula is a m-point Gauss-Christoffel quadrature formula with respect to the weight function $t \mapsto w_1(t) = w(\sqrt{t})/\sqrt{t}$ on $(0, a^2)$,

$$I(g; w_1) = \int_0^{a^2} g(t)\, w_1(t)\,\mathrm{d}t = \sum_{k=1}^{m} B_k^{(m)} g(\tau_k^{(m)}) + R_m^{\mathrm{GC}}(g; w_1), \tag{11}$$

with the nodes $0 < \tau_1^{(m)} < \cdots < \tau_m^{(m)} < a^2$ and the corresponding weight coefficients $B_k^{(m)}$ $(k = 1, \ldots, m)$. The remainder term $R_m^{\mathrm{GC}}(g; w_1) = 0$ for each $g \in \mathcal{P}_{2m-1}$.

Proposition 3.1. *The nodes $\tau_k^{(m)}$ $(k = 1, \ldots, m)$ in the formula* (11)*, that is, the zeros of the polynomial $p_m(t)$ in* (9)*, are the eigenvalues of the Jacobi matrix*

$$J_m(w_1) = \begin{bmatrix} \beta_1 & \sqrt{\beta_1\beta_2} & & & \mathbf{O} \\ \sqrt{\beta_1\beta_2} & \beta_2+\beta_3 & \sqrt{\beta_3\beta_4} & & \\ & \sqrt{\beta_3\beta_4} & \beta_4+\beta_5 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{2m-3}\beta_{2m-2}} \\ \mathbf{O} & & & \sqrt{\beta_{2m-3}\beta_{2m-2}} & \beta_{2m-2}+\beta_{2m-1} \end{bmatrix},$$

where β_k are the same values as in (2)*. Also, the weight coefficients $B_k^{(m)}$ are given by $B_k^{(m)} = \beta_0 v_{k,1}^2$, where $v_{k,1}$ is the first component of the eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,m}]^{\mathrm{T}})$ corresponding to the eigenvalue $\tau_k^{(m)}$ and normalized such that $\mathbf{v}_k^{\mathrm{T}}\mathbf{v}_k = 1$.*

3.2. Gauss-Radau quadrature formula with the weight $\mathbf{w_1(t)}$

The $(m+1)$-point Gauss-Radau quadrature formula with respect to the same weight function $w_1(t)$ as before and the new nodes $0 = \theta_0^{(m)} < \theta_1^{(m)} < \cdots < \theta_m^{(m)} < a^2$ and weight coefficients $C_k^{(m)}$ are given by

$$I(g; w_1) = \int_0^{a^2} g(t)\, w_1(t)\,\mathrm{d}t = C_0^{(m)} g(0) + \sum_{k=1}^{m} C_k^{(m)} g(\theta_k^{(m)}) + R_{m+1}^{\mathrm{GR}}(g; w_1). \tag{12}$$

It is clear that $R_{m+1}^{\mathrm{GR}}(g; w_1) = 0$ for each $g \in \mathcal{P}_{2m}$.

In order to construct the formula (12), we need to introduce a function h, $g(t) = g(0) + th(t)$, to get

$$I(g; w_1) = g(0) \int_0^{a^2} w_1(t)\,\mathrm{d}t + \int_0^{a^2} h(t)\, t\, w_1(t)\,\mathrm{d}t.$$

This means that

$$I(g; w_1) = \beta_0 g(0) + \int_0^{a^2} h(t) w_2(t)\,\mathrm{d}t = \beta_0 g(0) + I(h; w_2), \tag{13}$$

because $w_2(t) = tw_1(t)$ according to (7) and (8). To compute the integral $I(h; w_2)$ we can directly construct the Gauss-Christoffel rule with respect to the second weight function $w_2(t) = \sqrt{t}\, w(\sqrt{t})$ on $(0, a^2)$ as

$$I(h; w_2) = \int_0^{a^2} h(t)\, w_2(t)\,\mathrm{d}t = \sum_{k=1}^{m} D_k^{(m)} h(\theta_k^{(m)}) + R_m^{\mathrm{GC}}(h; w_2), \tag{14}$$

where $0 < \theta_1^{(m)} < \cdots < \theta_m^{(m)} < a^2$ and $D_k^{(m)}$ are the corresponding weight coefficients. Note in (14) that the remainder term $R_m^{\mathrm{GC}}(h; w_2) = 0$ for each $h \in \mathcal{P}_{2m-1}$. Thus, by noting (13) and (14) we first get

$$\begin{aligned} I(g; w_1) &= \beta_0 g(0) + \sum_{k=1}^{m} D_k^{(m)} h(\theta_k^{(m)}) + R_m^{\mathrm{GC}}(h; w_2) \\ &= \beta_0 g(0) + \sum_{k=1}^{m} D_k^{(m)} \frac{g(\theta_k^{(m)}) - g(0)}{\theta_k^{(m)}} + R_m^{\mathrm{GC}}(h; w_2) \\ &= \left(\beta_0 - \sum_{k=1}^{m} \frac{D_k^{(m)}}{\theta_k^{(m)}}\right) g(0) + \sum_{k=1}^{m} \frac{D_k^{(m)}}{\theta_k^{(m)}} g(\theta_k^{(m)}) + R_m^{\mathrm{GC}}(h; w_2), \end{aligned}$$

and then comparing this with (12), the weight coefficients of the Gauss-Radau quadrature (12) are

$$C_0^{(m)} = \beta_0 - \sum_{k=1}^{m} \frac{D_k^{(m)}}{\theta_k^{(m)}}, \quad C_k^{(m)} = \frac{D_k^{(m)}}{\theta_k^{(m)}} \quad (k = 1, \ldots, m), \tag{15}$$

and $R_{m+1}^{\mathrm{GR}}(g; w_1) = R_m^{\mathrm{GC}}(h; w_2)$ for $h(t) = (g(t) - g(0))/t$. This means that the nodes of the Gauss-Radau quadrature rule with respect to the weight function $w_1(t)$ are in fact the nodes of the Gauss-Christoffel formula with respect to the weight function $w_2(t) = t\, w_1(t)$ on $(0, a^2)$.

Proposition 3.2. *The nodes $\theta_k^{(m)}$ $(k = 1, \ldots, m)$ in the formula* (12)*, that is, the* *zeros of the polynomial* $q_m(t)$ *in* (10), *are the eigenvalues of the Jacobi matrix*

$$J_m(w_2) = \begin{bmatrix} \beta_1+\beta_2 & \sqrt{\beta_2\beta_3} & & & \mathbf{O} \\ \sqrt{\beta_2\beta_3} & \beta_3+\beta_4 & \sqrt{\beta_4\beta_5} & & \\ & \sqrt{\beta_4\beta_5} & \beta_5+\beta_6 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{2m-2}\beta_{2m-1}} \\ \mathbf{O} & & & \sqrt{\beta_{2m-2}\beta_{2m-1}} & \beta_{2m-1}+\beta_{2m} \end{bmatrix}.$$

where β_k *are the same values as in* (2) *and the weight coefficients* $C_k^{(m)}$ *are given by* (15), *where* $D_k^{(m)}$ *is determined by the first component* $v_{k,1}$ *of the normalized eigenvector* $\mathbf{v}_k$ $(= [v_{k,1} \ \dots \ v_{k,m}]^{\mathrm{T}})$ *of the Jacobi matrix* $J_m(w_2)$ *corresponding to the eigenvalue* $\theta_k^{(m)}$, *i.e.,* $D_k^{(m)} = \beta_0\beta_1 v_{k,1}^2$, $k = 1, \dots, m$.

REMARK 3.3. The quadratures (11) and (12) can be related to the basic quadrature (1), which allows a much simpler construction of these symmetric quadratures given in form

$$\int_{-a}^{a} f(x)w(x)\,\mathrm{d}x = Q_n(f;w) + R_n(f;w), \tag{16}$$

where $Q_n(f;w)$ is defined by (4). If we have the recursion coefficients β_k in the explicit form, in our construction we use Proposition 3.1 for even n and Proposition 3.2 for odd n. However, in the case of strong nonclassical weights, we first numerically construct the recursion coefficients a_ν and b_ν in (9), and c_ν and d_ν in (10), and then the Jacobi matrices $J_m(w_1)$ and $J_m(w_2)$ are given by

$$J_m(w_1) = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \mathbf{O} \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \sqrt{b_2} & a_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{b_{m-1}} \\ \mathbf{O} & & & \sqrt{b_{m-1}} & a_{m-1} \end{bmatrix}$$

and

$$J_m(w_2) = \begin{bmatrix} c_0 & \sqrt{d_1} & & & \mathbf{O} \\ \sqrt{d_1} & c_1 & \sqrt{d_2} & & \\ & \sqrt{d_2} & c_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{d_{m-1}} \\ \mathbf{O} & & & \sqrt{d_{m-1}} & c_{m-1} \end{bmatrix}.$$

Corollary 3.4. *The positive nodes $x_k^{(n)}$ of the symmetric quadrature rule* (16) *are given by $x_k^{(n)} = \sqrt{\tau_k^{(m)}}$ $(n = 2m)$ and $x_k^{(n)} = \sqrt{\theta_k^{(m)}}$ $(n = 2m+1)$, and the corresponding weight coefficients by*

$$1^\circ \ A_k^{(n)} = \frac{1}{2} B_k^{(m)} \quad (k = 1,\dots,m) \quad \text{for even} \ \ n = 2m,$$

and

$$2^\circ \ A_0^{(n)} = C_0^{(m)},\ A_k^{(n)} = \frac{1}{2} C_k^{(m)} \quad (k = 1,\dots,m) \quad \text{for odd} \ \ n = 2m+1,$$

where $\tau_k^{(m)}$ and $B_k^{(m)}$ and $\theta_k^{(m)}$ and $C_k^{(m)}$ are defined in Propositions 3.1 *and* 3.2*, respectively.*

Corollary 3.5. *Let $f : (-a,a) \to \mathbb{R}$ be an even function and $\varphi_1 : (0,a^2) \to \mathbb{R}$ and $\varphi_2 : (0,a^2) \to \mathbb{R}$ be defined by* (7) *and* (8) *respectively. The remainder term in* (16) *is given by*

$$R_n(f;w) = \begin{cases} R_m^{\mathrm{GC}}(\varphi_1; w_1) & n = 2m, \\ R_m^{\mathrm{GC}}(\varphi_2; w_2) & n = 2m+1, \end{cases}$$

where $R_m^{\mathrm{GC}}(\,\cdot\,; w_\nu)$ is the remainder term of the Gauss-Christoffel rule with respect to the weight function w_ν $(\nu = 1,2)$ on $(0,a^2)$.

Remark 3.4. Some fast variants of the Golub-Welsch algorithm for symmetric weight functions in Matlab have been considered in [**13**], including numerical experiments with Gegenbauer and Hermite weight functions.

4. GAUSSIAN RULES RELATED TO THE POLLACZEK WEIGHT

Recently De Bonis, Mastroianni, and Notarangelo [**5**] have considered Gaussian quadrature rules with respect to the Pollaczek-type weight $w(x;\lambda) = \mathrm{e}^{-(1-x^2)^{-\lambda}}$, $\lambda > 0$, on $(-1,1)$ in order to evaluate integrals of the form

$$I(f;\lambda) = \int_{-1}^{1} f(x)\mathrm{e}^{-(1-x^2)^{-\lambda}}\,\mathrm{d}x, \tag{17}$$

where f is a Riemann integrable function, in particular, f can increase exponentially at the endpoints ± 1. Also, their rule is useful for approximating integrals of functions that decay exponentially at ± 1 (e.g., when f is bounded or has a slower growth than exponential at the endpoints).

In [**5**], the authors use the first $2n$ moments

$$\mu_k = \int_{-1}^{1} x^k w(x;\lambda)\,\mathrm{d}x, \quad k = 0,1,\dots,2n-1,$$

in order to construct the first n recursive coefficients and the corresponding Gaussian quadratures with $\le n$ nodes, by the package `OrthogonalPolynomials` ([**1**]), which is downloadable from the web site `http://www.mi.sanu.ac.rs/~gvm/`.

Since $w(x;\lambda)$ is even on $(-1,1)$, in our construction, we can use the following weight functions on $(0,1)$

$$w_1(t;\lambda) = \frac{\mathrm{e}^{-(1-t)^{-\lambda}}}{\sqrt{t}} \quad \text{and} \quad w_2(t;\lambda) = \sqrt{t}\,\mathrm{e}^{-(1-t)^{-\lambda}}. \tag{18}$$

The weight functions $x \mapsto w(x;\lambda)$ on $(-1,1)$ and $x \mapsto w_1(x;\lambda)$ on $(0,1)$ for $\lambda = 1/10$, $\lambda = 1/2$ and $\lambda = 10$ are displayed in Fig. 1, left and right, respectively. Note

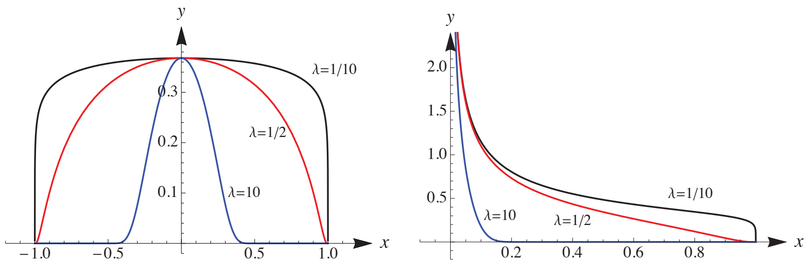

Figure 1: The weight functions $w(x;\lambda) = \mathrm{e}^{-(1-x^2)^{-\lambda}}$ (*left*) and $w_1(x;\lambda)$ (*right*) for three parameters $\lambda = 1/10$, $\lambda = 1/2$ and $\lambda = 10$.

that for a very small value of λ, the weight $x \mapsto w(x;\lambda)$ is very close to a constant value (Legendre weight) in $(-1,1)$ and tends exponentially to zero at the endponts ± 1.

According to results of Section 3, to construct Gaussian quadrature rules with respect to the weight w on $(-1,1)$, for n (or less) nodes, we need the corresponding Gaussian quadrature rules with respect to the weight function w_1 (and w_2) on $(0,1)$, but only for $[n/2]$ nodes. Thus, if we want to construct the quadrature sum $Q_n(f;w)$ for even number of nodes $\le n$ $(=2m)$, we should first compute the first m coefficients a_ν and b_ν for $\nu = 0,1,\ldots,m-1$ (see Remark 3.3), starting with the first $2m$ moments with respect to the weight function w_1, i.e.,

$$\mu_k^{(1)}(\lambda) = \int_0^1 t^{k-1/2}\mathrm{e}^{-(1-t)^{-\lambda}}\,\mathrm{d}t, \quad k = 0,1,\ldots,2m-1.$$

As an illustration, we take $m = 25$ (i.e., $n = 50$) and $\lambda = 10$. In this case, with the moments $\mu_k^{(1)}(10)$, $k = 0,1,\ldots,49$, calculated with `WorkingPrecision -> 80`, using Mathematica package `OrthogonalPolynomials` (see [**1, 17**]), we get the first 25 recursive coefficients a_ν and b_ν with maximal relative error less than 3.30×10^{-60}.

These coefficients enable us to establish the Gaussian quadrature rules (11) for each $m \le 25$, i.e., the symmetric quadratures (16) on $(-1,1)$ for each even $n = 2m \le 50$, according to Corollary 3.4.

Example 4.1. For a given function f, defined on $(-1,1)$ by

$$f(x) = \frac{3\mathrm{e}^{-\frac{1}{\sqrt{1-x^2}}} - 2\sin(3x) - x^2}{(1-x^2)^2}, \tag{19}$$

we consider the integral $I(f;\lambda)$ (with respect to the Pollaczek weight function), given by (17). For $\lambda = 1/2$ and $\lambda = 10$, the corresponding values are

$$I\left(f;\tfrac{1}{2}\right) = -0.100853578447701253704966132370110608871510279078813023527 0\ldots$$

$$I(f;10) = 0.182895219233483199388012214330942401509423267232629315052 76\ldots,$$

obtained in Mathematica with `WorkingPrecision -> 60`. Graphics of the function (19) and the corresponding integrands in (17) are presented in Fig. 2 and Fig. 3, respectively.

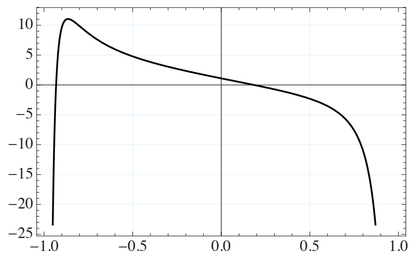

Figure 2: Graphic of the function $x \mapsto f(x)$ given by (19)

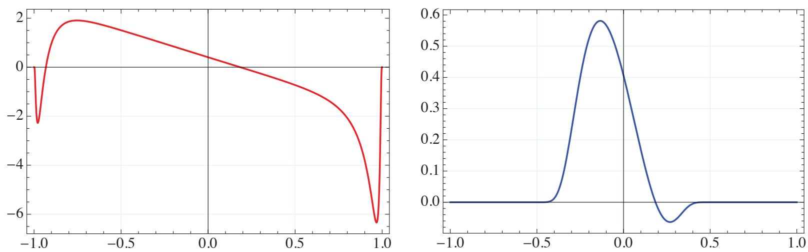

Figure 3: Integrand in $I(f;\lambda)$ for $\lambda = 1/2$ (*left*) and $\lambda = 10$ (*right*)

Now, let us apply Gauss-Pollaczek quadrature rule with $n = 10(10)50$ nodes to the integral $I(f;\lambda)$ and compare the results by ones obtained by the standard

Gauss-Legendre rules. Here, $Q_n(f;w)$ denotes the Gauss-Pollaczek quadrature sum defined by (4), and $r_n^P(f;\lambda)$ shows their relative errors,

$$r_n^P(f;\lambda) = \left| \frac{Q_n(f;w) - I(f;\lambda)}{I(f;\lambda)} \right|.$$

Relative errors for $\lambda = 1/2$ and $\lambda = 10$ are given in Table 1. Numbers in parentheses indicate decimal exponents. The corresponding relative errors in Gauss-Legendre sums are denoted by $r_n^L(f;\lambda)$. As we can see, for $\lambda = 1/2$ both quadratures are slow and have similar behaviour, while for a larger λ $(= 10)$ the advantage of the Gauss-Pollaczek quadrature is clearly evident.

Table 1: Relative errors in quadrature sums when $n = 10(10)50$

n	$r_n^P(f;1/2)$	$r_n^L(f;1/2)$	$r_n^P(f;10)$	$r_n^L(f;10)$
10	1.66	1.01	4.32(−13)	3.52(−2)
20	2.38(−1)	1.43(−1)	2.94(−24)	1.21(−3)
30	4.54(−2)	1.12(−2)	5.27(−35)	1.57(−5)
40	1.04(−2)	4.87(−3)	1.86(−45)	2.93(−6)
50	2.71(−3)	7.09(−4)	1.09(−55)	1.82(−7)

5. A CLASS OF SYMMETRIC WEIGHTS ON $\mathbb{R}$

In this section, we consider symmetric quadrature rules on $\mathbb{R}$ which play an important role in summation formulas of Abel-Plana type, which were intensively studied by Germund Dahlquist [**2, 3, 4**] (also see Milovanović [**14, 16**]). Such rules can be constructed in a simpler way if the corresponding formulas on $\mathbb{R}^+$ are first constructed. By noting the results of Sections 2 and 3, instead of the polynomials π_n orthogonal with respect to $x \mapsto w(x)$ on $\mathbb{R}$, we need the polynomials p_ν and q_ν, given by the recurrence relations (9) and (10), respectively. In other words, the recursive coefficients $\{a_\nu\}$ and $\{b_\nu\}$ for polynomials orthogonal with respect to the weight function $t \mapsto w(\sqrt{t})/\sqrt{t}$ on $\mathbb{R}^+$, as well as the coefficients $\{c_\nu\}$ and $\{d_\nu\}$ for polynomials orthogonal with respect to the weight function $t \mapsto \sqrt{t}\, w(\sqrt{t})$ on $\mathbb{R}^+$ must be computed.

In the sequel, let us mention some important cases of the symmetric weight $x \mapsto w(x)$ on $\mathbb{R}$.

1° In [**12**, p. 159] three interesting even weight functions on $\mathbb{R}$ are given, for which the recurrence coefficients β_k in (2) are known explicitly. They are respectively known as the *Abel weight*

$$w(x) = w^A(x) = \frac{x}{\mathrm{e}^{\pi x} - \mathrm{e}^{-\pi x}} = \frac{x}{2\sinh(\pi x)},$$

the *Lindelöf weight*

$$w(x) = w^L(x) = \frac{1}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}} = \frac{1}{2\cosh(\pi x)},$$

and the *logistic weight*

$$w(x) = w^{\log}(x) = \frac{\mathrm{e}^{-\pi x}}{(1+\mathrm{e}^{-\pi x})^2}.$$

The corresponding recurrence coefficients are

$$\beta_k^A = \frac{k(k+1)}{4}, \quad \beta_k^L = \frac{k^2}{4} \quad \text{and} \quad \beta_k^{\log} = \frac{k^4}{4k^2-1} \quad (k = 1, 2, \ldots),$$

with $\beta_0^A = 1/4$, $\beta_0^L = 1/2$ and $\beta_0^{\log} = 1/\pi$.

We mention also that $w^{\log}(x) = [w^L(x/2)]^2$.

For these weight functions, in the sequel we give the recurrence coefficients in (9) and (10) for polynomials orthogonal on $(0, \infty)$ with respect to $t \mapsto w(\sqrt{t})/\sqrt{t}$ and $t \mapsto w(\sqrt{t})\sqrt{t}$, respectively.

(i) In the Abel case we compute these coefficients as

$$\begin{aligned} a_\nu^A &= \frac{(2\nu+1)^2}{2} \quad (\nu \in \mathbb{N}_0), \quad b_0^A = \frac{1}{4}, \quad b_\nu^A = \frac{\nu^2(4\nu^2-1)}{4} \quad (\nu \in \mathbb{N}); \\ c_\nu^A &= 2(\nu+1)^2 \quad (\nu \in \mathbb{N}_0), \quad d_0^A = \frac{1}{8}, \quad d_\nu^A = \frac{\nu(\nu+1)(2\nu+1)^2}{4} \quad (\nu \in \mathbb{N}). \end{aligned}$$

(ii) Similarly in the Lindelöf case the corresponding coefficients are

$$\begin{aligned} a_\nu^L &= \frac{8\nu^2+4\nu+1}{4} \quad (\nu \in \mathbb{N}_0), \quad b_0^L = \frac{1}{2}, \quad b_\nu^A = \frac{\nu^2(4\nu^2-1)}{4} \quad (\nu \in \mathbb{N}); \\ c_\nu^L &= \frac{8\nu^2+12\nu+5}{4} \quad (\nu \in \mathbb{N}_0), \quad d_0^L = \frac{1}{8}, \quad d_\nu^L = \frac{\nu^2(2\nu+1)^2}{4} \quad (\nu \in \mathbb{N}). \end{aligned}$$

(iii) Finally, in the case of the logistic weight the recurrence coefficients in (9) are

$$\begin{aligned} a_\nu^{\log} &= \frac{32\nu^4+32\nu^3+8\nu^2-1}{(4\nu-1)(4\nu+3)} \quad (\nu \in \mathbb{N}_0), \\ b_0^{\log} &= \frac{1}{\pi}, \quad b_\nu^{\log} = \frac{16\nu^4(2\nu-1)^4}{(4\nu-3)(4\nu-1)^2(4\nu+1)} \quad (\nu \in \mathbb{N}), \end{aligned}$$

and in (10) they are

$$\begin{aligned} c_\nu^{\log} &= \frac{32\nu^4+96\nu^3+104\nu^2+48\nu+7}{16\nu^2+24\nu+5} \quad (\nu \in \mathbb{N}_0), \\ d_0^{\log} &= \frac{1}{3\pi}, \quad d_\nu^{\log} = \frac{16\nu^4(2\nu+1)^4}{(4\nu-1)(4\nu+1)^2(4\nu+3)} \quad (\nu \in \mathbb{N}). \end{aligned}$$

The first two weight functions appear in the so-called Abel-Plana summation formulas (cf. [**16**]). For example, under certain conditions for an analytic function

f in the complex plane, the finite sum $S_{n,m}(f) = \sum_{k=m}^{n}(-1)^k f(k)$ can be obtained from the *Abel summation formula*

$$S_{n,m}(f) = \frac{1}{2}\big((-1)^m f(m) + (-1)^n f(n+1)\big) - \int_{\mathbb{R}} h(x;m,n) w^A(x)\,\mathrm{d}x,$$

where

$$h(x;m,n) = (-1)^m \frac{f(m+\mathrm{i}x) - f(m-\mathrm{i}x)}{2\mathrm{i}x} + (-1)^n \frac{f(n+1+\mathrm{i}x) - f(n+1-\mathrm{i}x)}{2\mathrm{i}x}.$$

2° For other weight functions which also appear in summation formulas, since the explicit expressions of the coefficients β_k are not known, using the MATHEMATICA package `OrthogonalPolynomials` (see [**1, 17**]) enables us to obtain β_k in rational forms.

For instance, consider the Plana weight function

$$w(x) = w^P(x) = \frac{|x|}{\mathrm{e}^{|2\pi x|} - 1},$$

which appears in the so-called *Plana summation formula* (cf. [**19**], [**14**])

$$T_{m,n}(f) - \int_m^n f(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x;m,n) w^P(x)\,\mathrm{d}x$$

for the composite trapezoidal sum

$$T_{m,n}(f) = \sum_{k=m}^{n}{}'' f(k) = \frac{1}{2}f(m) + \sum_{k=m+1}^{n-1} f(k) + \frac{1}{2}f(n),$$

where

$$g(x;m,n) = \frac{f(n+\mathrm{i}x) - f(n-\mathrm{i}x)}{2\mathrm{i}x} - \frac{f(m+\mathrm{i}x) - f(m-\mathrm{i}x)}{2\mathrm{i}x}. \tag{20}$$

This formula holds for analytic functions in the strip $\Omega_{m,n} = \big\{z \in \mathbb{C} : m \le \operatorname{Re} z \le n\big\}$, such that

$$\int_0^{+\infty} |f(x+\mathrm{i}y) - f(x-\mathrm{i}y)| \mathrm{e}^{-|2\pi y|}\,\mathrm{d}y$$

exists, and $\lim_{|y|\to+\infty} \mathrm{e}^{-|2\pi y|}|f(x\pm \mathrm{i}y)| = 0$ uniformly in x, for every $m \le x \le n$ $(m,n \in \mathbb{N},\, m<n)$.

Using the package `OrthogonalPolynomials`, we can obtain the sequence of coefficients $\{\beta_k^P\}_{k\ge 0}$ in rational forms as

$$\beta_0^P = \frac{1}{12},\ \beta_1^P = \frac{1}{10},\ \beta_2^P = \frac{79}{210},\ \beta_3^P = \frac{1205}{1659},\ \beta_4^P = \frac{262445}{209429},\ \beta_5^P = \frac{33461119209}{18089284070},$$

$$\beta_6^P = \frac{361969913862291}{137627660760070},\ \beta_7^P = \frac{85170013927511392430}{24523312685049374477},\ \text{etc.}$$

When k increases, these values are becoming more complicated (see [**16**]).

The corresponding weights for polynomials p_ν and q_ν on $\mathbb{R}^+$ are

$$w_1(t) = w_1^P(t) = \frac{1}{e^{2\pi\sqrt{t}} - 1} \quad \text{and} \quad w_2(t) = w_2^P(t) = \frac{t}{e^{2\pi\sqrt{t}} - 1}, \tag{21}$$

respectively. It is interesting to mention that at the Helsinki International Congress of Mathematicians (1978), Nikishin [**18**] pointed out the importance of some classes of orthogonal polynomials different from classical ones. In particular, he proposed obtaining explicit forms of polynomials (if possible) orthogonal with respect to the weight function w_1^P.

Taking the moments

$$\mu_{1,\nu}^P = \int_0^{+\infty} t^\nu w_1^P(t)\,\mathrm{d}t = \frac{(2\nu+1)!\zeta(2\nu+2)}{2^{2\nu+1}\pi^{2\nu+2}}, \quad \nu = 0, 1, \ldots, 2m-1,$$

and

$$\mu_{2,\nu}^P = \int_0^{+\infty} t^\nu w_2^P(t)\,\mathrm{d}t = \mu_{1,\nu+1}^P \frac{(2\nu+3)!\zeta(2\nu+4)}{2^{2\nu+3}\pi^{2\nu+4}}, \quad \nu = 0, 1, \ldots, 2m-1,$$

we can obtain the corresponding coefficients in (9) as

$$a_0^P = \frac{1}{10}, \quad a_1^P = \frac{871}{790}, \quad a_2^P = \frac{1672667011}{539062030}, \quad a_3^P = \frac{5063448671781098710 7}{8296534235776787390},$$

$$a_4^P = \frac{32411158794986052698280155649496 09681}{32080132475162436080132763193341 5050}, \quad \text{etc.};$$

$$b_0^P = \frac{1}{12}, \quad b_1^P = \frac{79}{2100}, \quad b_2^P = \frac{1312225}{1441671}, \quad b_3^P = \frac{2491734801234609}{512172182993900},$$

$$b_4^P = \frac{276980623805265435471536707 00}{176955582231522908905742601 3}, \quad \text{etc.},$$

as well as in (10),

$$c_0^P = \frac{10}{21}, \quad c_1^P = \frac{110200}{55671}, \quad c_2^P = \frac{239533652610}{53469214601}, \quad c_3^P = \frac{312611606327029924743272 00}{391747872854992383570978 9},$$

$$c_4^P = \frac{2032299617271932287823786429182679246048749956 8690}{1628454245165190286597605307125063916376617814 289}, \quad \text{etc.};$$

$$d_0^P = \frac{1}{120}, \quad d_1^P = \frac{241}{882}, \quad d_2^P = \frac{423558471}{182722826}, \quad d_3^P = \frac{821210997517832607}{89904292554749621},$$

$$d_4^P = \frac{808764196606302105358539179685834 15257}{320659466284175189971489473039928 5285}, \quad \text{etc.},$$

but their explicit expressions (for each index) remains a mystery!

Another interesting summation formula is

$$\sum_{k=m}^{n} f(k) - \int_{m-1/2}^{n+1/2} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} g\bigl(x; m - \tfrac{1}{2}, n + \tfrac{1}{2}\bigr) w^M(x)\,\mathrm{d}x,$$

where the so-called *midpoint weight function* is defined by

$$w(x) = w^M(x) = \frac{|x|}{\mathrm{e}^{|2\pi x|} + 1},$$

and $g\big(x; m-\frac{1}{2}, n+\frac{1}{2}\big)$ by (20). As before, one can consider the polynomials p_ν and q_ν orthogonal on $\mathbb{R}^+$ with respect to the weight functions

$$w_1(t) = w_1^M(t) = \frac{1}{\mathrm{e}^{2\pi\sqrt{t}} + 1} \quad \text{and} \quad w_2(t) = w_2^M(t) = \frac{t}{\mathrm{e}^{2\pi\sqrt{t}} + 1}, \tag{22}$$

and in a similar way, one can obtain the corresponding coefficients in (9) and (10) in the following the rational forms

$$a_0^M = \frac{7}{40},\ a_1^M = \frac{97153}{82840},\ a_2^M = \frac{2143300949275717}{675664735216120},\ a_3^M = \frac{220953557093736349691768417054261}{358005012158232650133557971060 40},$$

$$a_4^M = \frac{130861346925393025853171746401175157050180563993602424972 07}{1286538803151559855777866179684631498656991773534847212200},\ \text{etc.};$$

$$b_0^M = \frac{1}{24},\quad b_1^M = \frac{2071}{33600},\quad b_2^M = \frac{15685119025}{15852295536},\quad b_3^M = \frac{5895324568676150049511881}{1170833101982789404702400},$$

$$b_4^M = \frac{919480999258696959661346213448241024976800075}{576540802597908800437584051097300398601002 12},\quad \text{etc.};$$

$$c_0^M = \frac{155}{294},\quad c_1^M = \frac{654837850}{323155833},\quad c_2^M = \frac{496471545892577 71035}{10966854047350313398},$$

$$c_3^M = \frac{543088581222807426712675575740027673298 00}{676531074327501862390841892603677460878 1},$$

$$c_4^M = \frac{2383807210883859864106057473176692820172732110851477347996900634325131 8055}{19027890078491705060617723955751917909302103587071623342058732934723211 34},\ \text{etc.};$$

$$d_0^M = \frac{7}{960},\quad d_1^M = \frac{199849}{691488},\quad d_2^M = \frac{366669459296427}{154646219485472},$$

$$d_3^M = \frac{26446525491560415511898191097 31}{286002885915941819991126155408},$$

$$d_4^M = \frac{7071951106108162652736604339756545328619345537100911995491 1}{2782343550785232136311735142019287634629029202932721468080},\quad \text{etc..}$$

Unfortunately, we were unable to discover their explicit forms!

6. A CLASS OF SYMMETRIC WEIGHTS WITH FOUR FREE PARAMETERS

In this section, we consider a special case of symmetric weight functions on $(-a, a)$ with four free parameters that covers many well-known classical weights such as Legendre, first and second kind Chebyshev, ultraspherical, generalized ultraspherical, Hermite and generalized Hermite weight, i.e.,

$$w(x) = \exp\left(\int_c^x \frac{rt^2 + s}{t(pt^2 + q)}\,\mathrm{d}t\right) = w(-x),$$

where p, q, r, s are real parameters and c is a constant in $(-a, a)$.

It is shown in [**11**] that a special solution of the differential equation

$$x^2(px^2+q)\Phi_n''(x)+x(rx^2+s)\Phi_n'(x)-\left(n(r+(n-1)p)x^2+\frac{1-(-1)^n}{2}\,s\right)\Phi_n(x)=0,$$

is the symmetric polynomial in the form

$$\begin{aligned}(23)\qquad \Phi_n(x) &= S_n\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right)\\ &= \sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{(2i-(-1)^n+2\,[n/2])\,p+r}{(2i-(-1)^n+2)\,q+s}\right)x^{n-2k},\end{aligned}$$

whose monic form is given by

$$\widehat{S}_n(x)=\widehat{S}_n\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right)=\prod_{i=0}^{[n/2]-1}\frac{\big(2i-(-1)^n+2\big)q+s}{\big(2i-(-1)^n+2[n/2]\big)p+r}\,S_n\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right).$$

For instance, we have

$$\begin{aligned}
\widehat{S}_2\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right) &= x^2+\frac{q+s}{p+r}\,,\\
\widehat{S}_3\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right) &= x^3+\frac{3q+s}{3p+r}x\,,\\
\widehat{S}_4\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right) &= x^4+2\frac{3q+s}{5p+r}x^2+\frac{(3q+s)(q+s)}{(5p+r)(3p+r)},\\
\widehat{S}_5\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right) &= x^5+2\frac{5q+s}{7p+r}x^3+\frac{(5q+s)(3q+s)}{(7p+r)(5p+r)}x\,.
\end{aligned}$$

According to [**11**], the monic form of these polynomials satisfies the three-term recurrence relation

$$(24)\qquad \widehat{S}_{k+1}(x)=x\widehat{S}_k(x)-\beta_k\begin{pmatrix} r & s \\ p & q\end{pmatrix}\widehat{S}_{k-1}(x)\qquad (k\ge 1),$$

where $\widehat{S}_0(x)=1$, $\widehat{S}_1(x)=x$, and

$$\beta_k\begin{pmatrix} r & s \\ p & q\end{pmatrix}=-\frac{pq\,k^2+\big((r-2p)q-(-1)^k ps\big)k+(r-2p)s(1-(-1)^k)/2}{(2pk+r-p)(2pk+r-3p)}.$$

This means that for the monic polynomials $\pi_k(x)=\widehat{S}_k(x)$, the coefficients

$$(25)\qquad \beta_k=\beta_k(p,q,r,s)=\beta_k\begin{pmatrix} r & s \\ p & q\end{pmatrix}\qquad (k\ge 1),$$

depend on four parameters p, q, r, s. Moreover, if $\beta_k(p,q,r,s) > 0$, the generic form of the orthogonality relation is as

$$\int_{-a}^{a} W\begin{pmatrix} r & s \\ p & q \end{pmatrix}\Big|\, x\Big) \widehat{S}_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}\Big|\, x\Big) \widehat{S}_k\begin{pmatrix} r & s \\ p & q \end{pmatrix}\Big|\, x\Big)\, \mathrm{d}x = \big(\beta_0\beta_1\cdots\beta_n\big)\delta_{n,k}, \tag{26}$$

where

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) = \exp\left(\int_c^x \frac{(r-2p)t^2+s}{t(pt^2+q)}\,\mathrm{d}t\right) \tag{27}$$

and

$$\beta_0 = \int_{-a}^{a} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) \mathrm{d}x.$$

Without loss of generality, we can assume only $a = 1$ for finite intervals and $a = \infty$ for the infinite interval.

Regarding [**11**], the function $(px^2+q)\, W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right)$ must vanish at $x = a$ in order to hold the orthogonality relation (26).

In general, there are four main sub-classes of distribution families (27) whose probability density functions are as follows (see [**11**])

$$K_1 W\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\, x\right) = \frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\beta+1)}|x|^{2\alpha}(1-x^2)^{\beta} \tag{28}$$

for $-1 \le x \le 1$, and

$$K_2 W\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\, x\right) = \frac{1}{\Gamma\left(\alpha+\frac{1}{2}\right)}|x|^{2\alpha}\mathrm{e}^{-x^2}, \tag{29}$$

$$K_3 W\left(\begin{matrix} -2\alpha-2\beta+2 & -2\alpha \\ 1 & 1 \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(\beta)}{\Gamma\left(\beta+\alpha-\frac{1}{2}\right)\Gamma\left(-\alpha+\frac{1}{2}\right)}\frac{|x|^{-2\alpha}}{(1+x^2)^{\beta}}, \tag{30}$$

$$K_4 W\left(\begin{matrix} -2\alpha+2 & 2 \\ 1 & 0 \end{matrix}\,\middle|\, x\right) = \frac{1}{\Gamma\left(\alpha-\frac{1}{2}\right)}|x|^{-2\alpha}\mathrm{e}^{-1/x^2} \tag{31}$$

for $-\infty < x < \infty$, where the values $\{K_i\}_{i=1}^{4}$ play the normalizing constant role in relations (28) to (31). Consequently, there are four sub-sequences of symmetric orthogonal polynomials (23).

According to (28), if $(p,q,r,s) = (-1, 1, -2\alpha-2\beta-2, 2\alpha)$ is substituted into (23), then

$$S_n\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \prod_{i=0}^{[n/2]-(k+1)} \frac{-2i-(2\beta+2\alpha+2-(-1)^n+2[n/2])}{2i+2\alpha+2-(-1)^n} x^{n-2k},$$

represents the explicit form of generalized ultraspherical polynomials (GUP). By noting (24) and (25), the recurrence relation of monic GUP takes the form

$$\widehat{S}_{k+1}(x) = x\,\widehat{S}_k(x) - \beta_k\begin{pmatrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{pmatrix}\widehat{S}_{k-1}(x),$$

in which

$$\beta_k\begin{pmatrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{pmatrix} = \frac{\big(k+(1-(-1)^k)\alpha\big)\big(k+(1-(-1)^k)\alpha+2\beta\big)}{(2k+2\alpha+2\beta-1)(2k+2\alpha+2\beta+1)}. \tag{32}$$

Hence, its orthogonality relation reads as

$$\int_{-1}^{1} |x|^{2\alpha}(1-x^2)^{\beta}\,\widehat{S}_n\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\, x\right)\widehat{S}_m\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\, x\right)\mathrm{d}x$$
$$= \int_{-1}^{1} |x|^{2\alpha}(1-x^2)^{\beta}\,\mathrm{d}x \prod_{i=1}^{n}\beta_i\begin{pmatrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{pmatrix}\delta_{n,m},$$

where

$$\int_{-1}^{1} |x|^{2\alpha}(1-x^2)^{\beta}\,\mathrm{d}x = B\Big(\alpha+\frac{1}{2},\beta+1\Big) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\beta+1)}{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}.$$

The above relation shows that the constraints on the parameters α and β should be $\alpha+1/2>0$ and $\beta+1>0$.

The second sub-class is the generalized Hermite polynomials

$$S_n\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\prod_{i=0}^{[n/2]-(k+1)}\frac{-2}{2i+(-1)^{n+1}+2+2\alpha}\,x^{n-2k},$$

satisfying the monic recurrence relation

$$\widehat{S}_{k+1}(x) = x\,\widehat{S}_k(x) - \beta_k\begin{pmatrix} -2 & 2\alpha \\ 0 & 1 \end{pmatrix}\widehat{S}_{k-1}(x),$$

with

$$\beta_k\begin{pmatrix} -2 & 2\alpha \\ 0 & 1 \end{pmatrix} = \frac{k}{2} + \frac{1-(-1)^k}{2}\,\alpha, \tag{33}$$

and the orthogonality relation

$$\int_{-\infty}^{\infty} |x|^{2\alpha}\mathrm{e}^{-x^2}\widehat{S}_n\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\, x\right)\widehat{S}_m\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\, x\right)\mathrm{d}x$$
$$= \left(\frac{1}{2^n}\prod_{i=1}^{n}\big((1-(-1)^i)\alpha+i\big)\right)\Gamma\Big(\alpha+\frac{1}{2}\Big)\delta_{n,m},$$

provided that $\alpha+1/2>0$. According to Favard's theorem [**7, 12**], if $\beta_n(p,q,r,s)>0$ holds only for a finite number of positive integers, i.e., $n=1,\dots,N$, then the related polynomials are finitely orthogonal. In this sense, there are two kinds of classical symmetric finite orthogonal polynomials.

The first finite class is orthogonal with respect to the weight function $x\mapsto |x|^{-2\alpha}(1+x^2)^{-\beta}$ on $(-\infty,\infty)$ with the initial vector $(p,q,r,s)=(1,1,-2\alpha-2\beta+2,-2\alpha)$, whose explicit form is as

$$\widehat{S}_n\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\,\middle|\, x\right)$$

$$=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\prod_{i=0}^{[n/2]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2\alpha-2b}{2i+(-1)^{n+1}+2-2\alpha}\,x^{n-2k},$$

and satisfies the recurrence relation (24) with

$$\beta_k\begin{pmatrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{pmatrix}=-\frac{\left[k-\alpha+(-1)^k\alpha\right]\left(k-(1-(-1)^k)\alpha-2\beta\right)}{(2k-2\alpha-2\beta+1)(2k-2\alpha-2\beta-1)}. \tag{34}$$

Hence, its orthogonality relation takes the form

$$\int_{-\infty}^{\infty}\frac{|x|^{-2\alpha}}{(1+x^2)^{\beta}}\,\widehat{S}_n\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\,\middle|\, x\right)\widehat{S}_m\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\,\middle|\, x\right)\mathrm{d}x$$

$$=\prod_{i=1}^{n}\beta_i\begin{pmatrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{pmatrix}\frac{\Gamma\left(\beta+\alpha-\frac12\right)\Gamma\left(-\alpha+\frac12\right)}{\Gamma(\beta)}\,\delta_{n,m},$$

if and only if

$$\beta_n\begin{pmatrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{pmatrix}>0;\quad \beta+\alpha>\frac12,\ \alpha<\frac12\quad\text{and}\quad\beta>0.$$

In other words, the finite polynomial set $\{S_n(1,1,-2\alpha-2\beta+2,-2\alpha\,;x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $|x|^{-2\alpha}(1+x^2)^{-\beta}$ on $(-\infty,\infty)$ if and only if $N\le\alpha+\beta-1/2$, $\alpha<1/2$ and $\beta>0$.

Similarly, the second finite class is orthogonal with respect to the weight $x\mapsto|x|^{-2\alpha}\mathrm{e}^{-1/x^2}$ on $(-\infty,\infty)$ with the initial vector $(p,q,r,s)=(1,0,-2\alpha+2,2)$, whose explicit form is as

$$\widehat{S}_n\left(\begin{matrix}-2\alpha+2 & 2\\ 1 & 0\end{matrix}\,\middle|\, x\right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\prod_{i=0}^{[n/2]-(k+1)}\left(i+\left[\frac n2\right]-\frac{(-1)^n}{2}+1-\alpha\right)x^{n-2k},$$

and satisfies the recurrence relation (24) with

$$\beta_k\begin{pmatrix}-2\alpha+2 & 2\\ 1 & 0\end{pmatrix}=\frac{2(-1)^k(k-\alpha)+2\alpha}{(2k-2\alpha+1)(2k-2\alpha-1)}, \tag{35}$$

and finally has the orthogonality relation

$$\int_{-\infty}^{\infty} |x|^{-2\alpha} \mathrm{e}^{-1/x^2} \widehat{S}_n \left(\begin{array}{cc|c} -2\alpha+2 & 2 & \\ 1 & 0 & x \end{array} \right) \widehat{S}_m \left(\begin{array}{cc|c} -2\alpha+2 & 2 & \\ 1 & 0 & x \end{array} \right) \mathrm{d}x$$

$$= \prod_{i=1}^{n} \beta_i \begin{pmatrix} -2\alpha+2 & 2 \\ 1 & 0 \end{pmatrix} \Gamma\Big(\alpha - \frac{1}{2}\Big) \delta_{n,m},$$

if and only if $N = \max\{m,n\} \le \alpha - 1/2$. This means that the finite polynomial set $\{S_n(1,0,-2\alpha+2,2\,;x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $|x|^{-2\alpha}\mathrm{e}^{-1/x^2}$ on $(-\infty,\infty)$ if $N \le \alpha - 1/2$. The following table summarizes the main characteristics of the four introduced sub-classes. For other symmetric orthogonal polynomials see e.g. [**9, 10**].

Table 2: Four special cases of $S_n(p,q,r,s;x)$

Definition	Weight	β_k
$S_n\left(\begin{array}{cc\|c} -2\alpha-2\beta-2 & 2\alpha & \\ -1 & 1 & x \end{array}\right)$	$\|x\|^{2\alpha}(1-x^2)^{\beta}$	$\frac{\big(k+\alpha-(-1)^k\alpha\big)\big(k+(1-(-1)^k)\alpha+2\beta\big)}{(2k+2\alpha+2\beta-1)(2k+2\alpha+2\beta+1)}$
$S_n\left(\begin{array}{cc\|c} -2 & 2\alpha & \\ 0 & 1 & x \end{array}\right)$	$\|x\|^{2\alpha}\mathrm{e}^{-x^2}$	$\frac{k}{2} + \frac{1-(-1)^k}{2}\,\alpha$
$S_n\left(\begin{array}{cc\|c} -2\alpha-2\beta+2 & -2\alpha & \\ 1 & 1 & x \end{array}\right)$	$\frac{\|x\|^{-2\alpha}}{(1+x^2)^{\beta}}$	$-\frac{\big(k-\alpha+(-1)^k\alpha\big)\big(k-(1-(-1)^k)\alpha-2\beta\big)}{(2k-2\alpha-2\beta+1)(2k-2\alpha-2\beta-1)}$
$S_n\left(\begin{array}{cc\|c} -2\alpha+2 & 2 & \\ 1 & 0 & x \end{array}\right)$	$\|x\|^{-2\alpha}\mathrm{e}^{-1/x^2}$	$\frac{2(-1)^k(k-\alpha)+2\alpha}{(2k-2\alpha+1)(2k-2\alpha-1)}$

In the last column of this table we give the explicit expressions for the recursion coefficients β_k in the three-term recurrence relation. We use them in the construction of the corresponding Jacobi matrices.

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Summation Formulas of Euler–Maclaurin and Abel–Plana: Old and New Results and Applications

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Abstract Summation formulas of the Euler–Maclaurin and Abel–Plana and their connections with several kinds of quadrature rules are studied. Besides the history of these formulas, several of their modifications and generalizations are considered. Connections between the Euler–Maclaurin formula and basic quadrature rules of Newton–Cotes type, as well as the composite Gauss–Legendre rule and its Lobatto modification are presented. Besides the basic Plana summation formula a few integral modifications (the midpoint summation formula, the Binet formula, Lindelöf formula) are introduced and analysed. Starting from the moments of their weight functions and applying the recent MATHEMATICA package `OrthogonalPolynomials`, recursive coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials are constructed, as well as the parameters (nodes and Christoffel numbers) of the corresponding Gaussian quadrature formula.

Keywords Summation • Euler–Maclaurin formula • Abel–Plana formula • Gaussian quadrature formula • Orthogonal polynomial • Three-term recurrence relation

Mathematics Subject Classification (2010): 33C45, 33C47, 41A55, 65B15, 65D30, 65D32

1 Introduction and Preliminaries

A summation formula was discovered independently by Leonhard Euler [18, 19] and Colin Maclaurin [35] plays an important role in the broad area of numerical analysis, analytic number theory, approximation theory, as well as in many applications in

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other fields. This formula, today known as the Euler–Maclaurin summation formula,

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,\mathrm{d}x + \frac{1}{2}(f(0)+f(n)) + \sum_{\nu=1}^{r} \frac{B_{2\nu}}{(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r(f), \tag{1}$$

was published first time by Euler in 1732 (without proof) in connection with the problem of determining the sum of the reciprocal squares,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots, \tag{2}$$

which is known as the *Basel problem*. The brothers Johann and Jakob Bernoulli, Leibnitz, Stirling, etc. also dealt intensively by such a kind of problems. In modern terminology, the sum (2) represents the zeta function of 2, where more generally

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad (s > 1).$$

Although at that time the theory of infinite series was not exactly based, it was observed a very slow convergence of this series, e.g. in order to compute directly the sum with an accuracy of six decimal places it requires taking into account at least a million first terms, because

$$\frac{1}{n+1} < \sum_{k=n+1}^{+\infty} \frac{1}{k^2} < \frac{1}{n}.$$

Euler discovered the remarkable formula with much faster convergence

$$\zeta(2) = \log^2 2 + \sum_{k=1}^{+\infty} \frac{1}{2^{k-1}k^2},$$

and obtained the value $\zeta(2) = 1.644944\ldots$ (with seven decimal digits). But the discovery of a general summation procedure (1) enabled Euler to calculate $\zeta(2)$ to 20 decimal places. For details see Gautschi [25, 26] and Varadarajan [61].

Using a generalized Newton identity for polynomials (when their degree tends to infinity), Euler [19] proved the exact result $\zeta(2) = \pi^2/6$. Using the same method he determined $\zeta(s)$ for even $s = 2m$ up to 12,

$$\zeta(4) = \frac{\pi^4}{90}, \ \zeta(6) = \frac{\pi^6}{945}, \ \zeta(8) = \frac{\pi^8}{9450}, \ \zeta(10) = \frac{\pi^{10}}{93555}, \ \zeta(12) = \frac{691\pi^{12}}{638512875}.$$

Sometime later, using his own partial fraction expansion of the cotangent function, Euler obtained the general formula

$$\zeta(2\nu) = (-1)^{\nu-1}\frac{2^{2\nu-1}B_{2\nu}}{(2\nu)!}\pi^{2\nu},$$

where $B_{2\nu}$ are the Bernoulli numbers, which appear in the general Euler–Maclaurin summation formula (1). Detailed information about Euler's complete works can be found in *The Euler Archive* (http://eulerarchive.maa.org).

We return now to the general Euler–Maclaurin summation formula (1) which holds for any $n, r \in \mathbb{N}$ and $f \in C^{2r}[0, n]$. As we mentioned before this formula was found independently by Maclaurin. While in Euler's case the formula (1) was applied for computing slowly converging infinite series, in the second one Maclaurin used it to calculate integrals. A history of this formula was given by Barnes [5], and some details can be found in [3, 8, 25, 26, 38, 61].

Bernoulli numbers B_k ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$) can be expressed as values at zero of the corresponding Bernoulli polynomials, which are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{+\infty} B_k(x)\frac{t^k}{k!}.$$

Similarly, Euler polynomials can be introduced by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{+\infty} E_k(x)\frac{t^k}{k!}.$$

Bernoulli and Euler polynomials play a similar role in numerical analysis and approximation theory like orthogonal polynomials. First few Bernoulli polynomials are

$$B_0(x) = 1,\ \ B_1(x) = x - \frac{1}{2},\ \ B_2(x) = x^2 - x + \frac{1}{6},\ \ B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2},$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},\ \ B_5(x) = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6},\ \ \text{etc.}$$

Some interesting properties of these polynomials are

$$B_n'(x) = nB_{n-1}(x),\ \ B_n(1-x) = (-1)^n B_n(x),\ \ \int_0^1 B_n(x)\,dx = 0\ \ (n \in \mathbb{N}).$$

The error term $E_r(f)$ in (1) can be expressed in the form (cf. [8])

$$E_r(f) = (-1)^r \sum_{k=1}^{+\infty} \int_0^n \frac{e^{i2\pi kt} + e^{-i2\pi kt}}{(2\pi k)^{2r}} f^{(2r)}(x)\,dx,$$

or in the form

$$E_r(f) = -\int_0^n \frac{B_{2r}(x - \lfloor x \rfloor)}{(2r)!} f^{(2r)}(x)\, \mathrm{d}x, \tag{3}$$

where $\lfloor x \rfloor$ denotes the largest integer that is not greater than x. Supposing $f \in C^{2r+1}[0, n]$, after an integration by parts in (3) and recalling that the odd Bernoulli numbers are zero, we get (cf. [28, p. 455])

$$E_r(f) = \int_0^n \frac{B_{2r+1}(x - \lfloor x \rfloor)}{(2r+1)!} f^{(2r+1)}(x)\, \mathrm{d}x. \tag{4}$$

If $f \in C^{2r+2}[0, n]$, using Darboux's formula one can obtain (1), with

$$E_r(f) = \frac{1}{(2r+2)!} \int_0^1 [B_{2r+2} - B_{2r+2}(x)] \left(\sum_{k=0}^{n-1} f^{(2r+2)}(k + x) \right) \mathrm{d}x \tag{5}$$

(cf. Whittaker and Watson [65, p. 128]). This expression for $E_r(f)$ can be also derived from (4), writting it in the form

$$\begin{aligned} E_r(f) &= \int_0^1 \frac{B_{2r+1}(x)}{(2r+1)!} \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k + x) \right) \mathrm{d}x \\ &= \int_0^1 \frac{B'_{2r+2}(x)}{(2r+2)!} \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k + x) \right) \mathrm{d}x, \end{aligned}$$

and then by an integration by parts,

$$\begin{aligned} E_r(f) = &\left[\frac{B_{2r+2}(x)}{(2r+2)!} \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k + x) \right) \right]_0^1 \\ &- \int_0^1 \frac{B_{2r+2}(x)}{(2r+2)!} \left(\sum_{k=0}^{n-1} f^{(2r+2)}(k + x) \right) \mathrm{d}x. \end{aligned}$$

Because of $B_{2r+2}(1) = B_{2r+2}(0) = B_{2r+2}$, the last expression can be represented in the form (5).

Since

$$(-1)^r [B_{2r+2} - B_{2r+2}(x)] \geq 0, \quad x \in [0, 1],$$

and

$$\int_0^1 [B_{2r+2} - B_{2r+2}(x)]\, \mathrm{d}t = B_{2r+2},$$

according to the *Second Mean Value Theorem for Integrals*, there exists $\eta \in (0,1)$ such that

$$E_r(f) = \frac{B_{2r+2}}{(2r+2)!}\left(\sum_{k=0}^{n-1} f^{(2r+2)}(k+\eta)\right) = \frac{nB_{2r+2}}{(2r+2)!} f^{(2r+2)}(\xi), \quad 0 < \xi < n. \tag{6}$$

The Euler–Maclaurin summation formula can be considered on an arbitrary interval (a,b) instead of $(0,n)$. Namely, taking $h = (b-a)/n$, $t = a + xh$, and $f(x) = f((t-a)/h) = \varphi(t)$, formula (1) reduces to

$$h\sum_{k=0}^{n} \varphi(a+kh) = \int_a^b \varphi(t)\,\mathrm{d}t + \frac{h}{2}\left[\varphi(a)+\varphi(b)\right]$$
$$+\sum_{\nu=1}^{r} \frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\varphi^{(2\nu-1)}(b) - \varphi^{(2\nu-1)}(a)\right] + E_r(\varphi), \tag{7}$$

where, according to (6),

$$E_r(\varphi) = (b-a)\frac{B_{2r+2}h^{2r+2}}{(2r+2)!}\varphi^{(2r+2)}(\xi), \quad a < \xi < b. \tag{8}$$

Remark 1. An approach in the estimate of the remainder term of the Euler–Maclaurin formula was given by Ostrowski [47].

Remark 2. The Euler–Maclaurin summation formula is implemented in MATHEMATICA as the function `NSum` with option `Method -> Integrate`.

2 Connections Between Euler–Maclaurin Summation Formula and Some Basic Quadrature Rules of Newton–Cotes Type

In this section we first show a direct connection between the Euler–Maclaurin summation formula (1) and the well-known composite trapezoidal rule,

$$T_n f := {\sum_{k=0}^{n}}{}'' f(k) = \frac{1}{2}f(0) + \sum_{k=1}^{n-1} f(k) + \frac{1}{2}f(n), \tag{9}$$

for calculating the integral

$$I_n f := \int_0^n f(x)\,\mathrm{d}x. \tag{10}$$

This rule for integrals over an arbitrary interval $[a, b]$ can be presented in the form

$$h \sum_{k=0}^{n}{}'' \varphi(a + kh) = \int_a^b \varphi(t)\,\mathrm{d}t + E^T(\varphi), \tag{11}$$

where, as before, the sign $\sum''$ denotes summation with the first and last terms halved, $h = (b-a)/n$, and $E^T(\varphi)$ is the remainder term.

Remark 3. In general, the sequence of the composite trapezoidal sums converges very slowly with respect to step refinement, because of $|E^T(\varphi)| = O(h^2)$. However, the trapezoidal rule is very attractive in numerical integration of analytic and periodic functions, for which $\varphi(t + b - a) = \varphi(t)$. In that case, the sequence of trapezoidal sums

$$T_n(\varphi; h) := h \sum_{k=0}^{n}{}'' \varphi(a + kh) = h \sum_{k=1}^{n} \varphi(a + kh) \tag{12}$$

converges geometrically when applied to analytic functions on periodic intervals or the real line. A nice survey on this subject, including history of this phenomenon, has been recently given by Trefethen and Weideman [59] (see also [64]). For example, when φ is a $(b-a)$-periodic and analytic function, such that $|\varphi(z)| \le M$ in the half-plane $\operatorname{Im} z > -c$ for some $c > 0$, then for each $n \ge 1$, the following estimate

$$|E^T(\varphi)| = \left| T_n(\varphi; h) - \int_a^b \varphi(t)\,\mathrm{d}t \right| \le \frac{(b-a)M}{\mathrm{e}^{2\pi cn/(b-a)} - 1}$$

holds. A similar result holds for integrals over $\mathbb{R}$.

It is well known that there are certain types of integrals which can be transformed (by changing the variable of integration) to a form suitable for the trapezoidal rule. Such transformations are known as *Exponential* and *Double Exponential Quadrature Rules* (cf. [44–46, 57, 58]). However, the use of these transformations could introduce new singularities in the integrand and the analyticity strip may be lost. A nice discussion concerning the error theory of the trapezoidal rule, including several examples, has been recently given by Waldvogel [63].

Remark 4. In 1990 Rahman and Schmeisser [51] gave a specification of spaces of functions for which the trapezoidal rule converges at a prescribed rate as $n \to +\infty$, where a correspondence is established between the speed of convergence and regularity properties of integrands. Some examples for these spaces were provided in [64].

In a general case, according to (1), it is clear that

$$T_n f - I_n f = \sum_{\nu=1}^{r} \frac{B_{2\nu}}{(2\nu)!} \left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0) \right] + E_r^T(f), \tag{13}$$

where $T_n f$ and $I_n f$ are given by (9) and (10), respectively, and the remainder term $E_r^T(f)$ is given by (6) for functions $f \in C^{2r+2}[0, n]$.

Similarly, because of (7), the corresponding formula on the interval $[a, b]$ is

$$h \sum_{k=0}^{n}{}'' \varphi(a + kh) - \int_a^b \varphi(t)\,\mathrm{d}t = \sum_{\nu=1}^{r} \frac{B_{2\nu} h^{2\nu}}{(2\nu)!} \left[\varphi^{(2\nu-1)}(b) - \varphi^{(2\nu-1)}(a)\right] + E_r^T(\varphi),$$

where $E_r^T(\varphi)$ is the corresponding remainder given by (8). Comparing this with (11) we see that $E^T(\varphi) = E_0^T(\varphi)$.

Notice that if $\varphi^{(2r+2)}(x)$ does not change its sign on (a, b), then $E_r^T(\varphi)$ has the same sign as the first neglected term. Otherwise, when $\varphi^{(2r+2)}(x)$ is not of constant sign on (a, b), then it can be proved that (cf. [14, p. 299])

$$|E_r^T(\varphi)| \le h^{2r+2} \frac{|2B_{2r+2}|}{(2r+2)!} \int_a^b |\varphi^{(2r+2)}(t)|\,\mathrm{d}t \approx 2\Big(\frac{h}{2\pi}\Big)^{2r+2} \int_a^b |\varphi^{(2r+2)}(t)|\,\mathrm{d}t,$$

i.e., $|E_r^T(\varphi)| = O(h^{2r+2})$. Supposing that $\int_a^{+\infty} |\varphi^{(2r+2)}(x)|\,\mathrm{d}x < +\infty$, this holds also in the limit case as $b \to +\infty$. This limit case enables applications of the Euler–Maclaurin formula in summation of infinite series, as well as for obtaining asymptotic formulas for a large b.

A standard application of the Euler–Maclaurin formula is in numerical integration. Namely, for a small constant h, the trapezoidal sum can be dramatically improved by subtracting appropriate terms with the values of derivatives at the endpoints a and b. In such a way, the corresponding approximations of the integral can be improved to $O(h^4)$, $O(h^6)$, etc.

Remark 5. Rahman and Schmeisser [52] obtained generalizations of the trapezoidal rule and the Euler–Maclaurin formula and used them for constructing quadrature formulas for functions of exponential type over infinite intervals using holomorphic functions of exponential type in the right half-plane, or in a vertical strip, or in the whole plane. They also determined conditions which equate the existence of the improper integral to the convergence of its approximating series.

Remark 6. In this connection an interesting question can be asked. Namely, what happens if the function $\varphi \in C^\infty(\mathbb{R})$ and its derivatives are $(b - a)$-periodic, i.e., $\varphi^{(2\nu-1)}(a) = \varphi^{(2\nu-1)}(b)$, $\nu = 1, 2, \ldots$? The conclusion that $T_n(\varphi; h)$, in this case, must be exactly equal to $\int_a^b \varphi(t)\,\mathrm{d}t$ is wrong, but the correct conclusion is that $E^T(\varphi)$ decreases faster than any finite power of h as n tends to infinity.

Remark 7. Also, the Euler–Maclaurin formula was used in getting an extrapolating method well-known as Romberg's integration (cf. [14, pp. 302–308 and 546–523] and [39, pp. 158–164]).

In the sequel, we consider a quadrature sum with values of the function f at the points $x = k + \frac{1}{2}$, $k = 0, 1, \ldots, n-1$, i.e., the so-called midpoint rule

$$M_n f := \sum_{k=0}^{n-1} f\Big(k + \frac{1}{2}\Big).$$

Also, for this rule there exists the so-called second Euler–Maclaurin summation formula

$$M_nf - I_nf = \sum_{\nu=1}^{r} \frac{(2^{1-2\nu}-1)B_{2\nu}}{(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r^M(f), \tag{14}$$

for which

$$E_r^M(f) = n\,\frac{(2^{-1-2r}-1)B_{2r+2}}{(2r+2)!}f^{(2r+2)}(\xi), \qquad 0 < \xi < n,$$

when $f \in C^{2r+2}[0,n]$ (cf. [39, p. 157]). As before, I_nf is given by (10).

The both formulas, (13) and (14), can be unified as

$$Q_nf - I_nf = \sum_{\nu=1}^{r} \frac{B_{2\nu}(\tau)}{(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r^Q(f),$$

where $\tau = 0$ for $Q_n \equiv T_n$ and $\tau = 1/2$ for $Q_n \equiv M_n$. It is true, because of the fact that [50, p. 765] (see also [10])

$$B_\nu(0) = B_\nu \quad \text{and} \quad B_\nu\left(\frac{1}{2}\right) = (2^{1-\nu}-1)B_\nu.$$

If we take a combination of T_nf and M_nf as

$$Q_nf = S_nf = \frac{1}{3}(T_nf + 2M_nf),$$

which is, in fact, the well-known classical composite Simpson rule,

$$S_nf := \frac{1}{3}\left[\frac{1}{2}f(0) + \sum_{k=1}^{n-1} f(k) + 2\sum_{k=0}^{n-1} f\left(k+\frac{1}{2}\right) + \frac{1}{2}f(n)\right],$$

we obtain

$$S_nf - I_nf = \sum_{\nu=2}^{r} \frac{(4^{1-\nu}-1)B_{2\nu}}{3(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r^S(f). \tag{15}$$

Notice that the summation on the right-hand side in the previous equality starts with $\nu = 2$, because the term for $\nu = 1$ vanishes. For $f \in C^{2r+2}[0,n]$ it can be proved that there exists $\xi \in (0,n)$, such that

$$E_r^S(f) = n\,\frac{(4^{-r}-1)B_{2r+2}}{3(2r+2)!}f^{(2r+2)}(\xi).$$

For some modification and generalizations of the Euler–Maclaurin formula, see [2, 7, 20–22, 37, 55, 60]. In 1965 Kalinin [29] derived an analogue of the Euler–Maclaurin formula for C^∞ functions, for which there is Taylor series at each positive integer $x = \nu$,

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=0}^{+\infty} \frac{\theta^{k+1} - (\theta-1)^{k+1}}{(k+1)!} h^{k+1} \sum_{\nu=1}^{n} f^{(k)}(a + (\nu-\theta)h),$$

where $h = (b-a)/n$, and used it to find some new expansions for the gamma function, the ψ function, as well as the Riemann zeta function.

Using Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$, in 1960 Keda [30] established a quadrature formula similar to the Euler–Maclaurin,

$$\int_0^1 f(x)\,\mathrm{d}x = T_n + \sum_{k=0}^{n-1} A_k \left[f^{(2k+2)}(0) + f^{(2k+2)}(1)\right] + R_n,$$

where

$$T_n = \frac{1}{n} \sum_{k=0}^{n}{}'' f\Big(\frac{k}{n}\Big), \quad A_k = \sum_{\nu=1}^{2k+2} \frac{B_\nu E_{2k+3-\nu}}{\nu!(2k+3-\nu)!n^\nu} \quad (k = 0, 1, \ldots, n-1),$$

and

$$R_n = f^{(2n+2)}(\xi) \sum_{m=1}^{n+1} \frac{2B_{2m}E_{2n-2m+3}}{(2m)!(2n-2m+3)!n^{2m}} \quad (0 \le \xi \le 1)$$

for $f \in C^{2n+2}[0,1]$, where $B_n = B_n(0)$ and $E_n = E_n(0)$. The convergence of Euler–Maclaurin quadrature formulas on a class of smooth functions was considered by Vaskevič [62].

Some periodic analogues of the Euler–Maclaurin formula with applications to number theory have been developed by Berndt and Schoenfeld [6]. In the last section of [6], they showed how the composite Newton–Cotes quadrature formulas (Simpson's parabolic and Simpson's three-eighths rules), as well as various other quadratures (e.g., Weddle's composite rule), can be derived from special cases of their periodic Euler–Maclaurin formula, including explicit formulas for the remainder term.

3 Euler–Maclaurin Formula Based on the Composite Gauss–Legendre Rule and Its Lobatto Modification

In the papers [15, 48, 56], the authors considered generalizations of the Euler–Maclaurin formula for some particular Newton–Cotes rules, as well as for 2- and 3-point Gauss–Legendre and Lobatto formulas (see also [4, 17, 33, 34]).

Recently, we have done [40] the extensions of Euler–Maclaurin formulas by replacing the quadrature sum Q_n by the composite Gauss–Legendre shifted formula, as well as by its Lobatto modification. In these cases, several special rules have been obtained by using the MATHEMATICA package `OrthogonalPolynomials` (cf. [9, 43]). Some details on construction of orthogonal polynomials and quadratures of Gaussian type will be given in Sect. 5.

We denote the space of all algebraic polynomials defined on $\mathbb{R}$ (or some its subset) by $\mathcal{P}$, and by $\mathcal{P}_m \subset \mathcal{P}$ the space of polynomials of degree at most m ($m \in \mathbb{N}$).

Let $w_\nu = w_\nu^G$ and $\tau_\nu = \tau_\nu^G$, $\nu = 1, \ldots, m$, be weights (Christoffel numbers) and nodes of the Gauss–Legendre quadrature formula on $[0, 1]$,

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{\nu=1}^{m} w_\nu^G f(\tau_\nu^G) + R_m^G(f). \tag{16}$$

Note that the nodes τ_ν are zeros of the shifted (monic) Legendre polynomial

$$\pi_m(x) = \binom{2m}{m}^{-1} P_m(2x-1).$$

Degree of its algebraic precision is $d = 2m-1$, i.e., $R_m^G(f) = 0$ for each $f \in \mathcal{P}_{2m-1}$. The quadrature sum in (16) we denote by $Q_m^G f$, i.e.,

$$Q_m^G f = \sum_{\nu=1}^{m} w_\nu^G f(\tau_\nu^G).$$

The corresponding composite Gauss–Legendre sum for approximating the integral $I_n f$, given by (10), can be expressed in the form

$$G_m^{(n)} f = \sum_{k=0}^{n-1} Q_m^G f(k + \cdot) = \sum_{\nu=1}^{m} w_\nu^G \sum_{k=0}^{n-1} f(k + \tau_\nu^G). \tag{17}$$

In the sequel we use the following expansion of a function $f \in C^s[0, 1]$ in Bernoulli polynomials for any $x \in [0, 1]$ (see Krylov [31, p. 15])

$$f(x) = \int_0^1 f(t)\,\mathrm{d}t + \sum_{j=1}^{s-1} \frac{B_j(x)}{j!}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] - \frac{1}{s!}\int_0^1 f^{(s)}(t) L_s(x,t)\,\mathrm{d}t, \tag{18}$$

where $L_s(x,t) = B_s^*(x-t) - B_s^*(x)$ and $B_s^*(x)$ is a function of period one, defined by

$$B_s^*(x) = B_s(x), \quad 0 \le x < 1, \quad B_s^*(x+1) = B_s^*(x). \tag{19}$$

Notice that $B_0^*(x) = 1$, $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer, and $B_s^*(x)$, $s > 1$, is a continuous function.

Suppose now that $f \in C^{2r}[0,n]$, where $r \ge m$. Since the all nodes $\tau_\nu = \tau_\nu^G$, $\nu = 1, \ldots, m$, of the Gaussian rule (16) belong to $(0,1)$, using the expansion (18), with $x = \tau_\nu$ and $s = 2r+1$, we have

$$f(\tau_\nu) = I_1 f + \sum_{j=1}^{2r} \frac{B_j(\tau_\nu)}{j!} \left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] \\ - \frac{1}{(2r+1)!} \int_0^1 f^{(2r+1)}(t) L_{2r+1}(\tau_\nu, t)\,\mathrm{d}t,$$

where $I_1 f = \int_0^1 f(t)\,\mathrm{d}t$.

Now, if we multiply it by $w_\nu = w_\nu^G$ and then sum in ν from 1 to m, we obtain

$$\sum_{\nu=1}^m w_\nu f(\tau_\nu) = \left(\sum_{\nu=1}^m w_\nu\right) I_1 f + \sum_{j=1}^{2r} \frac{1}{j!} \left(\sum_{\nu=1}^m w_\nu B_j(\tau_\nu)\right) \left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] \\ - \frac{1}{(2r+1)!} \int_0^1 f^{(2r+1)}(t) \left(\sum_{\nu=1}^m w_\nu L_{2r+1}(\tau_\nu, t)\right) \mathrm{d}t,$$

i.e.,

$$Q_m^G f = Q_m^G(1) \int_0^1 f(t)\,\mathrm{d}t + \sum_{j=1}^{2r} \frac{Q_m^G(B_j)}{j!} \left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] + E_{m,r}^G(f),$$

where

$$E_{m,r}^G(f) = -\frac{1}{(2r+1)!} \int_0^1 f^{(2r+1)}(t) Q_m^G\left(L_{2r+1}(\,\cdot\,, t)\right)\,\mathrm{d}t.$$

Since

$$\int_0^1 B_j(x)\,\mathrm{d}x = \begin{cases} 1, j = 0, \\ 0, j \ge 1, \end{cases}$$

and

$$Q_m^G(B_j) = \sum_{\nu=1}^m w_\nu B_j(\tau_\nu) = \begin{cases} 1, j = 0, \\ 0, 1 \le j \le 2m-1, \end{cases}$$

because the Gauss–Legendre formula is exact for all algebraic polynomials of degree at most $2m-1$, the previous formula becomes

$$Q_m^G f - \int_0^1 f(t)\,\mathrm{d}t = \sum_{j=2m}^{2r} \frac{Q_m^G(B_j)}{j!}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] + E_{m,r}^G(f). \tag{20}$$

Notice that for Gauss–Legendre nodes and the corresponding weights the following equalities

$$\tau_\nu + \tau_{m-\nu+1} = 1, \quad w_\nu = w_{m-\nu+1} > 0, \quad \nu = 1,\ldots,m,$$

hold, as well as

$$w_\nu B_j(\tau_\nu) + w_{m-\nu+1} B_j(\tau_{m-\nu+1}) = w_\nu B_j(\tau_\nu)(1 + (-1)^j),$$

which is equal to zero for odd j. Also, if m is odd, then $\tau_{(m+1)/2} = 1/2$ and $B_j(1/2) = 0$ for each odd j. Thus, the quadrature sum

$$Q_m^G(B_j) = \sum_{\nu=1}^{m} w_\nu B_j(\tau_\nu) = 0$$

for odd j, so that (20) becomes

$$Q_m^G f - \int_0^1 f(t)\,\mathrm{d}t = \sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(1) - f^{(2j-1)}(0)\right] + E_{m,r}^G(f). \tag{21}$$

Consider now the error of the (shifted) composite Gauss–Legendre formula (17). It is easy to see that

$$\begin{aligned} G_m^{(n)} f - I_n f &= \sum_{k=0}^{n-1}\left[Q_m^G f(k+\cdot) - \int_k^{k+1} f(t)\,\mathrm{d}t\right] \\ &= \sum_{k=0}^{n-1}\left[Q_m^G f(k+\cdot) - \int_0^1 f(k+x)\,\mathrm{d}x\right]. \end{aligned}$$

Then, using (21) we obtain

$$\begin{aligned} G_m^{(n)} f - I_n f &= \sum_{k=0}^{n-1}\left\{\sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(k+1) - f^{(2j-1)}(k)\right] + E_{m,r}^G(f(k+\cdot))\right\} \\ &= \sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(n) - f^{(2j-1)}(0)\right] + E_{n,m,r}^G(f), \end{aligned}$$

where $E^G_{n,m,r}(f)$ is given by

$$E^G_{n,m,r}(f) = -\frac{1}{(2r+1)!}\int_0^1 \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+t)\right) Q^G_m\left(L_{2r+1}(\cdot,t)\right)\,\mathrm{d}t. \tag{22}$$

Since $L_{2r+1}(x,t) = B^*_{2r+1}(x-t) - B^*_{2r+1}(x)$ and

$$B^*_{2r+1}(\tau_\nu) = B_{2r+1}(\tau_\nu), \quad B^*_{2r+1}(\tau_\nu - t) = -\frac{1}{2r+2}\frac{\mathrm{d}}{\mathrm{d}t}B^*_{2r+2}(\tau_\nu - t),$$

we have

$$\begin{aligned} Q^G_m\left(L_{2r+1}(\cdot,t)\right) &= Q^G_m\left(B^*_{2r+1}(\cdot - t)\right) - Q^G_m\left(B^*_{2r+1}(\cdot)\right) \\ &= -\frac{1}{2r+2}\,Q^G_m\left(\frac{\mathrm{d}}{\mathrm{d}t}B^*_{2r+2}(\cdot - t)\right), \end{aligned}$$

because $Q^G_m\left(B_{2r+1}(\cdot)\right) = 0$. Then for (22) we get

$$(2r+2)!E^G_{n,m,r}(f) = \int_0^1 \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+t)\right) Q^G_m\left(\frac{\mathrm{d}}{\mathrm{d}t}B^*_{2r+2}(\cdot - t)\right)\,\mathrm{d}t.$$

By using an integration by parts, it reduces to

$$(2r+2)!E^G_{n,m,r}(f) = F(t)Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)\Big|_0^1 - \int_0^1 Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)F'(t)\,\mathrm{d}t,$$

where $F(t)$ is introduced in the following way

$$F(t) = \sum_{k=0}^{n-1} f^{(2r+1)}(k+t).$$

Since $B^*_{2r+2}(\tau_\nu - 1) = B^*_{2r+2}(\tau_\nu) = B_{2r+2}(\tau_\nu)$, we have

$$\begin{aligned} F(t)Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)\Big|_0^1 &= \left(F(1) - F(0)\right)Q^G_m\left(B^*_{2r+2}(\cdot)\right) \\ &= Q^G_m\left(B_{2r+2}(\cdot)\right)\int_0^1 F'(t)\,\mathrm{d}t, \end{aligned}$$

so that

$$(2r+2)!E^G_{n,m,r}(f) = \int_0^1 \left[Q^G_m\left(B_{2r+2}(\cdot)\right) - Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)\right]F'(t)\,\mathrm{d}t.$$

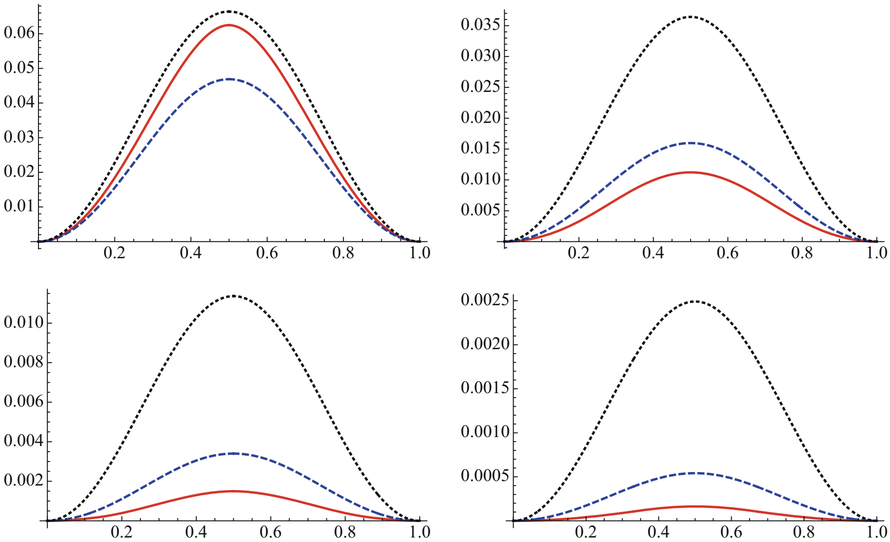

Fig. 1 Graphs of $t \mapsto g^G_{m,r}(t)$, $r = m$ (*solid line*), $r = m+1$ (*dashed line*), and $r = m+2$ (*dotted line*), when $m = 1$, $m = 2$ (*top*), and $m = 3$, $m = 4$ (*bottom*)

Since

$$g^G_{m,r}(t) := (-1)^{r-m} Q^G_m \left[B_{2r+2}(\cdot) - B^*_{2r+2}(\cdot - t)\right] > 0, \qquad 0 < t < 1, \tag{23}$$

there exists an $\eta \in (0,1)$ such that

$$(2r+2)!E^G_{n,m,r}(f) = F'(\eta) \int_0^1 Q^G_m \left[B_{2r+2}(\cdot) - B^*_{2r+2}(\cdot - t)\right] \mathrm{d}t.$$

Typical graphs of functions $t \mapsto g^G_{m,r}(t)$ for some selected values of $r \geq m \geq 1$ are presented in Fig. 1.

Because of continuity of $f^{(2r+2)}$ on $[0,n]$ we conclude that there exists also $\xi \in (0,n)$ such that $F'(\eta) = nf^{(2r+2)}(\xi)$.

Finally, because of $\int_0^1 Q^G_m \left[B^*_{2r+2}(\cdot - t)\right] \mathrm{d}t = 0$, we obtain that

$$(2r+2)!E^G_{n,m,r}(f) = nf^{(2r+2)}(\xi) \int_0^1 Q^G_m \left[B_{2r+2}(\cdot)\right] \mathrm{d}t.$$

In this way, we have just proved the Euler–Maclaurin formula for the composite Gauss–Legendre rule (17) for approximating the integral $I_n f$, given by (10) (see [40]):

Theorem 1. *For $n, m, r \in \mathbb{N}$ $(m \le r)$ and $f \in C^{2r}[0,n]$ we have*

$$G_m^{(n)} f - I_n f = \sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(n) - f^{(2j-1)}(0)\right] + E_{n,m,r}^G(f), \tag{24}$$

where $G_m^{(n)} f$ is given by (17), and $Q_m^G B_{2j}$ denotes the basic Gauss–Legendre quadrature sum applied to the Bernoulli polynomial $x \mapsto B_{2j}(x)$, i.e.,

$$Q_m^G(B_{2j}) = \sum_{\nu=1}^{m} w_\nu^G B_{2j}(\tau_\nu^G) = -R_m^G(B_{2j}), \tag{25}$$

where $R_m^G(f)$ is the remainder term in (16).

If $f \in C^{2r+2}[0,n]$, then there exists $\xi \in (0,n)$, such that the error term in (24) can be expressed in the form

$$E_{n,m,r}^G(f) = n\,\frac{Q_m^G(B_{2r+2})}{(2r+2)!} f^{(2r+2)}(\xi). \tag{26}$$

We consider now special cases of the formula (24) for some typical values of m. For a given m, by $G^{(m)}$ we denote the sequence of coefficients which appear in the sum on the right-hand side in (24), i.e.,

$$G^{(m)} = \left\{Q_m^G(B_{2j})\right\}_{j=m}^{\infty} = \left\{Q_m^G(B_{2m}), Q_m^G(B_{2m+2}), Q_m^G(B_{2m+4}), \ldots\right\}.$$

These Gaussian sums we can calculate very easily by using MATHEMATICA Package `OrthogonalPolynomials` (cf. [9, 43]). In the sequel we mention cases when $1 \le m \le 6$.

Case $m = 1$. Here $\tau_1^G = 1/2$ and $w_1^G = 1$, so that, according to (25),

$$Q_1^G(B_{2j}) = B_{2j}(1/2) = (2^{1-2j} - 1)B_{2j},$$

and (24) reduces to (14). Thus,

$$G^{(1)} = \left\{-\frac{1}{12}, \frac{7}{240}, -\frac{31}{1344}, \frac{127}{3840}, -\frac{2555}{33792}, \frac{1414477}{5591040}, -\frac{57337}{49152}, \frac{118518239}{16711680}, \ldots\right\}.$$

Case $m = 2$. Here we have

$$\tau_1^G = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \quad \tau_2^G = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad w_1^G = w_2^G = \frac{1}{2},$$

so that $Q_2^G(B_{2j}) = \frac{1}{2}\left(B_{2j}(\tau_1^G) + B_{2j}(\tau_2^G)\right) = B_{2j}(\tau_1^G)$. In this case, the sequence of coefficients is

$$G^{(2)} = \left\{-\frac{1}{180}, \frac{1}{189}, -\frac{17}{2160}, \frac{97}{5346}, -\frac{1291411}{21228480}, \frac{16367}{58320}, -\frac{243615707}{142767360}, \ldots\right\}.$$

Case $m = 3$. In this case

$$\tau_1^G = \frac{1}{10}\left(5 - \sqrt{15}\right), \quad \tau_2^G = \frac{1}{2}, \quad \tau_3^G = \frac{1}{10}\left(5 + \sqrt{15}\right)$$

and

$$w_1^G = \frac{5}{18}, \quad w_2^G = \frac{4}{9}, \quad w_3^G = \frac{5}{18},$$

so that

$$Q_3^G(B_{2j}) = \frac{5}{9}B_{2j}(\tau_1^G) + \frac{4}{9}B_{2j}(\tau_2^G)$$

and

$$G^{(3)} = \left\{-\frac{1}{2800}, \frac{49}{72000}, -\frac{8771}{5280000}, \frac{4935557}{873600000}, -\frac{15066667}{576000000}, \frac{3463953717}{21760000000}, \ldots\right\}.$$

Cases $m = 4, 5, 6$. The corresponding sequences of coefficients are

$$G^{(4)} = \left\{-\frac{1}{44100}, \frac{41}{565950}, -\frac{3076}{11704875}, \frac{93553}{75631500}, -\frac{453586781}{60000990000}, \frac{6885642443}{117354877500}, \ldots\right\},$$

$$G^{(5)} = \left\{-\frac{1}{698544}, \frac{205}{29719872}, -\frac{100297}{2880541440}, \frac{76404959}{352578272256}, -\frac{839025422533}{496513166929920}, \ldots\right\},$$

$$G^{(6)} = \left\{-\frac{1}{11099088}, \frac{43}{70436520}, -\frac{86221}{21074606784}, \frac{147502043}{4534139665440}, -\frac{1323863797}{4200045163776}, \ldots\right\}.$$

The Euler–Maclaurin formula based on the composite Lobatto formula can be considered in a similar way. The corresponding Gauss-Lobatto quadrature formula

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{\nu=0}^{m+1} w_\nu^L f(\tau_\nu^L) + R_m^L(f), \tag{27}$$

with the endnodes $\tau_0 = \tau_0^L = 0$, $\tau_{m+1} = \tau_{m+1}^L = 1$, has internal nodes $\tau_\nu = \tau_\nu^L$, $\nu = 1, \ldots, m$, which are zeros of the shifted (monic) Jacobi polynomial,

$$\pi_m(x) = \binom{2m+2}{m}^{-1} P_m^{(1,1)}(2x-1),$$

orthogonal on the interval $(0,1)$ with respect to the weight function $x \mapsto x(1-x)$. The algebraic degree of precision of this formula is $d = 2m+1$, i.e., $R_m^L(f) = 0$ for each $f \in \mathcal{P}_{2m+1}$.

For constructing the Gauss-Lobatto formula

$$Q_m^L(f) = \sum_{\nu=0}^{m+1} w_\nu^L f(\tau_\nu^L), \tag{28}$$

we use parameters of the corresponding Gaussian formula with respect to the weight function $x \mapsto x(1-x)$, i.e.,

$$\int_0^1 g(x)x(1-x)\,\mathrm{d}x = \sum_{\nu=1}^{m} \widehat{w}_\nu^G g(\widehat{\tau}_\nu^G) + \widehat{R}_m^G(g).$$

The nodes and weights of the Gauss-Lobatto quadrature formula (27) are (cf. [36, pp. 330–331])

$$\tau_0^L = 0, \quad \tau_\nu^L = \widehat{\tau}_\nu^G \quad (\nu = 1, \ldots, m), \quad \tau_{m+1}^L = 1,$$

and

$$w_0^L = \frac{1}{2} - \sum_{\nu=1}^{m} \frac{\widehat{w}_\nu^G}{\widehat{\tau}_\nu^G}, \quad w_\nu^L = \frac{\widehat{w}_\nu^G}{\widehat{\tau}_\nu^G(1-\widehat{\tau}_\nu^G)} \; (\nu = 1, \ldots, m), \quad w_{m+1}^L = \frac{1}{2} - \sum_{\nu=1}^{m} \frac{\widehat{w}_\nu^G}{1-\widehat{\tau}_\nu^G},$$

respectively. The corresponding composite rule is

$$\begin{aligned} L_m^{(n)} f &= \sum_{k=0}^{n-1} Q_m^L f(k+\cdot) = \sum_{\nu=0}^{m+1} w_\nu^L \sum_{k=0}^{n-1} f(k+\tau_\nu^L), \\ &= (w_0^L + w_{m+1}^L) \sum_{k=0}^{n}{}'' f(k) + \sum_{\nu=1}^{m} w_\nu^L \sum_{k=0}^{n-1} f(k+\tau_\nu^L). \end{aligned} \tag{29}$$

As in the Gauss–Legendre case, there exists a symmetry of nodes and weights, i.e.,

$$\tau_\nu^L + \tau_{m+1-\nu}^L = 1, \;\; w_\nu^L = w_{m+1-\nu}^L > 0 \quad \nu = 0, 1, \ldots, m+1,$$

so that the Gauss-Lobatto quadrature sum

$$Q_m^L(B_j) = \sum_{\nu=0}^{m+1} w_\nu^L B_j(\tau_\nu^L) = 0$$

for each odd j.

By the similar arguments as before, we can state and prove the following result.

Theorem 2. *For $n, m, r \in \mathbb{N}$ $(m \leq r)$ and $f \in C^{2r}[0, n]$ we have*

$$L_m^{(n)}f - I_nf = \sum_{j=m+1}^{r} \frac{Q_m^L(B_{2j})}{(2j)!} \left[f^{(2j-1)}(n) - f^{(2j-1)}(0)\right] + E_{n,m,r}^L(f), \tag{30}$$

where $L_m^{(n)}f$ is given by (29), and $Q_m^L B_{2j}$ denotes the basic Gauss-Lobatto quadrature sum (28) applied to the Bernoulli polynomial $x \mapsto B_{2j}(x)$, i.e.,

$$Q_m^L(B_{2j}) = \sum_{\nu=0}^{m+1} w_\nu^L B_{2j}(\tau_\nu^L) = -R_m^L(B_{2j}),$$

where $R_m^L(f)$ is the remainder term in (27).

If $f \in C^{2r+2}[0, n]$, then there exists $\xi \in (0, n)$, such that the error term in (30) can be expressed in the form

$$E_{n,m,r}^L(f) = n \frac{Q_m^L(B_{2r+2})}{(2r+2)!} f^{(2r+2)}(\xi).$$

In the sequel we give the sequence of coefficients $L^{(m)}$ which appear in the sum on the right-hand side in (30), i.e.,

$$L^{(m)} = \left\{Q_m^L(B_{2j})\right\}_{j=m+1}^{\infty} = \left\{Q_m^L(B_{2m+2}), Q_m^L(B_{2m+4}), Q_m^L(B_{2m+6}), \ldots\right\},$$

obtained by the Package `OrthogonalPolynomials`, for some values of m.

Case $m = 0$. This is a case of the standard Euler–Maclaurin formula (1), for which $\tau_0^L = 0$ and $\tau_1^L = 1$, with $w_0^L = w_1^L = 1/2$. The sequence of coefficients is

$$L^{(0)} = \left\{\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, -\frac{3617}{510}, \frac{43867}{798}, -\frac{174611}{330}, \frac{854513}{138}, \ldots\right\},$$

which is, in fact, the sequence of Bernoulli numbers $\{B_{2j}\}_{j=1}^{\infty}$.

Case $m = 1$. In this case $\tau_0^L = 0$, $\tau_1^L = 1/2$, and $\tau_2 = 1$, with the corresponding weights $w_0^L = 1/6$, $w_1^L = 2/3$, and $w_2^L = 1/6$, which is, in fact, the Simpson formula (15). The sequence of coefficients is

$$L^{(1)} = \left\{\frac{1}{120}, -\frac{5}{672}, \frac{7}{640}, -\frac{425}{16896}, \frac{235631}{2795520}, -\frac{3185}{8192}, \frac{19752437}{8355840}, -\frac{958274615}{52297728}, \ldots\right\}.$$

Case $m = 2$. Here we have

$$\tau_0^L = 0, \quad \tau_1^L = \frac{1}{10}(5 - \sqrt{5}), \quad \tau_2^L = \frac{1}{10}(5 + \sqrt{5}), \quad \tau_3^L = 1$$

and $w_0^L = w_3^L = 1/12$, $w_1^L = w_2^L = 5/12$, and the sequence of coefficients is

$$L^{(2)} = \left\{\frac{1}{2100}, -\frac{1}{1125}, \frac{89}{41250}, -\frac{25003}{3412500}, \frac{3179}{93750}, -\frac{2466467}{11953125}, \frac{997365619}{623437500}, \ldots\right\}.$$

Case $m = 3$. Here the nodes and the weight coefficients are

$$\tau_0^L = 0, \quad \tau_1^L = \frac{1}{14}(7-\sqrt{31}), \quad \tau_2^L = \frac{1}{2}, \quad \tau_3^L = \frac{1}{14}(7+\sqrt{31}), \quad \tau_4^L = 1$$

and

$$w_0^L = \frac{1}{20}, \quad w_1^L = \frac{49}{180}, \quad w_2^L = \frac{16}{45}, \quad w_3^L = \frac{49}{180}, \quad w_4^L = \frac{1}{20},$$

respectively, and the sequence of coefficients is

$$L^{(3)} = \left\{ \frac{1}{35280}, -\frac{65}{724416}, \frac{38903}{119857920}, -\frac{236449}{154893312}, \frac{1146165227}{122882027520}, \ldots \right\}.$$

Cases $m = 4, 5$. The corresponding sequences of coefficients are

$$L^{(4)} = \left\{ \frac{1}{582120}, -\frac{17}{2063880}, \frac{173}{4167450}, -\frac{43909}{170031960}, \frac{160705183}{79815002400}, -\frac{76876739}{3960744480}, \ldots \right\},$$

$$L^{(5)} = \left\{ \frac{1}{9513504}, -\frac{49}{68999040}, \frac{5453}{1146917376}, -\frac{671463061}{17766424811520}, \frac{1291291631}{3526568534016}, \ldots \right\}.$$

Remark 8. Recently Dubeau [16] has shown that an Euler–Maclaurin like formula can be associated with any interpolatory quadrature rule.

4 Abel–Plana Summation Formula and Some Modifications

Another important summation formula is the so-called Abel–Plana formula, but it is not so well known like the Euler–Maclaurin formula. In 1820 Giovanni (Antonio Amedea) Plana [49] obtained the summation formula

$$\sum_{k=0}^{+\infty} f(k) - \int_0^{+\infty} f(x)\,\mathrm{d}x = \frac{1}{2}f(0) + \mathrm{i}\int_0^{+\infty} \frac{f(\mathrm{i}y) - f(-\mathrm{i}y)}{\mathrm{e}^{2\pi y} - 1}\,\mathrm{d}y, \tag{31}$$

which holds for analytic functions f in $\Omega = \{z \in \mathbb{C} \,:\, \operatorname{Re} z \geq 0\}$ which satisfy the conditions:

1° $\lim\limits_{|y|\to+\infty} \mathrm{e}^{-|2\pi y|}|f(x \pm \mathrm{i}y)| = 0$ uniformly in x on every finite interval,

2° $\displaystyle\int_0^{+\infty} |f(x+\mathrm{i}y) - f(x-\mathrm{i}y)|\mathrm{e}^{-|2\pi y|}\,\mathrm{d}y$ exists for every $x \geq 0$ and tends to zero when $x \to +\infty$.

This formula was also proved in 1823 by Niels Henrik Abel [1]. In addition, Abel also proved an interesting *"alternating series version"*, under the same conditions,

$$\sum_{k=0}^{+\infty}(-1)^k f(k) = \frac{1}{2}f(0) + \mathrm{i}\int_0^{+\infty}\frac{f(\mathrm{i}y)-f(-\mathrm{i}y)}{2\sinh\pi y}\,\mathrm{d}y. \tag{32}$$

Otherwise, this formula can be obtained only from (31). Note that, by subtracting (31) from the same formula written for the function $z \mapsto 2f(2z)$, we get (32).

For the finite sum $S_{n,m}f = \sum_{k=m}^{n}(-1)^k f(k)$, (32) the *Abel summation formula* becomes

$$S_{n,m}f = \frac{1}{2}\big[(-1)^m f(m) + (-1)^n f(n+1)\big] - \int_{-\infty}^{+\infty}\big[(-1)^m\psi_m(y) + (-1)^n\psi_{n+1}(y)\big]w^A(y)\,\mathrm{d}y, \tag{33}$$

where the *Abel weight* on $\mathbb{R}$ and the function $\phi_m(y)$ are given by

$$w^A(x) = \frac{x}{2\sinh\pi x} \quad \text{and} \quad \phi_m(y) = \frac{f(m+\mathrm{i}y)-f(m-\mathrm{i}y)}{2\mathrm{i}y}. \tag{34}$$

The moments for the Abel weight can be expressed in terms of Bernoulli numbers as

$$\mu_k = \begin{cases} 0, & k \text{ odd}, \\ \left(2^{k+2}-1\right)\dfrac{(-1)^{k/2}B_{k+2}}{k+2}, & k \text{ even}. \end{cases} \tag{35}$$

A general Abel–Plana formula can be obtained by a contour integration in the complex plane. Let $m, n \in \mathbb{N}$, $m < n$, and $C(\varepsilon)$ be a closed rectangular contour with vertices at $m \pm \mathrm{i}b$, $n \pm \mathrm{i}b$, $b > 0$ (see Fig. 2), and with semicircular indentations of radius ε round m and n. Let f be an analytic function in the strip $\Omega_{m,n} = \{z \in \mathbb{C} : m \le \operatorname{Re} z \le n\}$ and suppose that for every $m \le x \le n$,

$$\lim_{|y|\to+\infty} \mathrm{e}^{-|2\pi y|}|f(x \pm \mathrm{i}y)| = 0 \quad \text{uniformly in } x,$$

and that

$$\int_0^{+\infty} |f(x+\mathrm{i}y) - f(x-\mathrm{i}y)|\mathrm{e}^{-|2\pi y|}\,\mathrm{d}y$$

exists.

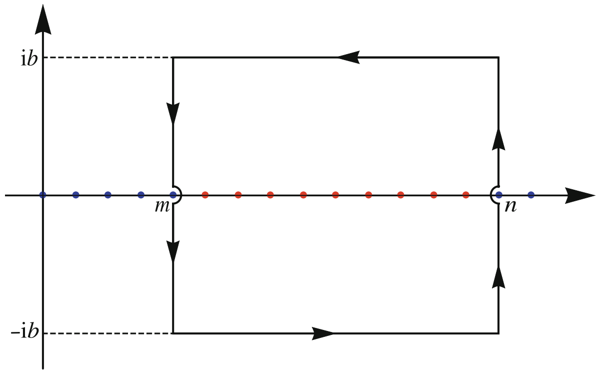

Fig. 2 Rectangular contour $C(\varepsilon)$

The integration

$$\int_{C(\varepsilon)} \frac{f(z)}{\mathrm{e}^{-\mathrm{i}2\pi z}-1}\,\mathrm{d}z,$$

with $\varepsilon \to 0$ and $b \to +\infty$, leads to the *Plana formula* in the following form (cf. [42])

$$T_{m,n}f - \int_m^n f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \big(\phi_n(y)-\phi_m(y)\big)w^P(y)\,\mathrm{d}y, \tag{36}$$

where

$$\phi_m(y) = \frac{f(m+\mathrm{i}y)-f(m-\mathrm{i}y)}{2\mathrm{i}y} \quad \text{and} \quad w^P(y) = \frac{|y|}{\mathrm{e}^{|2\pi y|}-1}. \tag{37}$$

Practically, the Plana formula (36) gives the error of the composite trapezoidal formula (like the Euler–Maclaurin formula). As we can see the formula (36) is similar to the Euler–Maclaurin formula, with the difference that the sum of terms

$$\frac{B_{2j}}{(2j)!}\big(f^{(2j-1)}(n) - f^{(2j-1)}(m)\big)$$

replaced by an integral. Therefore, in applications this integral must be calculated by some quadrature rule. It is natural to construct the Gaussian formula with respect to the *Plana* weight function $x \mapsto w^P(x)$ on $\mathbb{R}$ (see the next section for such a construction).

In order to find the moments of this weight function, we note first that if k is odd, the moments are zero, i.e.,

$$\mu_k(w^P) = \int_{\mathbb{R}} x^k w^P(x)\,\mathrm{d}x = \int_{\mathbb{R}} x^k \frac{|x|}{\mathrm{e}^{|2\pi x|}-1}\,\mathrm{d}x = 0.$$

For even k, we have

$$\mu_k(w^P) = 2\int_0^{+\infty} \frac{x^{k+1}}{e^{2\pi x}-1}\,dx = \frac{2}{(2\pi)^{k+2}}\int_0^{+\infty} \frac{t^{k+1}}{e^t-1}\,dt,$$

which can be exactly expressed in terms of the Riemann zeta function $\zeta(s)$,

$$\mu_k(w^P) = \frac{2(k+1)!\zeta(k+2)}{(2\pi)^{k+2}} = (-1)^{k/2}\frac{B_{k+2}}{k+2},$$

because the number $k+2$ is even. Thus, in terms of Bernoulli numbers, the moments are

$$\mu_k(w^P) = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2}\dfrac{B_{k+2}}{k+2}, & k \text{ is even}. \end{cases} \tag{38}$$

Remark 9. By the Taylor expansion for $\phi_m(y)$ (and $\phi_n(y)$) on the right-hand side in (36),

$$\phi_m(y) = \frac{f(m+iy)-f(m-iy)}{2iy} = \sum_{j=1}^{+\infty}\frac{(-1)^{j-1}y^{2j-2}}{(2j-1)!}f^{(2j-1)}(m),$$

and using the moments (38), the Plana formula (36) reduces to the Euler–Maclaurin formula,

$$\begin{aligned} T_{m,n}f - \int_m^n f(x)\,dx &= \sum_{j=1}^{+\infty}\frac{(-1)^{j-1}}{(2j-1)!}\mu_{2j-2}(w^P)\left(f^{(2j-1)}(n)-f^{(2j-1)}(m)\right) \\ &= \sum_{j=1}^{+\infty}\frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(n)-f^{(2j-1)}(m)\right), \end{aligned}$$

because of $\mu_{2j-2}(w^P) = (-1)^{j-1}B_{2j}/(2j)$. Note that $T_{m,n}f$ is the notation for the composite trapezoidal sum

$$T_{m,n}f := \sum_{k=m}^{n}{}'' f(k) = \frac{1}{2}f(m) + \sum_{k=m+1}^{n-1} f(k) + \frac{1}{2}f(n). \tag{39}$$

For more details see Rahman and Schmeisser [53, 54], Dahlquist [11–13], as well as a recent paper by Butzer, Ferreira, Schmeisser, and Stens [8].

A similar summation formula is the so-called *midpoint summation formula*. It can be obtained by combining two Plana formulas for the functions $z \mapsto f(z-1/2)$

and $z \mapsto f((z+m-1)/2)$. Namely,

$$T_{m,2n-m+2}f\Big(\frac{z+m-1}{2}\Big) - T_{m,n+1}f\Big(z-\frac{1}{2}\Big) = \sum_{k=m}^{n} f(k),$$

i.e.,

$$\sum_{k=m}^{n} f(k) - \int_{m-1/2}^{n+1/2} f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \big[\phi_{m-1/2}(y) - \phi_{n+1/2}(y)\big] w^M(y)\,\mathrm{d}y, \tag{40}$$

where the *midpoint weight function* is given by

$$w^M(x) = w^P(x) - w^P(2x) = \frac{|x|}{\mathrm{e}^{|2\pi x|}+1}, \tag{41}$$

and $\phi_{m-1/2}$ and $\phi_{n+1/2}$ are defined in (37), taking $m := m-1/2$ and $m := n+1/2$, respectively. The moments for the midpoint weight function can be expressed also in terms of Bernoulli numbers as

$$\mu_k(w^M) = \int_{\mathbb{R}} x^k \frac{|x|}{\mathrm{e}^{|2\pi x|}+1}\,\mathrm{d}x = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2}(1-2^{-(k+1)})\dfrac{B_{k+2}}{k+2}, & k \text{ is even}. \end{cases} \tag{42}$$

An interesting weight function and the corresponding summation formula can be obtained from the Plana formula, if we integrate by parts the right side in (36) (cf. [13]). Introducing the so-called *Binet* weight function $y \mapsto w^B(y)$ and the function $y \mapsto \psi_m(y)$ by

$$w^B(y) = -\frac{1}{2\pi}\log\big(1-\mathrm{e}^{-2\pi|y|}\big) \quad \text{and} \quad \psi_m(y) = \frac{f'(m+\mathrm{i}y)+f'(m-\mathrm{i}y)}{2}, \tag{43}$$

respectively, we see that $\mathrm{d}w^B(y)/\,\mathrm{d}y = -w^P(y)/y$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\Big\{\big[\phi_n(y)-\phi_m(y)\big]y\Big\} &= \frac{1}{2\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}y}\Big\{\big[f(n+\mathrm{i}y)-f(n-\mathrm{i}y)\big]-\big[f(m+\mathrm{i}y)-f(m-\mathrm{i}y)\big]\Big\} \\ &= \psi_n(y)-\psi_m(y),\end{aligned}$$

so that

$$\begin{aligned}\int_{-\infty}^{+\infty}\big[\phi_n(y)-\phi_m(y)\big]w^P(y)\,\mathrm{d}y &= \int_{-\infty}^{+\infty}\big[\phi_n(y)-\phi_m(y)\big](-y)\,\mathrm{d}w^B(y) \\ &= \int_{-\infty}^{+\infty}\big[\psi_n(y)-\psi_m(y)\big]w^B(y)\,\mathrm{d}y,\end{aligned}$$

because $w^B(y) = O(\mathrm{e}^{-2\pi|y|})$ as $|y| \to +\infty$. Thus, the *Binet summation formula* becomes

$$T_{m,n}f - \int_m^n f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \big[\psi_n(y) - \psi_m(y)\big] w^B(y)\,\mathrm{d}y. \tag{44}$$

Such a formula can be useful when $f'(z)$ is easier to compute than $f(z)$.

The moments for the Binet weight can be obtained from ones for w^P. Since

$$\mu_k(w^P) = \int_{\mathbb{R}} y^k w^P(y)\,\mathrm{d}y = \int_{\mathbb{R}} y^k(-y)\,\mathrm{d}w^B(y) = (k+1)\mu_k(w^B),$$

according to (38),

$$\mu_k(w^B) = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2} \dfrac{B_{k+2}}{(k+1)(k+2)}, & k \text{ is even}. \end{cases} \tag{45}$$

There are also several other summation formulas. For example, the *Lindelöf formula* [32] for alternating series is

$$\sum_{k=m}^{+\infty} (-1)^k f(k) = (-1)^m \int_{-\infty}^{+\infty} f(m - 1/2 + \mathrm{i}y) \frac{\mathrm{d}y}{2\cosh \pi y}, \tag{46}$$

where the *Lindelöf* weight function is given by

$$w^L(x) = \frac{1}{2\cosh \pi y} = \frac{1}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}}. \tag{47}$$

Here, the moments

$$\mu_k(w^L) = \int_{\mathbb{R}} \frac{x^k}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}}\,\mathrm{d}x$$

can be expressed in terms of the generalized Riemann zeta function $z \mapsto \zeta(z,a)$, defined by

$$\zeta(z,a) = \sum_{\nu=0}^{+\infty} (\nu + a)^{-z}.$$

Namely,

$$\mu_k(w^L) = \begin{cases} 0, & k \text{ odd}, \\ 2(4\pi)^{-k-1} k! \big[\zeta\left(k+1, \tfrac{1}{4}\right) - \zeta\left(k+1, \tfrac{3}{4}\right)\big], & k \text{ even}. \end{cases} \tag{48}$$

5 Construction of Orthogonal Polynomials and Gaussian Quadratures for Weights of Abel–Plana Type

The weight functions w ($\in \{w^P, w^M, w^B, w^A, w^L\}$) which appear in the summation formulas considered in the previous section are even functions on $\mathbb{R}$. In this section we consider the construction of (monic) orthogonal polynomials π_k ($\equiv \pi_k(w;\,\cdot)$) and corresponding Gaussian formulas

$$\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} A_\nu f(x_\nu) + R_n(w;f), \tag{49}$$

with respect to the inner product $(p,q) = \int_{\mathbb{R}} p(x)q(x)w(x)\,\mathrm{d}x$ $(p,q \in \mathcal{P})$. We note that $R_n(w;f) \equiv 0$ for each $f \in \mathcal{P}_{2n-1}$.

Such orthogonal polynomials $\{\pi_k\}_{k\in\mathbb{N}_0}$ and Gaussian quadratures (49) exist uniquely, because all the moments for these weights μ_k ($\equiv \mu_k(w)$), $k = 0,1,\ldots,$ exist, are finite, and $\mu_0 > 0$.

Because of the property $(xp,q) = (p,xq)$, these (monic) orthogonal polynomials π_k satisfy the fundamental *three–term recurrence relation*

$$\pi_{k+1}(x) = x\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 0,1,\ldots, \tag{50}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$, where $\{\beta_k\}_{k\in\mathbb{N}_0}$ ($= \{\beta_k(w)\}_{k\in\mathbb{N}_0}$) is a sequence of recursion coefficients which depend on the weight w. The coefficient β_0 may be arbitrary, but it is conveniently defined by $\beta_0 = \mu_0 = \int_{\mathbb{R}} w(x)\,\mathrm{d}x$. Note that the coefficients α_k in (50) are equal to zero, because the weight function w is an even function! Therefore, the nodes in (49) are symmetrically distributed with respect to the origin, and the weights for symmetrical nodes are equal. For odd n one node is at zero.

A characterization of the Gaussian quadrature (49) can be done via an eigenvalue problem for the symmetric tridiagonal Jacobi matrix (cf. [36, p. 326]),

$$J_n = J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

constructed with the coefficients from the three-term recurrence relation (50) (in our case, $\alpha_k = 0$, $k = 0,1,\ldots,n-1$).

The nodes x_ν are the eigenvalues of J_n and the weights A_ν are given by $A_\nu = \beta_0 v_{\nu,1}^2$, $\nu = 1,\ldots,n$, where β_0 is the moment $\mu_0 = \int_{\mathbb{R}} w(x)\,\mathrm{d}x$, and $v_{\nu,1}$ is the first component of the normalized eigenvector $\mathbf{v} = [v_{\nu,1} \ \cdots \ v_{\nu,n}]^T$ (with $\mathbf{v}_\nu^T\mathbf{v}_\nu = 1$)

corresponding to the eigenvalue x_ν,

$$J_n \mathbf{v}_\nu = x_\nu \mathbf{v}_\nu, \quad \nu = 1, \dots, n.$$

An efficient procedure for constructing the Gaussian quadrature rules was given by Golub and Welsch [27], by simplifying the well-known QR algorithm, so that only the first components of the eigenvectors are computed.

The problems are very sensitive with respect to small perturbations in the data.

Unfortunately, the recursion coefficients are known explicitly only for some narrow classes of orthogonal polynomials, as e.g. for the classical orthogonal polynomials (Jacobi, the generalized Laguerre, and Hermite polynomials). However, for a large class of the so-called *strongly non-classical polynomials* these coefficients can be constructed numerically, but procedures are very sensitive with respect to small perturbations in the data. Basic procedures for generating these coefficients were developed by Walter Gautschi in the eighties of the last century (cf. [23, 24, 36, 41]).

However, because of progress in symbolic computations and variable-precision arithmetic, recursion coefficients can be today directly generated by using the original Chebyshev method of moments (cf. [36, pp. 159–166]) in symbolic form or numerically in sufficiently high precision. In this way, instability problems can be eliminated. Respectively symbolic/variable-precision software for orthogonal polynomials and Gaussian and similar type quadratures is available. In this regard, the MATHEMATICA package `OrthogonalPolynomials` (see [9] and [43]) is downloadable from the web site http://www.mi.sanu.ac.rs/~gvm/. Also, there is Gautschi's software in MATLAB (packages `OPQ` and `SOPQ`). Thus, all that is required is a procedure for the symbolic calculation of moments or their calculation in variable-precision arithmetic.

In our case we calculate the first $2N$ moments in a symbolic form (list `mom`), using corresponding formulas (for example, (38) in the case of the Plana weight w^P), so that we can construct the Gaussian formula (49) for each $n \le N$. Now, in order to get the first N recurrence coefficients `{al,be}` in a symbolic form, we apply the implemented function `aChebyshevAlgorithm` from the Package `OrthogonalPolynomials`, which performs construction of these coefficients using Chebyshev algorithm, with the option `Algorithm->Symbolic`. Thus, it can be implemented in the MATHEMATICA package `OrthogonalPolynomials` in a very simple way as

```
<<orthogonalPolynomials`
 mom=Table[<expression for moments>,{k,0,199}];
 {al,be}=aChebyshevAlgorithm[mom,Algorithm->Symbolic]
 pq[n_]:=aGaussianNodesWeights[n,al,be,
           WorkingPrecision->65,Precision -> 60]
 xA = Table[pq[n],{n,5,40,5}];
```

where we put $N = 100$ and the `WorkingPrecision->65` in order to obtain very precisely quadrature parameters (nodes and weights) with `Precision->60`. These parameters are calculated for $n = 5(5)40$, so that `xA[[k]][[1]]` and `xA[[k]][[2]]` give lists of nodes and weights for five-point formula when `k=1`, for ten-point formula when `k=2`, etc. Otherwise, here we can calculate the n-point Gaussian quadrature formula for each $n \le N = 100$.

All computations were performed in MATHEMATICA, Ver. 10.3.0, on MacBook Pro (Retina, Mid 2012) OS X 10.11.2. The calculations are very fast. The running time is evaluated by the function `Timing` in MATHEMATICA and it includes only CPU time spent in the MATHEMATICA kernel. Such a way may give different results on different occasions within a session, because of the use of internal system caches. In order to generate worst-case timing results independent of previous computations, we used also the command `ClearSystemCache[]`, and in that case the running time for the Plana weight function w^P has been 4.2 ms (calculation of moments), 0.75 s (calculation of recursive coefficients), and 8 s (calculation quadrature parameters for $n = 5(5)40$).

In the sequel we mention results for different weight functions, whose graphs are presented in Fig. 3.

1. Abel and Lindelöf Weight Functions $\mathbf{w^A}$ and $\mathbf{w^L}$ These weight functions are given by (34) and (47), and their moments by (35) and (48), respectively. It is interesting that their corresponding coefficients in the three-term recurrence relation (50) are known explicitly (see [36, p. 159])

$$\beta_0^A = \mu_0^A = \frac{1}{4}, \quad \beta_k^A = \frac{k(k+1)}{4}, \quad k = 1, 2, \ldots,$$

and

$$\beta_0^L = \mu_0^L = \frac{1}{2}, \quad \beta_k^L = \frac{k^2}{4}, \quad k = 1, 2, \ldots.$$

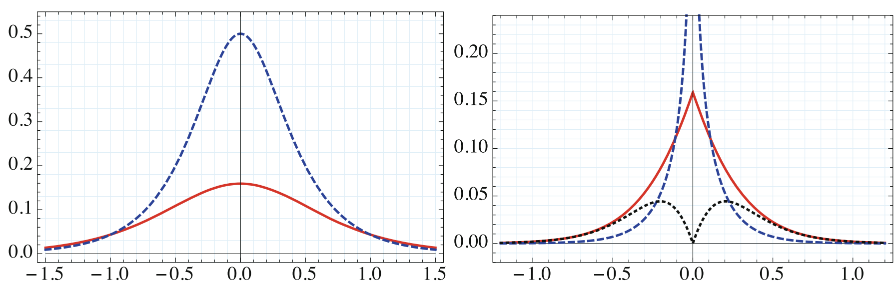

Fig. 3 Graphs of the weight functions: (*left*) w^A (*solid line*) and w^L (*dashed line*); (*right*) w^P (*solid line*), w^B (*dashed line*) and w^M (*dotted line*)

Thus, for these two weight functions we have recursive coefficients in the explicit form, so that we go directly to construction quadrature parameters.

2. Plana Weight Function $\mathbf{w^P}$ This weight function is given by (37), and the corresponding moments by (38). Using the Package `OrthogonalPolynomials` we obtain the sequence of recurrence coefficients $\{\beta_k^P\}_{k\geq 0}$ in the rational form:

$$\beta_0^P = \frac{1}{12},\ \beta_1^P = \frac{1}{10},\ \beta_2^P = \frac{79}{210},\ \beta_3^P = \frac{1205}{1659},\ \beta_4^P = \frac{262445}{209429},\ \beta_5^P = \frac{33461119209}{18089284070},$$

$$\beta_6^P = \frac{361969913862291}{137627660760070},\ \beta_7^P = \frac{85170013927511392430}{24523312685049374477},$$

$$\beta_8^P = \frac{1064327215185988443814288995130}{236155262756390921151239121153},$$

$$\beta_9^P = \frac{286789982254764757195675003870137955697117}{51246435664921031688705695412342990647850},$$

$$\beta_{10}^P = \frac{15227625889136643989610717434803027240375634452808081047}{2212147521291103911193549528920437912200375980011300650},$$

$$\beta_{11}^P = \frac{587943441754746283972138649821948554273878447469233852697401814148410885}{71529318090286333175985287358122471724664434392542372273400541405857921},$$

etc.

As we can see, the fractions are becoming more complicated, so that already β_{11}^P has the "form of complexity" $\{72/71\}$, i.e., it has 72 decimal digits in the numerator and 71 digits in the denominator. Further terms of this sequence have the "form of complexity" $\{88/87\}$, $\{106/05\}$, $\{129/128\}$, $\{152/151\}$, ..., $\{13451/13448\}$.

Thus, the last term β_{99}^P has more than 13 thousand digits in its numerator and denominator. Otherwise, its value, e.g. rounded to 60 decimal digits, is

$$\beta_{99}^P = 618.668116294139071216871819412846078447729830182674784697227.$$

3. Midpoint Weight Function $\mathbf{w^M}$ This weight function is given by (41), and the corresponding moments by (42). As in the previous case, we obtain the sequence of recurrence coefficients $\{\beta_k^M\}_{k\geq 0}$ in the rational form:

$$\beta_0^M = \frac{1}{24},\ \beta_1^M = \frac{7}{40},\ \beta_2^M = \frac{2071}{5880},\ \beta_3^M = \frac{999245}{1217748},\ \beta_4^M = \frac{21959166635}{18211040276},$$

$$\beta_5^M = \frac{108481778600414331}{55169934195679160},\ \beta_6^M = \frac{2083852396915648173441543}{813782894744588335008520},$$

$$\beta_7^M = \frac{25698543837390957571411809266308135}{7116536885169433586426285918882662},$$

$$\beta_8^M = \frac{202221739836050724659312728605015618097349555485}{45788344599633183797631374444694817538967629598},$$

$$\beta_9^M = \frac{14077564493254853375144075652878384268409784777236869234539068357}{2446087170499983327141705915330961521888001335934900402777402200},$$

etc. In this case, the last term β_{99}^{M} has slightly complicated the "form of complexity" $\{16401/16398\}$ than one in the previous case, precisely. Otherwise, its value (rounded to 60 decimal digits) is

$$\beta_{99}^{M} = 619.562819405146668677971154899553589896235540274133472854031.$$

4. Binet Weight Function $\mathbf{w}^{\mathbf{B}}$ The moments for this weight function are given in (38), and our Package `OrthogonalPolynomials` gives the sequence of recurrence coefficients $\{\beta_k^B\}_{k\geq 0}$ in the rational form:

$$\beta_0^B = \frac{1}{12},\quad \beta_1^B = \frac{1}{30},\quad \beta_2^B = \frac{53}{210},\quad \beta_3^B = \frac{195}{371},\quad \beta_4^B = \frac{22999}{22737},\quad \beta_5^B = \frac{29944523}{19733142},$$

$$\beta_6^B = \frac{109535241009}{48264275462},\quad \beta_7^B = \frac{29404527905795295658}{9769214287853155785},$$

$$\beta_8^B = \frac{455377030420113432210116914702}{113084128923675014537885725485},$$

$$\beta_9^B = \frac{26370812569397719001931992945645578779849}{5271244267917980801966553649147604697542},$$

$$\beta_{10}^B = \frac{152537496709054809881638897472985990866753853122697839}{24274291553105128438297398108902195365373879212227726},$$

$$\beta_{11}^B = \frac{100043420063777451042472529806266909090824649341814868347109676190691}{13346384670164266280033479022693768890138348905413621178450736182873},$$

etc. Numerical values of coefficients β_k^B for $k = 12, \ldots, 39$, rounded to 60 decimal digits, are presented in Table 1.

For this case we give also quadrature parameters x_ν^B and A_ν^B, $\nu = 1, \ldots, n$, for $n = 10$ (rounded to 30 digits in order to save space). Numbers in parenthesis indicate the decimal exponents (Table 2).

Table 1 Numerical values of the coefficients β_k^B, $k = 12, \ldots, 39$

k	β_k^B
12	9.040660234367726699531139360260481749336219635370722222675357
13	10.48930365450948227718837130459262952209723798938340499993209
14	12.2971936103862058639894371400919176597365509004516453610177
15	13.9828769539924301882597606512787300859080333154700506431789
16	16.0535514167049354697156163650062601783515764970917711361702
17	17.9766073998702775925694723076715543993147838556500117187847
18	20.3097620274416537438054147204948968937016485345196881526453
19	22.4704716399331324955179415715079221089953862901823520893038
20	25.0658465489459720291634003225063053682385176354570207084270
21	27.4644518250291336091755589826462226732286473857913864921713
22	30.3218212316730471268825993064057869944873787313809977426698
23	32.9585339299729872199940664514120882069601000999724796349878
24	36.0776989312992426451533209008554523367760033115543468301504
25	38.9527066823115557345443904104810462991593233805616588397077
26	42.3334900435769572113818539488560973399147861411953446717663
27	45.4469608500616210144241757375414510828484368311407665782656
28	49.0892031290125977081648833502750872924491998898068036677541
29	52.4412887514153373125698560469961084271478607455930155529787
30	56.3448453453418435384203659474761135421333046623523607025848
31	59.9356839071658582078525834927521121101345464090376940621335
32	64.1004227559203545279066118922379177529092202107679570943670
33	67.9301407880182211863677027451985358165225510069351193013587
34	72.3559405552117019696800529632362179107517585345562462880100
35	76.4246546268296897525850904222875264035700459112308348153069
36	81.1114032372479654848142309856834609745026942246296395824649
37	85.4192212764109726145856387173486827269888223681684704599999
38	90.3668147238641085955135745816833777807870911939721581625005
39	94.9138371000098879530762312919869274587678241868936940165561

Table 2 Gaussian quadrature parameters x_ν^B and A_ν^B, $\nu = 1, \ldots, n$, for ten-point rule

ν	$x_{\nu+5}^B$ $(= -x_{6-\nu}^B)$	$A_{\nu+5}^B$ $(= A_{6-\nu}^B)$
1	1.19026134410869931041299717296(−1)	3.95107541334705577733788440045(−2)
2	5.98589257742219693357956162107(−1)	2.10956883221363967243739596594(−3)
3	1.25058028819024934653033542222	4.60799503427397559669146065886(−5)
4	2.12020925569172605355904853247	2.63574272352001106479781030329(−7)
5	3.34927819645835833349223106504	1.76367377463777032308587486531(−10)

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A Nyström method for a class of Fredholm integral equations on the real semiaxis

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Abstract A class of Fredholm integral equations of the second kind, with respect to the exponential weight function $w(x) = \exp(-(x^{-\alpha} + x^{\beta}))$, $\alpha > 0$, $\beta > 1$, on $(0, +\infty)$, is considered. The kernel $k(x, y)$ and the function $g(x)$ in such kind of equations,

$$f(x) - \mu \int_0^{+\infty} k(x, y) f(y) w(y) \mathrm{d}y = g(x), \quad x \in (0, +\infty),$$

can grow exponentially with respect to their arguments, when they approach to 0^+ and/or $+\infty$. We propose a simple and suitable Nyström-type method for solving these equations. The study of the stability and the convergence of this numerical method in based on our results on weighted polynomial approximation and "truncated" Gaussian rules, recently published in Mastroianni and Notarangelo (Acta Math Hung,

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142:167–198, 2014), and Mastroianni, Milovanović and Notarangelo (IMA J Numer Anal 34:1654–1685, 2014) respectively. Moreover, we prove a priori error estimates and give some numerical examples. A comparison with other Nyström methods is also included.

Keywords Fredholm integral equation · Nyström method · Weighted polynomial approximation · Gaussian quadrature formula · Orthogonal polynomials · Truncation · Error estimate

Mathematics Subject Classification 65R10 · 65D30 · 65D32 · 41A55

1 Introduction

The aim of this paper is to approximate the solution of integral equations of the form

$$f(x) - \mu \int_0^{+\infty} k(x,y) f(y) w(y) \mathrm{d}y = g(x), \quad x \in (0,+\infty), \tag{1}$$

with the exponential weight function

$$w(x) = \exp\Big[-\Big(\frac{1}{x^\alpha} + x^\beta\Big)\Big], \quad \alpha > 0, \quad \beta > 1, \tag{2}$$

and the parameter $\mu \in \mathbb{R}$. The kernel $(x,y) \mapsto k(x,y)$ and the function $x \mapsto g(x)$ can grow exponentially with respect to their arguments, when they approach to 0^+ and/or $+\infty$.

The weight functions similar to (2) have been considered in statistics. Following Stoyanov [20, §7.1] we mention here a simple example with the inverse Gaussian distribution (IG) with "easy" parameters, say (1, 1), i.e., a random variable $\theta \sim$ IG, with density function

$$w(x) = \begin{cases} \dfrac{\mathrm{e}}{\sqrt{2\pi}}\, x^{-3/2} \exp\Big[-\dfrac{1}{2}\Big(x + \dfrac{1}{x}\Big)\Big], & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

In terms of the modified Bessel function of the second kind, its moments can be expressed in the form (cf. [17])

$$\int_0^{+\infty} x^k w(x)\, \mathrm{d}x = \mathrm{e}\sqrt{\frac{2}{\pi}}\, K_{k-1/2}(1), \quad k \in \mathbb{N}_0.$$

Also, this kind of weights on $(0,+\infty)$ were appeared in a consideration on expansions of confluent hypergeometric functions in terms of Bessel functions by Temme [21], as well as in the so-called Laurent–Hermite–Gauss quadrature rules investigated by Gustafson and Hagler [5] and Hagler [6].

Integral equations of the form (1), with proper assumptions on the kernel k and the function g, can occur in mathematical finance in computing distributions of geometrical brownian motion (see [3,7,8]).

However, as far as we know, the numerical treatment of this kind of integral equations does not appear in the literature. In this paper we are going to study integral equations of the form (1) in some suitable function space with weighted uniform metric and to approximate the solution by means of a Nyström interpolant. We will prove that this method is stable and convergent in the metric of the considered space. In order to prove the convergence of the method we will use some recent results on polynomial approximation with the weight w and related Gaussian quadrature rule, obtained by the authors in [13–16].

Therefore, the results in this paper are new.

For the sake of completeness, we observe that in the weight w, given by (2), the C^∞−function $\exp\left[-x^{-\alpha}\right]$ appears. Therefore one could think to introduce a new kernel function $k(x, y)\exp\left[-y^{-\alpha}\right]$ and, provided the function g fulfills some proper assumptions, to approximate the solution of equation (1) by using a Nyström interpolant based on Laguerre zeros, as in [10] (see also [4,9,11]). In Sect. 4 we will show that this procedure is in general more expensive and less precise. This fact is also one of the motivations of this paper.

The paper is structured as follows. In Sect. 2 we recall some basic facts and give some preliminary results. In Sect. 3 we introduce our numerical method and prove the main results. In Sect. 4 we will compare our method with the one based on Laguerre zeros. Finally, in Sect. 5 we show some numerical examples.

2 Basic facts and preliminary results

In the sequel c, $\mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$.

Moreover, the symbols $\|\cdot\|_I$ and $\|\cdot\|$ will denote the uniform norm in some interval I and in $(0, +\infty)$, respectively.

Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

2.1 Function spaces with weighted uniform metric

Letting

$$u(x) = (1+x)^\delta \sqrt{w(x)}, \qquad \delta > \frac{1}{2}, \tag{3}$$

where

$$w(x) = \mathrm{e}^{-\left(x^{-\alpha}+x^{\beta}\right)}, \quad \alpha > 0, \quad \beta > 1,$$

$x \in (0, +\infty)$, we introduce the function space

$$C_u := \left\{ f \in C^0(0, +\infty) : \lim_{x \to 0^+} f(x)u(x) = 0 = \lim_{x \to +\infty} f(x)u(x) \right\}, \tag{4}$$

with the norm

$$\|f\|_{C_u} := \|fu\| = \sup_{x \in (0,+\infty)} |f(x)u(x)| .$$

We emphasize that the space C_u contains functions, defined on the real semiaxis $(0, +\infty)$, which can grow exponentially both for $x \to 0^+$ and for $x \to +\infty$. Moreover, C_u is a Banach space.

For $1 \le r \in \mathbb{Z}$, we define the Sobolev-type spaces

$$W_r = W_r^{\infty}(u) = \left\{ f \in C_u : \ f^{(r-1)} \in AC(0, +\infty), \ \|f^{(r)}\varphi^r u\| < \infty \right\},$$

with the norm

$$\|f\|_{W_r} = \|fu\| + \|f^{(r)}\varphi^r u\|,$$

$\varphi(x) = \sqrt{x}$.

In order to define further function spaces, we introduce the following moduli of smoothness. For each $f \in C_u$ $r \ge 1$ and $0 < t < t_0$, we set

$$\Omega_{\varphi}^r(f, t)_u = \sup_{0<h\le t} \left\| \Delta_{h\varphi}^r (f)\, u \right\|_{\mathcal{I}_h(c)},$$

where

$$\mathcal{I}_h(c) = \left[h^{1/(\alpha+1/2)}, \frac{c}{h^{1/(\beta-1/2)}} \right],$$

$c > 1$ is a fixed constant, and

$$\Delta_{h\varphi}^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + (r-i)h\varphi(x)\right), \quad \varphi(x) = \sqrt{x}.$$

We remark that the behavior of this modulus of smoothness is independent on the constant c (see [14]).

Then we define the complete rth modulus of smoothness by

$$\omega_\varphi^r(f,t)_u = \Omega_\varphi^r(f,t)_u + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{(0,t^{2/(2\alpha+1)}]} + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{[c\,t^{-2/(2\beta-1)},+\infty)},$$

with $c>1$ a fixed constant.

For any $f\in W_r$, $r\geq 1$ and $t<t_0$, we have (see [14])

$$\Omega_\varphi^r(f,t)_u \leq \mathcal{C} \inf_{0<h\leq t} h^r \|f^{(r)}\varphi^r u\|_{\mathcal{I}_h(c)}\,, \tag{5}$$

where $\mathcal{C}$ is independent of f and t.

By means of the main part of the modulus of smoothness, we can define the Zygmund-type spaces

$$Z_s := Z_{s,r}^\infty(u) = \{f\in C_u:\ M_s(f)<\infty\}\,,$$

where

$$M_s(f) := \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_u}{t^s}\,,\quad r>s\,,\quad s\in\mathbb{R}^+\,,$$

with the norm

$$\|f\|_{Z_s} = \|f\|_{L_u^p} + M_s(f)\,.$$

We remark that, in the definition of Z_s, the main part of the rth modulus of smoothness $\Omega_\varphi^r(f,t)_u$ can be replaced by the complete modulus $\omega_\varphi^r(f,t)_u$, as shown in [14].

2.2 Weighted polynomial approximation

Let us denote by

$$E_m(f)_u = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|_p$$

the error of best weighted polynomial approximation of a function $f\in C_u$.

The following Jackson, weak Jackson and Stechkin inequalities have been proved in [14].

Theorem 1 *For any $f\in C_u$ and $m>r\geq 1$, we have*

$$E_m(f)_u \leq \mathcal{C}\,\omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_u\,,$$

where $a_m \sim m^{1/\beta}$. *Moreover, assuming* $\Omega_\varphi^r(f,t)_u\, t^{-1} \in L^1[0,1]$,

$$E_m(f)_u \le \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(f,t)_u}{t}\, \mathrm{d}t\,, \qquad r < m\,.$$

Finally, for any $f \in C_u$ *we get*

$$\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_u \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{i=0}^{m} \left(\frac{i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_u}{i+1}\,.$$

In any case $\mathcal{C}$ *is independent of* m *and* f.

In particular, by Theorem 1 and (5), for any $f \in W_r$ we get

$$E_m(f)_u \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\| \tag{6}$$

and, for any $f \in Z_s$, we have

$$E_m(f)_u \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s M_s(f)\,. \tag{7}$$

Moreover, the following equivalences (see [14])

$$f \in C_u \Leftrightarrow \lim_{t\to 0} \omega_\varphi(f,t)_u = 0 \Leftrightarrow \lim_{m\to\infty} E_m(f)_u = 0 \tag{8}$$

will be useful in the sequel.

2.3 Gaussian rules

Let $\{p_m(w)\}_m$ be the sequence of orthonormal polynomials related to $w(x) = \mathrm{e}^{-x^{-\alpha}-x^\beta}$. The zeros of $p_m(w)$ are located as follows

$$\varepsilon_m < x_1 < x_2 < \cdots < x_m < a_m\,,$$

where the Mhaskar–Rahmanov–Saff numbers a_m and ε_m fulfill $a_m \sim m^{1/\beta}$ and $\varepsilon_m \sim \left(\sqrt{a_m}/m\right)^{2/(2\alpha+1)}$.

For any continuous function f the Gaussian rule related to the weight w is given by

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{m} \lambda_k(w) f(x_k) + e_m(f)\,, \tag{9}$$

where x_k are the zeros of $p_m(w)$ and $\lambda_k(w)$ are the Christoffel numbers.

In order to introduce our numerical method for solving Eq. (1), we are going to consider a "truncated" gaussian rule. Fixed $\theta \in (0, 1)$, we define two indexes $j_1 = j_1(m)$ and $j_2 = j_2(m)$ as

$$\varepsilon_m < \varepsilon_{\theta m} \leq x_{j_1} < \cdots < x_{j_2} \leq a_{\theta m} < a_m \, .$$

To be more precise, with $\theta \in (0, 1)$, j_1 and j_2 are such that

$$x_{j_1} = \max_{1 \leq k \leq m} \{x_k \ : \ x_k \leq \varepsilon_{\theta m}\}\,, \quad x_{j_2} = \min_{1 \leq k \leq m} \{x_k \ : \ x_k \geq a_{\theta m}\}\,, \tag{10}$$

and, if $\{x_k \ : \ x_k \leq \varepsilon_{\theta m}\}$ or $\{x_k \ : \ x_k \geq a_{\theta m}\}$ is empty, we set $x_{j_1} = x_1$ or $x_{j_2} = x_m$, respectively.

Then we consider the following "truncated" Gaussian rule

$$\int_0^{+\infty} f(x) w(x)\, \mathrm{d}x = \sum_{i=j_1}^{j_2} \lambda_i(w) f(x_i) + e_m^*(f) \tag{11}$$

and for any $f \in C_{u^2}$ we have (see [15])

$$\left|e_m^*(f)\right| \leq \mathcal{C} \left\{E_M(f)_{u^2} + \mathrm{e}^{-cm^\nu} \left\| f u^2 \right\|\right\}, \tag{12}$$

where

$$M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor, \quad \nu = \left(1 - \frac{1}{2\beta}\right) \frac{2\alpha}{2\alpha+1}, \quad \mathcal{C} \neq \mathcal{C}(m, f), \quad \text{and} \quad c \neq c(m, f).$$

In particular, recalling the results in Sect. 2.2, for any $f \in W_r(u^2)$, we get

$$\left|e_m^*(f)\right| \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r(u^2)}$$

and, for any $f \in Z_s(u^2)$, we have

$$\left|e_m^*(f)\right| \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_s(u^2)} \, .$$

2.4 Compactness of linear operators in C_u

Let $A : C_u \to C_u$ be a linear operator. Then, following the Hausdorff definition, A is compact in C_u if and only if the limit condition

$$\lim_{m \to \infty} \sup_{\|f\|_{C_u}=1} E_m(Af)_u = 0 \tag{13}$$

holds. Taking into account (8), condition (13) can be rewritten in terms of moduli of smoothness (see [22, pp. 44, 93–94]) as follows

$$\lim_{t\to 0} \sup_{\|f\|_{C_u}=1} \omega_\varphi(Af,t)_u = 0\,.$$

Coming back to Eq. (1), let us consider the operator K defined by

$$(Kf)(x) = \mu \int_0^{+\infty} k(x,y)f(y)w(y)\mathrm{d}y\,, \quad x \in (0,+\infty)\,. \tag{14}$$

Then, letting $k(x,y) = k_y(x) = k_x(y)$, since

$$\omega_\varphi(Kf,t)_u \le |\mu|\|fu\| \sup_{y\in(0,+\infty)} \omega_\varphi(k_y,t)_u u(y) \int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{2\delta}}\,,$$

if

$$u(y)k_y \in C_u \quad \text{uniformly w.r.t.} \quad y, \tag{15}$$

then the operator K in C_u.

In an analogous way, the sequence of operators $\{A_m\}_m$ in C_u is collectively compact, i.e., the set

$$S = \{A_m f \in C_u \;:\; m \ge 1 \quad \text{and} \quad \|fu\| \le 1\}$$

is relatively compact in C_u, if and only if the limit condition

$$\lim_{N\to\infty} \sup_{\|f\|_{C_u}=1} \sup_{m\in\mathbb{N}} E_N(A_m f)_u = 0$$

holds, namely if and only if

$$\lim_{t\to 0} \sup_{\|f\|_{C_u}=1} \sup_{m\in\mathbb{N}} \omega_\varphi(A_m f,t)_u = 0\,.$$

In particular, for the sequence of operators

$$(K_m f)(x) = \mu \sum_{i=j_1}^{j_2} \lambda_i(w)k(x,x_i)f(x_i)\,, \tag{16}$$

obtained by applying the "truncated" Gaussian rule (11) to $(Kf)(x)$ given by (14), it is not difficult to show that the collective compactness follows from the assumption

$$u(x)k_x \in C_u \quad \text{uniformly w.r.t.} \quad x\,. \tag{17}$$

3 The numerical method

Let us now introduce our numerical method for solving Eq. (1), i.e.,

$$f(x) - \mu \int_0^{+\infty} k(x, y) f(y) w(y) \mathrm{d}y = g(x), \quad x \in (0, +\infty),$$

where $\mu \in \mathbb{R}$,

$$w(y) = \mathrm{e}^{-y^{-\alpha} - y^{\beta}}, \quad \alpha > 0, \ \beta > 1,$$

the given functions k and g can grow exponentially (w.r.t. x, y) when $x \to 0^+$ and/or $x \to +\infty$. Denoting the identity operator by I and the integral operator by K, we can rewrite this equation as

$$(I - K) f = g.$$

With u given by (3), we are going to study the Eq. (1) in the space C_u defined in Sect. 2.1. Under the assumptions (15), the Fredholm alternative holds true. So, if $\ker(I - K) = \{0\}$, Eq. (1) admits unique solution $f^* \in C_u$ for any fixed $g \in C_u$.

In order to approximate the solution of (1), we are going to use a Nyström method based on the "truncated" Gaussian rule defined in Sect. 2.3. To this end, we introduce the sequence of operators $\{K_m\}_m$,

$$(K_m f)(x) = \mu \sum_{i=j_1}^{j_2} \lambda_i(w) k(x, x_i) f(x_i) \tag{18}$$

which is obtained by applying the "truncated" Gaussian rule (11) to $(Kf)(x)$ given by (14). Then we are going to solve in C_u the equations

$$f_m(x) - (K_m f_m)(x) = g(x), \quad m = 1, 2, \ldots. \tag{19}$$

Multiplying both sides of (19) by $u(x)$, collocating at the quadrature knots and letting $a_i = (f_m u)(x_i)$, $b_i = (gu)(x_i)$, $i = j_1, \ldots, j_2$, for $m = 1, 2, \ldots$, we obtain the linear systems of equations

$$a_h - \mu \sum_{i=j_1}^{j_2} \lambda_i(w) k(x_h, x_i) \frac{u(x_h)}{u(x_i)} a_i = b_h, \quad h = j_1, \ldots, j_2,$$

in the unknowns a_h, i.e.,

$$\sum_{i=j_1}^{j_2} \left[\delta_{ih} - \mu \lambda_i(w) k(x_h, x_i) \frac{u(x_h)}{u(x_i)} \right] a_i = b_h, \quad h = j_1, \ldots, j_2. \tag{20}$$

If (20) is unisolvent and $(a_{j_1}^*, \dots, a_{j_2}^*)^T$ is its solution, then, by (18) and (19) , we can define the Nyström interpolant

$$f_m^*(x) = \mu \sum_{i=j_1}^{j_2} \frac{\lambda_i(w)}{u(x_i)} k(x, x_i) a_i^* + g(x) \tag{21}$$

which we will be an approximation of the solution f^* of equation (1) in C_u-metric.

Notice that, due to the choice of the "truncated" Gaussian rule in place of the ordinary Gaussian rule (9), the matrix of coefficients of the system of equations (20), in notation $V_m^{(j_1,j_2)}$, has dimension $j_2 - j_1 + 1$ instead of m and this produces a reduction of the computational cost.

Let us prove the stability and convergence of our method.

Theorem 2 *Let u be the weight in (3). Assume*

(i) $u(y)k_y \in C_u$ *uniformly w.r.t. y;*
(ii) $u(x)k_x \in C_u$ *uniformly w.r.t. x;*
(iii) $g \in C_u$.

If $\ker(I-K) = \{0\}$, the system of equations (20) is unisolvent and well-conditioned. Moreover, f_m^* converges to f^* in C_u and

$$\left\| \left(f_m^* - f^*\right) u \right\| \le \mathcal{C} \sup_{x \in (0,+\infty)} u(x) \left\{ E_M \left(f^* k_x\right)_{u^2} + \mathrm{e}^{-cm^\nu} \left\| f^* k_x u^2 \right\| \right\} \tag{22}$$

where

$$M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor, \quad \nu = \left(1 - \frac{1}{2\beta}\right) \frac{2\alpha}{2\alpha+1}, \quad \mathcal{C} \neq \mathcal{C}(m, f^*), \quad c \neq c(m, f^*).$$

Proof As already mentioned in Sect. 2.4, from the assumption (i), i.e., (15), the compactness of the operator $K : C_u \to C_u$ follows. So the Fredholm alternative holds for Eq. (1) and, if $\ker(I - K) = \{0\}$, Eq. (1) admits unique solution $f^* \in C_u$.

Now, using (12), we have

$$\|(Kf - K_m f)\, u\| \le \mathcal{C} \sup_{x \in (0,+\infty)} \left\{ E_M (f k_x)_{u^2} + \mathrm{e}^{-cm^\nu} \left\| f k_x u^2 \right\| \right\}, \tag{23}$$

i.e., the sequence $\{K_m\}_m$ strongly converges to the operator K.

Moreover, since $\{K_m\}_m$ is collectively compact by (ii), i.e., (17), it follows that

$$\lim_m \|(K - K_m) K_m\|_{C_u \to C_u} = 0$$

and, using [1, Theorem 4.1.1] or [19, Theorem 2.1], for $m \geq m_0$, the operators $(I - K_m)^{-1}$ exist and

$$\|(I - K_m)^{-1}\|_{C_u \to C_u} \leq \frac{1 + \|(I - K)^{-1}\|_{C_u \to C_u} \|K_m\|_{C_u \to C_u}}{1 - \|(I - K)^{-1}\|_{C_u \to C_u} \|(K - K_m)K_m\|_{C_u \to C_u}}$$
$$\leq \mathcal{C} < +\infty.$$

Then, proceeding as in [1, pp. 112–113], we deduce that the matrix $V_m^{(j_1, j_2)}$ of the coefficients of system (20) is well conditioned, i.e.,

$$\operatorname{cond}\left(V_m^{(j_1, j_2)}\right) \leq \operatorname{cond}(I - K_m) \leq \mathcal{C} < \infty\,, \quad \mathcal{C} \neq \mathcal{C}(m)\,.$$

Finally, the error estimate (22) immediately follows by (23). □

From (22) we deduce that the order of convergence of our method depends on the smoothness of the kernel k and the solution f^* of Eq. (1). Now, we want to show a more explicit error estimate, depending on the smoothness of the known functions k and g. In particular, from Theorem 2 we deduce the following corollary.

Corollary 1 *Let the assumptions of Theorem 2 be replaced by*

(a) $u(y)k_y \in W_r(u)$ *uniformly w.r.t. y;*
(b) $u(x)k_x \in W_r(u)$ *uniformly w.r.t. x;*
(c) $g \in W_r(u)$.

Then, for m sufficiently large, we have

$$\left\|\left(f_m^* - f^*\right) u\right\| = O\left(\left(\frac{\sqrt{a_m}}{m}\right)^r\right)\,,$$

where the constants in "O" are independent of m and f^*.

Proof We note that the assumptions on the given functions imply $f^* \in W_r(u)$ and then $f^* k_x \in W_r(u^2)$. Hence, by (22) and (6), we get (see for instance [12, Theorem 3.2])

$$\begin{aligned}\left\|\left(f_m^* - f^*\right) u\right\| &\leq \mathcal{C} \sup_{x\in(0,+\infty)} u(x)\left\{E_M\left(f^* k_x\right)_{u^2} + \mathrm{e}^{-cm^\nu}\left\|f^* k_x u^2\right\|\right\}\\ &\leq \mathcal{C} \sup_{x\in(0,+\infty)} u(x)\Big\{\left\|f^* u\right\| E_n\left(k_x\right)_u + \|k_x u\| E_n\left(f^*\right)_u\\ &\quad + \mathrm{e}^{-cm^\nu}\left\|f^* u\right\| \|k_x u\|\Big\},\end{aligned}$$

with $n = \lfloor M/2 \rfloor$ and the corollary follows from (6). □

We note that, by (7), an analogous corollary holds if we replace the Sobolev spaces W_r by the Zygmund spaces Z_s.

4 Comparison with the Nyström method based on Laguerre zeros

The following observation is crucial. The integral

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \int_0^{\infty} f(x)\mathrm{e}^{-x^{-\alpha}-x^{\beta}}\,\mathrm{d}x \tag{24}$$

can be evaluated by means of the Gaussian rule related to the weight $w(x) = \mathrm{e}^{-x^{-\alpha}-x^{\beta}}$, i.e.,

$$G_m(w, f) = \sum_{k=1}^{m} \lambda_k(w) f(x_k)\,,$$

as described in Sect. 2.3. On the other hand, this integral can be rewritten as

$$\int_0^{\infty} \left[f(x)\mathrm{e}^{-x^{-\alpha}}\right]\mathrm{e}^{-x^{\beta}}\,\mathrm{d}x = \int_0^{\infty} \left[f(x)\mathrm{e}^{-x^{-\alpha}}\right]\sigma(x)\,\mathrm{d}x$$

and evaluated by using the Gaussian rule related to the Laguerre-type weight $\sigma(x) = \mathrm{e}^{-x^{\beta}}$, i.e.,

$$\overline{G}_m(\sigma, g) = \sum_{k=1}^{m} \lambda_k(\sigma) g(t_k) = \sum_{k=1}^{m} \lambda_k(\sigma) f(t_k)\mathrm{e}^{-t_k^{-\alpha}}\,,$$

where $g(x) = f(x)\mathrm{e}^{-x^{-\alpha}}$, $t_k = t_{m,k}(\sigma)$ are the zeros of the mth Laguerre-type polynomial $p_m(\sigma)$, satisfying

$$\frac{\mathcal{C}}{m^{2-1/\beta}} \le t_1 < \cdots < t_m < \mathcal{C}m^{1/\beta}\,,$$

and $\lambda_k(\sigma)$ are the corresponding Christoffel numbers (see, e.g., [9,11]).

Now, considering the coefficients of the two Gaussian rules, we observe that for the first term of $G_m(w, f)$ we have

$$\lambda_1(w) \sim w(x_1)\Delta x_1 \sim \mathrm{e}^{-x_1^{-\alpha}}\Delta x_1 \sim \mathrm{e}^{-m^{\frac{\alpha(2\beta-1)}{\beta(2\alpha+1)}}}\,,$$

whereas the first term of $\bar{G}_m(\sigma, g)$ fulfills

$$\lambda_1(\sigma)\mathrm{e}^{-t_1^{-\alpha}} \sim \sigma(t_1)\Delta t_1 \mathrm{e}^{-t_1^{-\alpha}} \sim \Delta t_1 \mathrm{e}^{-t_1^{-\alpha}} \sim \mathrm{e}^{-m^{\frac{\alpha(2\beta-1)}{\beta}}}\,.$$

This last quantity is much smaller than $\lambda_1(w)$ for large values of m and also smaller than the ordinary tolerance usually adopted in computation. Therefore a certain number $\eta = \eta(m)$ of summands of $\overline{G}_m(\sigma, g)$ do not give any contribution. So, if $G_m(w, f)$ computes the integral with a certain error, one could obtain the same precision using

Table 1 Relative errors

m	Relative error of $\overline{G}_m(\sigma, g)$	Relative error of $G_m(w, f)$
2	3.077×10^{-1}	5.891×10^{-6}
7	1.222×10^{-2}	1.256×10^{-16}
30	9.005×10^{-7}	–
60	1.584×10^{-11}	–
110	6.984×10^{-16}	–

the Laguerre-type rule for larger values of m and with more evaluations of the function f. The following example confirms this fact.

Example 1 We apply the Gaussian quadrature rules w.r.t. the exponential weight $w(x) = e^{-1/x^3 - x^3}$ and the Laguerre weight $\sigma(x) = e^{-x^3}$ for calculating

$$\int_0^{+\infty} \arctan\left(\frac{1+x}{4}\right) e^{-1/x^3 - x^3}\, dx\,,$$

with $f(x) = \arctan\left(\frac{1+x}{4}\right)$ and $g(x) = \arctan\left(\frac{1+x}{4}\right) e^{-1/x^3}$. This integral can be evaluated with a high precision using the Mathematica function `NIntegrate`.

In Table 1 we compare the relative errors obtained applying the two rules for increasing values m, working in double arithmetic precision. We note that underflow phenomena occurred in the case of Laguerre weights, while in the case of w the symbol "–" means that the required precision has already been obtained and the relative error is of the order of the machine epsilon.

We also want to observe that a similar argument applies a fortiori if we compare the two truncated Gaussian rule related to w and σ. In fact, in $G_m(w, f)$ we can drop some terms related to the zeros close to $\varepsilon(w)$ and some other terms related to the zeros close to $a_m(w)$, but in $\overline{G}_m(\sigma, g)$ we can drop only some terms related to the largest zeros without loss of accuracy (see [11]).

Let us now compare the convergence of the two Gaussian rules. To this aim, letting

$$v(x) = (1+x)^\delta w(x) = (1+x)^\delta e^{-x^{-\alpha} - x^\beta}\,, \qquad \delta > 1\,,$$

$x \in (0, +\infty)$, we introduce the function space

$$C_v := \left\{ f \in C^0(0, +\infty) : \lim_{x \to 0^+} f(x)v(x) = 0 = \lim_{x \to +\infty} f(x)v(x) \right\},$$

with the norm

$$\|f\|_{C_v} := \|fv\| = \sup_{x \in (0, +\infty)} |f(x)v(x)|\,.$$

For more regular functions, we define the Sobolev-type spaces

$$W_r^\infty(v) = \left\{ f \in C_v : \ f^{(r-1)} \in AC(0,+\infty), \ \|f^{(r)}\varphi^r v\| < \infty \right\},$$

with the norm $\|f\|_{W_r^\infty(v)} = \|fv\| + \|f^{(r)}\varphi^r v\|$, where $\varphi(x) = \sqrt{x}$.

Then it is known that for any $f \in C_v$, the Gaussian rule $G_m(w, f)$ converges to the integral (24). Moreover, if $f \in W_r^\infty(v)$, $r \geq 1$, we have (see [14,15])

$$\left| G_m(w,f) - \int_0^\infty f(x)w(x)\,\mathrm{d}x \right| \leq \frac{\mathcal{C}}{\left(m^{1-1/(2\beta)}\right)^r} \|f\|_{W_r^\infty(v)}.$$

The Laguerre–Gaussian rule deals with functions belonging to the space

$$C_{\bar{v}} := \left\{ g \in C^0[0,+\infty) : \lim_{x\to+\infty} g(x)\bar{v}(x) = 0 \right\},$$

with $\bar{v}(x) = (1+x)^\delta e^{-x^\beta}$, $\delta > 1$,

$$\|g\|_{C_{\bar{v}}} := \|f\bar{v}\| = \sup_{x\in(0,+\infty)} |f(x)\bar{v}(x)| \,,$$

and/or to the Sobolev-type space

$$W_r^\infty(\bar{v}) = \left\{ g \in C_{\bar{v}} : \ g^{(r-1)} \in AC(0,+\infty), \ \|g^{(r)}\varphi^r \bar{v}\| < \infty \right\},$$

with

$$\|g\|_{W_r^\infty(\bar{v})} = \|gv\| + \|g^{(r)}\varphi^r \bar{v}\| \,,$$

$\varphi(x) = \sqrt{x}$ and $r \geq 1$. In analogy with the first Gaussian rule one has (see [9,11])

$$(\forall g \in C_{\bar{v}}) \quad \overline{G}_m(\sigma, g) \to \int_0^{+\infty} g(x)e^{-x^\beta}\,\mathrm{d}x \,, \quad m \to \infty \,, \tag{25}$$

and

$$\left| \overline{G}_m(\sigma, g) - \int_0^{+\infty} g(x)e^{-x^\beta}\,\mathrm{d}x \right| \leq \frac{\mathcal{C}}{\left(m^{1-1/(2\beta)}\right)^r} \|g\|_{W_r^\infty(v)} \,. \tag{26}$$

Nevertheless, if $g(x) = f(x)e^{-x^{-\alpha}}$ with $f \in C_v$ (i.e., in the case under consideration), the convergence relation (25) is true while the error estimate (26) is false in general. In fact, $f \in C_v$ implies $g \in C_{\bar{v}}$, but the norm $\|g\|_{W_r^\infty(\bar{v})}$ can be unbounded for $f \in W_r^\infty(v)$ (so, although inequality (26) holds). Therefore, the order of convergence of the Laguerre-Gaussian rule $\overline{G}_m(\sigma, g)$ is lower than the one of the rule $G_m(w, f)$.

From the previous observations we deduce that the Nyström interpolant obtained by approximating the integral

$$\int_0^{+\infty} \left[k(x, y) f(y) \mathrm{e}^{-y^{-\alpha}}\right] \mathrm{e}^{-y^\beta} \mathrm{d}y$$

by means of the Laguerre–Gaussian rule $\overline{G}_m(\sigma)$ will have a much larger number of summands with respect to the method proposed in this paper. This fact implies that the corresponding linear system will have a much larger order than the one in (20). We also want to emphasize that considering a "truncated" version of $\overline{G}_m(\sigma)$ would not solve this problem, since it is due to the exponential behaviour of the integrand close to 0 and the "truncated" rule wold drop only the terms related to the largest zeros.

In conclusion, solving integral equations of the form (1) by using a Nyström method based on the Laguerre–Gaussian rule would require a larger computational cost (with possible underflow/overflow phenomena) and a lower order of convergence, as shown in the next section.

5 Numerical examples

In the following examples the exact solutions are unknown and the corresponding tables show only the behaviour of the Nyström interpolants. As in Sect. 4, all computations were performed in MATHEMATICA, Ver. 8.0. In particular, for constructing the corresponding Gaussian rules (9) we use a procedure given in [15] and the MATHEMATICA package `OrthogonalPolynomials` (cf. [2,18]), which is freely downloadable from the Web Site:

http://www.mi.sanu.ac.rs/~gvm/.

For the sake of brevity we omit the description of the numerical procedures for the computation of the zeros of $p_m(w)$, the Christoffel numbers and the Mhaskar–Rahmanov–Saff numbers ε_m and a_m. The interested reader can find all the details about these procedures in [15, pp. 1676–1680].

Example 2 We consider the Fredholm integral equation of the second kind

$$f(x) - \frac{1}{10} \int_0^{+\infty} \cosh\left(\frac{y+1}{x+1}\right) f(y) \mathrm{e}^{-y^{-3}-y^3} \mathrm{d}y = \sinh(x+3), \quad x \in (0, +\infty),$$

with $k(x, y) = \cosh((y+1)/(x+1))$, $w(x) = \mathrm{e}^{-x^{-3}-x^3}$, and $g(x) = \sinh(x+3)$. By (3) we choose the weight $u(x) = (1+x)\,\mathrm{e}^{-(x^{-3}+x^3)/2}$ and consider the equation in the space C_u given by (4). Since $\|K\|_{C_u \to C_u} < 1$ this equation admits a unique solution in C_u.

On the other hand, if we consider the weight the Laguerre-type weight $\tilde{u}(x) = (1+x)\,\mathrm{e}^{-x^3/2}$ and the associated function space $C_{\tilde{u}}$, this equation admits a unique solution also in $C_{\tilde{u}}$. In Table 2 we compare the two associated Nyström methods, showing the correct decimal digits obtained in the Nyström interpolants at given points for the same values of m.

Table 2 Values of Nyström interpolants at $x = 0.5$, $x = 1$ and $x = 5$, for $m = 5(5)20$

m	$f_m^*(0.5)$ – exponential weight u	$\tilde{f}_m(0.5)$ – Laguerre weight $\tilde{u}$
5	**17.067206691**704378	**17.06**0042557600486
10	**17.067206693043214**	**17.067**92305565905
15	**17.067206693043214**	**17.0671**5565208371
20	**17.067206693043214**	**17.06720**0301955914
m	$f_m^*(1)$ – exponential weight u	$\tilde{f}_m(1)$ – Laguerre weight $\tilde{u}$
5	**27.67674918**6738305735	**27.67**13848546487190
10	**27.67674918738813**4070	**27.67**73392594205800
15	**27.676749187388135357**	**27.6767**086020279940
20	**27.676749187388135357**	**27.67674**35168185499
m	$f_m^*(5)$ – exponential weight u	$\tilde{f}_m(5)$ – Laguerre weight $\tilde{u}$
5	**1490.73103630**4753920402	**1490.7**275135882654744
10	**1490.73103630518894**8542	**1490.731**492523212215
15	**1490.731036305188949804**	**1490.7310**0663711270
20	**1490.731036305188949804**	**1490.73103**14344204

Using one of the two the Gaussian rules we obtain the corresponding Nyström interpolants $f_m^*(x)$, given by (21), and $\tilde{f}_m$. In Table 2 we give values of these interpolants at the points $x = 0.5$, $x = 1$ and $x = 5$. The same digits in $f_m^*(x)$ and $f_{25}^*(x)$ for $m = 5(5)20$ are bolded.

Since the kernel and the solution in this case are very smooth, we see a very fast convergence of Nyström interpolants $f_m^*(x)$, so that $f_{25}^*(x)$ can be taken as a very well approximation of the exact solution $f^*(x)$. On the other hand, the Nyström interpolant based on Laguerre-type nodes converges more slowly.

In both cases the matrices of the related linear systems are well-conditioned. For instance the condition numbers of the matrices in (20) $V_m \equiv V_m^{(j_1,j_2)}$, with $j_1 = 1$ and $j_2 = m$, for $m = 5,\ 10,\ 15\,,20$ (in the infinity-norm) are 1.0218, 1.0260, 1.0271, 1.0284, respectively.

Example 3 We consider the Fredholm integral equation of the second kind

$$f(x) - \int_0^{+\infty} \cos(x+y) f(y) \mathrm{e}^{-y^{-3}-y^3}\, \mathrm{d}y = \mathrm{e}^{1/x^2}, \quad x \in (0, +\infty),$$

with $k(x, y) = \cos(x + y)$, $w(x) = \mathrm{e}^{-x^{-3}-x^3}$, and $g(x) = \mathrm{e}^{1/x^2}$. By (3) we choose the weight $u(x) = (1 + x)\,\mathrm{e}^{-(x^{-3}+x^3)/2}$ and consider the equation in the space C_u given by (4). Since $\|K\|_{C_u \to C_u} < 1$ this equation admits a unique solution in C_u.

In this case the function g increases exponentially for $x \to 0^+$, so it does not belong to function spaces associated to generalized Laguerre weights. On the other hand, if we multiply both sides of the equation by e^{-1/x^2}, we obtain the equivalent equation

Table 3 Values of the approximate solution at $x = 0.5$, $x = 1$ and $x = 5$, for $m = 10(20)70$

m	$f_m^*(0.5)$ – exponential weight u	$\tilde{f}_m^*(0.5) \cdot \mathrm{e}^4$ – Laguerre weight $\tilde{u}$
10	**54.62464**753294463	**54.623**49877451629
30	**54.6246684277**2536	**54.624668**06769319
50	**54.62466842781927**	**54.62466843**181872
70	**54.62466842781927**	Underflow occurred in computation
m	$f_m^*(1)$ – exponential weight u	$\tilde{f}_m^*(1) \cdot \mathrm{e}$ – Laguerre weight $\tilde{u}$
10	**2.6340**588382504633	**2.633**7766902545059
30	**2.6340654174**597524	**2.634065**6449192323
50	**2.6340654174976796**	**2.634065**4201667359
70	**2.6340654174976796**	Underflow occurred in computation
m	$f_m^*(5)$ – exponential weight u	$\tilde{f}_m^*(5) \cdot \mathrm{e}^{0.04}$ – Laguerre weight $\tilde{u}$
10	**1.2543661133492144**	**1.25**31280168491409
30	**1.2543856822837072**	**1.25438**46501698619
50	**1.2543856823546275**	**1.25438568**32259046
70	**1.2543856823546275**	Underflow occurred in computation

$$\tilde{f}(x) - \int_0^{+\infty} \left[\cos(x+y)\mathrm{e}^{y^{-2}-y^{-3}-x^{-2}}\right] \tilde{f}(y)\mathrm{e}^{-y^3}\,\mathrm{d}y = 1, \quad x \in (0, +\infty),$$

with $\tilde{f}(x) = f(x)\,\mathrm{e}^{-1/x^2}$. This last equation admits a unique solution in the space $C_{\tilde{u}}$, with $\tilde{u}(x) = (1+x)\,\mathrm{e}^{-x^3/2}$. In Table 3 we compare the two associated Nyström methods, showing the correct decimal digits obtained in the approximate solution at given points for the same values of m.

The method proposed in Sect. 3 is stable and the condition numbers of the matrices in (20) $V_m \equiv V_m^{(j_1,j_2)}$, with $j_1 = 1$ and $j_2 = m$, for $m = 10, 30, 50, 70$ (in the infinity-norm) are 1.0955, 1.1066, 1.1110, 1.1134, respectively. On the other hand, the method based on Laguerre zeros is less precise and applicable only for small values of m.

Example 4 Now we consider the equation

$$f(x) - \int_0^{+\infty} |\cos(x+y)|^{5/4} f(y)\mathrm{e}^{-y^{-2}-y^2}\,\mathrm{d}y = \frac{\mathrm{e}^{(x^3+8)/(4x)}}{x+4}, \quad x \in (0, +\infty),$$

with $k(x, y) = |\cos(x+y)|^{5/4}$, $w(x) = \mathrm{e}^{-x^{-2}-x^2}$, and

$$g(x) = \frac{\mathrm{e}^{(x^3+8)/(4x)}}{x+4}.$$

Table 4 Absolute errors of the weighted Nyström interpolants $f_m^*(x)$ at $x = 1/2, 1, 4, 8$, for $m = 10$, $m = 100$ and $m = 200$

m	(j_1, j_2)	cond$(V_m^{(j_1,j_2)})$	$x = 1/2$	$x = 1$	$x = 4$	$x = 8$
10	$(1, 6)$	1.166	1.15 (−4)	7.93 (−4)	2.42 (−6)	8.51 (−17)
100	$(7, 45)$	1.212	3.59 (−5)	6.60 (−5)	1.11 (−7)	1.61 (−17)
200	$(11, 90)$	1.217	6.90 (−7)	3.78 (−6)	1.16 (−8)	9.61 (−19)

Table 5 Values of Nyström interpolants $f_m^*(x)$ at $x = 1/2, 1, 4, 8$, for $m = 10$, $m = 100$ and $m = 200$

m	$x = 1/2$	$x = 1$	$x = 4$	$x = 8$
10	**12.98**424	**1.99**0655	**11.32**452	**950832.875**282
100	**12.984**68	**1.991**643	**11.32**594	**950832.875**892
200	**12.984**88	**1.9917**28	**11.3260**2	**950832.876**026

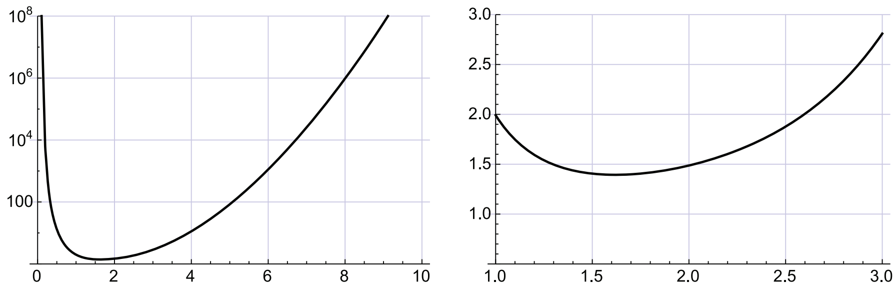

Fig. 1 The Nyström interpolant $f_m^*(x)$ for $0 \le x \le 10$ (*left*) and for $1 \le x \le 3$ (*right*), when $m = 300$, $j_1 = 15$, $j_2 = 134$

We consider this equation in C_u, where $u(x) = (1 + x)\,\mathrm{e}^{-(x^{-2}+x^2)/2}$. Since $\|K\|_{C_u \to C_u} < 1$ this equation admits a unique solution in C_u. By Theorem 2 and Corollary 1, since $u(y)k_y \in Z_{5/4}(u)$ uniformly w.r.t. y, $u(x)k_x \in Z_{5/4}(u)$ uniformly w.r.t. x, while g is a smooth function, we have

$$\left\|\left(f_m^* - f^*\right)u\right\| = O\left(\left(\frac{\sqrt{a_m}}{m}\right)^{5/4}\right) = O\left(m^{-15/16}\right),$$

taking into account that $a_m \sim m^{1/2}$.

Now, we apply the Gaussian quadratures for $m = 10(10)50$ and $m = 100(50)300$, with the corresponding truncation as in Example 8.2 in [15]. Following Table 2 from [15], we present here in Table 4 the indices j_2 and j_2 in "truncated sums" in (18) and (10) for $\theta = 1/20$, as well as the condition numbers of these reduced matrices $V_m^{(j_1,j_2)}$. Their dimensions are $j_2 - j_1 + 1$ instead of m as in the case of Gaussian formulae, dropping $c\,m^2$ terms, $c < 1$, in the matrix of coefficients in the system of linear equations.

The absolute errors of the corresponding weighted Nyström interpolants at some selected x are also given in the same table (we have considered as exact the approximated solution obtained for $m = 300$). Numbers in parentheses indicate decimal exponents, e.g., $1.15(-4)$ means 1.15×10^{-4}. Moreover, the values of the corresponding Nyström interpolants at the same selected points x are given in Table 5.

The Nyström interpolant $f_{300}^*(x)$ obtained with $j_1 = 15$ and $j_2 = 134$, for $0 \leq x \leq 10$ and $1 \leq x \leq 3$ is displayed in Fig. 1.

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OPTIMAL QUADRATURE FORMULAS FOR FOURIER COEFFICIENTS IN $W_2^{(m,m-1)}$ SPACE

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Abstract This paper studies the problem of construction of optimal quadrature formulas in the sense of Sard in the $W_2^{(m,m-1)}[0,1]$ space for calculating Fourier coefficients. Using S. L. Sobolev's method we obtain new optimal quadrature formulas of such type for $N+1 \geq m$, where $N+1$ is the number of the nodes. Moreover, explicit formulas for the optimal coefficients are obtained. We investigate the order of convergence of the optimal formula for $m=1$. The obtained optimal quadrature formula in the $W_2^{(m,m-1)}[0,1]$ space is exact for $\exp(-x)$ and $P_{m-2}(x)$, where $P_{m-2}(x)$ is a polynomial of degree $m-2$. Furthermore, we present some numerical results, which confirm the obtained theoretical results.

Keywords Fourier coefficients, optimal quadrature formulas, the error functional, extremal function, Hilbert space.

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1. Introduction

Numerical calculation of integrals of highly oscillating functions is one of the more important problems of numerical analysis, because such integrals are encountered in applications in many branches of mathematics as well as in other science such as quantum physics, flow mechanics and electromagnetism. Main examples of strongly oscillating integrands are encountered in different transformation, for example, the Fourier transformation and Fourier-Bessel transformation. Standard methods of numerical integration frequently require more computational works and they cannot be successfully applied. The earliest formulas for numerical integration of highly oscillatory functions were given by Filon [11] in 1928. Filon's approach for Fourier integrals

$$I[f;\omega] = \int_a^b \mathrm{e}^{\mathrm{i}\omega x} f(x)\,\mathrm{d}x$$

is based on piecewise approximation of $f(x)$ by arcs of the parabola on the integration interval. Then finite integrals on the subintervals are exactly integrated.

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Afterwards for integrals with different type highly oscillating functions many special effective methods such as Filon-type method, Clenshaw-Curtis-Filon type method, Levin type methods, modified Clenshaw-Curtis method, generalized quadrature rule, Gauss-Laguerre quadrature are worked out (see, for example, [4,15,20,40], for more review see, for instance, [21,23] and references therein).

In [22] the authors studied approximate computation of univariate oscillatory integrals (Fourier coefficients) for the standard Sobolev spaces H^s of periodic and non-periodic functions with an arbitrary integer $s \geq 1$. They found matching lower and upper bounds on the minimal worst case error of algorithms that use n function or derivative values. They also found sharp bounds on the information complexity which is the minimal n for which the absolute or normalized error is at most ε.

In the work [29] the weight lattice optimal cubature formulas in the periodic Sobolev's space $\tilde{L}_2^{(m)}(\Omega)$ were constructed. In particular, from the result of the work [29], in univariate case when the weight is the function $\exp(i\sigma x)$ (where $x \in [0, 2\pi]$ and σ is an integer), the Babuška optimal quadrature formula for Fourier coefficients was obtained [3].

Recently, some optimal quadrature formulas for Fourier coefficients in the Sobolev space $L_2^{(m)}(0,1)$ of non-periodic functions have been constructed in [6].

This paper is devoted to construction of optimal quadrature formulas for approximate calculation of Fourier integrals in a Hilbert space of non-periodic functions. Precisely, we study the problem of construction such optimal formulas in the sense of Sard in the $W_2^{(m,m-1)}[0,1]$ space.

We consider the following quadrature formula

$$\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} \varphi(x)\, \mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = \mathrm{e}^{2\pi \mathrm{i}\omega x} \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x - h\beta), \tag{1.2}$$

where C_β are the coefficients of formula (1.1), $h = 1/N$, $N \in \mathbb{N}$, $\mathrm{i}^2 = -1$, $\omega \in \mathbb{Z}$, $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$ and $\delta(x)$ is the Dirac delta-function. Functions φ belong to the space $W_2^{(m,m-1)}[0,1]$, where

$$W_2^{(m,m-1)}[0,1] = \left\{ \varphi : [0,1] \to \mathbb{C} \;\middle|\; \varphi^{(m-1)} \in AC[0,1] \text{ and } \varphi^{(m)} \in L_2[0,1] \right\}$$

is the Hilbert space of complex valued functions and in this space the inner product is defined by the equality

$$\langle \varphi, \psi \rangle = \int_0^1 \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x) \right) \left(\overline{\psi}^{(m)}(x) + \overline{\psi}^{(m-1)}(x) \right) \mathrm{d}x, \tag{1.3}$$

where $\overline{\psi}$ is the conjugate function to the function ψ and the norm of the function φ is correspondingly defined by the formula

$$\| \varphi | W_2^{(m,m-1)}[0,1] \| = \langle \varphi, \varphi \rangle^{1/2}$$

and
$$\int_0^1 \left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)\left(\overline{\varphi}^{(m)}(x)+\overline{\varphi}^{(m-1)}(x)\right)\,\mathrm{d}x<\infty.$$
We note that the coefficients C_β depend on ω, N and m, i.e., $C_\beta = C_\beta(\omega, N, m)$.

It should be noted that for a linear differential operator of order m, $L \equiv P_m(\mathrm{d}/\mathrm{d}x)$, Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces $K_2(P_m)$ in the context of generalized splines. Namely, with the inner product
$$\langle \varphi, \psi\rangle = \int_0^1 L\varphi(x)\cdot L\psi(x)\,\mathrm{d}x,$$
$K_2(P_m)$ is a Hilbert space if we identify functions that differ by a solution of $L\varphi = 0$. Also, such a type of spaces of periodic functions and optimal quadrature formulas were discussed in [8].

The difference
$$(\ell,\varphi)=\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\varphi(x)\,\mathrm{d}x-\sum_{\beta=0}^{N} C_\beta\varphi(x_\beta)=\int_{-\infty}^{\infty}\ell(x)\varphi(x)\,\mathrm{d}x \tag{1.4}$$
is called *the error* of the quadrature formula (1.1). The error of the formula (1.1) is a linear functional in $W_2^{(m,m-1)*}[0,1]$, where $W_2^{(m,m-1)*}[0,1]$ is the conjugate space to the space $W_2^{(m,m-1)}[0,1]$.

By the Cauchy-Schwarz inequality
$$|(\ell,\varphi)|\le \|\varphi|W_2^{(m,m-1)}[0,1]\|\cdot\|\ell|W_2^{(m,m-1)*}[0,1]\|.$$
So, the error (1.4) of formula (1.1) is estimated by the norm
$$\left\|\ell|W_2^{(m,m-1)*}[0,1]\right\| = \sup_{\|\varphi|W_2^{(m,m-1)}[0,1]\|=1}|(\ell,\varphi)|$$
of the error functional (1.2).

Thus, the estimation of the error of the quadrature formula (1.1) over functions of the space $W_2^{(m,m-1)}$ is reduced to finding the norm of the error functional ℓ in the conjugate space $W_2^{(m,m-1)*}$.

Clearly the norm of the error functional ℓ depends on the coefficients C_β. The problem of finding the minimum of the norm of the error functional ℓ by coefficients C_β when the nodes are fixed (in our case distances between neighbor nodes of formula (1.1) are equal, i.e., $x_\beta = h\beta$, $\beta = 0, 1, \ldots, N$, $h = 1/N$) is called *Sard's problem*. And the obtained formula is called *the optimal quadrature formula in the sense of Sard*. This problem was first investigated by A. Sard [24] in the space $L_2^{(m)}$ for some m. Here $L_2^{(m)}$ is the Sobolev space of functions which $(m-1)$-st derivative is absolutely continuous and m-th derivative is square integrable.

There are several methods for constructing of optimal quadrature formulas in the sense of Sard such as the spline method, the ϕ-function method (cf. [5], [25]) and Sobolev's method. Note that Sobolev's method is based on the construction of a discrete analogue to a linear differential operator (cf. [37–39]). In different spaces based on these methods, the Sard problem was investigated by many authors (see, for example, [2, 5, 7, 9, 10, 14, 16–19, 24–28, 30, 31, 33, 36–39, 41, 42] and references therein).

The main aim of the present paper is to solve the Sard problem for quadrature formulas (1.1) in the space $W_2^{(m,m-1)}[0,1]$ using S. L. Sobolev's method with $N+1 \geq m$, i.e., to look for the coefficients C_β that satisfy the following equality

$$\|\mathring{\ell}|W_2^{(m,m-1)*}[0,1]\| = \inf_{C_\beta}\|\ell\,|W_2^{(m,m-1)*}[0,1]\|. \tag{1.5}$$

Thus, to construct Sard's optimal quadrature formula of the form (1.1) in the space $W_2^{(m,m-1)}[0,1]$, we need to solve the following problems.

Problem 1. Find the norm of the error functional ℓ of quadrature formulas (1.1) in the space $W_2^{(m,m-1)*}[0,1]$.

Problem 2. Find the coefficients C_β that satisfy equality (1.5).

It should be noted that Problems 1 and 2 were solved in [34] for the case $\omega = 0$, i.e., in the work [34] the optimal quadrature formulas of the form

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta)$$

in the sense of Sard were constructed. In the sequel we will solve Problems 1 and 2 in the cases when $\omega \in \mathbb{Z}$ and $\omega \neq 0$.

The paper is organized as follows. In the second section the extremal function, which corresponds to the error functional ℓ, is given and, with its help, a representation of the norm of the error functional (1.2) is calculated, i.e., Problem 1 is solved. In Section 3 we obtain the system of linear equations for coefficients of the optimal quadrature formulas in the space $W_2^{(m,m-1)}[0,1]$. Moreover, the existence and uniqueness of the solution of this system are discussed. In Section 4, in the cases $m \geq 2$, the explicit formulas for the coefficients of the optimal quadrature formulas of the form (1.1) are found, i.e., Problem 2 is solved in the cases $m \geq 2$. The obtained optimal quadrature formulas are exact for any polynomial of order $\leq m-2$ and for the exponential function $\exp(-x)$. In Section 5 we solve Problem 2 in the case $m = 1$ and we calculate the norm of the error functional of the optimal quadrature formula in the $W_2^{(1,0)}[0,1]$ space. The obtained explicit formula for the norm of the error functional shows dependence on ω and h of the error of the optimal quadrature formula of the form (1.1) in $W_2^{(1,0)}[0,1]$ space. Finally, in Section 6 we present some numerical results which are confirm the obtained theoretical results of the present work.

2. Extremal function and norm of the error functional

To solve Problem 1, i.e., to get the explicit expression for the norm of the error functional (1.2) in the space $W_2^{(m,m-1)*}[0,1]$, we use the concept of the extremal function. The function ψ_ℓ is called *the extremal function* for the functional ℓ (see, [37]), if the following equality holds

$$(\ell, \psi_\ell) = \|\ell\,|\,W_2^{(m,m-1)*}[0,1]\| \cdot \|\psi_\ell\,|\,W_2^{(m,m-1)}[0,1]\|. \tag{2.1}$$

Since $W_2^{(m,m-1)}[0,1]$ is a Hilbert space, then the extremal function ψ_ℓ in this space, is found with the help of the general form of a linear continuous functional on Hilbert spaces given by the Riesz theorem. Then for the functional ℓ and for any $\varphi \in W_2^{(m,m-1)}[0,1]$ there exists the function $\psi_\ell \in W_2^{(m,m-1)}[0,1]$ for which the following equation holds

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle \,, \tag{2.2}$$

where

$$\langle \psi_\ell, \varphi \rangle = \int_0^1 \left(\overline{\psi}_\ell^{(m)}(x) + \overline{\psi}_\ell^{(m-1)}(x) \right) \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x) \right) \mathrm{d}x \tag{2.3}$$

is the inner product defined in the space $W_2^{(m,m-1)}[0,1]$.

From (2.2) taking into account (2.3) for the extremal function ψ_ℓ we get the following boundary value problem

$$\psi_\ell^{(2m)}(x) - \psi_\ell^{(2m-2)}(x) = (-1)^m \bar{\ell}(x), \tag{2.4}$$

$$\left(\psi_\ell^{(m+s)}(x) - \psi_\ell^{(m+s-2)}(x) \right) \Big|_{x=0}^{x=1} = 0, \quad s = \overline{1, m-1}, \tag{2.5}$$

$$\left(\psi_\ell^{(m)}(x) + \psi_\ell^{(m-1)}(x) \right) \Big|_{x=0}^{x=1} = 0, \tag{2.6}$$

where $\bar{\ell}$ is the conjugate to ℓ.

Theorem 2.1. *The solution of the boundary value problem* (2.4)–(2.6) *is the extremal function* ψ_ℓ *of the error functional* ℓ *and has the following form*

$$\psi_\ell(x) = (-1)^m \bar{\ell}(x) * G_m(x) + P_{m-2}(x) + d\, \mathrm{e}^{-x},$$

where

$$G_m(x) = \frac{\mathrm{sgn} x}{2} \left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!} \right) \tag{2.7}$$

is a solution of the equation

$$G_m^{(2m)}(x) - G_m^{(2m-2)}(x) = \delta(x), \tag{2.8}$$

d is any complex number and $P_{m-2}(x)$ is a polynomial of degree $m-2$ with complex coefficients, and $*$ is the operation of convolution.

Theorem 2.1 can be proved as Theorem 2.1 in [34].

For the error functional (1.2) to be defined on the space $W_2^{(m,m-1)}(0,1)$ it is necessary to impose the following conditions

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2, \ldots, m-2, \qquad (\ell, \mathrm{e}^{-x}) = 0. \tag{2.9}$$

Hence, it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N + 1 \geq m$ has to be met.

The equalities (2.9) mean that our quadrature formula is exact for the function e^{-x} and for any polynomial of degree $\leq m-2$.

Now, using Theorem 2.1 we will get the representation of the square of the norm of the error functional (1.2).

We recall that a convolution of two functions is defined by the formula

$$\varphi(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)\,\mathrm{d}y.$$

Taking into account the definition of convolution and equality (1.2) we calculate the convolution $\bar{\ell}(x) * G_m(x)$, i.e.,

$$\bar{\ell}(x)*G_m(x) = \int_{-\infty}^{\infty} \bar{\ell}(y) G_m(x-y)\,\mathrm{d}y = \int_0^1 \mathrm{e}^{-2\pi \mathrm{i}\omega y} G_m(x-y)\,\mathrm{d}y - \sum_{\beta=0}^{N} \bar{C}_\beta G_m(x-h\beta),$$

where $\bar{\ell}$ and $\bar{C}_\beta$ are conjugates to ℓ and C_β, respectively. Then keeping in mind (2.2), (2.3) and Theorem 2.1, we have

$$\|\ell\|^2 = (\ell, \psi_\ell) = \langle \psi_\ell, \psi_\ell \rangle = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)\,\mathrm{d}x = (-1)^m \int_{-\infty}^{\infty} \ell(x)\cdot\big(\bar{\ell}(x) * G_m(x)\big)\,\mathrm{d}x,$$

i.e.,

$$\|\ell\|^2 = (-1)^m \int_{-\infty}^{\infty} \left(\mathrm{e}^{2\pi \mathrm{i}\omega x}\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x-h\beta)\right) \\ \times\left(\int_0^1 \mathrm{e}^{-2\pi \mathrm{i}\omega y} G_m(x-y)\,\mathrm{d}y - \sum_{\gamma=0}^{N} \bar{C}_\gamma G_m(x-h\gamma)\right)\mathrm{d}x.$$

Hence we obtain

$$\|\ell\|^2 = (-1)^m \Bigg\{ \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta \overline{C}_\gamma G_m(h\beta - h\gamma) \\ - \sum_{\beta=0}^{N} \int_0^1 \left(\overline{C}_\beta \mathrm{e}^{2\pi \mathrm{i}\omega x} + C_\beta \mathrm{e}^{-2\pi \mathrm{i}\omega x}\right) G_m(x-h\beta)\,\mathrm{d}x \\ + \int_0^1\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\mathrm{e}^{-2\pi \mathrm{i}\omega y} G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y \Bigg\}. \tag{2.10}$$

Now we show that the right hand side of (2.10) is real. Really, let $C_\beta = C_\beta^R + \mathrm{i}C_\beta^I$, $\mathrm{i}^2 = -1$, where C_β^R and C_β^I are real. Using Euler's formula $\mathrm{e}^{2\pi \mathrm{i}\omega x} = \cos 2\pi\omega x + \mathrm{i}\sin 2\pi\omega x$, we get the following equalities

$$\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta \bar{C}_\gamma G_m(h\beta-h\gamma) = \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\left(C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I\right) G_m(h\beta-h\gamma),$$

$$\bar{C}_\beta \mathrm{e}^{2\pi \mathrm{i}\omega x} + C_\beta \mathrm{e}^{-2\pi \mathrm{i}\omega x} = 2C_\beta^R \cos 2\pi\omega x + 2C_\beta^I \sin 2\pi\omega x,$$

$$\int_0^1\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\mathrm{e}^{-2\pi \mathrm{i}\omega y} G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 \cos[2\pi\omega(x-y)] G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Keeping in mind the last three equalities, from (2.10) for the norm of the error functional we have

$$
\begin{aligned}
\|\ell\|^2 = (-1)^m \Bigg[& \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}(C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I)\ G_m(h\beta - h\gamma) \\
& -2\sum_{\beta=0}^{N} C_\beta^R \int_0^1 \cos 2\pi\omega x\ G_m(x-h\beta)\,\mathrm{d}x \\
& -2\sum_{\beta=0}^{N} C_\beta^I \int_0^1 \sin 2\pi\omega x\ G_m(x-h\beta)\,\mathrm{d}x \\
& + \int_0^1\int_0^1 \cos[2\pi\omega(x-y)]\ G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y \Bigg], \qquad (2.11)
\end{aligned}
$$

and from (2.9), we have the following equalities

$$\sum_{\beta=0}^{N} C_\beta^R (h\beta)^\alpha = \int_0^1 x^\alpha \cos 2\pi\omega x\,\mathrm{d}x, \quad \alpha = 0,1,2,\dots,m-2, \qquad (2.12)$$

$$\sum_{\beta=0}^{N} C_\beta^R \mathrm{e}^{-h\beta} = \int_0^1 \mathrm{e}^{-x} \cos 2\pi\omega x\,\mathrm{d}x, \qquad (2.13)$$

$$\sum_{\beta=0}^{N} C_\beta^I (h\beta)^\alpha = \int_0^1 x^\alpha \sin 2\pi\omega x\,\mathrm{d}x, \quad \alpha = 0,1,2,\dots,m-2, \qquad (2.14)$$

$$\sum_{\beta=0}^{N} C_\beta^I \mathrm{e}^{-h\beta} = \int_0^1 \mathrm{e}^{-x} \sin 2\pi\omega x\,\mathrm{d}x, \qquad (2.15)$$

Thus, Problem 1 is solved. Further in Sections 3 and 4 we solve Problem 2.

3. The system for coefficients of optimal quadrature formulas (1.1) in the space $W_2^{(m,m-1)}[0,1]$

To find the minimum of the expression (2.11) under the conditions (2.12)–(2.15) we apply the Lagrange method.

Consider the function

$$
\begin{aligned}
& \Psi(C_0^R,\dots,C_N^R,C_0^I,\dots,C_N^I,a_0^R,\dots,a_{m-2}^R,a_0^I,\dots,a_{m-2}^I,d^R,d^I) \\
&= \|\ell\|^2 - 2(-1)^m \sum_{\alpha=0}^{m-2} a_\alpha^R \left(\int_0^1 x^\alpha \cos 2\pi\omega x\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^R (h\beta)^\alpha\right) \\
&\quad -2(-1)^m \sum_{\alpha=0}^{m-2} a_\alpha^I \left(\int_0^1 x^\alpha \sin 2\pi\omega x\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^I (h\beta)^\alpha\right) \\
&\quad -2(-1)^m d^R \left(\int_0^1 \mathrm{e}^{-x} \cos 2\pi\omega x\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^R \mathrm{e}^{-h\beta}\right)
\end{aligned}
$$

$$-2(-1)^m d^I \left(\int_0^1 \mathrm{e}^{-x} \sin 2\pi\omega x \,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^I \mathrm{e}^{-h\beta} \right).$$

Equating to 0 the partial derivatives of Ψ with respect to C_β^R, C_β^I, $(\beta = \overline{0,N})$, a_α^R, a_α^I, $(\alpha = \overline{0,m-2})$, d^R, and d^I, we get the following system of linear equations, for $\alpha = 0,1,\ldots,m-2$ and $\beta = 0,1,\ldots,N$,

$$\sum_{\gamma=0}^{N} C_\gamma^R G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha^R (h\beta)^\alpha + d^R \mathrm{e}^{-h\beta} = \int_0^1 \cos 2\pi\omega x G_m(x - h\beta)\,\mathrm{d}x, \tag{3.1}$$

$$\sum_{\gamma=0}^{N} C_\gamma^R (h\gamma)^\alpha = \int_0^1 x^\alpha \cos 2\pi\omega x \,\mathrm{d}x, \tag{3.2}$$

$$\sum_{\gamma=0}^{N} C_\gamma^R \mathrm{e}^{-h\gamma} = \int_0^1 \mathrm{e}^{-x} \cos 2\pi\omega x \,\mathrm{d}x, \tag{3.3}$$

$$\sum_{\gamma=0}^{N} C_\gamma^I G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha^I (h\beta)^\alpha + d^I \mathrm{e}^{-h\beta} = \int_0^1 \sin 2\pi\omega x G_m(x - h\beta)\,\mathrm{d}x, \tag{3.4}$$

$$\sum_{\gamma=0}^{N} C_\gamma^I (h\gamma)^\alpha = \int_0^1 x^\alpha \sin 2\pi\omega x \,\mathrm{d}x, \tag{3.5}$$

$$\sum_{\gamma=0}^{N} C_\gamma^I \mathrm{e}^{-h\gamma} = \int_0^1 \mathrm{e}^{-x} \sin 2\pi\omega x \,\mathrm{d}x. \tag{3.6}$$

Now, multiplying both sides of (3.4), (3.5), and (3.6) by i and adding to both sides of (3.1), (3.2), and (3.3), respectively, using notations $C_\beta = C_\beta^R + \mathrm{i}C_\beta^I$ $(\beta = \overline{0,N})$, $a_\alpha = a_\alpha^R + \mathrm{i}a_\alpha^I$ $(\alpha = \overline{0,m-2})$, and $d = d^R + \mathrm{i}d^I$, for the coefficients of the optimal quadrature formulas of the form (1.1) we get the following system of $N+m+1$ linear equations, for $\alpha = 0,1,\ldots,m-2$ and $\beta = 0,1,\ldots,N$,

$$\sum_{\gamma=0}^{N} C_\gamma G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha (h\beta)^\alpha + d\,\mathrm{e}^{-h\beta} = f_m(h\beta), \tag{3.7}$$

$$\sum_{\gamma=0}^{N} C_\gamma (h\gamma)^\alpha = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x} x^\alpha \,\mathrm{d}x, \tag{3.8}$$

$$\sum_{\gamma=0}^{N} C_\gamma \mathrm{e}^{-h\gamma} = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x} \mathrm{e}^{-x} \,\mathrm{d}x, \tag{3.9}$$

where $G_m(x)$ is defined by equality (2.7),

$$f_m(h\beta) = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x} G_m(x - h\beta)\,\mathrm{d}x. \tag{3.10}$$

We note that the system (3.7)–(3.9) has a unique solution when $N+1 \geq m$ and this solution gives the minimum to $\|\ell\|^2$ under the conditions (3.8) and (3.9). The uniqueness of the solution of this system is obtained from Theorems 3.1 and 3.2 of [34].

From (2.11) and these theorems from [34], it follows that the square of the norm of the error functional ℓ, being a quadratic functions of the coefficients C_β has a unique minimum in some concrete value of $C_\beta = \mathring{C}_\beta$.

As it was said in the first section, the quadrature formula with the coefficients $\mathring{C}_\beta$ $(\beta = \overline{0,N})$, corresponding to this minimum, is called *the optimal quadrature formula in the sense of Sard*, and $\mathring{C}_\beta$ $(\beta = \overline{0,N})$ are called *the optimal coefficients*.

Below, for the purposes of convenience, the optimal coefficients $\mathring{C}_\beta$ will be denoted as C_β.

4. Coefficients of optimal quadrature formulas (1.1)

In the present section we solve the system (3.7)–(3.9) and we find the explicit formulas for the optimal coefficients C_β. Here we use a similar method to the one suggested by S. L. Sobolev [38] for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}(0,1)$. Here the main concept used is that of functions of discrete argument and operations on them. Theory of discrete argument functions is given in [37,39]. For the purposes of completeness we give some definitions about functions of discrete argument.

Suppose that $\varphi(x)$ and $\psi(x)$ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

Definition 4.1. A function $\varphi(h\beta)$ is called *function of discrete argument* if it is defined on some set of integer values of β.

Definition 4.2. We define *the inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ as the following number

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. We define *convolution* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ as the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

Now, we return to our problem.

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using the above mentioned definitions, we rewrite the system (3.7)–(3.9) in the following convolution form

$$G_m(h\beta) * C_\beta + P_{m-2}(h\beta) + d\,\mathrm{e}^{-h\beta} = f_m(h\beta), \quad \beta = 0, 1, \ldots, N, \tag{4.1}$$

$$\sum_{\beta=0}^{N} C_\beta \cdot (h\beta)^\alpha = g_\alpha, \quad \alpha = 0, 1, \ldots, m-2, \tag{4.2}$$

$$\sum_{\beta=0}^{N} C_\beta \cdot \mathrm{e}^{-h\beta} = \frac{\mathrm{e}^{-1} - 1}{2\pi \mathrm{i}\omega - 1}, \tag{4.3}$$

where $P_{m-2}(h\beta) = \sum_{\alpha=0}^{m-2} a_\alpha (h\beta)^\alpha$ is a polynomial of degree $m-2$,

$$f_m(h\beta) = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} G_m(x-h\beta)\,\mathrm{d}x, \tag{4.4}$$

$$g_\alpha = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} x^\alpha\,\mathrm{d}x = \frac{1}{2\pi\mathrm{i}\omega} + \sum_{k=1}^{\alpha-1} (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(2\pi\mathrm{i}\omega)^{k+1}} \tag{4.5}$$

for $\alpha = 1, 2, \ldots, m-2$, $g_0 = 0$, d is a constant, and $G_m(x)$ is defined by (2.7).

Consider the following problem:

Problem 3. Find a discrete function C_β, a polynomial $P_{m-2}(h\beta)$ of degree $m-2$ and a constant d which satisfy the system (4.1)–(4.3) for the given $f_m(h\beta)$.

Further we investigate Problem 3 and instead of C_β we introduce the functions

$$v(h\beta) = G_m(h\beta) * C_\beta \quad \text{and} \quad u(h\beta) = v\,(h\beta) + P_{m-2}(h\beta) + d\,\mathrm{e}^{-h\beta}. \tag{4.6}$$

In this statement it is necessary to express the coefficients C_β by the function $u(h\beta)$. For this, we need such an operator $D_m(h\beta)$ which satisfies the equality

$$D_m(h\beta) * G_m(h\beta) = \delta_{\mathrm{d}}(h\beta), \tag{4.7}$$

where $\delta_{\mathrm{d}}(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e., $\delta_{\mathrm{d}}(h\beta)$ is the discrete delta-function.

In [32,35] the discrete analogue $D_m(h\beta)$ of the operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$, which satisfies equation (4.7) is constructed and its some properties are investigated.

The following results are proved in [32,35].

Theorem 4.1. *The discrete analogues to the differential operator* $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$ *satisfying the equation* (4.7) *has the form*

$$D_m(h\beta) = \frac{1}{p_{2m-2}^{(2m-2)}} \begin{cases} \sum\limits_{k=1}^{m-1} A_k \lambda_k^{|\beta|-1}, & |\beta| \geq 2, \\ -2\mathrm{e}^h + \sum\limits_{k=1}^{m-1} A_k, & |\beta| = 1, \\ 2C + \sum\limits_{k=1}^{m-1} \frac{A_k}{\lambda_k}, & \beta = 0, \end{cases} \tag{4.8}$$

where

$$C = 1 + (2m-2)\mathrm{e}^h + \mathrm{e}^{2h} + \frac{\mathrm{e}^h p_{2m-3}^{(2m-2)}}{p_{2m-2}^{(2m-2)}},$$

$$A_k = \frac{2(1-\lambda_k)^{2m-2}[\lambda_k(\mathrm{e}^{2h}+1) - \mathrm{e}^h(\lambda_k^2+1)]p_{2m-2}^{(2m-2)}}{\lambda_k\ \mathcal{P}'_{2m-2}(\lambda_k)},$$

and

$$\mathcal{P}_{2m-2}(\lambda) = \sum_{s=0}^{2m-2} p_s^{(2m-2)} \lambda^s \tag{4.9}$$
$$= (1-\mathrm{e}^{2h})(1-\lambda)^{2m-2} - 2[\lambda(\mathrm{e}^{2h}+1) - \mathrm{e}^h(\lambda^2+1)]$$
$$\times \Big[h(1-\lambda)^{2m-4} + \frac{h^3(1-\lambda)^{2m-6}E_2(\lambda)}{3!} + \cdots + \frac{h^{2m-3}E_{2m-4}(\lambda)}{(2m-3)!}\Big].$$

Here, $p_{2m-2}^{(2m-2)}$ *and* $p_{2m-3}^{(2m-2)}$ *are the coefficients of the polynomial* $\mathcal{P}_{2m-2}(\lambda)$ *defined by equality* (4.9)*,* λ_k *are roots of the polynomial* $\mathcal{P}_{2m-2}(\lambda)$*,* $|\lambda_k| < 1$*, and* $E_k(\lambda)$ *is the Euler-Frobenius polynomial of degree* k (see [39]).

Theorem 4.2. *The discrete analogue* $D_m(h\beta)$ *of the differential operator*

$$\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$$

satisfies the following equalities

1) $D_m(h\beta) * \mathrm{e}^{h\beta} = 0$,
2) $D_m(h\beta) * \mathrm{e}^{-h\beta} = 0$,
3) $D_m(h\beta) * (h\beta)^n = 0$, $n \le 2m-3$,
4) $D_m(h\beta) * G_m(h\beta) = \delta_{\mathrm{d}}(h\beta)$.

Here, $G_m(h\beta)$ *is the function of discrete argument corresponding to the function* $G_m(x)$*, defined by equality* (2.7) *and* $\delta_{\mathrm{d}}(h\beta)$ *is the discrete delta function.*

Then taking into account (4.6), (4.7) and Theorems 4.1 and 4.2, for the optimal coefficients we have

$$C_\beta = D_m(h\beta) * u(h\beta). \tag{4.10}$$

Thus, if we find the function $u(h\beta)$, then the optimal coefficients can be obtained from equality (4.10).

To calculate this convolution, it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (4.1), we get that $u(h\beta) = f_m(h\beta)$ when $h\beta \in [0,1]$. Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D_m(h\beta) * u(h\beta) = 0$, $h\beta \notin [0,1]$.

Now, we calculate the convolution $v(h\beta) = G_m(h\beta) * C_\beta$ when $h\beta \notin [0,1]$.

Suppose $\beta < 0$ then, taking into account equalities (2.7), (4.2), (4.3), we have

$$v(h\beta) = G_m(h\beta) * C_\beta$$
$$= -\frac{1}{2}\sum_{\gamma=0}^{N} C_\gamma \left(\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{-h\beta+h\gamma}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!}\right)$$
$$= -\frac{\mathrm{e}^{h\beta}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega - 1} + De^{-h\beta} + R_{2m-3}(h\beta) + Q_{m-2}(h\beta), \tag{4.11}$$

where

$$R_{2m-3}(h\beta) = \frac{1}{2}\left(\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^{\alpha}}{(2k-1-\alpha)!\alpha!}g_{\alpha}\right.$$

$$\left. + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^{\alpha}}{(2k-1-\alpha)!\alpha!}g_{\alpha}\right) \tag{4.12}$$

is a polynomial of degree $2m-3$ in $(h\beta)$,

$$Q_{m-2}(h\beta) = \frac{1}{2}\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^{\alpha}}{(2k-1-\alpha)!\alpha!}\sum_{\gamma=0}^{N}C_{\gamma}(h\gamma)^{\alpha} \tag{4.13}$$

is an unknown polynomial of degree $m-2$ also in $(h\beta)$, and

$$D = \frac{1}{4}\sum_{\gamma=0}^{N}C_{\gamma}\mathrm{e}^{h\gamma}. \tag{4.14}$$

Similarly, in the case $\beta > N$, for the convolution $v(h\beta) = G_m(h\beta) * C_{\beta}$, we obtain

$$v(h\beta) = \frac{\mathrm{e}^{h\beta}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega - 1} - De^{-h\beta} - R_{2m-3}(h\beta) - Q_{m-2}(h\beta). \tag{4.15}$$

We denote

$$Q^{(-)}_{m-2}(h\beta) = P_{m-2}(h\beta) + Q_{m-2}(h\beta), \quad a^{-} = d + D, \tag{4.16}$$

$$Q^{(+)}_{m-2}(h\beta) = P_{m-2}(h\beta) - Q_{m-2}(h\beta), \quad a^{+} = d - D, \tag{4.17}$$

and, taking into account (4.11), (4.15), (4.6), we get the following problem.

Problem 4. Find the solution of the equation

$$D_m(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1], \tag{4.18}$$

having the form

$$u(h\beta) = \begin{cases} -\dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + a^{-}\mathrm{e}^{-h\beta} + R_{2m-3}(h\beta) + Q^{(-)}_{m-2}(h\beta), & \beta < 0, \\ f_m(h\beta), & 0 \le \beta \le N, \\ \dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + a^{+}\mathrm{e}^{-h\beta} - R_{2m-3}(h\beta) + Q^{(+)}_{m-2}(h\beta), & \beta > N. \end{cases}$$

Here, $Q^{(-)}_{m-2}(h\beta)$ and $Q^{(+)}_{m-2}(h\beta)$ are unknown polynomials of degree $m-2$ with respect to $(h\beta)$, a^{-} and a^{+} are unknown constants.

If we find $Q^{(-)}_{m-2}(h\beta)$, $Q^{(+)}_{m-2}(h\beta)$, a^{-} and a^{+}, then from (4.16), (4.17) we have

$$P_{m-2}(h\beta) = \frac{1}{2}\left(Q^{(-)}_{m-2}(h\beta) + Q^{(+)}_{m-2}(h\beta)\right), \quad d = \frac{1}{2}(a^{-} + a^{+}),$$

$$Q_{m-2}(h\beta)=\frac{1}{2}\left(Q^{(-)}_{m-2}(h\beta)-Q^{(+)}_{m-2}(h\beta)\right),\quad D=\frac{1}{2}(a^{-}-a^{+}).$$

Unknowns $Q^{(-)}_{m-2}(h\beta)$, $Q^{(+)}_{m-2}(h\beta)$, a^{-} and a^{+} can be found from the equation (4.18), using the function $D_m(h\beta)$. Then we can obtain the explicit form of the function $u(h\beta)$ and find the optimal coefficients C_β. Thus, Problem 4 and, respectively, Problem 3 can be solved.

But here we will not find $Q^{(-)}_{m-2}(h\beta)$, $Q^{(+)}_{m-2}(h\beta)$, a^{-} and a^{+}. Instead of them, using $D_m(h\beta)$ and $u(h\beta)$, taking into account (4.10), we find now the expressions for the optimal coefficients C_β when $\beta=1,\dots,N-1$.

We denote

$$a_k=\frac{A_k}{\lambda_k p}\sum_{\gamma=1}^{\infty}\lambda_k^{\gamma}\biggl(-\frac{\mathrm{e}^{-h\gamma}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1}+R_{2m-3}(-h\gamma)$$
$$+Q^{(-)}_{m-2}(-h\gamma)+a^{-}\mathrm{e}^{h\gamma}-f_m(-h\gamma)\biggr),\tag{4.19}$$

$$b_k=\frac{A_k}{\lambda_k p}\sum_{\gamma=1}^{\infty}\lambda_k^{\gamma}\biggl(\frac{\mathrm{e}^{h\gamma+1}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1}-R_{2m-3}(1+h\gamma)$$
$$+Q^{(+)}_{m-2}(1+h\gamma)+a^{+}\mathrm{e}^{-1-h\gamma}-f_m(1+h\gamma)\biggr),\tag{4.20}$$

where λ_k are roots and p is the leading coefficient of the polynomial $\mathcal{P}_{2m-2}(\lambda)$ of degree $2m-2$ defined by (4.9) and $|\lambda_k|<1$. The series in the notations (4.19), (4.20) are convergent.

The following statement holds:

Theorem 4.3 (Theorem 3, [31]). *The coefficients of optimal quadrature formulas in the sense of Sard of the form* (1.1) *in the space* $W_2^{(m,m-1)}[0,1]$ *have the following form*

$$C_\beta=D_m(h\beta)*f_m(h\beta)+\sum_{k=1}^{m-1}\left(a_k\lambda_k^{\beta}+b_k\lambda_k^{N-\beta}\right),\quad \beta=1,2,\dots,N-1,\tag{4.21}$$

where a_k *and* b_k *are unknowns and have the form* (4.19) *and* (4.20) *respectively,* λ_k *are the roots of the polynomial* $\mathcal{P}_{2m-2}(\lambda)$ *which is defined by equality* (4.9) *and* $|\lambda_k|<1$.

From Theorem 4.3, it is clear that to obtain the explicit forms of the optimal coefficients C_β in the space $W_2^{(m,m-1)}[0,1]$ it is sufficient to find a_k and b_k ($k=\overline{1,m-1}$). But here we will not calculate series (4.19) and (4.20). Instead of that substituting equality (4.21) into (4.1) we obtain the identity with respect to $(h\beta)$. Whence, equating the corresponding coefficients in the left and the right hand sides of equation (4.1) and using (4.2) when $\alpha=1,2,\dots,m-2$, we find a_k and b_k. The coefficients C_0 and C_N can be found from (4.2) when $\alpha=0$ and (4.3), respectively. Below we do it.

In the present section we solve the system (4.1)–(4.3) for any $m\ge 2$ and for natural N that $N+1\ge m$. As it was mentioned above, it is sufficient to find a_k and b_k ($k=\overline{1,m-1}$) in (4.21).

The case $m=1$ we consider in the next section. In the case $m\geq 2$ the following results hold:

Theorem 4.4. *The coefficients of optimal quadrature formulas of the form* (1.1) *with the error functional* (1.2) *and with equal spaced nodes in the space* $W_2^{(m,m-1)}[0,1]$ *when* $m\geq 2$, $N+1\geq m$ *and* $\omega h\notin\mathbb{Z}$ *are expressed by formulas*

$$C_0=\frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)}+\frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)}$$
$$+\sum_{k=1}^{m-1}\left(\frac{a_k\lambda_k^2}{(1-\lambda_k)(\mathrm{e}^h-\lambda_k)}+\frac{b_k\lambda_k^N}{(1-\lambda_k)(1-\lambda_k\mathrm{e}^h)}\right),$$

$$C_\beta=\mathrm{e}^{2\pi\mathrm{i}\omega h\beta}K+\sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta+b_k\lambda_k^{N-\beta}\right),\qquad \beta=\overline{1,N-1},$$

$$C_N=\frac{K\mathrm{e}^{h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)}+\frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)}$$
$$+\mathrm{e}^h\sum_{k=1}^{m-1}\left(\frac{a_k\lambda_k^N}{(1-\lambda_k)(\mathrm{e}^h-\lambda_k)}+\frac{b_k\lambda_k^2}{(1-\lambda_k)(1-\lambda_k\mathrm{e}^h)}\right),$$

where a_k *and* b_k $(k=\overline{1,m-1})$ *are defined by the following system of* $2m-2$ *linear equations*

$$\sum_{k=1}^{m-1}\frac{a_k\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)}+\sum_{k=1}^{m-1}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}$$
$$=\frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)}+\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})},$$

$$\sum_{k=1}^{m-1}\frac{a_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)}+\sum_{k=1}^{m-1}\frac{b_k\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}$$
$$=\frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)}+\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})},$$

$$\sum_{k=1}^{m-1}a_k\sum_{t=1}^{j}\frac{\lambda_k\Delta^t0^j}{(\lambda_k-1)^{t+1}}+\sum_{k=1}^{m-1}b_k\sum_{t=1}^{j}\frac{\lambda_k^{N+t}\Delta^t0^j}{(1-\lambda_k)^{t+1}}$$
$$=\frac{j!h}{(2\pi\mathrm{i}\omega h)^{j+1}}-\sum_{t=1}^{j}\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}\Delta^t0^j}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^{t+1}},\qquad j=\overline{1,m-2},$$

and

$$\sum_{k=1}^{m-1}a_k\left[h^j\sum_{t=1}^{j}\frac{\lambda_k^t\Delta^t0^j}{(1-\lambda_k)^{t+1}}-\sum_{i=1}^{j}h^iC_j^i\sum_{t=1}^{i}\frac{\lambda_k^{N+t}\Delta^t0^i}{(1-\lambda_k)^{t+1}}\right]$$
$$+\sum_{k=1}^{m-1}b_k\left[h^j\sum_{t=1}^{j}\frac{\lambda_k^{N+1}\Delta^t0^j}{(\lambda_k-1)^{t+1}}-\sum_{i=1}^{j}h^iC_j^i\sum_{t=1}^{i}\frac{\lambda_k\Delta^t0^i}{(\lambda_k-1)^{t+1}}\right]$$

$$= \sum_{k=1}^{j-1} (-1)^k \frac{j(j-1)\cdots(j-k+1)}{(2\pi i\omega)^{k+1}} + K \sum_{i=1}^{j-1} h^i C_j^i \sum_{t=1}^{i} \frac{e^{2\pi i\omega h t}\Delta^t 0^i}{(1-e^{2\pi i\omega h})^{t+1}},$$

for $j = \overline{1, m-2}$, *where*

$$K = \frac{L}{p_{2m-2}^{(2m-2)}} \left\{ \sum_{k=1}^{m-1} \left[\frac{2A_k}{\lambda_k} \cdot \frac{1-\lambda_k \cos(2\pi\omega h)}{\lambda_k^2 + 1 - 2\lambda_k \cos(2\pi\omega h)} - \frac{A_k}{\lambda_k} \right] - 4e^h \cos(2\pi\omega h) + 2C \right\},$$

$$L = \frac{1}{(2\pi i\omega)^2 - 1} - \sum_{k=1}^{m-1} \frac{1}{(2\pi i\omega)^{2k}},$$

λ_k *are the roots of the polynomial* (4.9), $|\lambda_k| < 1$, *and* $p_{2m-2}^{(2m-2)}$, A_k *and* C *are defined in Theorem* 4.1.

Theorem 4.5. *The coefficients of optimal quadrature formulas of the form* (1.1) *with the error functional* (1.2) *and with equal spaced nodes in the space* $W_2^{(m,m-1)}[0,1]$ *when* $m \geq 2$, $N+1 \geq m$ *and* $\omega h \in \mathbb{Z}$, $\omega \neq 0$, *are expressed by formulas*

$$C_0 = \frac{2\pi i\omega(1-e^h) - 1}{2\pi i\omega(1-2\pi i\omega)(1-e^h)} + \sum_{k=1}^{m-1} \left[\frac{a_k \lambda_k^2}{(1-\lambda_k)(e^h - \lambda_k)} + \frac{b_k \lambda_k^N}{(1-\lambda_k)(1-\lambda_k e^h)} \right],$$

$$C_\beta = \sum_{k=1}^{m-1} \left(a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \qquad \beta = \overline{1, N-1},$$

$$C_N = \frac{2\pi i\omega(e^h - 1) - e^h}{2\pi i\omega(1-2\pi i\omega)(1-e^h)} + e^h \sum_{k=1}^{m-1} \left[\frac{a_k \lambda_k^N}{(1-\lambda_k)(e^h - \lambda_k)} + \frac{b_k \lambda_k^2}{(1-\lambda_k)(1-\lambda_k e^h)} \right],$$

where a_k *and* b_k, $k = \overline{1, m-1}$, *are defined by the following system of* $2m-2$ *linear equations*

$$\sum_{k=1}^{m-1} \frac{a_k \lambda_k}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{m-1} \frac{b_k \lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k e^h - 1)} = \frac{1}{2\pi i\omega(1-2\pi i\omega)(1-e^h)},$$

$$\sum_{k=1}^{m-1} \frac{a_k \lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{m-1} \frac{b_k \lambda_k}{(\lambda_k - 1)(\lambda_k e^h - 1)} = \frac{1}{2\pi i\omega(1-2\pi i\omega)(1-e^h)},$$

$$\sum_{k=1}^{m-1} a_k \sum_{t=1}^{j} \frac{\lambda_k \Delta^t 0^j}{(\lambda_k - 1)^{t+1}} + \sum_{k=1}^{m-1} b_k \sum_{t=1}^{j} \frac{\lambda_k^{N+t} \Delta^t 0^j}{(1-\lambda_k)^{t+1}} = \frac{j! h}{(2\pi i\omega h)^{j+1}}, \quad j = \overline{1, m-2},$$

and

$$\sum_{k=1}^{m-1} a_k \left[h^j \sum_{t=1}^{j} \frac{\lambda_k^t \Delta^t 0^j}{(1-\lambda_k)^{t+1}} - \sum_{i=1}^{j} h^i C_j^i \sum_{t=1}^{i} \frac{\lambda_k^{N+t} \Delta^t 0^i}{(1-\lambda_k)^{t+1}} \right]$$

$$+ \sum_{k=1}^{m-1} b_k \left[h^j \sum_{t=1}^{j} \frac{\lambda_k^{N+1} \Delta^t 0^j}{(\lambda_k - 1)^{t+1}} - \sum_{i=1}^{j} h^i C_j^i \sum_{t=1}^{i} \frac{\lambda_k \Delta^t 0^i}{(\lambda_k - 1)^{t+1}} \right]$$

$$= \sum_{k=1}^{j-1} (-1)^k \frac{j(j-1)\cdots(j-k+1)}{(2\pi i\omega)^{k+1}}, \qquad j = \overline{1, m-2}.$$

Here, λ_k are the roots of the polynomial (4.9) *and* $|\lambda_k| < 1$, A_k *and* C *are defined in Theorem* 4.1.

In order to prove Theorem 4.4 we use the following formulas (cf. [13], [12])

$$\begin{aligned} \sum_{\gamma=0}^{n-1} q^\gamma \gamma^k &= \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k\Big|_{\gamma=n}, \\ \sum_{\gamma=0}^{\beta-1} \gamma^k &= \sum_{j=1}^{k+1} \frac{k!\, B_{k+1-j}}{j!\,(k+1-j)!}\beta^j, \end{aligned} \tag{4.22}$$

where $\Delta^i 0^k = \sum\limits_{\ell=1}^{i} (-1)^{i-\ell} C_i^\ell \ell^k$, $\Delta^i \gamma^k$ is the finite difference of order i of γ^k, and B_{k+1-j} are the Bernoulli numbers, as well as

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} C_\nu^p \Delta^\alpha 0^p x^{\nu-p}. \tag{4.23}$$

Proof of Theorem 4.4. Using the binomial formula in equality (4.4), for $f_m(h\beta)$ we deduce

$$\begin{aligned} f_m(h\beta) = {} & \frac{(\mathrm{e}+1)\mathrm{e}^{-h\beta}}{4(2\pi \mathrm{i}\omega+1)} - \frac{(1+\mathrm{e}^{-1})\mathrm{e}^{h\beta}}{4(2\pi \mathrm{i}\omega-1)} + \mathrm{e}^{2\pi \mathrm{i}\omega h\beta}\left[\frac{1}{(2\pi \mathrm{i}\omega)^2-1} - \sum_{k=1}^{m-1}\frac{1}{(2\pi \mathrm{i}\omega)^{2k}}\right] \\ & + \sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{2(2k-1-\alpha)!\alpha!}g_\alpha + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{2(2k-1-\alpha)!\alpha!}g_\alpha \\ & + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{2(2k-1-\alpha)!\alpha!}g_\alpha. \end{aligned} \tag{4.24}$$

Then, using (4.24), Definition 4.3 and Theorems 4.1 and 4.2, after certain calculations for the convolution $D_m(h\beta) * f_m(h\beta)$ we get

$$\begin{aligned} D_m(h\beta) * f_m(h\beta) &= D_m(h\beta) * \left[\mathrm{e}^{2\pi \mathrm{i}\omega h\beta}\left(\frac{1}{(2\pi i\omega)^2-1} - \sum_{k=1}^{m-1}\frac{1}{(2\pi i\omega)^{2k}}\right)\right] \\ &= \mathrm{e}^{2\pi \mathrm{i}\omega h\beta}\left[\frac{1}{(2\pi i\omega)^2-1} - \sum_{k=1}^{m-1}\frac{1}{(2\pi i\omega)^{2k}}\right]\sum_{\gamma=-\infty}^{\infty} D_m(h\gamma)\mathrm{e}^{2\pi \mathrm{i}\omega h\gamma} \\ &= K\mathrm{e}^{2\pi \mathrm{i}\omega h\beta}, \end{aligned}$$

where K is given in Theorem 4.4.

Therefore, from Theorem 4.3, taking into account the last equality, for coefficients C_β, $\beta = \overline{1, N-1}$, we have

$$C_\beta = K\mathrm{e}^{2\pi \mathrm{i}\omega h\beta} + \sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \quad \beta = 1, 2, \ldots, N-1. \tag{4.25}$$

For the convolution $G_m(h\beta) * C_\beta$ of equality (4.1) we have

$$S(h\beta) = C_0 \left(\frac{\mathrm{e}^{h\beta} - \mathrm{e}^{-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta)^{2k-1}}{(2k-1)!} \right) + S_1(h\beta) + S_2(h\beta), \tag{4.26}$$

where

$$S_1(h\beta) = \sum_{\gamma=1}^{\beta-1} C_\gamma \left(\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{h\gamma-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!} \right),$$

$$S_2(h\beta) = -\frac{1}{2} \sum_{\gamma=0}^{N} C_\gamma \left(\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{h\gamma-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!} \right).$$

Then, using (4.25), (4.22), (4.23) and taking into account that λ_k are roots of the polynomial (4.9), after some simplifications, we get

$$\begin{aligned}
S_1(h\beta) =& \mathrm{e}^{2\pi \mathrm{i}\omega h\beta} \Bigg[\frac{K\mathrm{e}^h}{2(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)} - \frac{K}{2(\mathrm{e}^{h+2\pi\mathrm{i}\omega h} - 1)} \\
& - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1} \sum_{k=1}^{m-1} \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{\Delta^t 0^{2l-1}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^t} \Bigg] \\
& - \frac{\mathrm{e}^{h\beta}}{2} \left[\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h} + \sum_{k=1}^{m-1} \left(\frac{a_k\lambda_k}{\lambda_k - \mathrm{e}^h} + \frac{b_k\lambda_k^N}{1 - \lambda_k \mathrm{e}^h} \right) \right] \\
& + \frac{\mathrm{e}^{-h\beta}}{2} \left[\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h+h}}{\mathrm{e}^{h+2\pi\mathrm{i}\omega h} - 1} + \sum_{k=1}^{m-1} \left(\frac{a_k\lambda_k\mathrm{e}^h}{\lambda_k\mathrm{e}^h - 1} + \frac{b_k\lambda_k^N\mathrm{e}^h}{\mathrm{e}^h - \lambda_k} \right) \right] \\
& + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1} \sum_{l=1}^{m-1} \frac{h^{2l-1}}{(2l-1)!} \sum_{j=0}^{2l-1} C_{2l-1}^j \beta^j \sum_{t=0}^{2l-1} \frac{\Delta^t 0^{2l-1-j}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^t} \\
& + \sum_{\ell=1}^{m-1} \frac{h^{2\ell-1}}{(2\ell-1)!} \sum_{j=0}^{2\ell-1} C_{2\ell-1}^j \beta^j \sum_{k=1}^{m-1} \frac{a_k\lambda_k}{\lambda_k - 1} \sum_{t=0}^{2\ell-1} \frac{\Delta^t 0^{2\ell-1-j}}{(\lambda_k - 1)^t} \\
& + \sum_{\ell=1}^{m-1} \frac{h^{2\ell-1}}{(2\ell-1)!} \sum_{j=0}^{2\ell-1} C_{2\ell-1}^j \beta^j \sum_{k=1}^{m-1} \frac{b_k\lambda_k^N}{1 - \lambda_k} \sum_{t=0}^{2\ell-1} \left(\frac{\lambda_k}{1 - \lambda_k} \right)^t \Delta^t 0^{2\ell-1-j}.
\end{aligned} \tag{4.27}$$

Now, using the binomial formula and equalities (4.2) and (4.3), we obtain

$$\begin{aligned}
S_2(h\beta) = \frac{1}{2} \Bigg\{ & \frac{(\mathrm{e}^{-1} - 1)\mathrm{e}^{h\beta}}{2(2\pi\mathrm{i}\omega - 1)} + \frac{\mathrm{e}^{-h\beta}}{2} \sum_{\gamma=0}^{N} C_\gamma \mathrm{e}^{h\gamma} \\
& + \Bigg[\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1} \sum_{\alpha=0}^{2k-1} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!\alpha!} g_\alpha + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1} \sum_{\alpha=0}^{m-2} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!\alpha!} g_\alpha \\
& + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1} \sum_{\alpha=m-1}^{2k-1} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!\alpha!} \sum_{\gamma=0}^{N} C_\gamma (h\gamma)^\alpha \Bigg] \Bigg\}.
\end{aligned} \tag{4.28}$$

Taking into account (4.27), (4.28) and putting (4.26), (4.24) into (4.1), we get the following identity with respect to $(h\beta)$:

$$S(h\beta) + P_{m-2}(h\beta) + d\,\mathrm{e}^{-h\beta} = f_m(h\beta). \tag{4.29}$$

As it was said above, equality (4.29) is the identity with respect to $(h\beta)$. Keeping in mind (4.27), (4.28), (4.24), equating the coefficients of $\mathrm{e}^{h\beta}$ and the terms which consist of $(h\beta)^\alpha$, $\alpha = \overline{m-1, 2m-3}$ in both sides of (4.29), we get the following equations for a_k and b_k

$$\sum_{k=1}^{m-1} \left\{ a_k \frac{\lambda_k - \lambda_k^{N+1}}{(e^h - \lambda_k)(1-\lambda_k)} + b_k \frac{\lambda_k - \lambda_k^{N+1}}{(\lambda_k e^h - 1)(1-\lambda_k)} \right\} = 0, \tag{4.30}$$

$$\begin{aligned} \sum_{\ell=[\frac{m+1}{2}]}^{m-1} \Bigg\{ & -C_0 \frac{(h\beta)^{2\ell-1}}{(2\ell-1)!} - \sum_{j=m-1}^{2\ell-1} \frac{(h\beta)^j h^{2\ell-j-1}}{j!\,(2\ell-j-1)!} \sum_{t=0}^{2l-1-j} \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}\Delta^t 0^{2l-1-j}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^{t+1}} \\ & + \sum_{j=m-1}^{2\ell-1} \frac{(h\beta)^j h^{2\ell-j-1}}{(2\ell-1-j)!j!} \sum_{k=1}^{m-1} a_k \sum_{t=0}^{2\ell-1} \frac{\lambda_k \Delta^t 0^{2\ell-1-j}}{(\lambda_k-1)^{t+1}} \\ & + \sum_{j=m-1}^{2\ell-1} (h\beta)^j \frac{h^{2\ell-j-1}}{(2\ell-1-j)!j!} \sum_{k=1}^{m-1} b_k \sum_{t=0}^{2\ell-1} \frac{\lambda_k^{N+i}\Delta^t 0^{2\ell-1-j}}{(1-\lambda_k)^{t+1}} \Bigg\} = 0. \end{aligned} \tag{4.31}$$

Unknown polynomial $P_{m-2}(h\beta)$ and the coefficient d can be found from (4.29) by equating the corresponding coefficients of $(h\beta)^\alpha$ when $\alpha = 0, 1, \ldots, m-2$ and $\mathrm{e}^{-h\beta}$, respectively.

Now, from equations (4.2) when $\alpha = 0$ and (4.3), taking into account (4.25), using identities (4.22) and (4.23), after some simplifications for the coefficients C_0 and C_N, we get the following expressions

$$\begin{aligned} C_0 = & \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h} - \frac{1}{2\pi\mathrm{i}\omega - 1} \\ & + \sum_{k=1}^{m-1} \Bigg\{ a_k \frac{\lambda_k(\mathrm{e}^h - \mathrm{e}) + \lambda_k^2(\mathrm{e}-1) + \lambda_k^{N+1}(1-\mathrm{e}^h)}{(\mathrm{e}-1)(1-\lambda_k)(\mathrm{e}^h-\lambda_k)} \\ & \qquad + b_k \frac{\lambda_k^{N+1}(\mathrm{e}^h - \mathrm{e}) + \lambda_k^N(\mathrm{e}-1) + \lambda_k(1-\mathrm{e}^h)}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} \Bigg\} \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} C_N = & \frac{K\mathrm{e}^h}{\mathrm{e}^h - \mathrm{e}^{2\pi i\omega h}} + \frac{1}{2\pi\mathrm{i}\omega - 1} \\ & + \sum_{k=1}^{m-1} \Bigg\{ a_k \frac{\lambda_k(\mathrm{e} - \mathrm{e}^{h+1}) + \lambda_k^N(\mathrm{e}^{h+1} - \mathrm{e}^h) + \lambda_k^{N+1}(\mathrm{e}^h - \mathrm{e})}{(\mathrm{e}-1)(1-\lambda_k)(\mathrm{e}^h-\lambda_k)} \\ & \qquad + b_k \frac{\lambda_k^{N+1}(\mathrm{e} - \mathrm{e}^{h+1}) + \lambda_k^2(\mathrm{e}^{h+1} - \mathrm{e}^h) + \lambda_k(\mathrm{e}^h - \mathrm{e})}{(\mathrm{e}-1)(1-\lambda_k)(1-\lambda_k\mathrm{e}^h)} \Bigg\}, \end{aligned} \tag{4.33}$$

respectively. Then, from (4.31), using (4.32), grouping the coefficients of same degrees of $(h\beta)$ and equating to zero, for a_k and b_k we obtain the following $m-1$

linear equations:

$$\sum_{k=1}^{m-1} a_k \left[\sum_{l=1}^{j} \frac{h^{2l-2}}{(2l-2)!} \sum_{t=0}^{2l-2} \frac{\lambda_k \Delta^t 0^{2l-2}}{(\lambda_k - 1)^{t+1}} - \frac{\lambda_k(\mathrm{e}^h - \mathrm{e}) + \lambda_k^2(\mathrm{e}-1) + \lambda_k^{N+1}(1-\mathrm{e}^h)}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} \right]$$

$$+ \sum_{k=1}^{m-1} b_k \left[\sum_{l=1}^{j} \frac{h^{2l-2}}{(2l-2)!} \sum_{t=0}^{2l-2} \frac{\lambda_k^{N+t} \Delta^t 0^{2l-2}}{(1-\lambda_k)^{t+1}} - \frac{\lambda_k(1-\mathrm{e}^h) + \lambda_k^N(\mathrm{e}-1) + \lambda_k^{N+1}(\mathrm{e}^h - \mathrm{e})}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k \mathrm{e}^h - 1)} \right]$$

$$= \sum_{l=1}^{j} \left[\frac{1}{(2\pi \mathrm{i}\omega)^{2l-1}} - \frac{h^{2l-2}}{(2l-2)!} \sum_{t=0}^{2l-2} \frac{K \mathrm{e}^{2\pi \mathrm{i}\omega h} \Delta^t 0^{2l-2}}{(\mathrm{e}^{2\pi \mathrm{i}\omega h} - 1)^{t+1}} \right] - \frac{K\mathrm{e}^{2\pi \mathrm{i}\omega h}}{\mathrm{e}^{2\pi i\omega h} - \mathrm{e}^h} - \frac{1}{2\pi \mathrm{i}\omega h - 1},$$

for $j = \overline{1, [m/2]}$, and

$$\sum_{k=1}^{m-1} a_k \left[\sum_{l=1}^{j} \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{\lambda_k \Delta^t 0^{2l-1}}{(\lambda_k - 1)^{t+1}} \right] + \sum_{k=1}^{m-1} b_k \left[\sum_{l=1}^{j} \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{\lambda_k^{N+t} \Delta^i 0^{2l-1}}{(1-\lambda_k)^{t+1}} \right]$$

$$= \sum_{l=1}^{j} \left[\frac{1}{(2\pi \mathrm{i}\omega)^{2l}} - \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{K \mathrm{e}^{2\pi \mathrm{i}\omega h} \Delta^t 0^{2l-1}}{(\mathrm{e}^{2\pi \mathrm{i}\omega h} - 1)^{t+1}} \right],$$

for $j = \overline{1, [(m-1)/2]}$.

Further, from (4.2) when $\alpha = 1, \dots, m-2$, using equalities (4.25), (4.33) and identities (4.22) and (4.23) for a_k and b_k we have the following $m-2$ linear equations:

$$\sum_{k=1}^{m-1} a_k \left\{ h^j \sum_{i=0}^{j} \frac{\lambda_k^i - \lambda_k^{N+i}}{(1-\lambda_k)^{i+1}} \Delta^i 0^j - \sum_{l=0}^{j-1} h^l C_j^l \sum_{i=0}^{l} \frac{\lambda_k^{N+i} \Delta^i 0^l}{(1-\lambda_k)^{i+1}} \right.$$
$$\left. + \frac{\lambda_k(\mathrm{e} - \mathrm{e}^{h+1}) + \lambda_k^N(\mathrm{e}^{h+1} - \mathrm{e}^h) + \lambda_k^{N+1}(\mathrm{e}^h - \mathrm{e})}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k - \mathrm{e}^h)} \right\}$$
$$+ \sum_{k=1}^{m-1} b_k \left\{ h^j \sum_{i=0}^{j} \frac{\lambda_k^{N+1} - \lambda_k}{(\lambda_k - 1)^{i+1}} \Delta^i 0^j - \sum_{l=0}^{j-1} h^l C_j^l \sum_{i=0}^{l} \frac{\lambda_k \Delta^i 0^l}{(\lambda_k-1)^{i+1}} \right.$$
$$\left. + \frac{\lambda_k^{N+1}(\mathrm{e} - \mathrm{e}^{h+1}) + \lambda_k(\mathrm{e}^h - \mathrm{e}) + \lambda_k^2(\mathrm{e}^{h+1} - \mathrm{e}^h)}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k \mathrm{e}^h - 1)} \right\}$$
$$= \frac{1}{2\pi \mathrm{i}\omega} + \sum_{l=1}^{j-1} (-1)^l \frac{j(j-1)(j-2)\cdots(j-l+1)}{(2\pi \mathrm{i}\omega)^{l+1}} - \frac{K\mathrm{e}^h}{\mathrm{e}^h - \mathrm{e}^{2\pi \mathrm{i}\omega h}}$$
$$- \frac{1}{2\pi \mathrm{i}\omega - 1} + K \sum_{l=0}^{j-1} h^l C_j^l \sum_{t=0}^{l} \frac{\mathrm{e}^{2\pi \mathrm{i}\omega t h} \Delta^t 0^l}{(1 - \mathrm{e}^{2\pi \mathrm{i}\omega h})^{t+1}},$$

where $j = \overline{1, m-2}$.

Finally, after some simplifications in (4.30) and the previous systems of equations for a_k and b_k, we get the system which is given in the assertion of this theorem. □

The proof of Theorem 4.5 is similar to one of Theorem 4.4. Only one difference is that $D_m(h\beta) * f_m(h\beta) = K = 0$.

For $m = 2,\ m = 3$ and $m = 4$, from Theorem 4.4 we have the following results:

Corollary 4.1. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h \notin \mathbb{Z}$ *in the space* $W_2^{(2,1)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{1}{1-\lambda_1}\left\{\frac{a_1\lambda_1^2}{\mathrm{e}^h-\lambda_1} + \frac{b_1\lambda_1^N}{1-\lambda_1\mathrm{e}^h}\right\},$$

$$C_\beta = \mathrm{e}^{2\pi\mathrm{i}\omega h\beta}K + a_1\lambda_1^\beta + b_1\lambda_1^{N-\beta}, \qquad \beta = \overline{1, N-1},$$

$$C_N = \frac{K\mathrm{e}^h}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{\mathrm{e}^h}{1-\lambda_1}\left\{\frac{a_1\lambda_1^N}{\mathrm{e}^h-\lambda_1} + \frac{b_1\lambda_1^2}{1-\lambda_1\mathrm{e}^h}\right\},$$

where

$$a_1 = \frac{(\mathrm{e}^h-\lambda_1)(1-\lambda_1)}{\lambda_1(\mathrm{e}^h-1)(\lambda_1^N+1)}\left[\frac{1}{2\pi i\omega(2\pi i\omega-1)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}(1-\mathrm{e}^h)}{(\mathrm{e}^h-\mathrm{e}^{2\pi\mathrm{i}\omega h})(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})}\right],$$

$$b_1 = \frac{(1-\mathrm{e}^h\lambda_1)(1-\lambda_1)}{\lambda_1(\mathrm{e}^h-1)(\lambda_1^N+1)}\left[\frac{1}{2\pi i\omega(2\pi i\omega-1)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}(1-\mathrm{e}^h)}{(\mathrm{e}^h-\mathrm{e}^{2\pi\mathrm{i}\omega h})(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})}\right],$$

$$\lambda_1 = \frac{h(\mathrm{e}^{2h}+1)-\mathrm{e}^{2h}+1-(\mathrm{e}^h-1)\sqrt{h^2(\mathrm{e}^h+1)^2+2h(1-\mathrm{e}^{2h})}}{1-\mathrm{e}^{2h}+2h\mathrm{e}^h}$$

$$K = \frac{L}{p_2^{(2)}}\left[\frac{2A_1}{\lambda_1}\cdot\frac{1-\lambda_1\cos(2\pi\omega h)}{\lambda_1^2+1-2\lambda_1\cos(2\pi\omega h)} - \frac{A_1}{\lambda_1} - 4\mathrm{e}^h\cos(2\pi\omega h) + 2C\right],$$

$$L = \frac{1}{(2\pi\omega)^2((2\pi\omega)^2+1)}, \quad A_1 = \frac{2(\lambda_1-1)(\lambda_1\mathrm{e}^h-1)(\mathrm{e}^h-\lambda_1)}{\lambda_1+1},$$

$$p_2^{(2)} = 1-\mathrm{e}^{2h}+2h\mathrm{e}^h, \quad C = \left(1+\mathrm{e}^h\right)^2 - \mathrm{e}^h\frac{\lambda_1^2+1}{\lambda_1}.$$

Remark 4.1. For λ_1 in Corollary 4.1 the following expansion

$$\lambda_1 = \sqrt{3}-2+\frac{2\sqrt{3}-3}{30}h^2 - \frac{3\sqrt{3}-1}{4200}h^4 + O\left(h^6\right)$$

holds.

Corollary 4.2. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h \notin \mathbb{Z}$ *in the space* $W_2^{(3,2)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \sum_{k=1}^{2}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^2}{\mathrm{e}^h-\lambda_k} + \frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h}\right\},$$

$$C_\beta = K\mathrm{e}^{2\pi\mathrm{i}\omega h\beta} + \sum_{k=1}^{2}\left(a_k\lambda_k^{\beta} + b_k\lambda_k^{N-\beta}\right), \qquad \beta = \overline{1, N-1},$$

$$C_N = \frac{K\mathrm{e}^h}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)} + \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h - 1) - \mathrm{e}^h}{2\pi\mathrm{i}\omega(1 - 2\pi\mathrm{i}\omega)(1 - \mathrm{e}^h)} + e^h \sum_{k=1}^{2} \frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^N}{\mathrm{e}^h - \lambda_k} + \frac{b_k\lambda_k^2}{1 - \lambda_k\mathrm{e}^h}\right\},$$

where a_k and b_k ($k = \overline{1,2}$) are defined by the following system of linear equations

$$\sum_{k=1}^{2}\frac{a_k\lambda_k}{(\lambda_k - 1)(\lambda_k - \mathrm{e}^h)} + \sum_{k=1}^{2}\frac{b_k\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k\mathrm{e}^h - 1)} = \frac{1}{2\pi i\omega(1 - 2\pi i\omega)(1 - \mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)(1 - \mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{2}\frac{a_k\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k - \mathrm{e}^h)} + \sum_{k=1}^{2}\frac{b_k\lambda_k}{(\lambda_k - 1)(\lambda_k\mathrm{e}^h - 1)} = \frac{1}{2\pi i\omega(1 - 2\pi i\omega)(1 - \mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)(1 - \mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{2}\frac{a_k\lambda_k}{(\lambda_k - 1)^2} + \sum_{k=1}^{2}\frac{b_k\lambda_k^{N+1}}{(\lambda_k - 1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^2};$$

$$\sum_{k=1}^{2}\frac{a_k\lambda_k^{N+1}}{(\lambda_k - 1)^2} + \sum_{k=1}^{2}\frac{b_k\lambda_k}{(\lambda_k - 1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^2}.$$

Here λ_k, $k = 1, 2$, are roots of the polynomial

$$\mathcal{P}_4(\lambda) = (1 - \mathrm{e}^{2h})(1-\lambda)^4 - 2\big[\lambda(\mathrm{e}^{2h} + 1) - \mathrm{e}^h(\lambda^2 + 1)\big]\big[h(1-\lambda)^2 + \frac{h^3}{6}(1 + 4\lambda + \lambda^2)\big],$$

for which $|\lambda_k| < 1$,

$$K = L\left\{\sum_{k=1}^{2}\left(\frac{2A_k}{\lambda_k}\cdot\frac{1 - \lambda_k\cos(2\pi\omega h)}{\lambda_k^2 + 1 - 2\lambda_k\cos(2\pi\omega h)} - \frac{A_k}{\lambda_k}\right) - 4\mathrm{e}^h\cos(2\pi\omega h) + 2C\right\},$$

$$L = \frac{1}{p_4^{(4)}}\left\{\frac{1}{(2\pi\mathrm{i}\omega)^2 - 1} - \sum_{k=1}^{2}\frac{1}{(2\pi\mathrm{i}\omega)^{2k}}\right\},$$

and A_k and C are defined in Theorem 4.1.

Corollary 4.3. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when $\omega h \notin \mathbb{Z}$ in the* *space $W_2^{(4,3)}[0,1]$, are expressed by formulas*

$$C_0 = \frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^2}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h}\right\},$$

$$C_\beta = \mathrm{e}^{2\pi\mathrm{i}\omega h\beta}K + \sum_{k=1}^{3}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \qquad \beta=\overline{1,N-1},$$

$$C_N = \frac{K\mathrm{e}^{h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \mathrm{e}^h\sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^N}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^2}{1-\lambda_k\mathrm{e}^h}\right\},$$

where a_k and b_k $(k=\overline{1,3})$ are defined by the following system of linear equations

$$\sum_{k=1}^{3}\frac{a_k\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{b_k\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k}{(\lambda_k-1)^2} + \sum_{k=1}^{3}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^2};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k^{N+1}}{(\lambda_k-1)^2} + \sum_{k=1}^{3}\frac{b_k\lambda_k}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^2};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k}{(\lambda_k-1)^3} + \sum_{k=1}^{3}\frac{b_k\lambda_k^{N+2}}{(1-\lambda_k)^3} - \frac{h}{(2\pi\mathrm{i}\omega h)^3} = -\frac{h}{2(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^3};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k^2(1-\lambda_k^N)}{(1-\lambda_k)^3} + \sum_{k=1}^{3}\frac{b_k\lambda_k(\lambda_k^N-1)}{(\lambda_k-1)^3} = \frac{(1-2h)K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{2h^2(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^2}.$$

Here λ_k, $k=1,2,3$ are the roots of the polynomial

$$\mathcal{P}_6(\lambda) = (1-\mathrm{e}^{2h})(1-\lambda)^6 - 2[\lambda(\mathrm{e}^{2h}+1)-\mathrm{e}^h(\lambda^2+1)]\Big[h(1-\lambda)^4 + \frac{h^3}{6}(1-\lambda)^2(1+4\lambda+\lambda^2) + \frac{h^5}{120}(1+26\lambda+66\lambda^2+26\lambda^3+\lambda^4)\Big],$$

for which $|\lambda_k| < 1$,

$$K = \frac{L}{p_6^{(6)}} \left\{ \sum_{k=1}^{3} \left(\frac{2A_k}{\lambda_k} \cdot \frac{1-\lambda_k \cos(2\pi\omega h)}{\lambda_k^2 + 1 - 2\lambda_k \cos(2\pi\omega h)} - \frac{A_k}{\lambda_k} \right) - 4\mathrm{e}^h \cos(2\pi\omega h) + 2C \right\},$$

$$L = \frac{1}{(2\pi\mathrm{i}\omega)^2 - 1} - \sum_{k=1}^{3} \frac{1}{(2\pi\mathrm{i}\omega)^{2k}},$$

$p_6^{(6)}$ *is the leading coefficient of the polynomial* $\mathcal{P}_6(\lambda)$, *and* A_k *and* C *are defined in Theorem* 4.1 *for* $m = 4$.

Now from Theorem 4.5 for $m = 2$, $m = 3$ and $m = 4$, we have the following corollaries:

Corollary 4.4. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h \in \mathbb{Z}$ *and* $\omega \neq 0$ *in the space* $W_2^{(2,1)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h) - 1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{1}{1-\lambda_1} \left\{ \frac{a_1\lambda_1^2}{\mathrm{e}^h - \lambda_1} + \frac{b_1\lambda_1^N}{1-\lambda_1\mathrm{e}^h} \right\},$$

$$C_\beta = a_1\lambda_1^\beta + b_1\lambda_1^{N-\beta}, \quad \beta = \overline{1, N-1},$$

$$C_N = \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h - 1) - \mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{\mathrm{e}^h}{1-\lambda_1} \left\{ \frac{a_1\lambda_1^N}{\mathrm{e}^h - \lambda_1} + \frac{b_1\lambda_1^2}{1-\lambda_1\mathrm{e}^h} \right\},$$

where

$$a_1 = \frac{(\mathrm{e}^h - \lambda_1)(1-\lambda_1)}{2\pi\mathrm{i}\omega(2\pi\mathrm{i}\omega - 1)\lambda_1(\mathrm{e}^h - 1)(\lambda_1^N + 1)},$$

$$b_1 = \frac{(1-\mathrm{e}^h\lambda_1)(1-\lambda_1)}{2\pi\mathrm{i}\omega(2\pi\mathrm{i}\omega - 1)\lambda_1(\mathrm{e}^h - 1)(\lambda_1^N + 1)},$$

and

$$\lambda_1 = \frac{h(\mathrm{e}^{2h}+1) - \mathrm{e}^{2h} + 1 - (\mathrm{e}^h - 1)\sqrt{h^2(\mathrm{e}^h+1)^2 + 2h(1-\mathrm{e}^{2h})}}{1 - \mathrm{e}^{2h} + 2h\mathrm{e}^h}.$$

Corollary 4.5. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h \in \mathbb{Z}$ *and* $\omega \neq 0$ *in the space* $W_2^{(3,2)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h) - 1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \sum_{k=1}^{2} \frac{1}{1-\lambda_k} \left\{ \frac{a_k\lambda_k^2}{\mathrm{e}^h - \lambda_k} + \frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h} \right\},$$

$$C_\beta = \sum_{k=1}^{2} \left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta} \right), \quad \beta = \overline{1, N-1},$$

$$C_N = \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h - 1) - \mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \mathrm{e}^h \sum_{k=1}^{2} \frac{1}{1-\lambda_k} \left\{ \frac{a_k\lambda_k^N}{\mathrm{e}^h - \lambda_k} + \frac{b_k\lambda_k^2}{1-\lambda_k\mathrm{e}^h} \right\},$$

where a_k and b_k $(k=\overline{1,2})$ are defined by the following system of linear equations

$$\sum_{k=1}^{2} a_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)}+\sum_{k=1}^{2} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}=\frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{2} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)}+\sum_{k=1}^{2} b_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}=\frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{2} a_k \frac{\lambda_k}{(\lambda_k-1)^2}+\sum_{k=1}^{2} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2}=\frac{h}{(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{2} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2}+\sum_{k=1}^{2} b_k \frac{\lambda_k}{(\lambda_k-1)^2}=\frac{h}{(2\pi\mathrm{i}\omega h)^2}.$$

Here λ_k, $k=1,2$, are roots of the polynomial

$$\mathcal{P}_4(\lambda)=(1-\mathrm{e}^{2h})(1-\lambda)^4-2\big[\lambda(\mathrm{e}^{2h}+1)-\mathrm{e}^h(\lambda^2+1)\big]\Big[h(1-\lambda)^2+\frac{h^3}{6}(1+4\lambda+\lambda^2)\Big]$$

for which $|\lambda_k|<1$.

Corollary 4.6. *The coefficients of optimal quadrature formulas of the form* (1.1)*, with the error functional* (1.2)*, and with equal spaced nodes when $\omega h\in\mathbb{Z}$ and $\omega\neq 0$ in the space $W_2^{(4,3)}[0,1]$, are expressed by formulas*

$$C_0=\frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)}+\sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^2}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h}\right\},$$

$$C_\beta=\sum_{k=1}^{3}\Big(a_k\lambda_k^\beta+b_k\lambda_k^{N-\beta}\Big),\quad \beta=\overline{1,N-1},$$

$$C_N=\frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)}+\mathrm{e}^h\sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^N}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^2}{1-\lambda_k\mathrm{e}^h}\right\},$$

where a_k and b_k $(k=\overline{1,3})$ are defined by the following system of linear equations

$$\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)}+\sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}=\frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)}+\sum_{k=1}^{3} b_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}=\frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k-1)^2}+\sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2}=\frac{h}{(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2}+\sum_{k=1}^{3} b_k \frac{\lambda_k}{(\lambda_k-1)^2}=\frac{h}{(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k-1)^3}+\sum_{k=1}^{3} b_k \frac{\lambda_k^{N+2}}{(1-\lambda_k)^3}=\frac{h}{(2\pi\mathrm{i}\omega h)^3}-\frac{h}{2(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k^2 - \lambda_k^{N+2}}{(1-\lambda_k)^3} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1} - \lambda_k}{(\lambda_k - 1)^3} = 0.$$

Here λ_k, $k = 1, 2, 3$, *are roots of the polynomial*

$$\mathcal{P}_6(\lambda) = (1 - \mathrm{e}^{2h})(1-\lambda)^6 - 2\big[\lambda(\mathrm{e}^{2h}+1) - \mathrm{e}^h(\lambda^2+1)\big]\Big[h(1-\lambda)^4 + \frac{h^3}{6}(1-\lambda)^2(1+4\lambda+\lambda^2) + \frac{h^5}{120}(1+26\lambda+66\lambda^2+26\lambda^3+\lambda^4)\Big],$$

for which $|\lambda_k| < 1$.

5. Coefficients and norm of the error functional of optimal quadrature formulas (1.1) in $W_2^{(1,0)}[0,1]$

Here we get the explicit expressions for coefficients and calculate the square of the norm of the error functional (1.2), of the optimal quadrature formula (1.1), on the space $W_2^{(1,0)}[0,1]$.

For $m = 1$ the system (4.1)–(4.3) takes the form

$$\sum_{\gamma=0}^{N} C_\gamma G_1(h\beta - h\gamma) + d\,\mathrm{e}^{-h\beta} = f_1(h\beta), \quad \beta = 0, 1, \ldots, N, \tag{5.1}$$

$$\sum_{\beta=0}^{N} C_\beta \mathrm{e}^{-h\beta} = \frac{e^{-1}-1}{2\pi\mathrm{i}\omega - 1}, \tag{5.2}$$

where

$$G_1(x) = \frac{\operatorname{sign}(x)}{4}\big(\mathrm{e}^x - \mathrm{e}^{-x}\big),$$

$$f_1(h\beta) = -\frac{\mathrm{e}^{h\beta}(\mathrm{e}^{2\pi\mathrm{i}\omega-1}+1)}{4(2\pi\mathrm{i}\omega-1)} + \frac{\mathrm{e}^{-h\beta}(\mathrm{e}^{2\pi\mathrm{i}\omega+1}+1)}{4(2\pi\mathrm{i}\omega+1)} + \frac{\mathrm{e}^{2\pi\mathrm{i}\omega h\beta}}{(2\pi\mathrm{i}\omega+1)(2\pi i\omega-1)}, \tag{5.3}$$

and C_β $(\beta = 0, 1, \ldots, N)$ and d are unknowns.

In this case Problem 4 is expressed as follows:

Problem 5. Find the solution of the equation

$$D_1(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1], \tag{5.4}$$

having the form

$$u(h\beta) = \begin{cases} -\dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + a^-\mathrm{e}^{-h\beta}, & \beta < 0, \\ f_1(h\beta), & 0 \le \beta \le N, \\ \dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + a^+\mathrm{e}^{-h\beta}, & \beta > N, \end{cases} \tag{5.5}$$

where $f_1(h\beta)$ is defined by (5.3), a^- and a^+ are unknowns.

For $m=1$, from Theorem 4.1 for $D_1(h\beta)$, we obtain

$$D_1(h\beta) = \frac{1}{1-\mathrm{e}^{2h}} \begin{cases} 0, & |\beta| \geq 2, \\ -2\mathrm{e}^{h}, & |\beta| = 1, \\ 2(1+\mathrm{e}^{2h}), & \beta = 0. \end{cases} \tag{5.6}$$

Now, taking into account (5.6), for the convolution $C_\beta = D_1(h\beta) * u(h\beta)$, we have

$$D_1(h\beta) * u(h\beta) = D_1(h)(u(h\beta - h) + u(h\beta + h)) + D_1(0)u(h\beta).$$

Hence, keeping in mind (5.4) for $\beta = -1$ and $\beta = N+1$, we get the following system

$$D_1(h)(u(-2h) + u(0)) + D_1(0)u(-h) = 0,$$
$$D_1(h)(u(Nh) + u(Nh+2h)) + D_1(0)u(Nh+h) = 0.$$

Whence, taking into account (5.5), (5.6) for a^- and a^+, we have

$$a^- = \frac{\mathrm{e}-1}{4(2\pi \mathrm{i}\omega + 1)}, \quad a^+ = -\frac{\mathrm{e}-1}{4(2\pi \mathrm{i}\omega + 1)}. \tag{5.7}$$

Then, using (5.7), from (4.16) and (4.17) we obtain

$$d = \frac{1}{2}(a^- + a^+) = 0, \quad D = \frac{1}{2}(a^- - a^+) = \frac{\mathrm{e}-1}{4(2\pi \mathrm{i}\omega + 1)}. \tag{5.8}$$

Substituting (5.7) into (5.5) for $u(h\beta)$ we have the following expression

$$u(h\beta) = \begin{cases} -\frac{\mathrm{e}^{h\beta}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi \mathrm{i}\omega - 1} + \frac{\mathrm{e}^{-h\beta}}{4}\frac{\mathrm{e}-1}{2\pi \mathrm{i}\omega + 1}, & \beta < 0, \\ f_1(h\beta), & 0 \leq \beta \leq N, \\ \frac{\mathrm{e}^{h\beta}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi \mathrm{i}\omega - 1} - \frac{\mathrm{e}^{-h\beta}}{4}\frac{\mathrm{e}-1}{2\pi \mathrm{i}\omega + 1}, & \beta > N. \end{cases} \tag{5.9}$$

Using (5.9) and (5.6), taking into account (5.3), by direct calculations for optimal coefficients $C_\beta = D_1(h\beta) * u(h\beta)$ $(\beta = 0, 1, \ldots, N)$ we obtain the following result:

Theorem 5.1. *Coefficients of the optimal quadrature formulas of the form* (1.1) *in the sense of Sard in the space* $W_2^{(1,0)}[0,1]$ *have the form*

$$C_0 = \frac{1 + \mathrm{e}^{2h} - 2\mathrm{e}^{2\pi \mathrm{i}\omega h + h} - 2\pi \mathrm{i}\omega(1 - \mathrm{e}^{2h})}{(\mathrm{e}^{2h} - 1)(4\pi^2\omega^2 + 1)},$$

$$C_\beta = \frac{2(1 + \mathrm{e}^{2h} - 2\mathrm{e}^{h}\cos 2\pi\omega h)}{(\mathrm{e}^{2h} - 1)(4\pi^2\omega^2 + 1)}\mathrm{e}^{2\pi \mathrm{i}\omega h\beta}, \quad \beta = 1, 2, \ldots, N-1,$$

$$C_N = \frac{1 + \mathrm{e}^{2h} - 2\mathrm{e}^{h - 2\pi \mathrm{i}\omega h} + 2\pi \mathrm{i}\omega(1 - \mathrm{e}^{2h})}{(\mathrm{e}^{2h} - 1)(4\pi^2\omega^2 + 1)}.$$

Note that, in Theorem 5.1, the formulas for the optimal coefficients C_β are decomposed into two parts – real and imaginary parts. Therefore, from Theorem 5.1 we get the following results:

Corollary 5.1. *Coefficients for the optimal quadrature formulas of the form*

$$\int_0^1 \varphi(x)\cos 2\pi\omega x\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta^R \varphi(h\beta)$$

in the sense of Sard in the space $W_2^{(1,0)}[0,1]$ *have the form*

$$C_0^R = \frac{1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},\quad C_N^R = \frac{1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},$$
$$C_\beta^R = \frac{2(1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h)}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)}\cos 2\pi\omega h\beta,\quad \beta=1,2,\ldots,N-1.$$

Corollary 5.2. *Coefficients for the optimal quadrature formulas of the form*

$$\int_0^1 \varphi(x)\sin 2\pi\omega x\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta^I \varphi(h\beta)$$

in the sense of Sard in the space $W_2^{(1,0)}[0,1]$ *have the form*

$$C_0^I = \frac{2\pi\omega(\mathrm{e}^{2h}-1)-2\mathrm{e}^h\sin 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},\quad C_N^I = -\frac{2\pi\omega(\mathrm{e}^{2h}-1)-2\mathrm{e}^h\sin 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},$$
$$C_\beta^I = \frac{2(1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h)}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)}\sin 2\pi\omega h\beta,\quad \beta=1,2,\ldots,N-1.$$

Remark 5.1. When $\omega=0$, Theorem 5.1 reduces to Theorem 4.4 from [34].

Theorem 5.2. *The square of the norm of the error functional* (1.2)*, of the optimal quadrature formula* (1.1)*, on the space* $W_2^{(1,0)}[0,1]$*, has the form*

$$\|\mathring{\ell}\|^2 = \frac{1}{(4\pi^2\omega^2+1)^2}\left(4\pi^2\omega^2+1-\frac{2(\mathrm{e}^{2h}+1-2\mathrm{e}^h\cos 2\pi\omega h)}{h(\mathrm{e}^{2h}-1)}\right). \tag{5.10}$$

Proof. For $m=1$ we rewrite the equality (2.11) in the following form

$$\begin{aligned}\|\ell\|^2 = -\Bigg[&\sum_{\beta=0}^{N} C_\beta^R\left(\sum_{\gamma=0}^{N} C_\gamma^R G_1(h\beta-h\gamma) - \int_0^1 \cos 2\pi\omega x\; G_1(x-h\beta)\,\mathrm{d}x\right)\\ &+\sum_{\beta=0}^{N} C_\beta^I\left(\sum_{\gamma=0}^{N} C_\gamma^I G_1(h\beta-h\gamma) - \int_0^1 \sin 2\pi\omega x\; G_1(x-h\beta)\,\mathrm{d}x\right)\\ &-\sum_{\beta=0}^{N} C_\beta^R\int_0^1 \cos 2\pi\omega x\; G_1(x-h\beta)\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^I\int_0^1 \sin 2\pi\omega x\; G_1(x-h\beta)\,\mathrm{d}x\\ &+\int_0^1\int_0^1 \cos[2\pi\omega(x-y)]G_1(x-y)\,\mathrm{d}x\,\mathrm{d}y\Bigg],\end{aligned}$$

where $G_1(x)$ is defined by (5.3).

Taking into account (5.8), from (5.1) we get

$$\sum_{\gamma=0}^{N} C_\gamma^R G_1(h\beta - h\gamma) - \int_0^1 \cos 2\pi\omega x \; G_1(x - h\beta)\,\mathrm{d}x = 0$$

and

$$\sum_{\gamma=0}^{N} C_\gamma^I G_1(h\beta - h\gamma) - \int_0^1 \sin 2\pi\omega x \; G_1(x - h\beta)\,\mathrm{d}x = 0.$$

Then, using the last two equalities, for $\|\ell\|^2$ we obtain

$$\begin{aligned}\|\ell\|^2 = &\sum_{\beta=0}^{N} C_\beta^R \int_0^1 \cos 2\pi\omega x \; G_1(x - h\beta)\,\mathrm{d}x + \sum_{\beta=0}^{N} C_\beta^I \int_0^1 \sin 2\pi\omega x \; G_1(x - h\beta)\,\mathrm{d}x \\ &- \int_0^1 \int_0^1 \cos[2\pi\omega(x-y)] G_1(x-y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Finally, calculating these integrals and using Corollaries 5.1 and 5.2, after some simplifications, we get (5.10). □

Remark 5.2. When $\omega = 0$, Theorem 5.2 reduces to Theorem 5.1 from [34].

6. Numerical results

In this section we give some numerical results of the upper bounds for the errors in the optimal quadrature formulas of the form (1.1), as well their analysis in the cases $m = 1$ and $m = 2$.

According to the Cauchy-Schwarz inequality, in the space $W_2^{(m,m-1)}[0,1]$ for the absolute value of the difference (1.4) we get

$$|(\mathring{\ell}, \varphi)| \leq \|\varphi\| \cdot \|\mathring{\ell}\|,$$

where $\|\mathring{\ell}\|$ is the norm of *the optimal error functional* which corresponds to the optimal quadrature formulas (1.1).

1° *First we consider the case* $m = 1$.

Using Theorem 5.2, for $\|\mathring{\ell}|W_2^{(1,0)*}[0,1]\|$, when $N = 1, 10, 10^2, 10^3, 10^4$ and $\omega = 1, 11, 101, 1001, 10001$, we get numerical results which are presented in Table 1. Numbers in parenthesis indicate the decimal exponents. From the first column of this table we see that order of convergence of our optimal quadrature formula is $O(N^{-1})$ and from the first row of Table 1 it is clear that the quantity $\|\mathring{\ell}\|$ converges as $O(|\omega|^{-1})$. From other columns and rows of Table 1 we conclude that order of convergence of our optimal quadrature formula in the case $m = 1$ is $O((N+|\omega|)^{-1})$.

Table 1. The numerical results for $\|\mathring{\ell}\|$ in the case $m = 1$ when $N = 10^k$, $k = 0, 1, 2, 3, 4$, and $\omega = 1, 11, 101, 1001, 10001$.

N	$\omega = 1$	$\omega = 11$	$\omega = 101$	$\omega = 1001$	$\omega = 10001$
1	1.5537(−1)	1.44657(−2)	1.5757878(−3)	1.589959433(−4)	1.5913902915020(−5)
10	2.8664(−2)	1.44078(−2)	1.5757130(−3)	1.589958665(−4)	1.5913902838027(−5)
10^2	2.8865(−3)	2.86386(−3)	1.5757104(−3)	1.589958638(−4)	1.5913902835341(−5)
10^3	2.8867(−4)	2.88652(−4)	2.8674495(−4)	1.589958638(−4)	1.5913902835314(−5)
10^4	2.8868(−5)	2.88675(−5)	2.8865576(−5)	2.867790858(−5)	1.5913902835314(−5)

Now, as an integrand we take the function $\varphi(x) = \mathrm{e}^{2x}$. Then for the actual error $R_N(\omega)$ of the optimal quadrature formula (1.1) we have the following estimate

$$
\begin{aligned}
R_N(\omega) = |(\mathring{\ell}, \mathrm{e}^{2x})| &= \left| \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} \mathrm{e}^{2x}\, \mathrm{d}x - \sum_{\beta=0}^{N} C_\beta \mathrm{e}^{2h\beta} \right| \\
&\le \|\mathrm{e}^{2x} | W_2^{(1,0)}[0,1]\| \cdot \|\mathring{\ell}\, | W_2^{(1,0)*}[0,1]\| \\
&= \frac{3}{2}\sqrt{\mathrm{e}^4 - 1}\; \|\mathring{\ell}\, | W_2^{(1,0)*}[0,1]\|.
\end{aligned}
$$

For the same values of N and ω, using formulas for the optimal coefficients C_β from Theorem 5.1 and formula (5.10), we get the numerical values for the actual error $R_N(\omega)$ and for the bound $B_N(\omega)$ on the right hand side in the previous inequality. These results are presented in Table 2.

Table 2. Numerical values of $R_N(\omega) = |(\mathring{\ell}, \mathrm{e}^{2x})|$ and $B_N(\omega) = \|\mathrm{e}^{2x}\| \|\mathring{\ell}\|$ in the case $m = 1$ for some selected values of N and ω.

N	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$		$\omega = 10001$	
	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$
1	2.1(−1)	1.7(0)	1.9(−3)	1.6(−1)	2.2(−5)	1.7(−2)	2.3(−7)	1.7(−3)	2.3(−9)	1.7(−4)
10	2.4(−3)	3.1(−1)	5.3(−4)	1.6(−1)	6.9(−6)	1.7(−2)	7.1(−8)	1.7(−3)	7.1(−10)	1.7(−4)
10^2	2.4(−5)	3.2(−2)	2.3(−6)	3.1(−2)	7.1(−6)	1.7(−2)	7.3(−8)	1.7(−3)	7.4(−10)	1.7(−4)
10^3	2.4(−7)	3.2(−3)	2.3(−8)	3.2(−3)	2.5(−9)	3.1(−3)	7.3(−8)	1.7(−3)	7.4(−10)	1.7(−4)
10^4	2.4(−9)	3.2(−4)	2.3(−10)	3.2(−4)	2.5(−11)	3.2(−4)	2.6(−12)	3.1(−4)	7.4(−10)	1.7(−4)

These numerical results confirm our theoretical results obtained in the previous sections.

2° *Now we consider the case* $m = 2$.

From (2.11), taking into account (2.7), after some calculations for the norm of the error functional of the optimal quadrature formula (1.1) we get the following expression

$$
\begin{aligned}
\|\mathring{\ell}\|^2 = &\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} (C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I) \frac{\operatorname{sgn}(h\beta - h\gamma)}{2} \big[\sinh(h\beta - h\gamma) - h\beta + h\gamma\big] \\
&+ \frac{4\pi^2\omega^2 + 2}{4\pi^2\omega^2(4\pi^2\omega^2 + 1)} - \sum_{\beta=0}^{N} C_\beta^R \left[\frac{(\mathrm{e}^{-1} + 1)\mathrm{e}^{h\beta}}{2(4\pi^2\omega^2 + 1)} + \frac{\cos(2\pi\omega h\beta)}{2\pi^2\omega^2(4\pi^2\omega^2 + 1)} \right] \\
&- \sum_{\beta=0}^{N} C_\beta^I \left[\frac{\pi\omega(\mathrm{e}^{-1} + 1)\mathrm{e}^{h\beta}}{4\pi^2\omega^2 + 1} + \frac{\sin(2\pi\omega h\beta)}{2\pi^2\omega^2(4\pi^2\omega^2 + 1)} - \frac{h\beta}{\pi\omega} \right]. \qquad (6.1)
\end{aligned}
$$

Hence, using the formulas for the optimal coefficients C_β which are given in Corollary 4.4 when $N = 1$ and $\omega = 1, 11, 101, 1001, 10001$, we get the results which are presented in the first row of Table 3. Using the formulas of the optimal coefficients C_β, which are given in Corollary 4.1, when $N = 10, 100, 1000, 10000$ and $\omega = 1, 11, 101, 1001, 10001$, we obtain the numerical results presented in other rows of Table 3. From the numerical results of the first column of Table 3 we see that order of convergence of the optimal quadrature formula (1.1) is $O(N^{-2})$. And from the results presented in the first row of Table 3 we conclude that order of convergence is $O(|\omega|^{-2})$. From the results which are given in other columns and rows of Table 3 we have that order of our optimal quadrature formula in this case is $O((N + |\omega|)^{-2})$.

Table 3. The numerical results for $\|\mathring{\ell}\|$ in the case $m = 2$ when $N = 10^k$, $k = 0, 1, 2, 3, 4$, and $\omega = 1, 11, 101, 1001, 10001$.

N	$\omega = 1$	$\omega = 11$	$\omega = 101$	$\omega = 1001$	$\omega = 10001$
1	3.5377(−2)	2.96022(−4)	3.5116561(−6)	3.575090999(−8)	3.581528459606(−10)
10	4.3982(−4)	2.15172(−4)	2.5538002(−6)	2.599946583(−8)	2.604628302387(−10)
10^2	3.7819(−6)	3.99301(−6)	2.4902722(−6)	2.535258220(−8)	2.539823327481(−10)
10^3	3.7322(−8)	3.73427(−8)	3.9122983(−8)	2.528700745(−8)	2.533254031754(−10)
10^4	3.7273(−10)	3.72734(−10)	3.7291046(−10)	3.904339277(−10)	2.532596167387(−10)

Now we consider the function $\varphi(x) = x^2$ as an integrand. Then for the error of the optimal quadrature formula (1.1) we have

$$|(\mathring{\ell}, x^2)| = \left| \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} x^2 \, \mathrm{d}x - \sum_{\beta=0}^{N} C_\beta (h\beta)^2 \right| \leq \|x^2 | W_2^{(2,1)}[0,1]\| \cdot \|\mathring{\ell} \,| W_2^{(2,1)*}[0,1]\|$$
$$\leq \frac{2}{3}\sqrt{21} \; \|\mathring{\ell} \,| W_2^{(2,1)*}[0,1]\|.$$

Using formulas for the optimal coefficients C_β which are given in Corollary 4.4 and formula (6.1) for the left and the right hand sides of the last inequality, respectively, when $N = 1$ and $\omega = 1, 11, 101, 1001, 10001$, we get the numerical results given in the first row of Table 4. The numerical results which are presented in other rows of Table 4 are obtained by using Corollary 4.1 and formula (6.1).

Table 4. Numerical values of $R_N(\omega) = |(\mathring{\ell}, x^2)|$ and $B_N(\omega) = \|x^2\|\|\mathring{\ell}\|$ in the case $m = 2$ for some selected values of N and ω.

N	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$		$\omega = 10001$	
	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$
1	7.5(−2)	1.1(−1)	6.3(−4)	9.0(−4)	7.4(−6)	1.1(−5)	7.6(−8)	1.1(−7)	7.6(−10)	1.1(−9)
10	1.5(−4)	1.3(−3)	3.6(−5)	6.6(−4)	4.3(−7)	7.8(−6)	4.3(−9)	7.9(−8)	4.3(−11)	8.0(−10)
10^2	1.4(−7)	1.2(−5)	1.5(−7)	1.2(−5)	4.4(−8)	7.6(−6)	4.4(−10)	7.7(−8)	4.4(−12)	7.8(−10)
10^3	1.4(−10)	1.1(−7)	1.4(−10)	1.1(−7)	1.5(−10)	1.2(−7)	4.5(−11)	7.7(−8)	4.4(−13)	7.7(−10)
10^4	1.4(−13)	1.1(−9)	1.4(−13)	1.1(−9)	1.4(−13)	1.1(−9)	1.5(−13)	1.2(−9)	4.5(−14)	7.7(−10)

Finally, for the function $x \mapsto \varphi(x) = x^2\mathrm{e}^{-x}$, we consider an example of calculating Fourier coefficients $\int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x}\varphi(x)\,\mathrm{d}x$ using the optimal quadrature formula in the space $W_2^{(2,1)}$. The real part of this integrand, $\cos(2\pi\omega x)\varphi(x)$, for $\omega = 80$ is presented in Figure 1 (left).

The exact value of the corresponding Fourier integral can be obtained in an

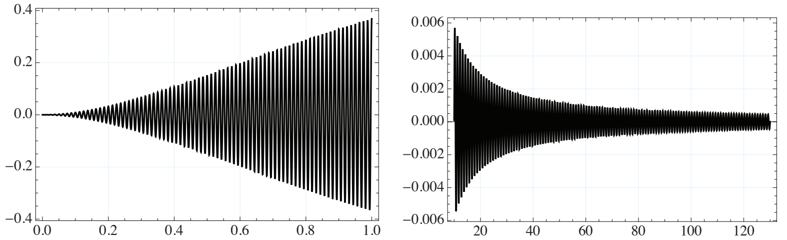

Figure 1. Graphics of the integrand $x \mapsto \cos(2\pi\omega x)\varphi(x)$ for $\omega = 80$ (left) and $\omega \mapsto \Re\{I(\omega)\}$ for $\omega \in [10, 130]$ (right).

analytic form,

$$I(\omega) = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\varphi(x)\,\mathrm{d}x = \frac{2\mathrm{e} + \mathrm{e}^{2\mathrm{i}\omega\pi}(-5 + 8\mathrm{i}\omega\pi + 4\omega^2\pi^2)}{\mathrm{e}(1 - 2\mathrm{i}\omega\pi)^3},$$

and therefore, we can calculate the actual relative errors

$$\mathrm{err}_N(\omega) = \Big|\frac{Q_N(\omega) - I(\omega)}{I(\omega)}\Big|$$

in our optimal quadrature sums $Q_N(\omega) = \sum\limits_{\beta=0}^{N} C_\beta \varphi(h\beta)$.

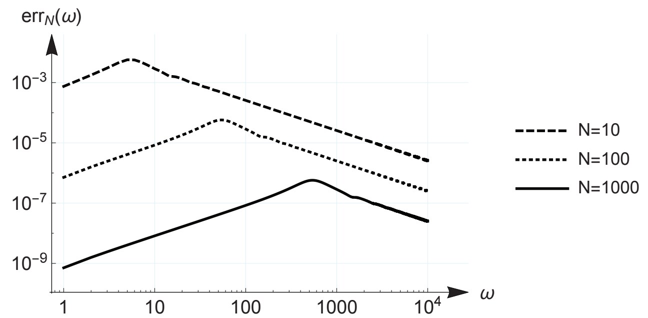

Figure 2. Relative errors $\omega \mapsto \mathrm{err}_N(\omega)$ for $N = 10, 100, 1000$.

The real part of the integral $I(\omega)$ is displayed in Figure 1 (right) for $\omega \in [10, 130]$.

Graphics of $\omega \mapsto \mathrm{err}_N(\omega)$ for $N = 10, 100, 1000$, when ω runs over $[1, 10^4]$, are presented in Figure 2 in log-log scale.

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Performance of SIM-MDPSK FSO Systems With Hardware Imperfections

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***Abstract*— This paper studies the error performance of free-space optical (FSO) systems, employing subcarrier intensity modulation (SIM) with M-ary differential phase-shift keying (MDPSK). Novel analytical expressions for the symbol error probability are derived, based on the Fourier series approach. The irradiance fluctuations of the received optical signal are modeled by considering both Gamma-Gamma atmospheric turbulence and pointing errors. In addition, hardware imperfections of DPSK demodulator, as the phase noise of local oscillator at the receiver, are considered. It is illustrated that the phase noise significantly degrades the system performance, especially when the optical signal transmission is impaired by weak atmospheric turbulence and weak pointing errors effect. Furthermore, the phase noise results in an unrecoverable error-rate floor, which is an important limiting factor for SIM-DPSK FSO systems.**

***Index Terms*— Atmospheric turbulence, free-space optics (FSO), Gamma-Gamma distribution, differential phase-shift keying (DPSK), phase noise, subcarrier intensity modulation (SIM), symbol error probability (SEP).**

I. Introduction

BESIDES the main advantages, as high data rate, wide bandwidth and license-free transmission, free-space optical (FSO) systems are also characterized by low-power and low-cost transmission, as well as easy and simple installation. Intensity-modulation/direct detection (IM/DD) with on-off keying (OOK) is usually employed in commercial FSO systems. However, in order to improve the system performance subcarrier intensity modulation (SIM) was proposed, where the radio-frequency (RF) subcarrier signal is firstly premodulated by the data sequence bearing information, and then it is used to modulate the intensity of the laser source [1]–[4].

Several well-known modulation techniques from the field of RF communications, were used to modulate a subcarrier signal in FSO systems. The SIM based FSO system employing quadrature amplitude modulation (QAM) were analysed in [5]–[9], while SIM with M-ary phase-shift keying (MPSK) was investigated in [4] and [9]–[14]. Furthermore, practical wireless communication systems also employ differential phase-shift keying (DPSK), which does not require the carrier phase estimation at the receiver. The performance of FSO systems with coherent detection and binary DPSK (BDPSK) was analyzed in [3] and [15]–[19], while the case of SIM-BDPSK was investigated in [20]–[26]. Furthermore, in order to increase the capacity (or the system throughput), an FSO system based on SIM and higher-order DPSK modulation was also proposed and analyzed in [12], [13], and [27]. Specifically, an expression in integral form for the bit error rate (BER) was presented in [12] and [13], while [27] compares the performance of different modulation formats, including BDPSK and quaternary DPSK (QDPSK), when space diversity is used at the reception.

The FSO system performance can be notably degraded due to the hardware imperfections. For example, the effects of the imperfect reference carrier signal phase recovery on error performance of SIM-MPSK FSO systems were examined in [28], considering weak atmospheric turbulence modeled by log-normal distribution. The effect of noisy reference signal extraction on error rate degradation of coherent BPSK FSO system in strong turbulence conditions was examined in [29]. Although the DPSK receiver does not require a carrier phase estimation, the hardware imperfections of the DPSK demodulator can seriously degrade the system performance. After optical-to-electrical signal conversion in SIM-DPSK receiver, it is necessary to down-convert the received DPSK signal. In other words, a local oscillator, used in DPSK receiver for down-conversion, generates signal, which is not ideal, in the sense that phase of this signal is a random process fluctuating over time. These fluctuations, which are in the same frequency band with the useful signal, have the influence on the detection process. This undesired phase is known as a *phase noise* [30], [31].

Scanning the open literature, to the best of the authors' knowledge, the effect of hardware imperfections as the phase noise on the performance of the FSO system employing SIM-MDPSK, has not been investigated so far. In this paper, we derive novel analytical expressions for the symbol error probability (SEP) of the SIM-MDPSK based FSO system, when hardware imperfections are considered, using the Fourier series method (FSM) [32]–[35]. The impact of hardware imperfections is represented through the phase noise,

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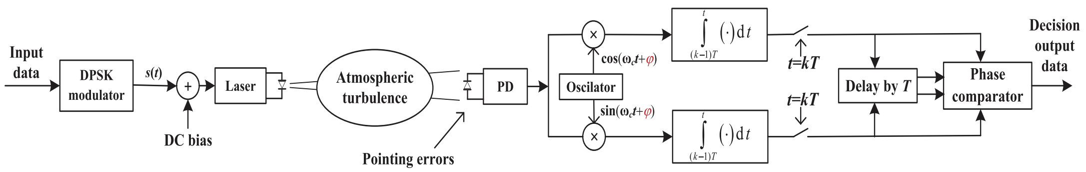

Fig. 1. Block diagram of a SIM-MDPSK FSO system.

which is modeled by the Tikhonov distribution [36]–[38], and is generated by the local oscillator of DPSK demodulator [30], [31], [36], [39], [40]. The intensity fluctuations of the received optical signal are assumed to originate from the combined effect of the Gamma-Gamma atmospheric turbulence and the pointing errors [16], [41]–[45]. The derived SEP expression is given in the convergent series form, whose upper bound for the truncation error is estimated. Furthermore, the derived expressions are simplified, when the pointing errors effect can be neglected. Finally, numerical results are presented and validated through Monte Carlo simulations.

The rest of the paper is organized as follows. Section II describes the system and channel model, while the error analysis is provided in Section III. Numerical results with discussion are presented in Section IV and some concluding remarks are given in Section V.

II. System and Channel Model

The block diagram of the SIM-MDPSK FSO system is presented in Fig. 1. The information data are differently encoded and PSK is applied in an RF domain [46, p. 333]. DC bias is added to avoid clipping and distortion, and resulting signal modulates the laser output, by using SIM. The radiated optical power is given by

$$P\left(t\right) = P_t\left(1 + ms\left(t\right)\right), \tag{1}$$

where P_t represents the transmitted optical power and m denotes the modulation index ($0 < m \leq 1$). The optical transmission via free space is influenced by atmospheric turbulence and pointing errors. At the receiver, direct detection is performed, DC bias is removed and an optical-to-electrical conversion is applied via a PIN photodetector. The electrical signal at the input of DPSK demodulator is expressed as

$$r_e\left(t\right) = I\eta P_t ms\left(t\right) + n(t), \tag{2}$$

where I is a random variable (RV), which follows Gamma-Gamma distribution and represents atmospheric turbulence and pointing errors, η denotes an optical-to-electrical conversion coefficient and $n(t)$ is an additive white Gaussian noise (AWGN), with zero mean and variance, σ_n^2. Finally, the electrical signal, $r_e\left(t\right)$, is recovered by the DPSK demodulator, presented in Fig. 1, assuming that hardware imperfections exist.

A. Modeling the Combined Effect of Atmospheric Turbulence and Pointing Errors

The well-known Gamma-Gamma distribution is used for describing the effect of atmospheric turbulence [41], while the pointing errors effect is described by the distribution which assumes the radial displacement of laser beam at receiver experiences Rayleigh distribution, with the jitter variance σ_s^2 [42, eq. (11)].

Based on (2), the instantaneous SNR is defined as $\gamma = I^2\eta^2 {P_t}^2 m^2/(2\sigma_n^2)$. The probability density function (PDF) of γ is [5]

$$f_\gamma\left(\gamma\right) = \frac{\xi^2}{2\Gamma(\alpha)\Gamma(\beta)\gamma} G_{1,3}^{3,0}\left(\alpha\beta\kappa\sqrt{\frac{\gamma}{\mu}}\,\middle|\, {\xi^2+1 \atop \xi^2,\,\alpha,\,\beta}\right), \tag{3}$$

where $G_{p,q}^{m,n}\left(\cdot\right)$ is the Meijer's G-function [47, (9.301)], and μ represents the average electrical SNR per symbol. The relation between μ and the average electrical SNR per bit, μ_b, is $\mu = \mu_b \log_2 M$. The average electrical SNR per bit is defined as $\mu_b = \eta^2 {P_t}^2 m^2 \kappa^2 A_0^2 I_l^2/(2\sigma_n^2)$, with $\kappa = \xi^2 \big/ (\xi^2+1)$ [5]. The atmospheric turbulence parameters are denoted by α and β, while ξ and A_0 represent the pointing errors parameters.

Assuming Gaussian plane wave propagation and zero inner scale, the parameters α and β are defined as $\alpha = (\exp\left[0.49\sigma_R^2(1+1.11\sigma_R^{12/5})^{-7/6}\right] - 1)^{-1}$ and $\beta = (\exp\left[0.51\sigma_R^2(1+0.69\sigma_R^{12/5})^{-5/6}\right] - 1)^{-1}$ [1], [41], with the Rytov variance $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$. The wave-number is $k = 2\pi/\lambda$ with the wavelength λ, L is the propagation distance, and the refractive index is denoted by C_n^2.

The pointing error represents the misalignment between the transmitter laser and the receiver photodetector. The parameter ξ is defined as the ratio between the equivalent beam radius at the receiver, $w_{L_{eq}}$, and the pointing error (jitter) standard deviation at the receiver as $\xi = w_{L_{eq}}/(2\sigma_s)$. The parameter $w_{L_{eq}}$ depends on the beam radius at distance L, w_L, as $w_{L_{eq}}^2 = w_L^2 \sqrt{\pi}\,\mathrm{erf}\left(\upsilon\right) \big/ \left(2\upsilon \exp\left(-\upsilon^2\right)\right)$, $\upsilon = \sqrt{\pi} a \Big/ \left(\sqrt{2} w_L\right)$ [42], where a is the radius of a circular detector aperture, $\mathrm{erf}\left(\cdot\right)$ is the error function [47, (8.250.1)], and $A_0 = \left[\mathrm{erf}\left(\upsilon\right)\right]^2$. Next, the parameter w_L is related with the beam radius at the waist, w_0, and the radius of curvature, F_0, by $w_L = w_0((\Theta_o + \Lambda_o)(1 + 1.63\sigma_R^{12/5}\Lambda_1))^{1/2}$, where $\Theta_o = 1 - L/F_0$, $\Lambda_o = 2L/(kw_0^2)$, $\Lambda_1 = \Lambda_o/(\Theta_o^2 + \Lambda_o^2)$ [44].

B. Phase Noise

After signal conversion from optical-to-electrical domain, classical signal detection is performed in electrical domain. During the process of down-conversion, electrical signal is multiplied by local oscillator output signal. The phase of the local oscillator signal (also known as a phase noise) is a random process fluctuating over time. Frequently local oscillator is embedded in frequency syntetyzator contained phase locked loop (PLL). The phase noise generated by PLL is well known to have a Tikhonov PDF [37, Ch. 2], [38]. Hence, the phase noise, φ, of the local oscillator is assumed to be a RV which follows Tikhonov PDF given by

$$f_\varphi(\varphi) = \frac{\exp(b\cos(\varphi))}{2\pi I_0(b)}, \quad |\varphi| \le \pi, \tag{4}$$

where $I_n(\cdot)$ is the nth order modified Bessel function of the first kind [47, (8.431)], $b = 1/\sigma_\varphi^2$, and σ_φ^2 is the variance of the phase noise.

Here, we use the Fourier series expansion of Tikhonov PDF, because Fourier series form is tractable for integration that will be necessary in mathematical derivations of SEP. We start with the Fourier expansion [48, (9.6.34)]

$$\mathrm{e}^{b\cos\varphi} = I_0(b) + 2\sum_{n=1}^{\infty} I_n(b)\cos(n\varphi), \quad |\varphi| \le \pi, \tag{5}$$

for a fixed $b > 0$.

Based on the expansion in (5), it is clear that Tikhonov PDF given by (4), can be expressed in the Fourier series as

$$f_\varphi(\varphi) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} c_n\cos(n\varphi), \quad |\varphi| \le \pi, \tag{6}$$

where

$$c_n = \frac{I_n(b)}{\pi I_0(b)}. \tag{7}$$

Proposition 1: The series in (6) is convergent. For the truncation error

$$E_N(\varphi; b) = \sum_{n=N+1}^{\infty} c_n\cos(n\varphi), \quad |\varphi| \le \pi, \tag{8}$$

the following estimate

$$|E_N(\varphi; b)| \le E_N(0; b) \le B_N \tag{9}$$

holds, where

$$B_N \equiv B_N(b) = \frac{1}{\pi I_0(b)}\left(I_{N+1}(b) + \int_{N+1}^{\infty} I_\nu(b)\,\mathrm{d}\nu\right). \tag{10}$$

Proof: See Appendix A.

In Fig. 2, we present the bounds B_N of the truncation errors for $N \le 40$ and different values of σ_φ. If we take a threshold for the errors, e.g., $\varepsilon = 10^{-8}$ (black line in Fig. 2), so that $B_N < \varepsilon$, we see that the corresponding number of terms should be $N = 35$, 18, 13 and 10 for $\sigma_\varphi = 10^\circ, 20^\circ, 30^\circ$ and 40°, respectively.

III. Error Performance

Since the decisions of the DPSK receiver are taken based on the composite phase difference between signals received during two consecutive symbol intervals, the decision variable of differential detector can be written as

$$\lambda' = [\psi_{k+1} - \psi_k] \bmod 2\pi, \tag{11}$$

where ψ_{k+1} and ψ_k are the composite phase of consecutive received signals, bearing the information at the

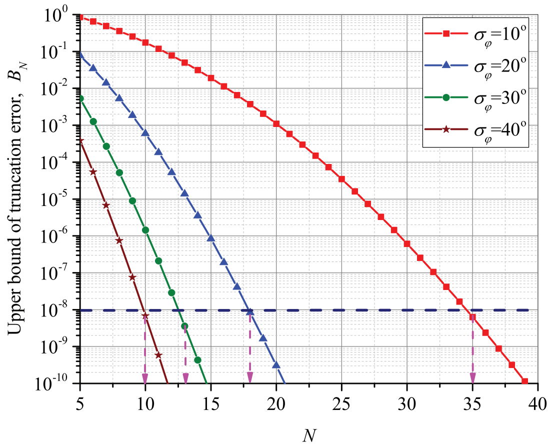

Fig. 2. Upper bound of truncation errors for $\sigma_\varphi = 10^\circ$ (*red*), $\sigma_\varphi = 20^\circ$ (*blue*), $\sigma_\varphi = 30^\circ$ (*green*), and $\sigma_\varphi = 40^\circ$ (*brown*) when $N \le 40$.

$(k+1)$-th and the k-th interval, respectively. The local oscillator imperfections are represented through the phase noise φ_{k+1} and φ_k at the $(k+1)$-th and the k-th intervals, respectively. Then, the decision variable of the differential detector is

$$\begin{aligned} \lambda &= [(\psi_{k+1} - \varphi_{k+1}) - (\psi_k - \varphi_k)] \bmod 2\pi \\ &= [(\psi_{k+1} - \psi_k) - (\varphi_{k+1} - \varphi_k)] \bmod 2\pi. \end{aligned} \tag{12}$$

The term, $(\psi_{k+1} - \psi_k)$, represents the difference of the composite phases, while, $(\varphi_{k+1} - \varphi_k)$, denotes the impact of the phase noise.

On the contrary to the situation at the transmitter, where the phase of RF carrier is constant, the composite phase of total received signal is a RV. The PDF of the resulting phase, ψ, of received signal in a signaling interval is presented in the Fourier series form as [32]–[34]

$$f_\psi(\psi) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} b_n\cos(n\psi), \tag{13}$$

where b_n represents the Fourier coefficient for the FSO channel influenced by the Gamma-Gamma atmospheric turbulence and pointing errors. In order to derive the Fourier coefficient for the considered scenario, the PDF of the received signal composite phase is written as

$$f_\psi(\psi) = \int_0^{\infty} f(\psi|\gamma) f_\gamma(\gamma)\,\mathrm{d}\gamma \tag{14}$$

where $f_\gamma(\gamma)$ is the PDF of the instantaneous SNR given in (3). The conditional PDF is defined through a Fourier series form of the received signal composite phase due to additive noise as [32]–[34]

$$f(\psi|\gamma) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} a_n(\gamma)\cos(n\psi), \tag{15}$$

where $a_n(\gamma)$ denotes the Fourier coefficient for AWGN channel defined as [34]

$$a_n(\gamma) = \frac{\Gamma\left(\frac{n}{2}+1\right)}{n!\pi}\gamma^{\frac{n}{2}}\exp(-\gamma)\,{}_1F_1\left(\frac{n}{2}+1; n+1; \gamma\right), \tag{16}$$

where ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function [47, (9.21)].

Proposition 2: After substituting (3), (15) and (16) into (14), the PDF of the phase ψ is given as

$$f_\psi(\psi) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{n 2^{\alpha+\beta-4}\xi^2}{\pi^2\Gamma(\alpha)\Gamma(\beta)} \cos(n\psi) \times G_{3,6}^{6,1}\left(\frac{\alpha^2\beta^2\kappa^2}{16\mu} \,\middle|\, \begin{matrix} 1-\frac{n}{2}, 1+\frac{n}{2}, \frac{\xi^2+2}{2} \\ \frac{\xi^2}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, 0 \end{matrix} \right). \quad (17)$$

Proof: See Appendix B.

Based on (13) and (17), the Fourier coefficient for FSO channel influenced by Gamma-Gamma atmospheric turbulence and pointing errors is determined as

$$b_n = \frac{n 2^{\alpha+\beta-4}\xi^2}{\pi^2\Gamma(\alpha)\Gamma(\beta)} \times G_{3,6}^{6,1}\left(\frac{\alpha^2\beta^2\kappa^2}{16\mu} \,\middle|\, \begin{matrix} 1-\frac{n}{2}, 1+\frac{n}{2}, \frac{\xi^2+2}{2} \\ \frac{\xi^2}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, 0 \end{matrix} \right). \quad (18)$$

When the considered scenario assumes the pointing errors to be very small, it can be neglected ($\xi \to \infty$). In this case, the optical link suffers only from atmospheric turbulence, and the Fourier coefficient can be found by taking the limit of (18) for $\xi \to \infty$. After applying [49, (07.34.25.0007.01), (07.34.25.0006.01) and (06.05.16.0002.01)], the Fourier coefficient can be derived as

$$b_n^{GG} = \lim_{\xi\to\infty} b_n = \frac{n 2^{\alpha+\beta-3}}{\pi^2\Gamma(\alpha)\Gamma(\beta)} \times G_{2,5}^{5,1}\left(\frac{\alpha^2\beta^2}{16\mu} \,\middle|\, \begin{matrix} 1-\frac{n}{2}, 1+\frac{n}{2} \\ \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, 0 \end{matrix} \right). \quad (19)$$

For further analysis, it is required to find the PDF of the decision variable λ, defined in (12). Firstly, we will introduce the following rule related to the PDFs presented in the Fourier series form.

Proposition 3: If the variables x_1 and x_2 are RVs with the PDFs given in the Fourier series form, with coefficients z_{1n} and z_{2n}, respectively, as

$$f_{x_1}(x) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} z_{1n}\cos(nx), \quad |x| \le \pi,$$
$$f_{x_2}(x) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} z_{2n}\cos(nx), \quad |x| \le \pi, \quad (20)$$

then, the PDF of $y = [x_1 - x_2] \bmod 2\pi$, is

$$f_y(y) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \pi z_{1n} z_{2n}\cos(ny), \quad |y| \le \pi. \quad (21)$$

Proof: The proof can be found in [36], [50], and [51].

A. Error Analysis Without Considering Hardware Imperfections

If no hardware imperfections are assumed, the decision variable λ' is defined in (11). Based on Proposition 3, after replacing x_1 and x_2 with ψ_{k+1} and ψ_k, respectively, and both z_{1n} and z_{2n} with b_n, the PDF of λ' can be easily obtained as

$$f_{\lambda'}(\lambda') = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \pi b_n^2 \cos(n\lambda'), \quad |\lambda'| \le \pi. \quad (22)$$

The detection is performed in the manner to find the closest possible transmitted phase compared with received composite phase λ'. The probability of wrong symbol detection is given by

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} f_{\lambda'}(\lambda')\, d\lambda'. \quad (23)$$

By substituting (22) into (23), the average SEP can be found as

$$P_s = 1 - \frac{1}{M} - \sum_{n=1}^{\infty} \frac{2\pi b_n^2}{n} \sin\left(\frac{n\pi}{M}\right). \quad (24)$$

In [13], an expression for the average BER was derived in integral form, assuming that the intensity fluctuations of the optical signal are modeled by the log-normal and Gamma-Gamma distributions. In the region of high average electrical SNR values, the bit error probability could be approximated by $BER \approx P_s / \log_2 M$ [40, p. 271]. By using this approximation and SEP in (24) with the Fourier coefficient of (19), the numerical results from [13, Fig. 2] can be obtained.

B. Error Analysis in the Presence of Phase Noise

In the presence of the phase noise, the decision variable λ is defined as in (12). The PDF of the variable, $\delta = \varphi_{k+1} - \varphi_k$, can be found by utilization of Proposition 3. Since the Tikhonov PDF of the phase noise is given in the Fourier series form by (6), the PDF of the variable δ is found as

$$f_\delta(\delta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \pi c_n^2 \cos(n\delta), \quad |\delta| \le \pi, \quad (25)$$

with the Fourier coefficient c_n previously defined by (7).

Taking into consideration that the variables ψ and φ are statistically independent, based on (22) and (25), and Proposition 3, the PDF of λ is

$$f_\lambda(\lambda) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \pi^3 b_n^2 c_n^2 \cos(n\lambda), \quad |\lambda| \le \pi. \quad (26)$$

When the Gamma-Gamma atmospheric turbulence, pointing errors and phase noise are assumed, the average SEP of the SIM-MDPSK FSO system can be written as

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} f_\lambda(\lambda)\, d\lambda = 1 - \frac{1}{M} - \sum_{n=1}^{\infty} \frac{2\pi^3 b_n^2 c_n^2}{n} \sin\left(\frac{n\pi}{M}\right), \quad (27)$$

where the Fourier coefficients b_n and c_n are previously defined in (18) and (7), respectively.

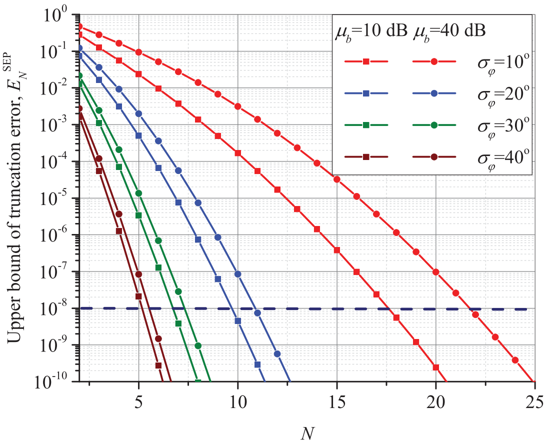

Fig. 3. Upper bound of truncation errors for $\sigma_\varphi = 10^\circ$ (*red*), $\sigma_\varphi = 20^\circ$ (*blue*), $\sigma_\varphi = 30^\circ$ (*green*), and $\sigma_\varphi = 40^\circ$ (*brown*) when $N \leq 25$.

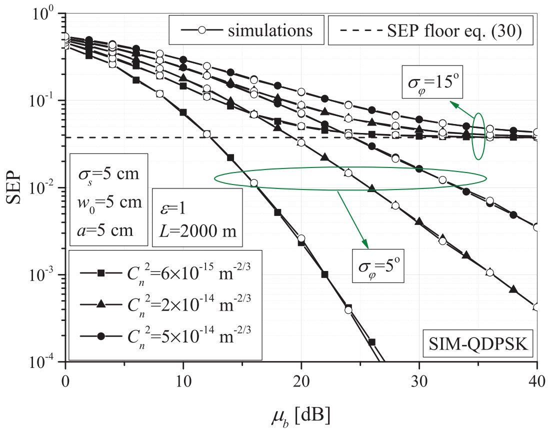

Fig. 4. SIM-QDPSK SEP versus average electrical SNR for different values of the phase noise standard deviation in various atmospheric turbulence conditions.

Proposition 4: The series in (27) is convergent and the following estimate

$$\left| P_s - 1 + \frac{1}{M} + \sum_{n=1}^{N} \frac{2\pi^3 b_n^2 c_n^2}{n} \sin\left(\frac{n\pi}{M}\right) \right| \leq E_N^{\mathrm{SEP}} \quad (28)$$

holds, with the bound of truncation error

$$E_N^{\mathrm{SEP}} = \frac{2\pi b_{N+1}^2}{I_0(b)^2} \left(\frac{I_{N+1}(b)^2}{N+1} + \int_{N+1}^{\infty} \frac{I_\nu(b)^2}{\nu} \mathrm{d}\nu \right). \quad (29)$$

Proof: See Appendix C.

This truncation error is illustrated in Fig. 3. To achieve the given truncation error, the higher number of terms in summation is required if the standard deviation is lower. In order to achieve truncation error less than 10^{-8}, for $\mu = 10$ dB the required number of terms in summation is $N = 18, 10, 7$ and 6, when $\sigma_\varphi = 10^o, 20^o, 30^o$ and 40^o, respectively. In addition, the convergence rate decreases with increasing the electrical SNR. In other words, the proposed series expression converges better in low electrical SNR regime compared to high electrical SNR regime.

As it will be shown in the next Section, the existence of the phase noise results in the unrecoverable error-rate floor, which is a meaningful limiting factor in SIM-DPSK based FSO systems. This error-rate floor represents the constant value of the average SEP, which occurs at the high average electrical SNR. With a further increase in the transmitted optical power, the improvement of the SEP performance will not be achieved.

Proposition 5: The unrecoverable error-rate floor can be expressed as

$$P_s^{floor} = 1 - \frac{1}{M} - \sum_{n=1}^{\infty} \frac{2\pi c_n^2}{n} \sin\left(\frac{n\pi}{M}\right). \quad (30)$$

Proof: See Appendix D.

It can be noticed that the SEP floor is independent on the FSO channel state (atmospheric turbulence and pointing errors). On the other hand, the value of the SEP floor depends on the phase noise standard deviation and order of DPSK modulation, as it will be presented in the next Section.

IV. Numerical Results and Discussion

Based on derived expressions for the average SEP, numerical results are obtained and validated by Monte Carlo simulations. Monte Carlo simulations have been performed using MATLAB® software package. Since intensity fluctuations originate from both atmospheric turbulence and pointing errors, the resulting optical signal intensity, I, is obtained as a product of two different RVs, i.e., $I = I_a \times I_p$. The intensity fluctuations, I_a, due to atmospheric turbulence are modeled by Gamma-Gamma distribution. The corresponding RV, I_a, is generated as a product of two independent Gamma-distributed RVs with shaping parameters α and β. Command for generating Gamma-distributed RV is built-in into MATLAB®. The RVs relating to the pointing errors, I_p, are generated based on [42, (9)], employing built-in command for generating Rayleigh RV. The Tikhonov-distributed samples of phase noise are generated using the modified acceptance/rejection method, explained in [52, p. 382]. Modulation and demodulation is simulated based on [46, p. 333-335]. The average SEP values are estimated using 10^7 transmitted symbols.

In order to obtain the numerical results, the atmospheric turbulence strength is determined by the refractive index structure parameter as: $C_n^2 = 6 \times 10^{-15}$ m$^{-2/3}$ for weak, $C_n^2 = 2 \times 10^{-14}$ m$^{-2/3}$ for moderate and $C_n^2 = 5 \times 10^{-14}$ m$^{-2/3}$ for strong turbulence conditions. The impact of the phase noise is specified by the phase noise standard deviation.

The average SEP dependence on the average electrical SNR of the FSO system employing SIM-QDPSK is presented in Fig. 4, assuming different atmospheric turbulence conditions and phase noise standard deviation $\sigma_\varphi = 5^o$ or $\sigma_\varphi = 15^o$. Lower values of the phase noise standard deviation correspond to the weaker phase noise and better system performance.

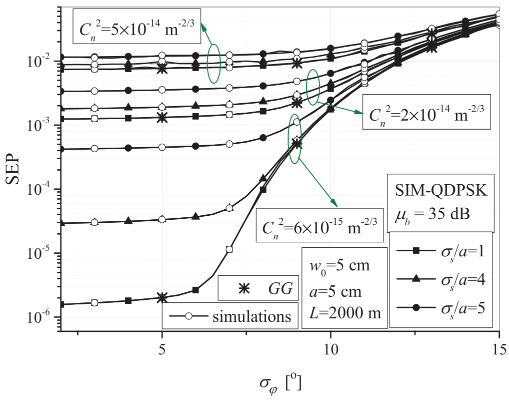

Fig. 5. SIM-QDPSK SEP versus the phase noise standard deviation for different values of the normalized jitter standard deviation, in various atmospheric turbulence conditions.

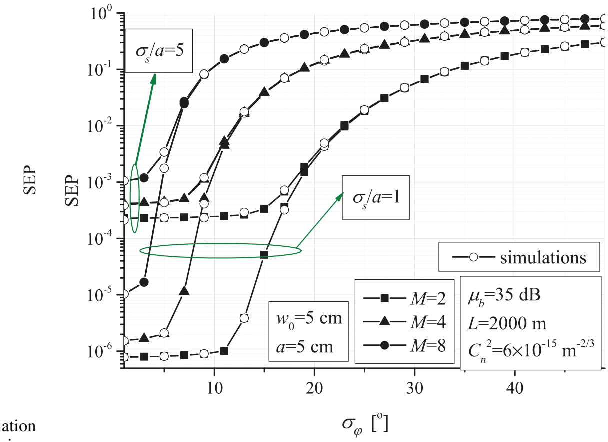

Fig. 6. SIM-MDPSK SEP versus the phase noise standard deviation for different values of the normalized jitter standard deviations.

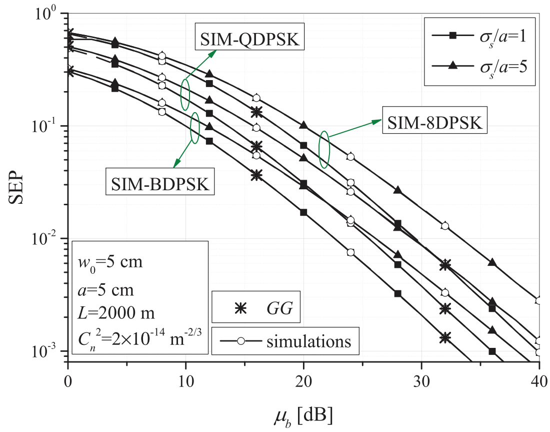

Fig. 7. SIM-MDPSK SEP versus average electrical SNR of the FSO system without hardware imperfections.

Furthermore, the impact of the atmospheric turbulence conditions is stronger when the value of σ_φ is lower. On the other hand, when the effect of phase noise is very strong, the atmospheric turbulence conditions has minor influence on the SEP performance. In addition, the existence of the unrecoverable error-rate floor is noticed in Fig. 4, meaning that the DPSK hardware imperfections presented through phase noise are an important limiting factor for SIM-DPSK systems.

This SEP floor appears at lower values of average electrical SNR in weak atmospheric turbulence, as well as when the value of σ_φ is greater (stronger impact of the phase noise). The SEP floor results based on (30) for $\sigma_\varphi = 5^{\circ}$ are not visible in Fig. 4 due to very low value. It can be concluded that the SEP floor is not dependent on atmospheric turbulence conditions, which is in agreement with mathematical derivation (see (41) and (30)).

Fig. 5 presents the SIM-QDPSK SEP dependence on the phase noise standard deviation for different values of the normalized jitter standard deviation, in various atmospheric turbulence conditions. It can be observed that lower values of the normalized jitter standard deviation reflects in better system performance. It means that the positioning of the FSO apertures is better and the pointing errors effect is weaker. Also, the pointing error effect is stronger in weak compared to moderate and strong atmospheric turbulence. When the optical signal transmission suffers from very strong atmospheric turbulence, the pointing errors effect has less impact on the SEP performance.

In addition, the results for the FSO system when the pointing errors effect is neglected, obtained by using (27) and (19), are also presented. These results are in agreement with those when $\sigma_s/a = 1$. Hence, very low values of the normalized jitter standard deviation means that the pointing errors effect is very weak and can be neglected.

When the DPSK demodulator hardware imperfections are dominant, and the phase noise is quite strong, the value of σ_φ is large. In that case, the FSO channel state (atmospheric turbulence and pointing errors) does not play a major role in the SEP performance. When $\sigma_\varphi \to 0$, the impact of the phase noise is very weak and can be neglected. For these phase noise standard deviation values, the SEP takes constant values, which are approximately the same as the SEP values for the FSO system without phase noise. Also, atmospheric turbulence and pointing errors have very strong impact on the SEP performance, when σ_φ is low.

Fig. 6 represents the SIM-MDPSK SEP dependence on the phase noise standard deviation. The impact of the phase noise on SEP is stronger when higher order SIM-MDPSK is employed. For example, for $\sigma_s/a = 1$, in the case of $M = 2$, the SEP is independent on phase noise up to $\sigma_\varphi = 10^{\circ}$, while for $M = 8$, SEP drastically increases even starting from $\sigma_\varphi = 2^{\circ}$. In addition, the weaker the pointing errors, the stronger is the effect of phase noise on SEP. It can be observed that the efect of DPSK order has minor influence on the SEP performance when the impact of the phase noise is very strong.

The SEP dependence on the average electrical SNR of the FSO system without hardware imperfections is presented in Fig. 7. The results are obtained based on (24) with the Fourier coefficient in (18), or in (19) when the pointing errors are neglected. Different DPSK formats are observed: SIM employing BDPSK, QDPSK and 8DPSK. As it is expected, FSO system based on SIM-DPSK with higher modulation format has worse SEP performance, but the larger amount of information can be transmitted. Also, consistent with previous conclusions, greater value of the normalized jitter standard deviations means worse system performance due to stronger pointing errors. Agreement of the results based on (18) for $\sigma_s/a = 1$ and (19) is noticed, meaning that very low jitter standard deviation leads to weak pointing errors.

V. Conclusion

We have derived novel analytical expressions for the average SEP of FSO system employing SIM-MDPSK. The irradiance fluctuations at the received signal originate from the Gamma-Gamma atmospheric turbulence and pointing errors. Based on derived SEP expressions, numerical results have been presented and confirmed by Monte Carlo simulations.

From the illustrated results, we have found that the hardware imperfections result in the significant deterioration of the FSO system performance. The phase noise is dominant system factor, which causes the SEP performance damaging, especially when optical signal transmission is influenced by favorable conditions (weak atmospheric turbulence and weak pointing errors effect). Similarly, when the impact of the phase noise is very strong, atmospheric turbulence and pointing errors effect has minor effect on the system performance. Furthermore, the SIM based FSO system with higher DPSK format is more sensitive to the existence of the phase noise. Further, the existence of the phase noise leads to the unrecoverable SEP floor, being meaningful limiting factor for SIM-DPSK systems. It is observed that the SEP floor is not dependent on the FSO channel state, but it is highly dependent on the phase noise standard deviation and the DPSK modulation order.

Appendix A
Proof of Proposition 1

The series in (6) is a uniformly convergent series, because the numerical series with positive terms,

$$\sum_{n=1}^{\infty} I_n(b), \tag{31}$$

is convergent, which can be proved using the inequality [53]

$$\left(1+\frac{\nu}{b}\right) I_{\nu+1}(b) < I_\nu(b) \quad (\nu \geq -1,\ b > 0). \tag{32}$$

Namely, the series (31) is convergent if for a fixed $m = \lceil b \rceil$ ($\lceil x \rceil$ denotes the smallest integer greater than or equal to x) the series $\sum_{n=m}^{\infty} I_n(b)$ converges. According to (32), for $n > m$ we have

$$I_n(b) < \frac{I_{n-1}(b)}{1+(n-1)/b} < \frac{I_{n-1}(b)}{2} < \cdots < \frac{I_m(b)}{2^{n-m}}, \tag{33}$$

so that

$$\sum_{n=m}^{\infty} I_n(b) < I_m(b) \sum_{n=m}^{\infty} \frac{1}{2^{n-m}} = 2I_m(b),$$

wherefrom we conclude that the series $\sum_{n=m}^{\infty} I_n(b)$ and (31) are convergent. The sum of (31) is $S = \frac{1}{2}(\mathrm{e}^b - I_0(b))$ [54, p. 254].

According to Cauchy's integral test (cf. [55, p. 120] or [56, p. 159]), for the numerical series (31) we can give the following estimates for the remainder term

$$\sum_{n=N+1}^{\infty} I_n(b) \leq I_{N+1}(b) + \int_{N+1}^{\infty} I_\nu(b)\,\mathrm{d}\nu, \tag{34}$$

where $\nu \mapsto I_\nu(a)$ is a decreasing positive continuous function on $(0,\infty)$ [53].

Thus, for the truncation error $E_N(\varphi; b)$ given by (8) we obtain $|E_N(\varphi; b)| \leq E_N(0; b) = (\pi I_0(b))^{-1} \sum_{n=N+1}^{\infty} I_n(b)$, i.e., (9), where B_N is given by (10), because of (34).

Appendix B
Proof of Proposition 2

After substituting (3), (15) and (16) into (14), the PDF of the phase ψ is re-written as

$$\begin{aligned} f_\psi(\psi) = \frac{1}{2\pi} &+ \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{n!\pi} \frac{\xi^2}{2\Gamma(\alpha)\Gamma(\beta)} \cos(n\psi) \\ &\times \int_0^\infty \gamma^{\frac{n}{2}-1} \exp(-\gamma)\, {}_1F_1\left(\frac{n}{2}+1; n+1; \gamma\right) \\ &\times G_{1,3}^{3,0}\left(\alpha\beta\kappa\sqrt{\frac{\gamma}{\mu}} \,\middle|\, \begin{matrix} \xi^2+1 \\ \xi^2, \alpha, \beta \end{matrix}\right) \mathrm{d}\gamma. \end{aligned} \tag{35}$$

Based on [49, (07.20.26.0015.01)], the product of exponential and confluent hypergeometric function is presented in terms of the Meijer's G-function as

$$\exp(-\gamma)\, {}_1F_1\left(\frac{n}{2}+1; n+1; \gamma\right) = \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}\right)} G_{1,2}^{1,1}\left(\gamma \,\middle|\, \begin{matrix} 1-\frac{n}{2} \\ 0, -n \end{matrix}\right). \tag{36}$$

After substituting (36) into (35) and applying [49, (06.05.16.0002.01) and (06.05.03.0001.01)], the PDF of the phase ψ is

$$\begin{aligned} f_\psi(\psi) &= \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{n\xi^2}{4\pi\,\Gamma(\alpha)\Gamma(\beta)} \cos(n\psi) \\ &\times \int_0^\infty \gamma^{\frac{n}{2}-1} G_{1,2}^{1,1}\left(\gamma \,\middle|\, \begin{matrix} 1-\frac{n}{2} \\ 0, -n \end{matrix}\right) G_{1,3}^{3,0}\left(\alpha\beta\kappa\sqrt{\frac{\gamma}{\mu}} \,\middle|\, \begin{matrix} \xi^2+1 \\ \xi^2, \alpha, \beta \end{matrix}\right) \mathrm{d}\gamma. \end{aligned} \tag{37}$$

The integral in (37) can be evaluated in closed-form by using [49, (07.34.21.0013.01)], so the PDF of the phase ψ is derived as

$$\begin{aligned} f_\psi(\psi) = \frac{1}{2\pi} &+ \sum_{n=1}^{\infty} \frac{n 2^{\alpha+\beta-4}\xi^2}{\pi^2\Gamma(\alpha)\Gamma(\beta)} \cos(n\psi) \\ &\times G_{4,7}^{7,1}\left(\frac{\alpha^2\beta^2\kappa^2}{16\mu} \,\middle|\, \begin{matrix} 1-\frac{n}{2}, 1+\frac{n}{2}, \frac{\xi^2+1}{2}, \frac{\xi^2+2}{2} \\ \frac{\xi^2}{2}, \frac{\xi^2+1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, 0 \end{matrix}\right). \end{aligned} \tag{38}$$

After the permutation of the parameters via [49, (07.34.04.0003.01) and (07.34.04.0004.01)], and the transformation of the Meijer's G-function by [49, (07.34.03.0002.01)], the final form of the PDF of the phase ψ is presented in (17).

APPENDIX C
PROOF OF PROPOSITION 4

We observe the series in (27) given by

$$S = \sum_{n=1}^{\infty} \frac{2\pi^3 b_n^2 c_n^2}{n} \sin\left(\frac{n\pi}{M}\right), \tag{39}$$

for which we can prove its absolute convergence. As in APPENDIX A we use the inequalities (32) and (33) and consider the series $\sum_{n=m}^{\infty} 2\pi^3\, b_n^2 c_n^2/n$, where $m = \lceil b \rceil$. Since b_n is a decreasing sequence, we can write

$$\sum_{n=m}^{\infty} \frac{2\pi^3\, b_n^2 c_n^2}{n} < 2\pi^3\, b_m^2 \sum_{n=m}^{\infty} \frac{c_n^2}{n} = \frac{2\pi b_m^2}{I_0(b)^2} \sum_{n=m}^{\infty} \frac{I_n(b)^2}{n}.$$

Now, using (33) we conclude that

$$\sum_{n=m}^{\infty} \frac{I_n(b)^2}{n} < I_m(b)^2 \sum_{n=m}^{\infty} \frac{1}{n 4^{n-m}} < \frac{4 I_m(b)^2}{3m},$$

i.e.,

$$\sum_{n=m}^{\infty} \frac{2\pi^3\, b_n^2 c_n^2}{n} < \frac{8\pi b_m^2}{3m} \left(\frac{I_m(b)}{I_0(b)}\right)^2 < +\infty.$$

Thus, the series (39) is absolutely convergent, and also convergent. For its truncation error we obtain the following estimate

$$\begin{aligned} \left| \sum_{n=N+1}^{\infty} \frac{2\pi^3\, b_n^2 c_n^2}{n} \sin\left(\frac{n\pi}{M}\right) \right| &\le \sum_{n=N+1}^{\infty} \frac{2\pi^3\, b_n^2 c_n^2}{n} \\ &\le \frac{2\pi b_{N+1}^2}{I_0(b)^2} \sum_{n=N+1}^{\infty} \frac{I_n(b)^2}{n}. \end{aligned}$$

Based on Cauchy's criteria, as in APPENDIX A, it follows

$$\sum_{n=N+1}^{\infty} \frac{I_n(b)^2}{n} \le \frac{I_{N+1}(b)^2}{N+1} + \int_{N+1}^{\infty} \frac{I_\nu(b)^2}{\nu} \mathrm{d}\nu$$

so that we get (28), with (29).

APPENDIX D
PROOF OF PROPOSITION 5

In order to determine the value of the SEP floor, it is necessary to take the limit of (27) for $\mu \to \infty$, i.e.,

$$P_s^{floor} = \lim_{\mu\to\infty} P_s = \lim_{\mu\to\infty} \left\{ 1 - \frac{1}{M} - \sum_{n=1}^{\infty} \frac{2\pi^3 b_n^2 c_n^2}{n} \sin\left(\frac{n\pi}{M}\right) \right\}. \tag{40}$$

Since the Fourier coefficient b_n is the only term in (27), which depends on the average electrical SNR, after following derivation in this Appendix, the limit of b_n for $\mu \to \infty$ is derived as

$$b_n^{\mu\to\infty} = \lim_{\mu\to\infty} b_n = \frac{1}{\pi}. \tag{41}$$

The term $b_n^{\mu\to\infty}$ is derived by following

$$\begin{aligned} \lim_{\mu\to\infty} b_n &= \lim_{\mu\to\infty} \frac{n 2^{\alpha+\beta-4} \xi^2}{\pi^2 \Gamma(\alpha)\Gamma(\beta)} \\ &\quad \times G_{3,6}^{6,1}\left(\frac{\alpha^2\beta^2\kappa^2}{16\mu} \middle| \begin{matrix} 1-\frac{n}{2}, 1+\frac{n}{2}, \frac{\xi^2+2}{2} \\ \frac{\xi^2}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, 0 \end{matrix} \right) \\ &= \lim_{z\to 0} \frac{n 2^{\alpha+\beta-4} \xi^2}{\pi^2 \Gamma(\alpha)\Gamma(\beta)} \\ &\quad \times G_{3,6}^{6,1}\left(z \middle| \begin{matrix} 1-\frac{n}{2}, 1+\frac{n}{2}, \frac{\xi^2+2}{2} \\ \frac{\xi^2}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}, 0 \end{matrix} \right). \end{aligned} \tag{42}$$

The first step in finding $\lim_{\mu\to\infty} b_n$ is applying [49, (07.34.06.0001.01)] to represent the Meijer's G-function in (42) in series form. Since $z \to 0$, higher order terms in the series representation of Meijer's G-function can be neglected, and $b_n^{\mu\to\infty}$ is determined as

$$\begin{aligned} b_n^{\mu\to\infty} &= \lim_{\mu\to\infty} b_n \approx \frac{1}{2^2\pi^2} \\ &\quad \times \frac{2^\alpha \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma(\alpha)} \frac{2^\beta \Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma(\beta)}. \end{aligned} \tag{43}$$

After utilizing [49, (06.05.03.0002.01) and (06.01.16.0006.01)], it is proved that holds

$$\frac{2^x \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)}{\Gamma(x)} = 2\sqrt{\pi}, \tag{44}$$

so the final form of $b_n^{\mu\to\infty}$ is derived as

$$b_n^{\mu\to\infty} = \frac{1}{\pi}. \tag{45}$$

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Generating Functions for Special Polynomials and Numbers Including Apostol-Type and Humbert-Type Polynomials

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Abstract. The aim of this paper is to give generating functions and to prove various properties for some new families of special polynomials and numbers. Several interesting properties of such families and their connections with other polynomials and numbers of the Bernoulli, Euler, Apostol–Bernoulli, Apostol–Euler, Genocchi and Fibonacci type are presented. Furthermore, the Fibonacci-type polynomials of higher order in two variables and a new family of special polynomials $(x, y) \mapsto \mathbb{G}_d(x, y; k, m, n)$, including several particular cases, are introduced and studied. Finally, a class of polynomials and corresponding numbers, obtained by a modification of the generating function of Humbert's polynomials, is also considered.

Mathematics Subject Classification. 05A15, 11B39, 11B68, 11B73, 11B83.

Keywords. Generating function, Fibonacci polynomials, Humbert polynomials, Bernoulli polynomials and numbers, Euler polynomials and numbers, Apostol–Bernoulli polynomials and numbers, Apostol–Euler polynomials and numbers, Genocchi polynomials, Stirling numbers.

1. Introduction and Preliminaries

The special polynomials and numbers play an important role in many branches of mathematics and their development is always actual. Many papers and books were published in this very wide area. We mention only a few books connected with our results in this work (cf. [4,7,27,28]).

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In this paper we consider some new families of numbers and polynomials, including their generating functions, several interesting properties, as well as their connections with other polynomials and numbers of the Bernoulli, Euler, Apostol–Bernoulli, Apostol–Euler, Genocchi, Fibonacci and Lucas type. In order to give our results, we need to mention several special classes of polynomials and numbers with their generating functions.

1° *The Bernoulli polynomials* of higher order $B_d^{(h)}(x)$ are defined by means of the following generating function

$$F_{Bh}(x,t;h)=\left(\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t-1}\right)^h=\sum_{d=0}^{\infty}B_d^{(h)}(x)\frac{t^d}{d!}. \tag{1.1}$$

For $h=1$, (1.1) reduces to the generating function of the classical Bernoulli polynomials, $B_d^{(1)}(x)=B_d(x)$. Furthermore, for $x=0$, this gives the well-known Bernoulli numbers $B_d=B_d(0)$. For details see [1–7,13–22,29].

2° *The Apostol–Bernoulli polynomials* were introduced in 1951 by Apostol [1] by means of the following generating function

$$F_{AB}(x,t;\lambda)=\frac{t\mathrm{e}^{xt}}{\lambda\mathrm{e}^t-1}=\sum_{d=0}^{\infty}\mathcal{B}_d(x,\lambda)\frac{t^d}{d!}, \tag{1.2}$$

where $|t+\log\lambda|<2\pi$ (for details see [1–7,13–22,29]). Several their interesting properties, formulas and extensions have been obtained by Srivastava [26] (see also the recent book [27]). Using the suitable generating functions several authors have obtained different generalizations and unifications of these numbers and polynomials (cf. [2,5,13,14,16,17,22,29]).

Substituting $x=0$ in (1.2), for $\lambda\neq1$, we get the Apostol–Bernoulli numbers $\mathcal{B}_d(\lambda)$,

$$\mathcal{B}_d(\lambda)=\mathcal{B}_d(0,\lambda), \tag{1.3}$$

and they can be expressed it terms of Stirling numbers of the second kind [1, Eq. (3.7)]. Setting $\lambda=1$ in (1.2), we get the classical Bernoulli polynomials $B_d(x)=\mathcal{B}_d(x,1)$.

Alternatively, the Apostol–Bernoulli numbers can be expressed in the form

$$\mathcal{B}_0(\lambda)=0,\quad \mathcal{B}_d(\lambda)=(-1)^{d-1}d\,\frac{\lambda\varphi_{d-2}(\lambda)}{(\lambda-1)^d},\quad d\geq1, \tag{1.4}$$

where $\varphi_k(\lambda)$ are monic polynomials in λ and of degree k and $\varphi_k(0)=1$. Using the generating function (1.2) for $x=0$ and (1.4), it is easy to prove that the polynomials $\varphi_k(\lambda)$ are self-inversive (cf. [20, pp. 16–18]), i.e., $\lambda^k\varphi_k(1/\lambda)\equiv\varphi_k(\lambda)$. Also, we can prove that

$$\varphi_k(\lambda)=(1-\lambda)^k+\lambda\sum_{\nu=1}^{k}\binom{k+1}{\nu}(1-\lambda)^{\nu-1}\varphi_{k-\nu}(\lambda),\quad k\geq0, \tag{1.5}$$

as well as the following determinant form

$$\varphi_k(\lambda) = (-1)^k \lambda^k \begin{vmatrix} -1/\lambda & 0 & 0 & \cdots & 0 & 1 \\ \binom{2}{1} & -1/\lambda & 0 & \cdots & 0 & \xi \\ \binom{3}{1}\xi & \binom{3}{2} & -1/\lambda & \cdots & 0 & \xi^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{k}{1}\xi^{k-2} & \binom{k}{2}\xi^{k-3} & \binom{k}{3}\xi^{k-4} & \cdots & -1/\lambda & \xi^{k-1} \\ \binom{k+1}{1}\xi^{k-1} & \binom{k+1}{2}\xi^{k-2} & \binom{k+1}{3}\xi^{k-3} & \cdots & \binom{k+1}{k} & \xi^k \end{vmatrix},$$

where $\xi = 1 - \lambda$. For example, we have

$$\varphi_0(\lambda) = 1, \quad \varphi_1(\lambda) = \lambda + 1, \quad \varphi_2(\lambda) = \lambda^2 + 4\lambda + 1,$$
$$\varphi_3(\lambda) = \lambda^3 + 11\lambda^2 + 11\lambda + 1, \quad \varphi_4(\lambda) = \lambda^4 + 26\lambda^3 + 66\lambda^2 + 26\lambda + 1,$$
$$\varphi_5(\lambda) = \lambda^5 + 57\lambda^4 + 302\lambda^3 + 302\lambda^2 + 57\lambda + 1,$$
$$\varphi_6(\lambda) = \lambda^6 + 120\lambda^5 + 1191\lambda^4 + 2416\lambda^3 + 1191\lambda^2 + 120\lambda + 1,$$

etc. Using (1.5) we can conclude that $\varphi_k(1) = (k+1)!$.

3° *The Apostol–Euler polynomials of the first kind* $\mathcal{E}_d(x,\lambda)$ are defined by means of the generating function

$$F_{AE}(x,t;\lambda) = \frac{2\mathrm{e}^{xt}}{\lambda \mathrm{e}^t + 1} = \sum_{d=0}^{\infty} \mathcal{E}_d(x,\lambda)\frac{t^d}{d!}, \tag{1.6}$$

where $|2t + \log\lambda| < \pi$ (cf. [1–7,22,29]). For $\lambda \neq 1$, substituting $x = 1/2$ in (1.6) and making some arrangement, we obtain the Apostol–Euler numbers. Setting $\lambda = 1$ in (1.6), we get the first kind Euler polynomials $E_d(x) = \mathcal{E}_d(x,1)$.

4° *The Apostol–Euler polynomials of the second kind* are defined by means of the generating function

$$\frac{2}{\lambda \mathrm{e}^t + \lambda^{-1}\mathrm{e}^{-t}}\mathrm{e}^{tx} = \sum_{d=0}^{\infty} \mathcal{E}_d^*(x,\lambda)\frac{t^d}{d!} \tag{1.7}$$

(cf. [25]). A special kind of these polynomials for $\lambda = 1$ is denoted by $\mathcal{E}_d^*(x) = \mathcal{E}_d^*(x,1)$, and the corresponding numbers by $\mathcal{E}_d^* = \mathcal{E}_d^*(0)$. By using (1.6) and (1.7), for $x = 0$, we have the following relation

$$\mathcal{E}_d^*(0,\lambda) = 2^d \lambda \mathcal{E}_d\left(\frac{1}{2}, \lambda^2\right).$$

The second kind Euler numbers E_d^* are defined by the special case of the first kind Euler polynomials, $E_d^* = 2^d E_d(1/2)$.

5° *The Euler polynomials of higher order* $E_d^{(h)}(x)$ are defined by means of the following generating function

$$F_{Eh}(x,t;h) = \left(\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t + 1}\right)^h = \sum_{d=0}^{\infty} E_d^{(h)}(x)\frac{t^d}{d!}, \tag{1.8}$$

so that, obviously, $E_d^{(1)}(x) = E_d(x)$.

6° *The Genocchi numbers and polynomials and their generalizations.* The Genocchi numbers G_d are defined by the generating function

$$F_g(t) = \frac{2t}{\mathrm{e}^t + 1} = \sum_{d=0}^{\infty} G_d \frac{t^d}{d!}, \tag{1.9}$$

where $|t| < \pi$ (cf. [13,16,22,29]).

In general, for these numbers we have $G_0 = 0$, $G_1 = 1$, and $G_{2d+1} = 0$ for $d \in \mathbb{N}$. Some relations between the Genocchi, Bernoulli and Euler numbers are given by $G_{2d} = 2(1 - 2^{2d})B_{2d}$ and $G_{2d} = 2dE_{2d-1}$. The sequence of Genocchi numbers is

$$\{g_d\}_{d\geq 0} = \{0, 1, -1, 0, 1, 0, -3, 0, 17, 0, -155, 0, \ldots\}.$$

The Genocchi polynomials $G_d(x)$ are defined by the following generating function

$$F_g(x;t) = F_g(t)\mathrm{e}^{xt} = \sum_{d=0}^{\infty} G_d(x) \frac{t^d}{d!}, \tag{1.10}$$

where $|t| < \pi$. Using (1.10), it is easy to see that

$$G_d(x) = \sum_{k=0}^{d} \binom{d}{k} G_k x^{d-k}$$

The first seven Genocchi polynomials are

$$\begin{aligned}
&G_0(x) = 0, \quad G_1(x) = 1, \quad G_2(x) = 2x - 1, \quad G_3(x) = 3x^2 - 3x,\\
&G_4(x) = 4x^3 - 6x^2 + 1, \quad G_5(x) = 5x^4 - 10x^3 + 5x,\\
&G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3.
\end{aligned}$$

The Apostol–Genocchi polynomials $g_d(x, \lambda)$ are defined by the generating function

$$\frac{2t}{\lambda\mathrm{e}^t + 1}\mathrm{e}^{xt} = \sum_{d=0}^{\infty} G_d(x, \lambda) \frac{t^d}{d!}, \tag{1.11}$$

where $|2t + \log\lambda| < \pi$. Setting $\lambda = 1$ in (1.11), we get the classical Genocchi polynomials $G_d(x) = G_d(x, 1)$, which reduce to the classical Genocchi numbers $G_d = G_d(0)$ for $x = 0$.

Substituting $x = 0$ in (1.11), for $\lambda \neq 1$, we obtain the Apostol–Genocchi numbers $G_d(\lambda) = G_d(0, \lambda)$. For some details, properties and other generalizations see [11,13,16,22,26,27,29].

7° *The Stirling numbers of the second kind* $S_2(n, k; \lambda)$ are defined by means of the following generating function (cf. [3,24,26]):

$$F_S(t, k; \lambda) = \frac{(\lambda\mathrm{e}^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k; \lambda) \frac{t^n}{n!}, \tag{1.12}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$.

The generalized Stirling numbers and polynomials have been defined by means of the following generating function (cf. [3]):

$$\frac{(\mathrm{e}^t - 1)^k}{k!}\mathrm{e}^{t\alpha} = \sum_{n=0}^{\infty} S^{(\alpha)}(n, k) \frac{t^n}{n!}. \tag{1.13}$$

Several combinatorial properties of these polynomials have been proved in [3].

Simsek [24] has modified the generating function (1.13), defining the so-called λ-array polynomials $S_k^n(x;\lambda)$ by means of the following generating function

$$F_A(t,x,k;\lambda) = \frac{(\lambda \mathrm{e}^t - 1)^k}{k!}\mathrm{e}^{tx} = \sum_{n=0}^{\infty} S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{1.14}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Substituting $\lambda = 1$, the λ-array polynomials reduce to the array polynomials, $S^{(\alpha)}(n,k) = S_k^n(\alpha;1)$ (cf. [3,24]).

8° *The Humbert polynomials* $\{\Pi_{n,m}^{\lambda}\}_{n=0}^{\infty}$ were defined in 1921 by Humbert [12]. Their generating function is

$$(1 - mxt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} \Pi_{n,m}^{\lambda}(x)t^n. \tag{1.15}$$

This function satisfies the following recurrence relation (cf. [7,18,19] and references therein):

$$(n+1)\Pi_{n+1,m}^{\lambda}(x) - mx(n+\lambda)\Pi_{n,m}^{\lambda}(x) - (n + m\lambda - m + 1)\Pi_{n-m+1,m}^{\lambda}(x) = 0.$$

A special case of these polynomials are the Gegenbauer polynomials given as follows [8]:

$$C_n^{\lambda}(x) = \Pi_{n,2}^{\lambda}(x)$$

and also the Pincherle polynomials given as follows (see [12,23]):

$$\mathcal{P}_n(x) = \Pi_{n,3}^{-1/2}(x).$$

Later, Gould [9] studied a class of generalized Humbert polynomials, $P_n(m,x,y,p,C)$, defined by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,C)t^n,$$

where $m \geq 1$ is an integer and the other parameters are unrestricted in general (cf. [7], [10]).

Some special cases of the generalized Humbert polynomials, $P_n(m,x,y,p,C)$, can be given as follows (cf. [12]):

$$\begin{aligned}
P_n\left(2,x,1,-\frac{1}{2},1\right) &= \mathrm{P}_n(x) && \text{Legendre (1784)},\\
P_n(2,x,1,-\nu,1) &= C_n^{\nu}(x) && \text{Geganbauer (1874)},\\
P_n\left(3,x,1,-\frac{1}{2},1\right) &= \mathcal{P}_n(x) && \text{Pincherle (1890)},\\
P_n(m,x,1,-\nu,1) &= h_{n,m}^{\nu}(x) && \text{Humbert (1921)}.
\end{aligned}$$

9° *The Fibonacci-type polynomials in two variables* $(x,y) \mapsto \mathcal{G}_j(x,y;k,m,n)$ has been recently defined by Ozdemir and Simsek [21] by the following generating function

$$H(t;x,y;k,m,n)=\sum_{j=0}^{\infty}\mathcal{G}_j(x,y;k,m,n)t^j=\frac{1}{1-x^kt-y^mt^{m+n}}, \tag{1.16}$$

where $k,m,n\in\mathbb{N}_0$. An explicit formula for the polynomials $\mathcal{G}_j(x,y;k,m,n)$, $j=0,1,\ldots$, can be done in the following form [21]

$$\mathcal{G}_j(x,y;k,m,n)=\sum_{c=0}^{\left[\frac{j}{m+n}\right]}\binom{j-c(m+n-1)}{c}y^{mc}x^{jk-mck-nck},$$

where $[a]$ is the largest integer $\leq a$.

In this paper we give some new identities for the previous classes of polynomials and investigate some new properties of these polynomials. Moreover, by using their generating functions, we give some applications which are associated with the Fibonacci-type polynomials of higher order in two variables

The paper is organized as follows. Fibonacci-type polynomials of higher order in two variables and a new family of special polynomials $(x,y)\to\mathbb{G}_d(x,y;k,m,n)$, which includes several special cases, are introduced and studied in Sects. 2 and 3, respectively. Finally, Sect. 4 is devoted to a class of polynomials and corresponding numbers, obtained by a modification of the generating function of Humbert's polynomials.

2. Fibonacci-Type Polynomials of Higher Order in Two Variables

In this section we give a new generalization of the Fibonacci-type polynomials in two variables.

Definition 2.1. Two variable Fibonacci-type polynomials of higher order $(x,y)\mapsto\mathcal{G}_j^{(h)}(x,y;k,m,n)$ are defined by the following generating function

$$\sum_{j=0}^{\infty}\mathcal{G}_j^{(h)}(x,y;k,m,n)t^j=\frac{1}{(1-x^kt-y^mt^{n+m})^h}, \tag{2.1}$$

where h is a positive integer.

Observe that

$$\mathcal{G}_j^{(1)}(x,y;k,m,n)=\mathcal{G}_j(x,y;k,m,n).$$

We give now a computation formula of two variable Fibonacci-type polynomials of higher order h in the following statement.

Theorem 2.2. *We have*

$$\mathcal{G}_j^{(h_1+h_2)}(x,y;k,m,n)=\sum_{\ell=0}^{j}\mathcal{G}_\ell^{(h_1)}(x,y;k,m,n)\mathcal{G}_{j-\ell}^{(h_2)}(x,y;k,m,n). \tag{2.2}$$

Proof. Setting $h = h_1 + h_2$ into (2.1), we start with

$$\sum_{j=0}^{\infty} \mathcal{G}_j^{(h_1+h_2)}(x,y;k,m,n)t^j = \frac{1}{(1-x^k t - y^m t^{n+m})^{h_1}} \cdot \frac{1}{(1-x^k t - y^m t^{n+m})^{h_2}},$$

and then, using again (2.1), we get

$$\sum_{j=0}^{\infty} \mathcal{G}_j^{(h_1+h_2)}(x,y;k,m,n)t^j = \sum_{j=0}^{\infty} \mathcal{G}_j^{(h_1)}(x,y;k,m,n)t^j \sum_{j=0}^{\infty} \mathcal{G}_j^{(h_2)}(x,y;k,m,n)t^j.$$

Now, by using the Cauchy product in the right-hand side of the above equation, we obtain

$$\sum_{j=0}^{\infty} \mathcal{G}_j^{(h_1+h_2)}(x,y;k,m,n)t^j = \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \mathcal{G}_\ell^{(h_1)}(x,y;k,m,n)\mathcal{G}_{j-\ell}^{(h_2)}(x,y;k,m,n)t^j.$$

Finally, comparing the coefficients of t^j on both sides in the previous equality, we arrive at the desired result (2.2). □

Remark 2.3. Setting $h_1 = h_2 = 1$ in (2.2), we obtain the following formula for computing two variable Fibonacci-type polynomials of the second order,

$$\mathcal{G}_j^{(2)}(x,y;k,m,n) = \sum_{\ell=0}^{j} \mathcal{G}_\ell(x,y;k,m,n)\mathcal{G}_{j-\ell}(x,y;k,m,n).$$

If we take $x := ax$, $y = -1$, $k = 1$, $m = 1$, $n = a - 1$, (2.1) reduces to

$$\sum_{j=0}^{\infty} \mathcal{G}_j^{(h)}(ax,-1;1,1,a-1)t^j = \frac{1}{(1-axt+t^a)^h} = \sum_{j=0}^{\infty} \Pi_{j,a}^h(x)t^j.$$

Comparing the coefficients of t^j on both sides of the above equality, we obtain the following result.

Corollary 2.4. *A relation between two variable Fibonacci-type polynomials of higher order* $\mathcal{G}_j^{(h)}(x,y;k,m,n)$ *and Humbert polynomials* $\Pi_{n,m}^h(x)$ *is given by*

$$\mathcal{G}_j^{(h)}(ax,-1;1,1,a-1) = \Pi_{j,a}^h(x).$$

3. Special Polynomials Including Two Variable Fibonacci-Type Polynomials and Bernoulli and Euler-Type Polynomials

In this section, in order to introduce a new family of polynomials, we modify and unify the generating functions of the Fibonacci-type polynomials in two variables. By using these generating functions, we derive some relations and identities including the Apostol–Bernoulli numbers, the Bernoulli-type polynomials, the Humbert polynomials and the Genocchi polynomials. These relations and identities also include the Fibonacci-type polynomials in two variables.

Now, we introduce the generating function for these new special polynomials in two variables $(x, y) \mapsto \mathbb{G}_d(x, y; k, m, n)$, $d \geq 0$, with the three free parameters k, m, n.

Definition 3.1. The polynomials $\mathbb{G}_d(x, y; k, m, n)$ are defined by means of the following generating function

$$\begin{aligned} \mathbb{F}(z; x, y; k, m, n) &= \frac{1 - x^k - y^m}{1 - x^k \mathrm{e}^z - y^m \mathrm{e}^{z(m+n)}} \\ &= \sum_{d=0}^{\infty} \frac{\mathbb{G}_d(x, y; k, m, n)}{d!} \left(\frac{z}{1 - x^k - y^m} \right)^d . \end{aligned} \tag{3.1}$$

A recurrence relation for the polynomials $\mathbb{G}_d(x, y; k, m, n)$ can be proved.

Theorem 3.2. *Let $\mathbb{G}_0(x, y; k, m, n) = 1$ and d be a positive integer. Then we have*

$$\begin{aligned} \mathbb{G}_d(x, y; k, m, n) &= x^k \sum_{j=0}^{d} \binom{d}{j} \mathbb{G}_j(x, y; k, m, n)(1 - x^k - y^m)^{d-j} \\ &\quad + y^m \sum_{j=0}^{d} \binom{d}{j} \mathbb{G}_j(x, y; k, m, n)(m + n)^{d-j} (1 - x^k - y^m)^{d-j}. \end{aligned}$$

Proof. By applying the umbral calculus methods to (3.1), we get

$$\begin{aligned} 1 - x^k - y^m &= \sum_{d=0}^{\infty} \mathbb{G}_d(x, y; k, m, n) \frac{z^d}{(1 - x^k - y^m)^d d!} \\ &\quad - x^k \sum_{d=0}^{\infty} (\mathbb{G}(x, y; k, m, n) + 1 - x^k - y^m)^d \frac{z^d}{(1 - x^k - y^m)^d d!} \\ &\quad - y^m \sum_{d=0}^{\infty} \left(\mathbb{G}(x, y; k, m, n) + (m + n) \left(1 - x^k - y^m \right) \right)^d \frac{z^d}{(1 - x^k - y^m)^d d!}, \end{aligned}$$

with the usual convention of replacing $\mathbb{G}^d(x, y; k, m, n)$ by $\mathbb{G}_d(x, y; k, m, n)$. Comparing the coefficients of z^d on the both sides of the previous equality, we arrive at the desired result. □

A few first polynomials are

$$\begin{aligned} &\mathbb{G}_0(x, y; k, m, n) = 1, \quad \mathbb{G}_1(x, y; k, m, n) = x^k + (m + n)y^m, \\ &\mathbb{G}_2(x, y; k, m, n) = [x^k + (m + n)y^m]^2 - (m + n - 1)^2 x^k y^m + x^k + (m + n)^2 y^m, \end{aligned}$$

etc.

Now we consider special cases of the polynomials $\mathbb{G}_d(x, y; k, m, n)$. By using the generating function from (3.1), we derive some new identities and relations, which include the polynomials $\mathbb{G}_d(x, y; k, m, n)$, the Apostol–Bernoulli and the Apostol–Euler polynomials and numbers, as well as the classical Bernoulli, Euler and Genocchi polynomials and numbers.

Theorem 3.3. *Let $d \geq 1$. The polynomials $\mathbb{G}_d(x, y; k, m, n)$, $d \geq 1$, are connected with the Apostol–Bernoulli numbers $\mathcal{B}_d(\lambda)$ in the following way*

$$\mathbb{G}_{d-1}(x, y; k, 1, 0) = -\frac{(1 - x^k - y)^d}{d} \mathcal{B}_d(x^k + y), \quad d \geq 1. \tag{3.2}$$

Proof. First, according to (3.1) and (1.2), we have the following relation

$$\mathbb{F}(z; x, y; k, 1, 0) = -\frac{1 - x^k - y}{z} F_{AB}(0, z; x^k + y),$$

i.e.,

$$\sum_{d=0}^{\infty} \frac{\mathbb{G}_d(x, y; k, 1, 0)}{(1 - x^k - y)^d} \frac{z^d}{d!} = -\frac{1 - x^k - y}{z} \sum_{d=0}^{\infty} \mathcal{B}_d(x^k + y) \frac{z^d}{d!},$$

where we also used (1.3). However, since

$$\begin{aligned} \frac{z}{1 - x^k - y} \sum_{d=0}^{\infty} \frac{\mathbb{G}_d(x, y; k, 1, 0)}{(1 - x^k - y)^d} \cdot \frac{z^d}{d!} &= \sum_{d=0}^{\infty} \frac{(d+1)\mathbb{G}_d(x, y; k, 1, 0)}{(1 - x^k - y)^{d+1}} \cdot \frac{z^{d+1}}{(d+1)!} \\ &= \sum_{d=1}^{\infty} \frac{d\mathbb{G}_{d-1}(x, y; k, 1, 0)}{(1 - x^k - y)^d} \cdot \frac{z^d}{d!}, \end{aligned}$$

we have

$$\sum_{d=1}^{\infty} \frac{d\mathbb{G}_{d-1}(x, y; k, 1, 0)}{(1 - x^k - y)^d} \cdot \frac{z^d}{d!} = -\sum_{d=1}^{\infty} \mathcal{B}_d(x^k + y) \frac{z^d}{d!}, \tag{3.3}$$

because $\mathcal{B}_0(\lambda) = 0$.

Comparing the coefficients of $z^d/d!$ on both sides in (3.3), we obtain (3.2). □

By using the Apostol–Bernoulli numbers and the equality (3.2) we get another computation formula for the polynomials $\mathbb{G}_d(x, y; k, m, n)$. Thus,

$$\begin{aligned} \mathbb{G}_1(x, y; k, 1, 0) &= -\frac{(1 - x^k - y)^2}{2} \mathcal{B}_2(x^k + y) = x^k + y, \\ \mathbb{G}_2(x, y; k, 1, 0) &= -\frac{(1 - x^k - y)^3}{3} \mathcal{B}_3(x^k + y) = x^{2k} + 2x^k y + x^k + y^2 + y, \\ \mathbb{G}_3(x, y; k, 1, 0) &= -\frac{(1 - x^k - y)^4}{4} \mathcal{B}_4(x^k + y) = (x^k + y)[(x^k + y)^2 + 4(x^k + y) + 1], \\ \mathbb{G}_4(x, y; k, 1, 0) &= -\frac{(1 - x^k - y)^5}{5} \mathcal{B}_5(x^k + y) \\ &= (x^k + y)[(x^k + y)^3 + 11(x^k + y)^2 + 11(x^k + y) + 1], \quad \text{etc.} \end{aligned}$$

Theorem 3.4. *Let $d \geq 0$. The relation between the polynomials $\mathbb{G}_d(x, y; k, m, n)$ and the Apostol–Bernoulli polynomials $\mathcal{B}_d(x, \lambda)$ is given by*

$$\mathcal{B}_d(x, x^k) = -d \sum_{j=0}^{d-1} \binom{d-1}{j} \frac{x^{d-1-j}}{(1 - x^k)^{j+1}} \mathbb{G}_j(x, 0; k, m, n). \tag{3.4}$$

Proof. Starting with (1.6) and (3.1) for $y = 0$ $m \neq 0$, we conclude that

$$ze^{xz}\mathbb{F}(z;x,0;k,m,n) = ze^{xz}\frac{1-x^k}{1-x^k e^z} = (x^k-1)F_{AE}(x,z;x^k), \tag{3.5}$$

i.e.,

$$z\sum_{d=0}^{\infty}\frac{(xz)^d}{d!}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(x,0;k,m,n)}{(1-x^k)^{d+1}}\frac{z^d}{d!} = -\sum_{d=0}^{\infty}\mathcal{B}_d(x,x^k)\frac{z^d}{d!},$$

after replacing by the corresponding series representations. Now, using the Cauchy product on the left-hand side of the above equality, we obtain

$$\sum_{d=0}^{\infty}\sum_{j=0}^{d}\binom{d}{j}x^{d-j}\frac{\mathbb{G}_j(x,0;k,m,n)}{(1-x^k)^{j+1}}\frac{z^{d+1}}{d!} = -\sum_{d=0}^{\infty}\mathcal{B}_d(x,x^k)\frac{z^d}{d!},$$

i.e., (3.4). □

Remark 3.5. By using (3.5), the equality (3.4) can be also given in the following form:

$$\mathcal{B}_d(x,x^k) = -\sum_{j=1}^{d}\binom{d}{j}x^{d-j}\frac{j\,\mathbb{G}_{j-1}(x,0;k,m,n)}{(1-x^k)^j}.$$

Theorem 3.6. *The Euler polynomials $E_d(x)$ can be expressed in terms of the polynomials $\mathbb{G}_d(x,y;k,m,n)$ as*

$$E_d(x) = \sum_{j=0}^{d}\binom{d}{j}x^{d-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}. \tag{3.6}$$

Proof. As in the proof of Theorem 3.4, we assume that $m \neq 0$ and start with a special case of the generating function in (3.1), with $x = -1$, $y = 0$ and $k = 1$, i.e.,

$$\mathbb{F}(z;-1,0;1,m,n) = F_{AE}(0,t;1).$$

Then, by the generating function of the Euler polynomials $E_d(x)$ given by (1.8) (for $h = 1$), we conclude that

$$e^{xz}\,\mathbb{F}(z;-1,0;1,m,n) = F_{Eh}(x,z;1),$$

i.e.,

$$\sum_{d=0}^{\infty}\frac{(xz)^d}{d!}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(-1,0;1,m,n)}{2^d}\frac{z^d}{d!} = \sum_{d=0}^{\infty}E_d(x)\frac{z^d}{d!}$$

or

$$\sum_{d=0}^{\infty}\sum_{j=0}^{d}\binom{d}{j}x^{d-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}\frac{z^d}{d!} = \sum_{d=0}^{\infty}E_d(x)\frac{z^d}{d!},$$

from which we obtain (3.6). □

Theorem 3.7. *The relation between the polynomials $\mathbb{G}_d(x, y; k, m, n)$ and the Genocchi polynomials $G_d(x)$ is given by*

$$G_d(x) = d\sum_{j=0}^{d-1}\binom{d-1}{j}x^{d-1-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}. \tag{3.7}$$

Proof. Assuming that $m \neq 0$ and using (3.1) and (1.10), we have

$$z\mathrm{e}^{xz}\mathbb{F}(z;-1,0;1,m,n) = F_g(x;t),$$

i.e.,

$$z\sum_{d=0}^{\infty}\frac{(xz)^d}{d!}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(-1,0;1,m,n)}{2^d}\frac{z^d}{d!} = \sum_{d=0}^{\infty}G_d(x)\frac{z^d}{d!}.$$

Since $G_0(x) = 0$, after some standard manipulations, we obtain

$$\sum_{d=1}^{\infty}\left(d\sum_{j=0}^{d-1}\binom{d-1}{j}x^{d-1-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}\right)\frac{z^d}{d!} = \sum_{d=1}^{\infty}G_d(x)\frac{z^d}{d!},$$

i.e., (3.7). □

Remark 3.8. The relation (3.7) can be also expressed in the following form

$$G_d(x) = \sum_{j=1}^{d}\binom{d}{j}x^{d-j}\frac{j\,\mathbb{G}_{j-1}(-1,0;1,m,n)}{2^{j-1}}.$$

4. Modified Humbert Polynomials

In this section we modify the generating function of the Humbert polynomials in order to obtain the generating functions for some other families of special polynomials and numbers. We investigate certain properties of these generating functions and derive a few identities and relations which include the Apostol–Bernoulli and the Apostol–Euler numbers and polynomials, as well as the Bernoulli numbers of higher order, the array polynomials, and some other special numbers and polynomials.

First, we introduce a two-parameter family of the numbers $\{Y_n(\lambda;a)\}_{n\geq 0}$ by a generating function obtained from one of Humbert polynomials (1.15), by the substitution $(m, x, t, \lambda) \to (a, \lambda, \mathrm{e}^z, 1)$.

Definition 4.1. A family of the numbers $\{Y_n(\lambda;a)\}_{n\geq 0}$ is defined by

$$F(z;\lambda,a) = \frac{1}{1-a\lambda\mathrm{e}^z+\mathrm{e}^{az}} = \sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!}. \tag{4.1}$$

4.1. Computing Some Special Values of the Numbers $Y_n(\lambda, a)$

Here, we consider two special cases.

Case $a = 2$. Substituting $\lambda = 1$ and $a = 2$ into (4.1), after multiplication by z^2, we obtain

$$\sum_{n=0}^{\infty} Y_n(1,2)\frac{z^{n+2}}{n!} = \left(\frac{z}{e^z - 1}\right)^2 = \sum_{n=0}^{\infty} B_n^{(2)}\frac{z^n}{n!},$$

i.e.,

$$\sum_{n=2}^{\infty} Y_{n-2}(1,2)n(n-1)\frac{z^n}{n!} = \sum_{n=0}^{\infty} B_n^{(2)}\frac{z^n}{n!},$$

after using the series representation (1.1). Therefore, we have

$$B_n^{(2)} = n(n-1)Y_{n-2}(1,2), \quad n \neq 2,$$

where $B_n^{(2)}$ denotes the Bernoulli numbers of the second order.

Now, we are interested for a case when $\lambda = -\frac{1}{2}(\beta + \beta^{-1})$, where $\beta > 1$.

Theorem 4.2. *If $\beta > 1$ we have*

$$Y_n\left(-\tfrac{1}{2}(\beta + \beta^{-1}), 2\right) = \frac{1}{4}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j(0,\beta)\mathcal{E}_{n-j}(0,\beta^{-1}). \tag{4.2}$$

Proof. Starting from (4.1), for $a = 2$, $\lambda = -\frac{1}{2}(\beta + \beta^{-1})$, and $\beta > 1$, i.e.,

$$F\left(z; -\frac{1}{2}(\beta + \beta^{-1}), 2\right) = \frac{1}{4}F_{AE}(0,z;\beta)F_{AE}(0,z;\beta^{-1}),$$

and using the corresponding series representations (4.1) and (1.6), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Y_n\left(-\tfrac{1}{2}(\beta + \beta^{-1}), 2\right)\frac{z^n}{n!} &= \frac{1}{4}\left(\sum_{i=0}^{\infty}\mathcal{E}_i(0,\beta)\frac{z^i}{i!}\right)\left(\sum_{j=0}^{\infty}\mathcal{E}_j(0,\beta^{-1})\frac{z^j}{j!}\right)\\ &= \frac{1}{4}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j(0,\beta)\mathcal{E}_{n-j}(0,\beta^{-1})\right)\frac{z^n}{n!},\end{aligned}$$

i.e., (4.2). □

In a particular case for $\beta = 2$, the equality (4.2) reduces to the following identity

$$Y_n\left(-\tfrac{5}{4}, 2\right) = \frac{1}{4}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j(0,2)\mathcal{E}_{n-j}\left(0, \tfrac{1}{2}\right).$$

We give the following functional equations related to the numbers $Y_n(\lambda, a)$:

$$zF(z;\lambda,1) = -F_{AB}(0,z;\lambda - 1)$$

and

$$2F(z;\lambda,1) = F_{AE}(0,z;1-\lambda).$$

Combining the above equations with (4.1), (1.2) and also (1.6), we get

$$Y_{n-1}(\lambda,1) = -\frac{1}{n}\mathcal{B}_n(\lambda-1) \quad \text{and} \quad Y_n(\lambda,1) = \frac{1}{2}\mathcal{E}_n(0,1-\lambda).$$

4.2. A Recurrence Relation for the Numbers $Y_n(\lambda, a)$

By applying the Umbral calculus methods to (4.1), we find a recurrence relation for these numbers.

Theorem 4.3. *Let* $2 \neq a\lambda$ *and*

$$Y_0(\lambda,a) = \frac{1}{2-a\lambda}.$$

Then, for $n \geq 1$, *we have*

$$Y_n(\lambda,a) = \sum_{j=0}^{n}\binom{n}{j}(a\lambda - a^{n-j})Y_j(\lambda,a). \tag{4.3}$$

Proof. Starting from (4.1), we get

$$\sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!} - a\lambda\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!} + \sum_{n=0}^{\infty}\frac{(az)^n}{n!}\sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!} = 1.$$

Now, using the Cauchy product rule in the left-hand side of this equality, we obtain

$$\sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!} - a\lambda\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}Y_j(\lambda,a)\frac{z^n}{n!} + \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Y_j(\lambda,a)\frac{z^n}{n!} = 1.$$

Therefore,

$$\sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!} = 1 + \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(a\lambda - a^{n-j})Y_j(\lambda,a)\right)\frac{z^n}{n!}.$$

Comparing the coefficients of $z^n/n!$ on both sides of the above equality, we arrive at the desired result. □

According to (4.3), we can recursively compute the values of the numbers $Y_n(\lambda,a)$ for $a\lambda \neq 2$,

$$Y_n(\lambda,a) = \frac{1}{2-a\lambda}\sum_{j=0}^{n-1}\binom{n}{j}(a\lambda - a^{n-j})Y_j(\lambda,a).$$

This formula gives

$$\begin{aligned}
Y_0(\lambda,a) &= \frac{1}{2-a\lambda},\\
Y_1(\lambda,a) &= \frac{a\lambda - a}{(2-a\lambda)^2},\\
Y_2(\lambda,a) &= \frac{a^2\lambda^2 + (2-4a+a^2)a\lambda}{(2-a\lambda)^3},\\
Y_3(\lambda,a) &= \frac{a^3\lambda^3 + (8-12a+6a^2-a^3)a^2\lambda^2 + (4-12a+6a^2-2a^3)a\lambda + 2a^3}{(2-a\lambda)^4},
\end{aligned}$$

etc.

Remark 4.4. All numbers $Y_n(\lambda, a)$ are rational functions of real parameters a and λ, with a pole $\lambda = 2/a$ of order $n+1$.

4.3. A New Family of Polynomials $P_n(x; \lambda, a)$

By (4.1), we can define a new family of polynomials $P_n(x; \lambda, a)$ by means of the following generating function:

$$G(z; x; \lambda, a) = \mathrm{e}^{xz} F(z; \lambda, a),$$

i.e.,

$$G(z; x; \lambda, a) = \sum_{n=0}^{\infty} P_n(x; \lambda, a) \frac{z^n}{n!} = \frac{\mathrm{e}^{xz}}{1 - a\lambda \mathrm{e}^z + \mathrm{e}^{az}}. \tag{4.4}$$

Using (4.1), (4.4), as well as the numbers $Y_j(\lambda, a)$, we obtain the following representation of the polynomials $P_n(x; \lambda, a)$.

Theorem 4.5. *For $n \in \mathbb{N}_0$ we have*

$$P_n(x; \lambda, a) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} Y_j(\lambda, a).$$

Proof. According to (4.4), we have

$$\sum_{n=0}^{\infty} P_n(x; \lambda, a) \frac{z^n}{n!} = \left(\sum_{n=0}^{\infty} x^n \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} Y_n(\lambda, a) \frac{z^n}{n!} \right),$$

i.e.,

$$\sum_{n=0}^{\infty} P_n(x; \lambda, a) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} x^{n-j} Y_j(\lambda, a) \right) \frac{z^n}{n!}.$$

The last equality gives the desired result. □

Theorem 4.6. *For $n \geq 1$ we have*

$$\frac{\partial}{\partial x} P_n(x; \lambda, a) = n P_{n-1}(x; \lambda, a).$$

Proof. By differentiating the generating function (4.4) with respect to x, we conclude that

$$\frac{\partial}{\partial x} G(z; x; \lambda, a) = z G(z; x; \lambda, a).$$

Then, using the corresponding series representation, we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} P_n(x; \lambda, a) \frac{z^n}{n!} = \sum_{n=1}^{\infty} n P_{n-1}(x; \lambda, a) \frac{z^n}{n!},$$

from which the desired result directly follows. □

Theorem 4.7. *The following identities*

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} P_{n-2k}(x; \lambda, a) = \frac{1}{2} \left(P_n(x+1; \lambda, a) + P_n(x-1; \lambda, a) \right) \tag{4.5}$$

and

$$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} P_{n-2k-1}(x;\lambda,a) = \frac{1}{2}(P_n(x+1;\lambda,a) - P_n(x-1;\lambda,a)) \tag{4.6}$$

hold.

Proof. According to (4.4), we find that

$$G(z;x+y;\lambda,a) = \mathrm{e}^{yz} G(z;x;\lambda,a),$$

as well as the following equality

$$\sum_{n=0}^{\infty} P_n(x+y;\lambda,a)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} y^{n-j} P_j(x;\lambda,a)\right)\frac{z^n}{n!},$$

i.e.,

$$P_n(x+y;\lambda,a) = \sum_{j=0}^{n}\binom{n}{j} y^j P_{n-j}(x;\lambda,a).$$

Now, substituting $y = 1$ and $y = -1$ into this equality, we obtain

$$P_n(x+1;\lambda,a) = \sum_{j=0}^{n}\binom{n}{j} P_{n-j}(x;\lambda,a)$$

and

$$P_n(x-1;\lambda,a) = \sum_{j=0}^{n}(-1)^j\binom{n}{j} P_{n-j}(x;\lambda,a),$$

respectively. Finally, adding and subtracting these equalities we get the identities (4.5) or (4.6). □

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Some Notes on Weak Subdifferential

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Abstract. Some necessary conditions for having nonempty weak subdifferential of a function are presented and the positively homogeneous of the weak subdifferential operator is proved. Necessary and sufficient conditions for achieving a global minimum of a weak subdifferentiable function is stated, as well as a link between subdifferential and the Fréchet differential with a weak subdifferential. A result about the equality of the fuzzy sum rule inclusion is also investigated. Finally, some examples are included.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The notion of weak subdifferential which is a generalization of the classic subdifferential, is introduced by Azimov and Gasimov [1]. It uses explicitly defined supporting conic surfaces instead of supporting hyperplanes. Recall that a convex set has a supporting hyperplane at each boundary point. This leads to one of the central notions in convex analysis, that of a subgradient of a possible nonsmooth even extended real valued function [4]. The main reason of difficulties arising when passing from the convex analysis to the nonconvex one is that the nonconvex cases may arise in many different forms and each case may require a special approach. The main ingredient is the method of supporting the given nonconvex set. Subgradients plays an important role in deriving of optimality conditions and duality theorems. The first canonical generalized gradient was introduced by Clarke [4]. He applied the generalized gradient systematically to nonsmooth problems in a variety of problems. Since a nonconvex set has no supporting hyperline at each boundary point, the notion of subgradient have been generalized by most researchers on optimality conditions for nonconvex problems [3, 4]. By using the notion of subgradients, a collection of zero duality gap conditions for a wide class of nonconvex optimization problems was derived [1]. In this study we give some important properties of the weak subdifferentials. By using the definition and properties of the weak subdifferential which are described in [1, 2, 10, 11], we present some facts concerning weak subdifferential in the nonsmooth and nonconvex analysis. It is also obtained Necessary and sufficient optimality conditions by using the weak subdifferential.

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This paper is organized as follows. The definition and some preliminaries of the weak subdifferential are given in Section 2. In Section 3, some theorems connecting operations on the weak subdifferential in the non-smooth and non-convex analysis are provided. Also, a necessary condition in which a function attains its global minimum by applying weak subdifferential is stated.

2. Preliminaries

Throughout this paper let X be a real normed space and let X^* be the topological dual of X. By $\|\cdot\|$ we denote the norm of X and by $\langle x^*, x\rangle$ the value of the linear functional $x^* \in X^*$ at the point $x \in X$.

Definition 2.1 ([10, 11]). *Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. The set*

$$\partial f(\bar{x}) = \left\{x^* \in X^* : (\forall x \in X)\ \ f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle\right\}$$

is called the subdifferential of f at $\bar{x} \in X$.

Definition 2.2 ([10, 11]). *Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. A pair $(x^*, c) \in X^* \times \mathbb{R}^+$ where $\mathbb{R}^+$, the set of nonnegative real numbers, is called the weak subgradient of f at $\bar{x} \in X$ if the following inequality holds:*

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

The set

$$\partial^w f(\bar{x}) = \left\{(x^*, c) \in X^* \times \mathbb{R}^+ : (\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|\ \right\}$$

of all weak subgradients of f at $\bar{x} \in X$ is called the weak subdifferential of f at $\bar{x} \in X$. If $\partial^w f(\bar{x}) \neq \emptyset$, then f is called weakly subdifferentiable at $\bar{x}$.

Remark 2.3. *It is obvious from the definition of weak subgradient that if $\partial^w f(\bar{x})$ is nonempty then it contains uncountable members. Because if $(x^*, \bar{c}) \in \partial^w f(\bar{x})$, then we have*

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - \bar{c}\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X, \forall c \geq \bar{c}) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

which the last inequality means that $(x^*, c) \in \partial^w f(\bar{x})$. This completes proof of the assertion.

Remark 2.4. *It is clear that when f is subdifferentiable at $\bar{x}$, then f is also weakly subdifferentiable at $\bar{x}$; that is, if $x^* \in \partial f(\bar{x})$, then by the definition of weak subgradient we get $(x^*, c) \in \partial^w f(\bar{x})$ for every $c \geq 0$. But the following example shows that the converse may fail.*

Example 2.5. *Let $X = \mathbb{R}$ and $f(x) = -|x|$. Then it follows from the definition of weak subdifferential that*

$$(a, c) \in \partial^w f(0) \Longleftrightarrow (a, c) \in \mathbb{R} \times \mathbb{R}^+ \quad \textit{and} \quad (\forall x \in X) \quad -|x| \geq ax - c|x|.$$

Hence the weak subdifferential can be explicitly written as

$$\partial^w f(0) = \left\{(a, c) \in \mathbb{R} \times \mathbb{R}^+;\ |a| \leq c - 1\right\}.$$

On the other hand, it follows from the definition of the subdifferential that $\partial f(0) = \emptyset$.

Remark 2.6. *It follows from Definition 2.2 that the pair $(x^*, c) \in X^* \times \mathbb{R}^+$ is a weak subdifferential of f at $\bar{x} \in X$ if and only if there exists a continuous (super linear) concave function*

$$g(x) = \langle x^*, x - \bar{x}\rangle + f(\bar{x}) - c\|x - \bar{x}\|,$$

such that

$$(\forall x \in X) \quad g(x) \leq f(x) \quad \textit{and} \quad g(\bar{x}) = f(\bar{x}).$$

The class of weakly subdifferentiable functions are wider than the class of subdifferentiable functions. The weak subdifferential is a strong tool for studying nonconvex optimization problems, for instance, see [1, 12]. It is worth noting that the calculation of weak subdifferential by using its definition is not easy in general. The calculation of weak subdifferential for some functions is given in [14]. M.Kucuk, et al. presented the very useful method for calculation of weak subdifferential of functions that represented as the infimum of support functions, the functions that represented as difference of two sublinear functions , and convex functions.

Definition 2.7 ([13]). *A function $f : X \to \mathbb{R}$ is called locally Lipschitz at $\bar{x} \in X$ if there exist a nonnegative number L (Lipschitz constant) and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that*

$$(\forall x \in N(\bar{x})) \qquad |f(x) - f(\bar{x})| \leq L\|x - \bar{x}\|.$$

If the above inequality holds for all $x \in X$, then f is called Lipschitz with the Lipschitz constant L.

Theorem 2.8 ([10]). *Let the weak subdifferential of $f : X \to \mathbb{R}$ at $\bar{x}$ be nonempty. Then the set $\partial^w f(\bar{x})$ is closed and convex.*

3. Main Result

In this section we follow the main results given in [10]. In the sequel we need the following definition .

Definition 3.1 ([13]). *A function $f : X \to (-\infty, +\infty]$ is lower semicontinuous at $\bar{x} \in X$ if*

$$x_n \to \bar{x} \ \to \ \liminf f(x_n) \geq f(\bar{x}).$$

It is worth noting that Definition 3.1 was called sequentially lower semicontinuity by some authors while they defined the lower semicontinuity of f at the point $\bar{x} \in X$ as

$$\liminf_{x \to \bar{x}} \ f(x) \geq f(\bar{x}).$$

It is clear that the lower semicontinuity at a point implies the sequentially lower semicontinuity at the point.

The next result provides a necessary condition for weak subdifferentiability of a function at a point.

Proposition 3.2. *Let f be weak subdifferentiable at $\bar{x} \in X$. Then f is lower semicontinuous at $\bar{x} \in X$.*

Proof. The weak subdifferentiability of f at $\bar{x}$ implies that $\partial^w f(\bar{x}) \neq \emptyset$. Hence there exists the pair $(x^*, c) \in X^* \times \mathbb{R}^+$ such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

The result follows by taking the limit inferior of the both sides of the last inequality when $x \to \bar{x}$. □

The following example shows that the converse of Proposition 3.2 may fail.

Example 3.3. *Let $X = \mathbb{R}$ and $f(x) = -x^2$. It is easy to see that $\partial^w f(0) = \emptyset$ while f is a continuous function.*

The next definition is important in this paper.

Definition 3.4 ([7]). *Let $f : X \to \mathbb{R}$ be a function. If there is a continuous linear map $f'(\bar{x}) : X \to \mathbb{R}$ with the property*

$$\lim_{\|h\| \to 0} \frac{|f(\bar{x} + h) - f(\bar{x}) - \langle f'(\bar{x}), h\rangle|}{\|h\|} = 0,$$

then $f'(\bar{x}) : X \to \mathbb{R}$ is called the Fréchet derivative of f at $\bar{x} \in X$ and f is called the Fréchet differentiable at $\bar{x}$.

The next conclusion provides a link between Fréchet differentiability and weak subdifferentiability of a function.

Proposition 3.5. *Assume $f : X \to \mathbb{R}$ is subdifferentiable and Fréchet differentiable at $\bar{x}$. Then*

$$\{(f'(\bar{x}), c); c \geq 0\} \subset \partial^w f(\bar{x}).$$

Proof. Since f is subdifferentiable at $\bar{x} \in X$, then there exists $x^* \in \partial f(\overline{x}) \subset X^*$ such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle.$$

By taking

$$x = \bar{x} + te \ \text{ s.t. } \ t \geq 0, \ \ e \in X, \ \ \|e\| = 1,$$

we get

$$f(\bar{x} + te) - f(\bar{x}) \geq \langle x^*, te \rangle.$$

Therefore,

$$\frac{f(\bar{x} + te) - f(\bar{x})}{t} \geq \frac{\langle x^*, te \rangle}{t}.$$

Now it is obvious from Fréchet differentiability of f at $\bar{x}$, by letting $t \to 0^+$, that

$$\langle x^* - f'(\bar{x}), e \rangle \leq 0.$$

Hence $x^* = f'(\bar{x})$ and $f'(\bar{x}) \in \partial f(\bar{x})$. Then $f'(\bar{x}) \in \partial f(\overline{x})$ and so it follows from Remark 2.4 that

$$\{(f'(\bar{x}), c); c \geq 0\} \subset \partial^w f(\bar{x}).$$

This completes the proof. □

The following example shows that the conclusion in Proposition 3.5 may be strict.

Example 3.6. *Let $X = \mathbb{R}, f \equiv 0$ and $\bar{x} = 0$. Then by the definition of weak subdifferential and Fréchet differentiability of f at $\bar{x}$ we have, respectively,*

$$\partial^w f(0) = \{(a, c) \in \mathbb{R} \times \mathbb{R}^+ \, ; |a| \leq c\}$$

and

$$A = \{(f'(0), c); c \geq 0\} = \{(0, c); c \geq 0\}.$$

It is clear that $A \subsetneq \partial^w f(0)$.

The following example shows that the subdifferentiability of f at $\bar{x}$ in Proposition 3.5 is essential.

Example 3.7. *Let $X = \mathbb{R}$ and $f(x) = -x^2$. Then it is easy to verify that*

$$\partial f(0) = \emptyset, \quad \partial^w f(\bar{x}) = \emptyset \quad \text{and} \quad f'(0) = 0.$$

Remark 3.8. *It is well known that if f is convex and Fréchet differentiable at $\bar{x}$ then $\partial f(\bar{x}) = \left\{f'(\bar{x})\right\}$. Hence by Proposition 3.5 f is weak subdifferentiable at $\bar{x}$.*

The next result gives a characterization of having global minimum for a weakly subdifferentiable function.

Proposition 3.9. *Suppose $f : X \to (-\infty, +\infty]$ is weakly subdifferentiable at $\bar{x} \in X$. Then f has a global minimum at $\bar{x}$ if and only if $(0, c) \in \partial^w f(\bar{x})$ for all $c \geq 0$.*

Proof. The proof directly follows from the definition of weak subdifferentiability of f at $\bar{x} \in X$. □

The next conclusion asserts that the operator weak subdifferential (∂^w) is positively homogeneous.

Proposition 3.10. *Let $f : X \to \mathbb{R}$ be weakly subdifferentiable at $\bar{x} \in X$. Then*

$$(\forall \alpha > 0) \qquad \partial^w(\alpha f)(\bar{x}) = \alpha \partial^w f(\bar{x}).$$

Proof. If $(x^*, c) \in \alpha \partial^w f(\bar{x})$, then

$$\frac{1}{\alpha}(x^*, c) \in \partial^w f(\bar{x}).$$

Hence

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \left\langle \frac{x^*}{\alpha}, x - \bar{x} \right\rangle - \frac{c}{\alpha} \|x - \bar{x}\|.$$

Thus,

$$(\forall x \in X) \qquad \alpha f(x) - \alpha f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \bar{c} \|x - \bar{x}\|.$$

This means that $\alpha \partial^w f(\bar{x}) \subset \partial^w \alpha f(\bar{x})$. Now we prove that the converse of the inclusion. Since αf with the first part of proof is weakly subdifferentiable at $\bar{x}$, then there exists a pair $(x^*, c) \in \partial^w \alpha f(\bar{x})$ such that

$$(\forall x \in X) \qquad \alpha f(x) - \alpha f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \bar{c} \|x - \bar{x}\|.$$

Hence

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \left\langle \frac{x^*}{\alpha}, x - \bar{x} \right\rangle - \frac{c}{\alpha} \|x - \bar{x}\|.$$

This implies that

$$\left(\frac{x^*}{\alpha}, \frac{c}{\alpha} \right) \in \partial^w f(\bar{x}).$$

Consequently, $(x^*, c) \in \alpha \partial^w f(\bar{x})$ and therefore $\partial^w \alpha f(\bar{x}) \subset \alpha \partial^w f(\bar{x})$. This completes the proof. □

Remark 3.11. *Note that $\partial^w(f(\alpha \bar{x})) = \partial^w \alpha f(\bar{x})$ may drop. Consider $X = \mathbb{R}$, $\bar{x} = 1$, $\alpha = \sqrt{2}$, and define*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c, \\ 0, & x \in \mathbb{Q}. \end{cases}$$

Then we have

$$\partial^w f(1) = \left\{ (a, c) \in \mathbb{R} \times \mathbb{R}^+ ; \, |a| \leq c \right\}, \quad \partial^w f(\sqrt{2}) = \emptyset.$$

Now we are interested to find a sufficient condition that the following equality holds.

Proposition 3.12. *If f is a positively homogeneous function and weak subdifferentiable at $\bar{x}$ and $\alpha \bar{x}$, where α is a positive real number, then*

$$\partial^w(f(\alpha \bar{x})) = \partial^w f(\bar{x}).$$

Proof. It follows from the hypothesis that

$$\begin{aligned}(x^*, c) \in \partial^w f(\alpha\bar{x}) &\iff f(\alpha x) - f(\alpha\bar{x}) \geq \langle x^*, \alpha x - \alpha\bar{x}\rangle - c\|\alpha x - \alpha\bar{x}\| \\ &\iff \alpha(f(x) - f(\bar{x})) \geq \alpha(\langle x^*, x - \bar{x}\rangle - \bar{c}\|x - \bar{x}\|) \\ &\iff (x^*, c) \in \partial^w f(\bar{x}).\end{aligned}$$

This completes the proof. □

In the next we recall the fuzzy sum rule and we investigate sufficient condition which the equality holds.

Proposition 3.13 ([10]). *If $f_1 : X \to \mathbb{R}$ and $f_2 : X \to \mathbb{R}$ are weak subdifferential at $\bar{x}$, then $f_1 + f_2$ is weak subdifferential at $\bar{x}$ and*

$$\partial^w f_1(\bar{x}) + \partial^w f_2(\bar{x}) \subseteq \partial^w (f_1 + f_2)(\bar{x}).$$

Remark 3.14. *The simple example $X = \mathbb{R}$, $f_1(x) = \sin x$, $f_2(x) = -\sin x$, $\bar{x} = 0$, shows that the inclusion of Proposition 3.13 may be strict.*

The following proposition provides sufficient conditions in which the equality of Proposition 3.13 holds.

Proposition 3.15. *Assume that $f_1 : X \to \mathbb{R}$ is weak subdifferentiable at $\bar{x}$, $f_2 : X \to \mathbb{R}$ is subdifferentiable and Fréchet differentiable at $\bar{x}$ and $-f_2$ is subdifferentiable at $\bar{x}$. Then*

$$\partial^w f_1(\bar{x}) + \partial^w f_2(\bar{x}) = \partial^w (f_1 + f_2)(\bar{x}).$$

Proof. If $(x^*, c) \in \partial^w (f_1 + f_2)(\bar{x})$, then

$$(\forall x \in X) \qquad (f_1 + f_2)(x) - (f_1 + f_2)(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Since $f_2 : X \to \mathbb{R}$ is Fréchet differentiable at $\bar{x}$ and $-f_2$ is subdifferentiable at $\bar{x}$, we get, see Proposition 3.5,

$$(\forall x \in X) \qquad -f_2(x) + f_2(\bar{x}) \geq \langle -f_2'(\bar{x}), x - \bar{x}\rangle.$$

It follows from the first inequality that

$$(\forall x \in X) \qquad (f_1(x) - f_1(\bar{x})) + (f_2(x) - f_2(\bar{x})) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X) \qquad (f_1(x) - f_1(\bar{x})) \geq -(f_2(x) - f_2(\bar{x})) + \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Now the hypothesis implies

$$f_1(x) - f_1(\bar{x}) \geq \langle(-f_2'(\bar{x}), x - \bar{x})\rangle + \langle x^*, x - \bar{x}\rangle \ - c\|x - \bar{x}\|.$$

Therefore

$$(f_1(x) - f_1(\bar{x})) \geq \langle x^* - f_2'(\bar{x}), x - \bar{x})\rangle - c\|x - \bar{x}\|.$$

Then

$$(x^* - f_2'(\bar{x}), c) \in \partial^w f_1(\bar{x}) \ , \ (f_2'(\bar{x}), 0) \in \partial^w f_2(\bar{x}).$$

This means that

$$\partial^w (f_1 + f_2)(\bar{x}) \subseteq \partial^w f_1(\bar{x}) + \partial^w f_2(\bar{x}).$$

The reverse side of the inclusion follows from Proposition 3.13 and so the proof is completed. □

Corollary 3.16. *If for all but at most one of the weak subdifferentiable functions f_i at $\bar{x}$, $f_i, -f_i$ are Fréchet differentiable and subdifferentiable at $\bar{x}$, then*

$$\sum_{i=1}^{n} \partial^{w} f_i(\bar{x}) = \partial^{w}\left(\sum_{i=1}^{n} f_i\right)(\bar{x}).$$

Remark 3.17. *It is easy to check that if $f : X \to \mathbb{R}$ is Fréchet differentiable at $\bar{x}$, then $f, -f$ are subdifferentiable at $\bar{x}$ if and only if*

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) = \langle f'(\bar{x}), x - \bar{x}\rangle.$$

Proposition 3.18. *Let $f_1 : X \to \mathbb{R}$ be a function, $-f_1$ be Fréchet differentiable and subdifferentiable at $\bar{x}$ and $f_2 : X \to \mathbb{R}$ be a function. If $f_1 + f_2$ attains a global minimum at $\bar{x}$, then $(-f_1'(\bar{x}), 0) \in \partial^{w} f_2(\bar{x})$.*

Proof. Since $f_1 + f_2$ attains a global minimum at $\bar{x}$ then

$$(\forall x \in X) \qquad (f_1 + f_2)(x) \geq (f_1 + f_2)(\bar{x})$$

and so we can rewrite the inequality as

$$(\forall x \in X) \qquad f_2(x) - f_2(\bar{x}) \geq f_1(\bar{x}) - f_1(x).$$

Hence the subdifferentability and Fréchet differentiabability of $-f_1$, similar to the proof of Proposition 3.15, imply that

$$(\forall x \in X) \qquad f_2(x) - f_2(\bar{x}) \geq \langle -f_1'(\bar{x}), x - \bar{x}\rangle.$$

This means that

$$(-f_1'(\bar{x}), 0) \in \partial^{w} f_2(\bar{x})$$

and so the proof is completed. □

Proposition 3.19. *Let $f : X \to \mathbb{R}$ be weak subdifferentiable at $\bar{x}$ and $g - f$ attain a global minimum at $\bar{x}$. Then*

$$\partial^{w} f(\bar{x}) \subset \partial^{w} g(\bar{x}).$$

Proof. The weak subdifferentiability of f at $\bar{x}$ implies that $\partial^{w} f(\bar{x}) \neq \emptyset$. Hence there exists $(x^*, c) \in X^* \times \mathbb{R}^+$ such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Since $g - f$ attains a global minimum at $\bar{x}$ then

$$(\forall x \in X) \qquad (g - f)(x) \geq (g - f)(\bar{x}).$$

Therefore,

$$(\forall x \in X) \qquad g(x) - g(\bar{x}) \geq f(x) - f(\bar{x}).$$

Consequently, the above inequalities imply that

$$(\forall x \in X) \qquad g(x) - g(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

This means that $(x^*, c) \in \partial^{w} g(\bar{x})$, which is the desired result and the proof is completed. □

Corollary 3.20. *If f attains a global minimum at $\bar{x}$ then $\partial^{w} f(\bar{x})$ contains the weak subdifferentiable of the zero function at $\bar{x}$, that is*

$$\partial^{w} 0(\bar{x}) \subset \partial^{w} f(\bar{x}).$$

Example 3.21. *The example, $f(x) = -|x|$ for all $x \in \mathbb{R}$ and $\overline{x} = 0$ shows that the condition $\overline{x}$ is a global minimum of f in the previous corollary is essential.*

Proposition 3.22. *If $g - f$ is a constant function on X, then*

$$(\forall \bar{x} \in X) \qquad \partial^w f(\bar{x}) = \partial^w g(\bar{x}).$$

Proof. From

$$(\forall x \in X) \qquad (f-g)(x) \geq (f-g)(\bar{x})$$

by Proposition 3.19 we have

$$\partial^w g(\bar{x}) \subset \partial^w f(\bar{x}).$$

Similarly it follows from

$$(\forall x \in X) \qquad (g-f)(x) \geq (g-f)(\bar{x})$$

that

$$\partial^w f(\bar{x}) \subset \partial^w g(\bar{x}).$$

This completes the proof. □

Let Y be a real normed space and Y^* denote the topological dual space of Y. For any $y^* \in Y^*$, we consider the scalar function $\langle y^*, h\rangle$ is defined by the equality

$$(\forall u \in X) \qquad \langle y^*, h\rangle(u) = \langle y^*, h(u)\rangle,$$

where $h : X \to Y$ is a function and X is a real normed space.

Let $g : Y \to \mathbb{R}$ be a function and $\bar{y} = h(\bar{x})$. In the next result we will concentrate on the composition $f(u) = g(h(u)), u \in X$, and the projection operator $\pi : X^* \times \mathbb{R} \to X^*$, such that $\pi(x^*, t) = x^*$ for all $(x^*, t) \in X^* \times \mathbb{R}$.

Proposition 3.23. *Assume that g is weak subdifferentiable at $\bar{y}$ and $\langle y^*, h\rangle$ is weak subdifferentiable at $\bar{x}$ for some $y^* \in \pi(\partial^w g(\bar{y}))$. If h is locally Lipschitz at $\bar{x}$ with the constant Lipschitz L, then f is weak subdifferentiable at $\bar{x}$ and*

$$\pi(\partial^w \langle y^*, h\rangle(\bar{x})) \subset \pi(\partial^w f(\bar{x})).$$

Proof. If $w \in \pi(\partial^w \langle y^*, h\rangle(\bar{x}))$ then there exists a nonnegative number c such that

$$(\forall x \in X) \quad \langle y^*, h\rangle(x) - \langle y^*, h\rangle(\bar{x}) \geq \langle w, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Since $y^* \in \pi(\partial^w g(\bar{y}))$ then there exists $\bar{c} \geq 0$ such that

$$(\forall y \in Y) \quad g(y) - g(\bar{y}) \geq \langle y^*, y - \bar{y}\rangle - \bar{c}\|y - \bar{y}\|,$$

and so

$$(\forall x \in X) \quad g(h(x)) - g(h(\bar{x})) \geq \langle y^*, h(x) - h(\bar{x})\rangle - \bar{c}\|h(x) - h(\bar{x})\|.$$

This means that

$$\begin{aligned} f(x) - f(\bar{x}) &\geq \langle y^*, h(x) - h(\bar{x})\rangle - \bar{c}\|h(x) - h(\bar{x})\| \\ &\geq \langle w, x - \bar{x}\rangle - c\|x - \bar{x}\| - \bar{c}L\|x - \bar{x}\| \\ &= \langle w, x - \bar{x}\rangle - (c + \bar{c}L)\|x - \bar{x}\|, \end{aligned}$$

then $(w, c + \bar{c}L) \in \partial^w f(\bar{x})$. Consequently, $w \in \pi(\partial^w f(\bar{x}))$. This completes the proof. □

It is worth noting that the conclusion of Proposition 3.23 can be rewritten in the following form:

$$\bigcup\Big\{\pi(\partial^w\langle y^*,h\rangle(\bar{x})) : y^*\in\partial^* g(\bar{y})\Big\}\subset\pi(\partial^w f(\bar{x})).$$

Proposition 3.24. *If f and $-g$ is weak subdifferentiable, respectively, at $\bar{x}$ and $\bar{y}$. If h is Lipschitz function with the constant Lipschitz L, then for any $y^*\in\pi(\partial^w(-g(\bar{y})))$ the function $\langle y^*,h\rangle$ is weak subdifferential at $\bar{x}$ and*

$$\pi(\partial^w f(\bar{x}))\subset\pi(\partial^w\langle -y^*,h\rangle(\bar{x})).$$

Proof. If $x^*\in\pi(\partial^w f(\bar{x}))$, then there exists a nonnegative number c such that

$$(\forall x\in X)\qquad f(x)-f(\bar{x})\geq\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|.$$

Also if $(y^*,\bar{c})\in\partial^w(-g)(\bar{y})$, then we have

$$(\forall y\in Y)\qquad -g(y)+g(\bar{y})\geq\langle y^*,y-\bar{y}\rangle-\bar{c}\|y-\bar{y}\|.$$

Consequently,

$$(\forall y\in Y)\qquad -\langle y^*,h\rangle(x)+\langle y^*,h\rangle(\bar{x})\ \geq\ g(y)-g(\bar{y})-\bar{c}\|y-\bar{y}\|.$$

Therefore,

$$\begin{aligned}-\langle y^*,h\rangle(x)+\langle y^*,h\rangle(\bar{x}) &\geq f(x)-f(\bar{x})-\bar{c}\|h(x)-h(\bar{x})\|\\ &\geq \langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|-\bar{c}\|h(x)-h(\bar{x})\|.\end{aligned}$$

Thus,

$$\begin{aligned}-\langle y^*,h\rangle(x)+\langle y^*,h\rangle(\bar{x}) &\geq \langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|-\bar{c}L\|x-\bar{x}\|\\ &= \langle x^*,x-\bar{x}\rangle-(c+\bar{c}L)\|x-\bar{x}\|.\end{aligned}$$

This means that $(x^*,c+\bar{c}L)\in\partial^w(\langle -y^*,h\rangle(\bar{x}))$. Hence

$$x^*\in\pi(\partial^w(\langle -y^*,h\rangle(\bar{x})).$$

This completes the proof. □

By combining Propositions 3.23 and 3.24 we obtain the following result.

Corollary 3.25. *Let f be weak subdifferentiable at $\bar{x}$ and g be Fréchet differentiable at $\bar{y}$, and g , $-g$ is subdifferentiable at $\bar{y}$. If h is locally Lipschitz function with the constant Lipschitz L at $\bar{x}$, then*

$$\pi(\partial f(\bar{x}))=\pi(\partial^w\langle g'(\bar{y}),h\rangle(\bar{x})).$$

In the following we present some examples.

Example 3.26. *The example*

$$f(x)=\begin{cases}1, & x\in\mathbb{Q}^c,\\ 0, & x\in\mathbb{Q},\end{cases}\qquad g(x)=\begin{cases}0, & x\in\mathbb{Q}^c,\\ 1, & x\in\mathbb{Q},\end{cases}$$

shows that the weak subdifferentability of $f\circ g$ at $\bar{x}$ may not imply the weak subdifferentability of f and g at $\bar{x}$.

The next example shows that the composition of two weak subdifferentiable functions is not necessarily weak subdifferentiable.

Example 3.27. *Take $f(x) = x^2$ and $g(x) = -x$. Then f, g are weak subdifferentiable at $\bar{x} = 0$, but $(g \circ f)(x) = -x^2$ is not weak subdifferentiable at $\bar{x} = 0$.*

The next example shows that the product of two weak subdifferentiable functions is not necessarily weak subdifferentiable.

Example 3.28. *Let $f(x) = x$, $g(x) = -x$. Then f, g are weak subdifferentiable at $\bar{x} = 0$ while $(fg)(x) = -x^2$ is not weak subdifferentiable at $\bar{x} = 0$.*

The next example shows that the weak subdifferentability of fg at $\bar{x}$ may not imply the weak subdifferentability of f and g at $\bar{x}$.

Example 3.29. *Consider*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c, \\ 0, & x \in \mathbb{Q}, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & x \in \mathbb{Q}^c, \\ 1, & x \in \mathbb{Q}. \end{cases}$$

Then g is not weak subdifferentiable at $x = 0$ while $(fg)(x) = 0$ is weak subdifferentiable at each point of the real number.

Proposition 3.30. *If all f_i, $i \in I$ (I is a finite nonempty set) and $f(u) = \sup_{i \in I} f_i(u)$, $u \in X$, are finite at $\bar{x}$, then the closure of the convex hull of the set $\bigcup_{i \in I_0(\bar{x})} \partial^w f_i(\bar{x})$ is a subset of $\partial^w f(\bar{x})$, i.e.,*

$$\mathrm{cl}\left(\mathrm{co}\left(\bigcup_{i \in I_0(\bar{x})} \partial^w f_i(\bar{x})\right)\right) \subset \partial^w f(\bar{x}),$$

where $I_0(\bar{x}) = \Big\{ i \in I \,:\, f_i(\bar{x}) = f(\bar{x}) \Big\}$.

Proof. Suppose that

$$\sum_{i \in I_0(\bar{x})} \alpha_i (x_i^*, c_i) \in \mathrm{co} \bigcup_{i \in I_0(\bar{x})} \partial^w f_i(\bar{x}),$$

such that $\sum_{i \in I_0(\bar{x})} \alpha_i = 1$, $\alpha_i \geq 0$, $(x_i^*, c_i) \in \partial^w f_i(\bar{x})$. Then we have

$$\Big(\forall x \in X,\ \forall i \in I_0(\bar{x})\Big) \qquad f_i(x) - f_i(\bar{x}) \geq \langle x_i{}^*, x - \bar{x} \rangle - c_i \|x - \bar{x}\|.$$

Therefore,

$$(\forall x \in X) \qquad \sum_{i \in I_0(\bar{x})} \alpha_i f_i(x) - \sum_{i \in I_0(\bar{x})} \alpha_i f_i(\bar{x}) \geq \sum_{i \in I_0(\bar{x})} \alpha_i \langle x_i{}^*, x - \bar{x} \rangle - \sum_{i \in I_0(\bar{x})} \alpha_i c_i \|x - \bar{x}\|.$$

Since $f(x) = \sup_{i \in I} f_i(x)$, $x \in X$, we have $I_0(\bar{x}) = \Big\{ i \in I \,:\, f_i(\bar{x}) = f(\bar{x}) \Big\}$, so that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \Big\langle \sum_{i \in I_0(\bar{x})} \alpha_i x_i{}^*, x - \bar{x} \Big\rangle - \sum_{i \in I_0(\bar{x})} \alpha_i c_i \|x - \bar{x}\|$$

and

$$\sum_{i \in I_0(\bar{x})} \alpha_i (x_i{}^*, c_i) \in \partial^w f(\bar{x}).$$

Consequently,

$$\mathrm{co}\left(\bigcup_{i\in I_0(x)} \partial^w f_i(\bar{x})\right) \subset \partial^w f(\bar{x}).$$

Now the clossedness of the set $\partial^w f(\bar{x})$ completes the proof. □

The next proposition states necessary conditions that with them a weakly subdifferentiable function obtains a global maximum.

Proposition 3.31. *Let f at $\bar{x}$ attain a global maximum. If $(x^*, c) \in \partial^w f(\bar{x})$, then $\|x^*\| \leq c$.*

Proof. Since f has a global maximum at $\bar{x}$, then we have

$$(\forall x \in X) \qquad f(x) \leq f(\bar{x}).$$

It follows from $(x^*, c) \in \partial^w f(\bar{x})$, that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X) \qquad 0 \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

Consequently

$$(\forall x \in X) \qquad \langle x^*, x - \bar{x}\rangle \leq c\|x - \bar{x}\|,$$

and so

$$\|x^*\| \leq c.$$

This completes the proof. □

Recall that in [11, 12, 14, 15], the well-known theorem about the representation of the directional derivative of the convex functions as a point wise maximum of subgradients of that function is generalized to a nonconvex case by using the notion of a subgradient. They worked on special class of invex functions that includes the class of convex functions. It should be noted that the results given in [12] is a generalization of the results presented in [11] for a special class of invex functions .The optimality condition formulated in [12], guarantees the existence of the weak subgradient, that is the pair consisting of some linear functional and some real number such that the graph of the homogeneous function defined by this paper, is a conical surface which separates the optimal point from the given (non convex) set. In the sequel we establish a new version of the main result of [12], for the Fréchet differentiable functions in the setting of infinite dimensional normed spaces.

Proposition 3.32. *If f is subdifferentiable and Fréchet differentiable at $\bar{x}$, then f has a global minimum at $\bar{x}$ if and only if*

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x - \bar{x}\rangle = 0.$$

Proof. Suppose that f has a global minimum at $\bar{x}$, then we have

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq 0.$$

From the Fréchet differentiability of f at $\bar{x}$, we get

$$\lim_{\|h\|\to 0} \frac{|f(\bar{x} + h) - f(\bar{x}) - \langle f'(\bar{x}), h\rangle|}{\|h\|} = 0.$$

If we take

$$h = \lambda(x - \bar{x}),$$

then we obtain

$$0 = \lim_{\lambda \to 0^+} \frac{f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) - \langle f'(\bar{x}), \lambda(x - \bar{x})\rangle}{\|\lambda(x - \bar{x})\|}$$

and so, since $\bar{x}$ is a global minimum of f, we have

$$0 \geq \lim_{\lambda \to 0^+} \frac{-\langle f'(\bar{x}), \lambda(x - \bar{x})\rangle}{\|\lambda(x - \bar{x})\|}.$$

Consequently, by the linearity of $f'(\bar{x})$, we can deduce that

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x - \bar{x}\rangle = 0.$$

Conversely, by using our assumptions, we have

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle f'(\bar{x}), x - \bar{x}\rangle = 0.$$

Then

$$(\forall x \in X) \qquad f(x) \geq f(\bar{x})$$

and this shows that $\bar{x}$ is a global minimum of f. Hence the proof is completed. □

Proposition 3.33. *If f is subdifferentiable and Fréchet differentiable at $\bar{x}$, then f is weakly subdifferentiable at $\bar{x}$ if and only if $f'(\bar{x})$ is weakly subdifferentiable at 0, the zero element of X, and*

$$\partial^w(f(\bar{x})) = \partial^w(f'(\bar{x}))(0).$$

Proof. From the Fréchet differentiability f at $\bar{x}$, we have

$$\lim_{\|h\| \to 0} \frac{|f(\bar{x} + h) - f(\bar{x}) - \langle f'(\bar{x}), h\rangle|}{\|h\|} = 0.$$

By taking

$$h = \lambda(x - \bar{x})$$

and by using the weak subdifferentiability of f at $\bar{x}$, there exist $(x^*, c) \in \partial^w f(\bar{x})$, such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \;\geq\; \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Hence

$$0 = \lim_{\lambda \to 0^+} \frac{f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) - \langle f'(\bar{x}), \lambda(x - \bar{x})\rangle}{\|\lambda(x - \bar{x})\|}$$

and from the weak subdifferentiability of f at $\bar{x}$ we get

$$(\forall x \in X) \qquad 0 \geq \lim_{\lambda \to 0^+} \frac{\langle x^*, \lambda(x - \bar{x})\rangle - c\|\lambda(x - \bar{x})\| - \langle f'(\bar{x}), \lambda(x - \bar{x})\rangle}{\|\lambda(x - \bar{x})\|}$$

and equally

$$(\forall x \in X) \qquad \frac{\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| - \langle f'(\bar{x}), x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0.$$

Therefore,

$$(\forall x \in X) \qquad \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| - \langle f'(\bar{x}), x - \bar{x}\rangle \leq 0$$

and so by taking $z = x - \bar{x}$, we obtain

$$(\forall z \in X) \qquad \langle f'(\bar{x}), z\rangle \geq \langle x^*, z\rangle - c\|z\|.$$

Now, it follows from $f'(\bar{x})(0) = 0$, that $(x^*, c) \in \partial^w(f'(\bar{x}))(0)$. Conversely, if $(x^*, c) \in \partial^w(f'(\bar{x}))(0)$, then we can write

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x\rangle \geq \langle x^*, x\rangle - c\|x\|.$$

Hence

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x - \bar{x}\rangle \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

and by applying the subdifferentiability and Fréchet differentiability f at $\bar{x}$, we get

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle f'(\bar{x}), x - \bar{x}\rangle.$$

Then

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

This means that $(x^*, c) \in \partial^w(f(\bar{x}))$ and proof is completed. □

Proposition 3.34. *If f is subdifferentiable and Fréchet differentiable at $\bar{x}$, then*

$$\langle f'(\bar{x}), x - \bar{x}\rangle = \sup\Big\{\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\}.$$

Proof. From the hypothesis and by using a similar proof as in Proposition 3.33, we deduce that

$$\langle f'(\bar{x}), x - \bar{x}\rangle \geq \sup\Big\{\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\}.$$

Since $(f'(\bar{x}), 0) \in \partial^w f(\bar{x})$, then

$$\langle f'(\bar{x}), x - \bar{x}\rangle \in \Big\{\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\},$$

and the desired equality is obtained. □

Corollary 3.35. *We note that under above assumptions, if f attains a global minimum at $\bar{x}$, then*

$$\sup\Big\{\{\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\} = 0.$$

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An Extension of Pochhammer's Symbol and its Application to Hypergeometric Functions

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Abstract. By using a special property of the gamma function, we first define a productive form of gamma and beta functions and study some of their general properties in order to define a new extension of the Pochhammer symbol. We then apply this extended symbol for generalized hypergeometric series and study the convergence problem with some illustrative examples in this sense. Finally, we introduce two new extensions of Gauss and confluent hypergeometric series and obtain some of their general properties.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ respectively denote the sets of real and complex numbers and z be an arbitrary complex variable. The well known (Euler's) gamma function is defined, for $\mathrm{Re}(z) > 0$, as

$$\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x,$$

and for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$, as

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{k=0}^{n-1}(z+k)} \quad (n \in \mathbb{N}).$$

The limit definition of the gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{\prod_{k=0}^{n}(z+k)}, \tag{1}$$

is valid for all complex numbers except the non-positive integers. An alternative definition is the productive form of the gamma function, i.e.,

$$\Gamma(z) = \frac{1}{z} \prod_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{z} \left(1+\frac{z}{k}\right)^{-1}. \tag{2}$$

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When $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, the (Euler's) beta function [4] has a close relationship with the classical gamma function as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x). \tag{3}$$

The generalized binomial coefficient may be defined (for real or complex parameters a and b) by

$$\binom{a}{n} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \binom{a}{a-b} \qquad (a,b \in \mathbb{C}),$$

which is reduced to the following special case when $b = n$ $(n \in \mathbb{N} \cup \{0\})$:

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!} = \frac{(-1)^n(-a)_n}{n!},$$

where $(a)_b$ $(a,b \in \mathbb{C})$ denotes the Pochhammer symbol [19] given, in general, by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & (b=0,\ a \in \mathbb{C}\backslash\{0\}), \\ a(a+1)\cdots(a+n-1) & (b \in \mathbb{N},\ a \in \mathbb{C}). \end{cases}$$

A remarkable property of the gamma function, which is provable via the limit definition (1), is

$$\overline{\Gamma(z)} = \Gamma(\bar{z}) \overset{(z=p+\mathrm{i}q)}{\Rightarrow} \Gamma(p+\mathrm{i}q)\,\Gamma(p-\mathrm{i}q) \in \mathbb{R}. \tag{4}$$

In this paper, we exploit the property (4) to introduce an extension of the Pochhammer symbol in order to apply it in the hypergeometric series of any arbitrary order. Then, we study the convergence problem of the involved hypergeometric series with some illustrative examples. Finally, we introduce two new extensions of Gauss and confluent hypergeometric series and obtain some of their general properties. For this purpose, we first define a productive form of the gamma function, by referring to the property (4), as follows

$$\Pi(p,q) = \frac{\Gamma(p+\mathrm{i}q)\,\Gamma(p-\mathrm{i}q)}{\Gamma(p)} \qquad (p>0,\ q \in \mathbb{R}). \tag{5}$$

For analogous extensions of the gamma function see e.g. [2, 14]. The limit definition of (5) can be derived from (1), so that we have

$$\Pi(p,q) = \frac{1}{\Gamma(p)} \lim_{n\to\infty} \frac{(n!)^2\, n^{2p}}{\prod\limits_{k=0}^{n}\left((p+k)^2+q^2\right)} = \lim_{n\to\infty} n!\, n^p \prod_{k=0}^{n} \frac{p+k}{(p+k)^2+q^2}. \tag{6}$$

Also, the limit relation (1) implies that relation (6) is written as

$$\Pi(p,q) = \Gamma(p) \prod_{k=0}^{\infty} \frac{(p+k)^2}{(p+k)^2+q^2}. \tag{7}$$

The result (7) shows that for any $p > 0$ and $q \in \mathbb{R}$ we respectively have

$$0 \le \Pi(p,q) \le \Gamma(p),$$

and

$$\lim_{q\to\infty} \Pi(p,q) = \lim_{q\to\infty} \frac{\Gamma(p+\mathrm{i}q)\,\Gamma(p-\mathrm{i}q)}{\Gamma(p)} = 0.$$

In order to obtain an integral representation for $\Pi(p,q)$, we should first study the real function $\Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q)$. Hence, we consider the second kind of (Cauchy's) beta function [4], which says that if $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$ and $\mathrm{Re}(c+d) > 1$ then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(a+\mathrm{i}t)^c(b-\mathrm{i}t)^d} = \frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)}\,(a+b)^{1-(c+d)}. \tag{8}$$

One of the consequences of (8) is the definite integral

$$\int_{-\pi/2}^{\pi/2} \mathrm{e}^{st}(\cos t)^r\,\mathrm{d}t = \frac{2^{-r}\Gamma(r+1)\,\pi}{\Gamma\left(1+\frac{r+\mathrm{i}s}{2}\right)\Gamma\left(1+\frac{r-\mathrm{i}s}{2}\right)}, \tag{9}$$

which can be derived from the well-known identity

$$(a-\mathrm{i}t)^{p+\mathrm{i}q}(a+\mathrm{i}t)^{p-\mathrm{i}q} = (a^2+t^2)^p\exp\left(2q\arctan\frac{t}{a}\right).$$

The simplified version of (9) is as

$$\Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q) = \frac{\pi\,2^{2-2p}\Gamma(2p-1)}{\int\limits_{-\pi/2}^{\pi/2}\mathrm{e}^{2qt}(\cos t)^{2p-2}\,\mathrm{d}t}. \tag{10}$$

On the other hand, since

$$\frac{\Gamma(p+\mathrm{i}q)\Gamma(p-\mathrm{i}q)}{\Gamma(2p)} = \mathrm{B}(p+\mathrm{i}q,p-\mathrm{i}q) = \int_0^1 (x-x^2)^{p-1}\left(\frac{x}{1-x}\right)^{\mathrm{i}q}\mathrm{d}x \tag{11}$$

$$= \int_0^1 (x-x^2)^{p-1}\cos\left(q\log\frac{x}{1-x}\right)\mathrm{d}x + \mathrm{i}\int_0^1 (x-x^2)^{p-1}\sin\left(q\log\frac{x}{1-x}\right)\mathrm{d}x, \tag{12}$$

is a real value, for any $p > 0$ and $q \in \mathbb{R}$ we can conclude that

$$\int_0^1 (x-x^2)^{p-1}\sin\left(q\log\frac{x}{1-x}\right)\mathrm{d}x = 0.$$

Therefore, by noting relations (10) and (11), two integral representations of $\Pi(p,q)$ are as

$$\Pi(p,q) = \frac{\Gamma(2p)}{\Gamma(p)}\int_0^1 (x-x^2)^{p-1}\cos\left(q\log\frac{x}{1-x}\right)\mathrm{d}x = \frac{\pi\,2^{2-2p}\Gamma(2p-1)}{\Gamma(p)\int\limits_{-\pi/2}^{\pi/2}\mathrm{e}^{2qt}(\cos t)^{2p-2}\,\mathrm{d}t}. \tag{13}$$

Note that the definite integral in the second equality of (13) can be computed in terms of a series. In fact, since

$$(\cos t)^a = 2^{-a}(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})^a = 2^{-a}\sum_{k=0}^{\infty}\binom{a}{k}\mathrm{e}^{(a-2k)\mathrm{i}t} = 2^{-a}\sum_{k=0}^{\infty}\binom{a}{k}\cos(a-2k)t,$$

and

$$\int \mathrm{e}^{pt}\cos qt\,\mathrm{d}t = \mathrm{e}^{pt}\frac{p\cos qt + q\sin qt}{p^2+q^2} + c,$$

so we have

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} e^{2qt}(\cos t)^{2p-2}\,dt &= 2^{-2p+2}\int_{-\pi/2}^{\pi/2} e^{2qt}\sum_{k=0}^{\infty}\binom{2p-2}{k}\cos(2p-2-2k)t\,dt\\ &= 2^{-2p+2}\sum_{k=0}^{\infty}\binom{2p-2}{k}\int_{-\pi/2}^{\pi/2} e^{2qt}\cos(2p-2-2k)t\,dt\\ &= 2^{-2p+2}\sum_{k=0}^{\infty}(-1)^k\binom{2p-2}{k}\frac{q\sinh(q\pi)\cos((p-1)\pi)+(p-1-k)\cosh(q\pi)\sin((p-1)\pi)}{q^2+(p-1-k)^2}.\end{aligned}$$

Remark 1.1. By noting the well-known identity $\Gamma(z+1) = z\,\Gamma(z)$, since

$$\Gamma(p+1+iq) = (p+iq)\,\Gamma(p+iq) \quad \text{and} \quad \Gamma(p+1-iq) = (p-iq)\,\Gamma(p-iq),$$

so

$$\Pi(p+1,q) = \frac{p^2+q^2}{p}\,\Pi(p,q). \tag{14}$$

Similarly, the approach (14) can be followed for e.g. the Legendre duplication formula [5, 15]

$$(2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz}\Gamma(mz) = \Gamma(z)\,\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right),$$

(when $m = 2$) so that we can eventually obtain

$$\Pi(2p,2q) = \frac{2^{2p-1}}{\sqrt{\pi}}\,\Pi(p,q)\,\Pi\left(p+\frac{1}{2},q\right).$$

Remark 1.2. When $m \in \mathbb{N}$, to compute $\Pi(m+1,q)$ we can again use the recurrence relation $\Gamma(z+1) = z\,\Gamma(z)$ to finally obtain

$$\begin{aligned}\Pi(m+1,q) &= \frac{(m+iq)(m-1+iq)\cdots(1+iq)\,\Gamma(1+iq)\,(m-iq)(m-1-iq)\cdots(1-iq)\,\Gamma(1-iq)}{m!}\\ &= \frac{q\pi}{m!\ \sinh(q\pi)}\prod_{k=0}^{m-1}\left((m-k)^2+q^2\right).\end{aligned}$$

One of the other applications of (4) is to define the productive form of the beta function as follows

$$B(r,s;q) = \frac{B(r+iq,s-iq)\,B(r-iq,s+iq)}{B(r,s)} = \frac{\Pi(r,q)\,\Pi(s,q)}{\Gamma(r+s)}. \tag{15}$$

For analogous extensions of the family of beta functions see e.g. [6, 13]. By referring to relation (7), the productive form of (15) can be obtained as

$$B(r,s;q) = B(r,s)\prod_{k=0}^{\infty}\frac{(r+k)^2(s+k)^2}{((r+k)^2+q^2)((s+k)^2+q^2)}. \tag{16}$$

Clearly the latter relation (16) shows that if $r,s > 0$ and $q \in \mathbb{R}$ then

$$\left|B(r,s;q)\right| \le B(r,s).$$

2. An Extension of Pochhammer's Symbol and its Application to Hypergeometric Functions

The generalized hypergeometric series appear in a wide variety of applied mathematics and engineering sciences [1, 3, 12, 18]. For instance, there is a large set of hypergeometic-type polynomials whose variable is located in one or more of the parameters of the corresponding hypergeometric functions [8–10]. These polynomials are of great importance in mathematics as well as in many areas of physics. A few examples of their applications are discussed by Nikiforov, Suslov and Uvarov [16]. See also [5, 15]. Hence, it seems that any change in hypergeometric series, especially in Gauss and confluent hypergeometric functions, can be notable in various branches of mathematics. In recent years, some new extensions are given in this direction, e.g. [7, 17]. A main reason for introducing and developing the generalized hypergeometric series is that many special functions [4, 9, 11] can be represented in terms of them and therefore their initial properties can be directly found via the initial properties of hypergeometric functions. Also, they appear as solutions of many important ordinary differential equations [9, 11, 15]. The generalized hypergeometric function

$$ {}_pF_q\left(\begin{array}{c} a_1,\ a_2,\ \ldots\ ,\ a_p \\ b_1,\ b_2,\ \ldots\ ,\ b_q \end{array} \middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{17} $$

in which $(r)_k = \prod\limits_{j=0}^{k-1} (r+j)$ denotes the same as Pochhammer symbol and z may be a complex variable is indeed a Taylor series expansion for a function, say f, as $\sum\limits_{k=0}^{\infty} c_k^* z^k$ with $c_k^* = f^{(k)}(0)/k!$ for which the ratio of successive terms can be written as

$$ \frac{c_{k+1}^*}{c_k^*} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}. $$

According to the ratio test [4], the series (17) is convergent for any $p \le q+1$. In fact, it converges in $|z|<1$ for $p = q+1$, converges everywhere for $p < q+1$ and converges nowhere ($z \neq 0$) for $p > q+1$. Moreover, for $p = q+1$ it absolutely converges for $|z| = 1$ if the condition

$$ A^* = \operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \right) > 0, \tag{18} $$

holds and is conditionally convergent for $|z| = 1$ and $z \neq 1$ if $-1 < A^* \le 0$ and is finally divergent for $|z| = 1$ and $z \neq 1$ if $A^* \le -1$.

There are two important cases of the series (16) arising in many physical problems [3, 8, 12, 15]. The first case is the Gauss hypergeometric function convergent in $|z| \le 1$ that is denoted by

$$ y = {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, $$

and satisfies the differential equation

$$ z\,(z-1)\,y'' + ((a+b+1)z - c)\,y' + ab\,y = 0. \tag{19} $$

Particular choices of the parameters in the linearly independent solutions of the differential equation (19) yield 24 special cases. The ${}_2F_1$ can be given an integral representation as

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle|\, z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t \quad \Big(\operatorname{Re} c > \operatorname{Re} b > 0;\ |\arg(1-z)| < \pi\Big). \tag{20} $$

By using a series expansion of $(1-tz)^{-a}$ in (20), one can also write the ${}_2F_1$ in terms of the beta function as

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}. \tag{21} $$

The second case, which converges everywhere, is the confluent hypergeometric function

$$ y = {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!}, $$

as a basis solution of the differential equation

$$ z\,y'' + (c-z)\,y' - b\,y = 0, $$

which is a degenerate form of equation (19) where two of the three regular singularities merge into an irregular singularity. The ${}_1F_1$ has an integral form as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \mathrm{e}^{zt}\,\mathrm{d}t \qquad \big(\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ |\arg(1-z)| < \pi\big), $$

and can be represented in terms of the beta function as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{\mathrm{B}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}. \tag{22} $$

Now, we can introduce an extension of the Pochhammer symbol in order to apply it in the generalized hypergeometric series of any arbitrary order. Let us reconsider the gamma form of the Pochhammer symbol

$$ (s)_k = \frac{\Gamma(s+k)}{\Gamma(s)}. \tag{23} $$

By noting (5), a real extension of (23) may be defined as

$$ [s;q]_k = \frac{\Pi(s+k, q)}{\Pi(s, q)} = \frac{(s+\mathrm{i}q)_k (s-\mathrm{i}q)_k}{(s)_k} = \prod_{j=0}^{k-1} \frac{(s+j)^2 + q^2}{s+j}. $$

Subsequently, a real extension of the hypergeometric functions may be defined as

$$ {}_pF_q\left(\begin{array}{c} [a_1;\alpha_1 r],\ [a_2;\alpha_2 r],\ \ldots,\ [a_p;\alpha_p r] \\ [b_1;\beta_1 r],\ [b_2;\beta_2 r],\ \ldots,\ [b_q;\beta_q r] \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{[a_1;\alpha_1 r]_k [a_2;\alpha_2 r]_k \cdots [a_p;\alpha_p r]_k}{[b_1;\beta_1 r]_k [b_2;\beta_2 r]_k \cdots [b_q;\beta_q r]_k} \frac{z^k}{k!}, \tag{24} $$

where $\{a_k, \alpha_k\}_{k=1}^{p}, \{b_k, \beta_k\}_{k=1}^{q} \in \mathbb{R}$ and $r \in \mathbb{R}$. On the other hand, the definition

$$ [s;q]_k = \frac{(s+\mathrm{i}q)_k (s-\mathrm{i}q)_k}{(s)_k}, $$

implies that the fraction term of (24) is expanded as

$$ \frac{[a_1;\alpha_1 r]_k [a_2;\alpha_2 r]_k \cdots [a_p;\alpha_p r]_k}{[b_1;\beta_1 r]_k [b_2;\beta_2 r]_k \cdots [b_q;\beta_q r]_k} = \prod_{j=1}^{p} \frac{(a_j + \mathrm{i}\alpha_j r)_k (a_j - \mathrm{i}\alpha_j r)_k}{(a_j)_k} \prod_{j=1}^{q} \frac{(b_j)_k}{(b_j + \mathrm{i}\beta_j r)_k (b_j - \mathrm{i}\beta_j r)_k}. $$

This means that the real series (24) can be transformed to a standard hypergeometric function as follows

$$
{}_pF_q\left(\begin{array}{c} [a_1;\alpha_1 r],\,[a_2;\alpha_2 r],\,\ldots,\,[a_p;\alpha_p r] \\ [b_1;\beta_1 r],\,[b_2;\beta_2 r],\,\ldots,\,[b_q;\beta_q r] \end{array}\,\middle|\, z\right)
$$

$$
= {}_{2p+q}F_{2q+p}\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ \ldots,\ a_p+\mathrm{i}\alpha_p r,\ a_p+\mathrm{i}\alpha_p r,\ b_1,\,b_2,\ldots,b_q \\ b_1+\mathrm{i}\beta_1 r,\ b_1-\mathrm{i}\beta_1 r,\ \ldots,\ b_q+\mathrm{i}\beta_q r,\ b_q-\mathrm{i}\beta_q r,\ a_1,\,a_2,\ldots,a_p \end{array}\,\middle|\, z\right). \tag{25}
$$

Hence, the convergence radius of (24) would directly depend on the convergence radius of ${}_{2p+q}F_{2q+p}$ in (25) as the following illustrative examples show.

Example 2.1. Let $(p,q)=(2,1)$. In this case, (24) is reduced to

$$
{}_2F_1\left(\begin{array}{c} [a_1;\alpha_1 r],\ [a_2;\alpha_2 r] \\ [b_1;\beta_1 r] \end{array}\,\middle|\, z\right) = {}_5F_4\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ a_2+\mathrm{i}\alpha_2 r,\ a_2-\mathrm{i}\alpha_2 r,\ b_1 \\ b_1+\mathrm{i}\beta_1 r,\quad b_1-\mathrm{i}\beta_1 r,\ a_1,\ a_2 \end{array}\,\middle|\, z\right), \tag{26}
$$

whose convergence radius is $|z|<1$. Moreover, according to (18), if $a_1+a_2<b_1$ in (26), then the convergence radius is extended to $|z|\le 1$.

As a particular case of (26), taking $r=0$ gives the same as classical ${}_2F_1$ and $(\alpha_1,\alpha_2)=(0,0)$ with $\beta_1 r=q$ yields

$$
{}_2F_1\left(\begin{array}{c} [a_1;0],\ [a_2;0] \\ [b_1;q] \end{array}\,\middle|\, z\right) = {}_3F_2\left(\begin{array}{c} a_1,\quad a_2,\quad b_1 \\ b_1+\mathrm{i}q,\ \ b_1-\mathrm{i}q \end{array}\,\middle|\, z\right),
$$

which is convergent in $|z|\le 1$ if $a_1+a_2<b_1$. Finally, if $\beta_1=0$, (26) is reduced to

$$
{}_2F_1\left(\begin{array}{c} [a_1;\alpha_1 r],\ [a_2;\alpha_2 r] \\ [b_1;0] \end{array}\,\middle|\, z\right) = {}_4F_3\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ a_2+\mathrm{i}\alpha_2 r,\ a_2-\mathrm{i}\alpha_2 r \\ b_1,\ a_1,\ a_2 \end{array}\,\middle|\, z\right),
$$

convergent in $|z|\le 1$ when $a_1+a_2<b_1$.

Example 2.2. Let $(p,q)=(1,1)$. Then (24) changes to

$$
{}_1F_1\left(\begin{array}{c} [a_1;\alpha_1 r] \\ [b_1;\beta_1 r] \end{array}\,\middle|\, z\right) = {}_3F_3\left(\begin{array}{c} a_1+\mathrm{i}\alpha_1 r,\ a_1-\mathrm{i}\alpha_1 r,\ b_1 \\ b_1+\mathrm{i}\beta_1 r,\ b_1-\mathrm{i}\beta_1 r,\ a_1 \end{array}\,\middle|\, z\right), \tag{27}
$$

which is convergent everywhere. For instance, if $\alpha_1=0$ and $\beta_1 r=q$ in (27), then the following real series converges everywhere

$$
{}_1F_1\left(\begin{array}{c} [a_1;0] \\ [b_1;q] \end{array}\,\middle|\, z\right) = {}_2F_2\left(\begin{array}{c} a_1,\ \ b_1 \\ b_1+\mathrm{i}q,\ b_1-\mathrm{i}q \end{array}\,\middle|\, z\right).
$$

Example 2.3. An interesting case of (24) is when $(p,q)=(1,0)$, because the real series

$$
y = {}_1F_0\left(\begin{array}{c} [a;q] \\ - \end{array}\,\middle|\, z\right) = {}_2F_1\left(\begin{array}{c} a+\mathrm{i}q,\ a-\mathrm{i}q \\ a \end{array}\,\middle|\, z\right) = \sum_{k=0}^{\infty}\left(\prod_{j=0}^{k-1}(a+j)^2+q^2\right)\frac{z^k}{(a)_k k!}, \tag{28}
$$

satisfies the second order differential equation

$$
z\,(z-1)\,y'' + ((2a+1)z-a)\,y' + (a^2+q^2)\,y = 0.
$$

Note that the more general case of (28) is indeed the real series

$$
y = {}_2F_1\left(\begin{array}{c} a+\mathrm{i}q,\ a-\mathrm{i}q \\ b \end{array}\,\middle|\, z\right) = \sum_{k=0}^{\infty}\left(\prod_{j=0}^{k-1}(a+j)^2+q^2\right)\frac{z^k}{(b)_k k!},
$$

which satisfies the differential equation

$$
z\,(z-1)\,y'' + ((2a+1)z-b)\,y' + (a^2+q^2)\,y = 0.
$$

2.1. A new extension of Gauss and confluent hypergeometric functions

Since many special functions of mathematical physics can be represented in terms of ${}_2F_1$ or ${}_1F_1$ by special choices of the parameters, they play a unifying role in the theory of special functions. Hence, any significant generalization of them may be useful. In this section, we apply the generalized beta function (15) for two relations (21) and (22) to respectively extend the functions ${}_2F_1$ and ${}_1F_1$. First, by noting two relations (15) and (21), the proposed extension of ${}_2F_1$ may be considered as

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z;q\right) = \sum_{k=0}^{\infty} (a)_k \frac{\mathrm{B}\,(b+k, c-b;q)}{\mathrm{B}\,(b,c-b)} \frac{z^k}{k!}, \tag{29} $$

which reduces to the same as ${}_2F_1$ for $q=0$. Since $\left|\mathrm{B}(r,s;q)\right| \le \mathrm{B}(r,s)$ for any $r,s>0$ and $q\in\mathbb{R}$, the extended series (29) converges in $|z|\le 1$ if $c>b>0$ and c is not a negative integer or zero. Now, the integral representation of (29) can be derived by (13) and (15) as follows

$$ \begin{aligned} {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z;q\right) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{(a)_k\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Pi(c-b,q)}{\Gamma(c)\,(c)_k} \frac{\Gamma(2b+2k)}{\Gamma(b+k)} \int_0^1 \left(x-x^2\right)^{b+k-1} \cos\left(q\log\frac{x}{1-x}\right) \mathrm{d}x \\ &= \frac{\Gamma(c-b+\mathrm{i}q)\Gamma(c-b-\mathrm{i}q)\Gamma(2b)}{\Gamma^2(b)\Gamma^2(c-b)} \int_0^1 \left(x-x^2\right)^{b-1} \cos\left(q\log\frac{x}{1-x}\right) \left(\sum_{k=0}^{\infty} \frac{(a)_k(b+1/2)_k}{(c)_k} \frac{(4zx(1-x))^k}{k!}\right) \mathrm{d}x \\ &= \frac{\Gamma(c-b+\mathrm{i}q)\Gamma(c-b-\mathrm{i}q)\Gamma(2b)}{\Gamma^2(b)\Gamma^2(c-b)} \int_0^1 \left(x-x^2\right)^{b-1} \cos\left(q\log\frac{x}{1-x}\right) {}_2F_1\left(\begin{array}{c} a,\ b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. \end{aligned} \tag{30} $$

Note that $q=0$ in (30) gives a new representation for ${}_2F_1$ so that we have

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z\right) = \frac{\Gamma(2b)}{\Gamma^2(b)} \int_0^1 \left(x-x^2\right)^{b-1} {}_2F_1\left(\begin{array}{c} a,\ b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. $$

Similarly, for the extension of ${}_1F_1$ we can define

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array}\middle|\, z;q\right) = \sum_{k=0}^{\infty} \frac{\mathrm{B}\,(b+k, c-b;q)}{\mathrm{B}\,(b,c-b)} \frac{z^k}{k!}, \tag{31} $$

which reduces to the same as ${}_1F_1$ for $q=0$. Again since $\left|\mathrm{B}(r,s;q)\right| \le \mathrm{B}(r,s)$ for any $r,s>0$ and $q\in\mathbb{R}$, the generalized series (31) converges everywhere if $c>b>0$ and c is not a negative integer or zero. Also, the integral representation of (31) is derived in a similar way as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array}\middle|\, z;q\right) = \frac{\Gamma(c-b+\mathrm{i}q)\Gamma(c-b-\mathrm{i}q)\Gamma(2b)}{\Gamma^2(b)\Gamma^2(c-b)} \int_0^1 \left(x-x^2\right)^{b-1} \cos\left(q\log\frac{x}{1-x}\right) {}_1F_1\left(\begin{array}{c} b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. \tag{32} $$

Finally for $q=0$, (32) reduces to a new representation for the series ${}_1F_1$ as

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array}\middle|\, z\right) = \frac{\Gamma(2b)}{\Gamma^2(b)} \int_0^1 \left(x-x^2\right)^{b-1} {}_1F_1\left(\begin{array}{c} b+1/2 \\ c \end{array}\middle|\, 4zx(1-x)\right) \mathrm{d}x\,. $$

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Some Reverse Hölder Type Inequalities Involving $(k, s)-$Riemann-Liouville Fractional Integrals

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Abstract. In this paper, we aim to present the improved version of the reverse Hölder type inequalities by taking $(k, s)-$Riemann-Liouville fractional integrals. Furthermore, we also discuss some applications of Theorem 1 using some types of fractional integrals.

Keywords: $(k, s)-$Riemann-Liouville fractional integrals · Holder inequality · Reverse Holder inequality

1 Introduction

Fractional integral inequalities involving $(k, s)-$ type integrals attract the attentions of many researchers due their diverse applications see, for examples, [1–4]. In [5], Farid *et al.* an integral inequality obtained by Mitrinovic and Pecaric was generalized to measure space as follows.

Theorem 1. *Let $(\Omega_1, \Sigma_1, \mu_1)$,$(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with $\sigma-$finite measures and let $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, 2, 3, 4$ be non-negative functions. Let g be the function having representation*

$$g(x) = \int_{\Omega_1} k(x, t) f(t) d\mu_1(t),$$

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where $k : \Omega_2 \times \Omega_1 \to \mathbb{R}$ is a general non-negative kernel and $f : \Omega_1 \to \mathbb{R}$ is real-valued function, and μ_2 is a non-decreasing function. If p, q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then

$$\int_{\Omega_2} f_1(x) f_2(x) g(x) d\mu_2(x) \tag{1}$$
$$\leq C \left(\int_{\Omega_2} f_3(x) g(x) d\mu_2(x) \right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_4(x) g(x) d\mu_2(x) \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_a^b k(x,t) f_1(x) f_2(x) d\mu_2(x) \right) \right. \tag{2}$$
$$\left. \left(\int_a^b k(x,t) f_3(x) d\mu_2(x) \right)^{\frac{-1}{p}} \left(\int_a^b k(x,t) f_4(x) d\mu_2(x) \right)^{\frac{-1}{q}} \right\}.$$

The following definitions and results are also required.

2 Preliminaries

Recently fractional integral inequalities are considered to be an important tool of applied mathematics and their many applications described by a number of researchers. As well as, the theory of fractional calculus is used in solving differential, integral and integro-differential equations and also in various other problems involving special functions [6–8].

We begin by recalling the well-known results.

1. The Pochhammer k-symbol $(x)_{n,k}$ and the k-gamma function Γ_k are defined as follows (see [9]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) \quad (n \in \mathbb{N};\ k > 0) \tag{3}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!\, k^n\, (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad \left(k > 0;\ x \in \mathbb{C} \backslash k\mathbb{Z}_0^-\right), \tag{4}$$

where $k\mathbb{Z}_0^- := \left\{kn : n \in \mathbb{Z}_0^-\right\}$. It is noted that the case $k = 1$ of equation ((3)) and equation ((4)) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^\infty t^{x-1}\, e^{-\frac{t^k}{k}}\, dt \quad (\Re(x) > 0). \tag{5}$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties: (i) $\Gamma_k(x+k) = x\,\Gamma_k(x)$; (ii) $\Gamma_k(k) = 1$; (iii) $\Gamma_k(x)$ is logarithmically convex. It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\frac{x}{k}\right) \quad (\Re(x) > 0;\ k > 0)\,. \tag{6}$$

2. Mubeen and Habibullah [10] introduced k-fractional integral of the Riemann-Liouville type of order α as follows:

$${}_kJ_a^\alpha\left[f(t)\right] = \frac{1}{\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)\,d\tau, (\alpha > 0, x > 0, k > 0)\,, \tag{7}$$

which, upon setting $k = 1$, is seen to yield the classical Riemann-Liouville fractional integral of order α:

$$J_a^\alpha\left\{f(t)\right\} := {}_1J_a^\alpha\left\{f(t)\right\} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau \quad (\alpha > 0;\ t > a)\,. \tag{8}$$

3. Sarikaya *et al.* [11] presented (k,s)-fractional integral of the Riemann-Liouville type of order α, which is a generalization of the k-fractional integral (7), defined as follows:

$${}_k^sJ_a^\alpha\left[f(t)\right] := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s f(\tau)\,d\tau,\ \tau \in [a,b]\,, \tag{9}$$

where $k > 0, s \in \mathbb{R}\backslash\{-1\}$ and which, upon setting $s = 0$, immediately reduces to the k-integral (7).

4. In [11], the following results have been obtained. For f be continuous on $[a,b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then,

$${}_k^sJ_a^\alpha\left[{}_k^sJ_a^\beta f(t)\right] = {}_k^sJ_a^{\alpha+\beta} f(t) = {}_k^sJ_a^\beta\left[{}_k^sJ_a^\alpha f(t)\right], \tag{10}$$

and

$${}_k^sJ_a^\alpha\left[\left(x^{s+1} - a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+\beta)}\left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1},$$

for all $\alpha, \beta > 0$, $x \in [a,b]$ and Γ_k denotes the $k-$gamma function.

5. Also, in [12], Akkurt *et al.* introduced $(k,H)-$fractional integral. Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by

$$\begin{aligned}&\left({}_kJ_{a^+,h}^\alpha f\right)(x) \\ &:= \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left[h(x) - h(t)\right]^{\frac{\alpha}{k}-1} h'(t) f(t)dt,\ k > 0\ , \Re(\alpha) > 0\end{aligned} \tag{11}$$

$$\left({}_kJ^{\alpha}_{b^-,h} f \right)(x) \tag{12}$$
$$:= \frac{1}{k\Gamma_k(\alpha)} \int_x^b [h(x)-h(t)]^{\frac{\alpha}{k}-1} h'(t) f(t) dt,\ k>0\ , \Re(\alpha)>0.$$

Recently, Tomar and Agarwal [13] obtained following results for $(k,s)-$fractional integrals.

Theorem 2 (Hölder Inequality for (k,s)-fractional integrals). *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, for all $t>0,\ k>0,\ \alpha>0, s\in\mathbb{R}-\{-1\}$,*

$${}^s_kJ^{\alpha}_a \mid fg(t)| \le \left[{}^s_kJ^{\alpha}_a |f(t)|^p\right]^{\frac{1}{p}} \left[{}^s_kJ^{\alpha}_a |g(t)|^q\right]^{\frac{1}{q}}. \tag{13}$$

Lemma 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions and $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,k>0$ and $s\in\mathbb{R}-\{-1\}$, such that for $t\in[a,b]$, ${}^s_kJ^{\alpha}_a f^p(t)<\infty$, ${}^s_kJ^{\alpha}_a g^q(t)<\infty$. If*

$$0\le m\le \frac{f(\tau)}{g(\tau)} \le M<\infty, \tau\in[a,b], \tag{14}$$

then the inequality

$$\left[{}^s_kJ^{\alpha}_a f(t)\right]^{\frac{1}{p}} \left[{}^s_kJ^{\alpha}_a g(t)\right]^{\frac{1}{q}} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}} {}^s_kJ^{\alpha}_a \left[f^{\frac{1}{p}}(t) g^{\frac{1}{q}}(t)\right] \tag{15}$$

holds.

Lemma 2. *Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions $\alpha,k>0$ and $s\in\mathbb{R}-\{-1\}$, such that for $t\in[a,b]$, ${}^s_kJ^{\alpha}_a f^p(t)<\infty$, ${}^s_kJ^{\alpha}_a g^q(t)<\infty$. If*

$$0\le m\le \frac{f^p(\tau)}{g^q(\tau)} \le M<\infty, \tau\in[a,b], \tag{16}$$

then we have

$$\left[{}^s_kJ^{\alpha}_a f^p(t)\right]^{\frac{1}{p}} \left[{}^s_kJ^{\alpha}_a g^q(t)\right]^{\frac{1}{q}} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}} {}^s_kJ^{\alpha}_a \left(f(t)g(t)\right), \tag{17}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by this work, we establish in this paper some new extensions of the reverse Hölder type inequalities by taking $(k,s)-$Riemann-Liouville fractional integrals.

3 Reverse Hölder Type Inequalites

In this section we prove our main results (Theorems 3 and 4).

Theorem 3. *Let $f(x)$ and $g(x)$ be integrable functions and let $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequality holds*

$$ {}_k^sJ_a^\alpha \left|fg(t)\right| \geq {}_k^sJ_a^\alpha \left|f^p(t)\right|^{\frac{1}{p}} \; {}_k^sJ_a^\alpha \left|f^q(t)\right|^{\frac{1}{q}}. \tag{18}$$

Proof. *Set $c = \frac{1}{p}$, $q = -pd$. Then we have $d = \frac{c}{c-1}$. By the Hölder inequality for $(k,s)-$fractional integrals, we have*

$$\begin{aligned} {}_k^sJ_a^\alpha \left|f^p(t)\right| &= {}_k^sJ_a^\alpha \left|fg(t)\right|^p \left|g^{-p}(t)\right| \\ &\leq \left[{}_k^sJ_a^\alpha \left|fg(t)\right|^{pc}\right]^{\frac{1}{c}} \left[{}_k^sJ_a^\alpha \left|g(t)\right|^{-pd}\right]^{\frac{1}{d}} \\ &= \left[{}_k^sJ_a^\alpha \left|fg(t)\right|\right]^{\frac{1}{c}} \left[{}_k^sJ_a^\alpha \left|g(t)\right|^q\right]^{1-p}. \end{aligned} \tag{19}$$

In equation (19), multiplying both sides by $({}_k^sJ_a^\alpha \left|g^q(t)\right|)^{p-1}$, we obtain

$$\begin{aligned} &{}_k^sJ_a^\alpha \left|f^p(t)\right| ({}_k^sJ_a^\alpha \left|g^q(t)\right|)^{p-1} \\ &\leq \left[{}_k^sJ_a^\alpha \left|fg(t)\right|\right]^p. \end{aligned} \tag{20}$$

Inequality (20) implies inequality

$$ {}_k^sJ_a^\alpha \left|fg(t)\right| \geq {}_k^sJ_a^\alpha \left|f^p(t)\right|^{\frac{1}{p}} \; {}_k^sJ_a^\alpha \left|f^q(t)\right|^{\frac{1}{q}} \tag{21}$$

which completes this theorem.

Theorem 4. *Suppose $p, q, l > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{l} = 1$. If f, g and h are positive functions such that*

i.) $0 < m \leq \frac{f^{\frac{p}{s}}}{g^{\frac{q}{s}}} \leq M < \infty$ *for some $l > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$,*

ii.) $0 < m \leq \frac{(fg)^s}{h^r} \leq M < \infty$,

then

$$\begin{aligned} &({}_k^sJ_a^\alpha f^p(t))^{\frac{1}{p}} ({}_k^sJ_a^\alpha f^q(t))^{\frac{1}{q}} ({}_k^sJ_a^\alpha f^r(t))^{\frac{1}{r}} \\ &\leq \left(\frac{M}{m}\right)^{\frac{1}{sr} + \frac{pq}{s^3}} \; {}_k^sJ_a^\alpha (fgh)(t). \end{aligned} \tag{22}$$

Proof. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ for some $s > 0$. Thus, $\frac{s}{p} + \frac{s}{q} = 1$ and $\frac{1}{s} + \frac{1}{r} = 1$. If we use ii and Lemma 2 for $H = fg$ and h, then we get

$$({}_k^sJ_a^\alpha H^s(t))^{\frac{1}{s}} ({}_k^sJ_a^\alpha h^r(t))^{\frac{1}{r}} \leq \left(\frac{M}{m}\right)^{\frac{1}{sr}} ({}_k^sJ_a^\alpha (Hh)(t)) \tag{23}$$

which is equivalent to

$$\left({}_{k}^{s}J_{a}^{\alpha}[f^{s}(t)g^{s}(t)]\right)^{\frac{1}{s}}\left({}_{k}^{s}J_{a}^{\alpha}h^{r}(t)\right)^{\frac{1}{r}} \leq \left(\frac{M}{m}\right)^{\frac{1}{sr}}\left({}_{k}^{s}J_{a}^{\alpha}(fgh)(t)\right). \tag{24}$$

Now, using i and the fact that $\frac{s}{p}+\frac{s}{q}=1$, and applying Lemma 2 to f^{s} and g^{s}, we also have

$$\left({}_{k}^{s}J_{a}^{\alpha}f^{p}(t)\right)^{\frac{s}{p}}\left({}_{k}^{s}J_{a}^{\alpha}g^{q}(t)\right)^{\frac{s}{q}} \leq \left(\frac{M}{m}\right)^{\frac{pq}{s^{2}}}\left({}_{k}^{s}J_{a}^{\alpha}f^{s}(t)g^{s}(t)\right) \tag{25}$$

which is equivalent to

$$\left({}_{k}^{s}J_{a}^{\alpha}f^{p}(t)\right)^{\frac{1}{p}}\left({}_{k}^{s}J_{a}^{\alpha}g^{q}(t)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{pq}{s^{3}}}\left({}_{k}^{s}J_{a}^{\alpha}f^{s}(t)g^{s}(t)\right)^{\frac{1}{s}}. \tag{26}$$

Combining equations (24) and (26), we obtain desired inequality equation (22), which is complete the proof.

4 Applications for Some Types Fractional Integrals

Here in this section, we discuss some applications of Theorem 1 in the terms of Theorems 5-7 and Corollary 1-5.

Theorem 5. *Let p,q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and let f be continuous on $[a,b]$, $k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then*

$$\begin{aligned}&\int_{a}^{b} f_{1}(x)f_{2}(x){}_{k}^{s}J_{a}^{\alpha}f(x)dx \\ &\leq C\left(\int_{a}^{b} f_{3}(x){}_{k}^{s}J_{a}^{\alpha}f(x)dx\right)^{\frac{1}{p}}\left(\int_{a}^{b} f_{4}(x){}_{k}^{s}J_{a}^{\alpha}f(x)dx\right)^{\frac{1}{q}},\end{aligned} \tag{27}$$

where

$$\begin{aligned}C=\sup_{t\in[a,b]}&\left\{\left(\int_{a}^{b}\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_{1}(x)f_{2}(x)dx\right)\right.\\ &\left.\left(\int_{a}^{b}\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_{3}(x)dx\right)^{\frac{-1}{p}}\left(\int_{a}^{b}\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_{4}(x)dx\right)^{\frac{-1}{q}}\right\}.\end{aligned} \tag{28}$$

Proof. In Theorem 1, if we take $\Omega_{1}=\Omega_{2}=(a,b)$, $d\mu_{1}(t)=dt, d\mu_{2}(x)=dx$ and the kernel

$$k(x,t)=\begin{cases}\frac{(s+1)^{1-\frac{\alpha}{k}}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s}}{k\Gamma_{k}(\alpha)} & \text{if } a\leq t\leq x\\ 0 & \text{if } x<t\leq b,\end{cases}$$

then $g(x)$ becomes ${}_{k}^{s}J_{a}^{\alpha}f(t)$ and so we get desired inequality (27). This completes the proof of Theorem 5.

Corollary 1. *In Theorem 5, if we take $s = 0$, then we get*

$$\int_a^b f_1(x) f_2(x)_k J_a^\alpha f(x) dx \tag{29}$$
$$\leq C \left(\int_a^b f_3(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_4(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1(x) f_2(x) dx \right) \right. \tag{30}$$
$$\left. \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_3(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_4(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Remark 1. In Corollary 1, $\alpha = k = 1$, Theorem 1 reduces to Theorem 3.1 in [5].

Corollary 2. *In Theorem 5, if we take $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get*

$$\int_a^b f_1(x) f_2(x)_k^s J_a^\alpha f(x) dx \tag{31}$$
$$\leq C \left(\int_a^b f_1^p(x)_k^s J_a^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x)_k^s J_a^\alpha f(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} f_1(x) f_2(x) dx \right) \right. \tag{32}$$
$$\left. \left(\int_a^b \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} f_1^p(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Corollary 3. *In Corollary 2, if we take $s = 0$, then we get*

$$\int_a^b f_1(x) f_2(x)_k J_a^\alpha f(x) dx \tag{33}$$
$$\leq C \left(\int_a^b f_1^p(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1(x) f_2(x) dx \right) \right. \tag{34}$$
$$\left. \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1^p(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Remark 2. In Corollary 3, $\alpha = k = 1$, Corollary 3 reduces to Corollary 3.2 in [5].

Theorem 6. *Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). Also, let p,q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and let f be continuous on $[a,b]$, $k>0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then*

$$\int_a^b f_1(x)f_2(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx \tag{35}$$
$$\leq C\left(\int_a^b f_3(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_4(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_1(x)f_2(x)dx\right)\right.$$
$$\times\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_3(x)dx\right)^{\frac{-1}{p}}$$
$$\left.\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_4(x)dx\right)^{\frac{-1}{q}}\right\}. \tag{36}$$

Proof. Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(t) = dt, d\mu_2(x) = dx$ and the kernel

$$k(x,t) = \begin{cases} \frac{(h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)}{k\Gamma_k(\alpha)} & \text{if } a \leq t \leq x \\ 0 & \text{if } x < t \leq b, \end{cases}$$

then $g(x)$ becomes $\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)$ and so we get desired inequality (35). This completes the proof of Theorem 6.

Corollary 4. *In Theorem 6, setting $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, we get*

$$\int_a^b f_1(x)f_2(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx \tag{37}$$
$$\leq C\left(\int_a^b f_1^p(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_2^q(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1^p(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{38}$$

Theorem 7. *Under the assumptions of Theorem 6, we have*

$$\int_a^b f_1(x) f_2(x) \left({}_kJ^{\alpha}_{b-,h} f \right)(x) dx \tag{39}$$
$$\leq C \left(\int_a^b f_3(x) \left({}_kJ^{\alpha}_{b-,h} f \right)(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_4(x) \left({}_kJ^{\alpha}_{b-,h} f \right)(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_3(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_4(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{40}$$

Proof. In contrast to Theorem 6, if we take the kernel

$$k(x,t) = \begin{cases} \frac{(h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t)}{k\Gamma_k(\alpha)} & \text{if } x \leq t \leq b \\ 0 & \text{if } a < t \leq x, \end{cases}$$

we obtain desired inequality.

Corollary 5. *In Theorem 7, setting $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, we get*

$$\int_a^b f_1(x) f_2(x) \left({}_kJ^{\alpha}_{b-,h} f \right)(x) dx \tag{41}$$
$$\leq C \left(\int_a^b f_1^p(x) \left({}_kJ^{\alpha}_{b-,h} f \right)(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x) \left({}_kJ^{\alpha}_{b-,h} f \right)(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1^p(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{42}$$

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On weighted Adams-Bashforth rules

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Abstract. One class of linear multistep methods for solving the Cauchy problems of the form $y' = F(x, y)$, $y(x_0) = y_0$, contains Adams-Bashforth rules of the form $y_{n+1} = y_n + h\sum_{i=0}^{k-1} B_i^{(k)} F(x_{n-i}, y_{n-i})$, where $\{B_i^{(k)}\}_{i=0}^{k-1}$ are fixed numbers. In this paper, we propose an idea for a weighted type of Adams-Bashforth rules for solving the Cauchy problem for singular differential equations,

$$A(x)y' + B(x)y = G(x, y), \quad y(x_0) = y_0,$$

where A and B are two polynomials determining the well-known classical weight functions in the theory of orthogonal polynomials. Some numerical examples are also included.

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1. Introduction

In this paper, we present an idea for constructing weighted Adams-Bashforth rules for solving Cauchy problems for singular differential equations.

There are two main approaches to increase the accuracy of a numerical method for ordinary non-singular differential equations. In the first approach (i.e., multistep methods), the accuracy is increased by considering previous information, while in the second one (i.e., multistage methods or more precisely Runge-Kutta methods), the accuracy is increased by approximating the solution at several internal points.

Multistep methods were originally proposed by Bashforth and Adams [2] (see also [1, 3, 4]), where the approximate solution of the initial value problem

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0, \tag{1}$$

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is considered as

$$y_{n+1} = y_n + h \sum_{i=0}^{k-1} B_i^{(k)} F(x_{n-i}, y_{n-i}). \tag{2}$$

Many years later, Moulton [12] (see also, [3, 4]) developed a class of implicit multistep methods, the so-called Adams-Moulton methods,

$$y_{n+1} = y_n + h \sum_{i=-1}^{k-1} \alpha_i^{(k)} F(x_{n-i}, y_{n-i}), \tag{3}$$

which have some better characteristics than the previous ones.

The unknown coefficients $B_i^{(k)}$ and $\alpha_i^{(k)}$ in relations (2) and (3) are chosen in such a way that they have the highest possible accuracy order. These formulas are indeed special cases of the so-called linear multistep methods denoted by

$$y_n = \sum_{j=1}^{k_1} \eta_j y_{n-j} + h \sum_{i=0}^{k_2} \gamma_i F(x_{n-i}, y_{n-i}).$$

Other special cases of linear multistep methods were derived by Nyström and Milne [1, 4]. The idea of Predictor-Corrector methods was proposed by Milne [4] in which y_n is predicted by the Adams-Bashforth methods and then corrected by the Adams-Moulton methods.

It is not fair to talk about linear multistep methods without mentioning the name of Germund Dahlquist. In 1956, he [6] established some basic concepts such as consistency, stability and convergence in numerical methods and showed that if a numerical method is consistent and stable, then it is necessarily convergent.

However, it should be noted that the above-mentioned methods are valid only for non-singular problems of type (1). In other words, if equation (1) is considered as an initial value problem on (a, b) in the form

$$A(x)y' = H(x, y), \quad y(a) = y_0, \tag{4}$$

such that

$$A(a) = 0 \quad \text{or} \quad A(b) = 0,$$

then it is no longer possible to use usual Adams-Bashforth methods or other numerical techniques. For this purpose, in this paper, we gave an idea for using a weighted Adams-Bashforth rule.

For constructing these weighted rules we use a similar procedure as in the case of non-weighted formulas. Therefore, in Section 2, we give a short account of constructing the usual Adams-Bashforth methods by using linear difference operators and the backward Newton interpolation formula. Such a procedure is applied in Section 3 for obtaining the weighted rules. By introducing the weighted local truncation error of such rules, we determine their order. Finally, in order to illustrate the efficiency of such weighted rules, we give some numerical examples in Section 4.

2. Computing the usual Adams-Bashforth methods

In this section, we obtain the explicit forms of the coefficients $\{B_i^{(k)}\}_{i=0}^{k-1}$ in (2) using the backward Newton interpolation formula for $F(x,y) = F(x,y(x))$ at equidistant nodes $x_{n-\nu} = x_n - \nu h$, $\nu = 0,1,\ldots,k-1$, and in the next section we apply such an approach in order to get the corresponding weighted type of Adams-Bashforth methods. Here we use standard linear difference operators ∇ (the backward-difference operator), E (the shifting operator), and 1 (the identity operator), defined by

$$\nabla f(x) = f(x) - f(x-h), \qquad Ef(x) = f(x+h) \quad \text{and} \quad 1f(x) = f(x).$$

Since $E^\lambda = (1-\nabla)^{-\lambda}$, we have

$$E^\lambda = \sum_{\nu=0}^{+\infty} (-1)^\nu \binom{-\lambda}{\nu} \nabla^\nu = \sum_{\nu=0}^{+\infty} \frac{(\lambda)_\nu}{\nu!} \nabla^\nu, \tag{5}$$

where

$$(\lambda)_\nu = \lambda(\lambda+1)\cdots(\lambda+\nu-1) = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

is Pochhammer's symbol. Assuming $F_{n-\nu} \equiv F(x_{n-\nu}, y(x_{n-\nu}))$ for $\nu = 0,1,\ldots,k-1$ and taking the first k terms of (5) for $x = x_n + \lambda h$ we get

$$F(x,y(x)) = E^\lambda F_n = \sum_{\nu=0}^{k-1} \frac{(\lambda)_\nu}{\nu!} \nabla^\nu F_n + r_k(F_n), \tag{6}$$

where $r_k(F_n)$ denotes the corresponding error term. Using (6) we have

$$\begin{aligned}
\sum_{\nu=0}^{k-1} \frac{(\lambda)_\nu}{\nu!} \nabla^\nu F_n &= \sum_{\nu=0}^{k-1} \frac{(\lambda)_\nu}{\nu!} \sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} E^{-i} F_n \\
&= \sum_{i=0}^{k-1} \left(\frac{(-1)^i}{i!} \sum_{\nu=i}^{k-1} \frac{(\lambda)_\nu}{(\nu-i)!} \right) F_{n-i} = \sum_{i=0}^{k-1} C_i^{(k)}(\lambda) F_{n-i},
\end{aligned} \tag{7}$$

where $\lambda = (x - x_n)/h$ and

$$C_i^{(k)}(\lambda) = \frac{(-1)^i}{i!} \sum_{\nu=i}^{k-1} \frac{(\lambda)_\nu}{(\nu-i)!} = (-1)^i \frac{(\lambda)_i}{i!} \binom{\lambda+k-1}{k-1-i}, \tag{8}$$

because, based on induction, we have

$$\begin{aligned}
\sum_{\nu=i}^{k} \frac{(\lambda)_\nu}{(\nu-i)!} &= \sum_{\nu=i}^{k-1} \frac{(\lambda)_\nu}{(\nu-i)!} + \frac{(\lambda)_k}{(k-i)!} \\
&= (\lambda)_i \binom{\lambda+k-1}{k-1-i} + (\lambda)_i \binom{\lambda+k-1}{k-i} = (\lambda)_i \binom{\lambda+k}{k-i}.
\end{aligned}$$

Now, integrating (1) over (x_n, x_{n+1}) and approximating $F(x, y)$ by its backward Newton interpolation polynomial (7) yield

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} F(x,y)\mathrm{d}x \approx h\sum_{i=0}^{k-1} B_i^{(k)} F_{n-i}, \tag{9}$$

where

$$B_i^{(k)} = \int_0^1 C_i^{(k)}(\lambda)\mathrm{d}\lambda = \frac{(-1)^i}{i!}\int_0^1 (\lambda)_i \binom{\lambda+k-1}{k-1-i}\mathrm{d}\lambda. \tag{10}$$

Table 1 shows the values $B_i^{(k)}$ in (2) for $k = 2, 3, \ldots, 8$.

k	2	3	4	5	6	7	8
$B_0^{(k)}$	$\frac{3}{2}$	$\frac{23}{12}$	$\frac{55}{24}$	$\frac{1901}{720}$	$\frac{4277}{1440}$	$\frac{198721}{60480}$	$\frac{16083}{4480}$
$B_1^{(k)}$	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{59}{24}$	$-\frac{1387}{360}$	$-\frac{2641}{480}$	$-\frac{18637}{2520}$	$-\frac{1152169}{120960}$
$B_2^{(k)}$		$\frac{5}{12}$	$\frac{37}{24}$	$\frac{109}{30}$	$\frac{4991}{720}$	$\frac{235183}{20160}$	$\frac{242653}{13440}$
$B_3^{(k)}$			$-\frac{3}{8}$	$-\frac{637}{360}$	$-\frac{3649}{720}$	$-\frac{10754}{945}$	$-\frac{296053}{13440}$
$B_4^{(k)}$				$\frac{251}{720}$	$\frac{959}{480}$	$\frac{135713}{20160}$	$\frac{2102243}{120960}$
$B_5^{(k)}$					$-\frac{95}{288}$	$-\frac{5603}{2520}$	$-\frac{115747}{13440}$
$B_6^{(k)}$						$\frac{19087}{60480}$	$\frac{32863}{13440}$
$B_7^{(k)}$							$-\frac{5257}{17280}$

Table 1: *The coefficients of usual Adams-Bashforth formulae*

By assuming that all previous values y_{n-i}, $i = 0, 1, \ldots, k-1$, are exact, i.e., $y_{n-i} = y(x_{n-i})$, $i = 0, 1, \ldots, k-1$, (9) gives the k-step method (2). This k-step method, known also as *the kth-order Adams-Bashforth method*, can be written in the form

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k-1} \beta_j^{(k)} F_{n+j},$$

where $\beta_j^{(k)} = B_{k-1-j}^{(k)}$, $j = 0, 1, \ldots, k-1$.

According to (6), *the local truncation error* of this method at the point $x_{n+k} \in [a, b]$ can be expressed in the form

$$(T_h)_{n+k} = \frac{y(x_{n+k}) - y(x_{n+k-1})}{h} - \sum_{j=0}^{k-1} \beta_j^{(k)} y'(x_{n+j}) = \int_0^1 r_k(F_{n+k-1})\mathrm{d}\lambda, \tag{11}$$

where $x \mapsto y(x)$ is the exact solution of the Cauchy problem (1). If $y \in C^{k+2}[a, b]$, then (11) can be expressed as (cf. [7, pp. 409–410])

$$(T_h)_{n+k} = C_k y^{(k+1)}(\xi_k)h^k = C_k y^{(k+1)}(x_n)h^k + O(h^{k+1}), \tag{12}$$

where $x_n < \xi_k < x_{n+k-1}$. In the simplest case ($k = 1$), we have the well-known Euler method, $y_{n+1} - y_n = hF_n$. The so-called *error constants* C_k in *the main term of the local truncation error* for $k = 1, 2, 3, 4$ and 5 are

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{5}{12}, \quad C_3 = \frac{3}{8}, \quad C_4 = \frac{251}{720}, \quad C_5 = \frac{95}{288}, \tag{13}$$

respectively. Details on multistep methods, including convergence, stability and estimation of global errors $e_n = y_n - y(x_n)$, can be found in [4, 7, 11].

Remark 1. These coefficients $B_i^{(k)}$ can also be expressed in terms of the first kind Stirling numbers $\mathbf{S}(n, k)$, which are defined by

$$\prod_{i=0}^{n-1} (x - i) = \sum_{k=0}^{n} \mathbf{S}(n, k) x^k,$$

(see [5, 8, 9, 13]). Namely, for each $k \in \mathbb{N}$, coefficients (10) can be explicitly represented in terms of the first kind Stirling numbers as

$$B_i^{(k)} = \sum_{\nu=i}^{k-1} \left(\frac{\displaystyle\sum_{j=0}^{\nu} \frac{(-1)^j}{j+1} \mathbf{S}(\nu, j)}{\displaystyle(-1)^\nu \nu! + \sum_{j=1}^{\nu} i^j (j+1) \mathbf{S}(\nu+1, j+1)} \right), \quad i = 0, 1, \ldots, k-1. \tag{14}$$

3. Weighted type of Adams-Bashforth methods

In this section, we study the Cauchy problem for a special type of differential equations of the first order given on a finite interval, on a half line or on the real line, which can be considered, without loss of generality, as $(-1, 1)$, $(0, +\infty)$, and $(-\infty, +\infty)$. Thus, we consider the following initial value problem on (a, b)

$$A(x)y' + B(x)y = G(x, y), \quad y(x_0) = y_0, \tag{15}$$

where A and B are two polynomials determining the well-known classical weight functions in the theory of orthogonal polynomials (cf. [10, p. 122]). Such polynomials and weight functions are given in Table 2, where $\alpha, \beta, \gamma > -1$.

(a, b)	$w(x)$	$A(x)$	$B(x)$
$(-1, 1)$	$(1-x)^\alpha (1+x)^\beta$	$1 - x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$
$(0, +\infty)$	$x^\gamma \mathrm{e}^{-x}$	x	$\gamma + 1 - x$
$(-\infty, +\infty)$	e^{-x^2}	1	$-2x$

Table 2: *Classical weight functions and corresponding polynomials A and B*

Let again $\{x_k\}$ be a system of equidistant nodes with the step h, i.e., $x_k = x_0 + kh \in [a, b]$.

Since the differential equation of the weight function is as (cf. [10, p. 122])

$$(Aw)' = Bw,$$

after multiplying by $w(x)$, our initial value problem (15) becomes

$$A(x)w(x)y' + B(x)w(x)y = w(x)G(x,y), \quad y(x_0) = y_0,$$

which is equivalent to

$$(A(x)w(x)y)' = w(x)G(x,y), \quad y(x_0) = y_0. \tag{16}$$

Now, integrating from both sides of (16) over $[x_n, x_{n+1}]$ yields

$$A(x_{n+1})w(x_{n+1})y(x_{n+1}) - A(x_n)w(x_n)y(x_n) = \int_{x_n}^{x_{n+1}} w(x)G(x,y)\mathrm{d}x. \tag{17}$$

Let $x = x_n + \lambda h$. Similar to relations (6), (7) and (9), the right-hand side of (17) can be written in the form

$$\int_{x_n}^{x_{n+1}} w(x)G(x,y)\mathrm{d}x = h\int_0^1 w(x_n+\lambda h)\left\{\sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\nabla^\nu G_n + r_k(G_n)\right\}\mathrm{d}\lambda, \tag{18}$$

and approximated as

$$\begin{aligned}\int_{x_n}^{x_{n+1}} w(x)G(x,y)\mathrm{d}x &\approx h\int_0^1 w(x_n+\lambda h)\sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\left(\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i}G_{n-i}\right)\mathrm{d}\lambda\\ &= h\sum_{i=0}^{k-1}\left(\frac{(-1)^i}{i!}\sum_{\nu=i}^{k-1}\frac{1}{(\nu-i)!}\int_0^1 w(x_n+\lambda h)(\lambda)_\nu \mathrm{d}\lambda\right)G_{n-i}\\ &= h\sum_{i=0}^{k-1}B_i^{(k)}(h,x_n)G_{n-i},\end{aligned}$$

where

$$B_i^{(k)}(h,x_n) = \int_0^1 w(x_n+\lambda h)C_i^{(k)}(\lambda)\mathrm{d}\lambda, \tag{19}$$

$G_{n-i} \equiv G(x_{n-i}, y(x_{n-i}))$, $C_i^{(k)}(\lambda)$ is given by (8) and $r_k(G_n)$ is the error term in the corresponding backward Newton interpolation formula for $G(x,y) = G(x,y(x))$ at equidistant nodes $x_{n-i} = x_n - ih$, $i = 0,1,\dots,k-1$. Hence, the approximate form of (17) becomes

$$A(x_{n+1})w(x_{n+1})y(x_{n+1}) - A(x_n)w(x_n)y(x_n) = h\sum_{i=0}^{k-1}B_i^{(k)}(h,x_n)G_{n-i}, \tag{20}$$

where the coefficients $B_i^{(k)}(h,x_n)$ depend on h and x_n. As in the case of the standard Adams-Bashforth methods, by assuming that all previous values y_{n-i},

$i = 0, 1, \ldots, k-1$, are exact, i.e., $y_{n-i} = y(x_{n-i})$, $i = 0, 1, \ldots, k-1$, (20) gives our *weighted k-step method*

$$A(x_{n+1})w(x_{n+1})y_{n+1} - A(x_n)w(x_n)y_n = h\sum_{i=0}^{k-1} B_i^{(k)}(h, x_n)G_{n-i}, \quad n \geq k-1, \tag{21}$$

where $G_{n-i} \equiv G(x_{n-i}, y_{n-i})$, $i = 0, 1, \ldots, k-1$.

The mentioned dependence of the coefficients $B_i^{(k)}(h, x_n)$ on the stepsize h and x_n makes these methods fundamentally different from the standard ones.

Similarly to (11), we can here define the corresponding *weighted local truncation error* at the point $x_{n+k} \in [a, b]$ as

$$(T_h^w)_{n+k} = \left\{\frac{1}{h}\left[A(x_{n+k})w(x_{n+k})y(x_{n+k}) - A(x_{n+k-1})w(x_{n+k-1})y(x_{n+k-1})\right]\right.$$
$$\left. - \sum_{j=0}^{k-1} B_{k-j-1}^{(k)}(h, x_{n+k-1})G(x_{n+j}, y(x_{n+j}))\right\}\frac{1}{A(x_{n+k})w(x_{n+k})},$$

where $x \mapsto y(x)$ is the exact solution of the Cauchy problem (15).

Then, according to (17), with $n := n + k - 1$, and using (18) and (20), we obtain

$$(T_h^w)_{n+k} = \frac{1}{A(x_{n+k})w(x_{n+k})h}\int_{x_{n+k-1}}^{x_{n+k}} w(x)r_k(G_{n+k-1})\mathrm{d}x.$$

The first term omitted in the summation on the right-hand side in (18) is a good approximation of the truncation error. We will call this quantity *the main term of the truncation error* and denote by $(\widehat{T}_h^w)_{n+k}$.

Proposition 1. *Let the exact solution of the singular Cauchy problem* (15) *be sufficiently smooth, as well as the function $x \mapsto g(x) = G(x, y(x))$. Then, the main term of the truncation error at the point x_{n+k} can be expressed in the form*

$$(\widehat{T}_h^w)_{n+k} = \frac{h^k g^{(k)}(\xi_k)}{A(x_{n+k})w(x_{n+k})}\int_0^1 \binom{\lambda + k - 1}{k} w(x_{n+k-1} + \lambda h)\mathrm{d}\lambda, \tag{22}$$

where

$$g^{(k)}(x) = A(x)y^{(k+1)} + \left[B(x) + kA'(x)\right]y^{(k)} + k\Big[B'(x) + \frac{1}{2}(k-1)A''(x)\Big]y^{(k-1)} \tag{23}$$

and $x_{n-1} < \xi_k < x_{n+k-1}$.

Proof. According to (18), we have

$$(\widehat{T}_h^w)_{n+k} = \frac{\nabla^k G_{n+k-1}}{A(x_{n+k})w(x_{n+k})k!}\int_0^1 w(x_{n+k-1} + \lambda h)(\lambda)_k \mathrm{d}\lambda,$$

where the factor in front of the integral can be expressed in terms of divided differences as (cf. [7, p. 410])

$$\frac{\nabla^k g_{n+k-1}}{k!} = h^k\left[x_{n+k-1}, x_{n+k-2}, \ldots, x_{n-1}\right]g.$$

On the other hand, supposing that the function $x \mapsto g(x) = G(x, y(x))$ is sufficiently smooth, we can write

$$[x_{n+k-1}, x_{n+k-2}, \dots, x_{n-1}]\, g = \frac{g^{(k)}(\xi_k)}{k!},$$

where ξ_k is between the smallest and the largest of these points. In order to calculate these derivatives,

$$g'(x) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} y', \quad g''(x) = \frac{\partial^2 G}{\partial x^2} + \left[2\frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 G}{\partial y^2} y'\right] y' + \frac{\partial G}{\partial y} y'', \quad \text{etc.}$$

we use the relation $g(x) = G(x, y(x)) = A(x)y'(x) + B(x)y(x)$. Since $A(x)$ is a polynomial of degree at most two and $B(x)$ is a polynomial of first degree, these derivatives can be calculated much simpler for each $k \geq 0$ in the form (23).

In this way, we obtain (22). □

Formally, (22) is of the same form as (12), i.e., $C_k g^{(k)}(\xi_k) h^k$, where

$$C_k = C_k(h, x_n) = \frac{1}{A(x_{n+k})w(x_{n+k})} \int_0^1 \binom{\lambda + k - 1}{k} w(x_{n+k-1} + \lambda h) \mathrm{d}\lambda$$

and $g^{(k)}$ is given by (23). In the case of standard Adams-Bashforth methods, C_k are the error constants given by (13) and they are independent of the stepsize h and x_n. Also, instead of $g^{(k)}$, there is only the derivative $y^{(k+1)}$ in (12). Because of these differences, the actual order of the weighted methods can be reduced (see examples in Section 4).

In the sequel, we consider the three cases given previously in Table 2.

3.1. Case $(a, b) = (-1, 1)$

Consider the Cauchy problem of Jacobi type

$$(1 - x^2)y' + (\beta - \alpha - (\alpha + \beta + 2)x)\, y = G(x, y), \qquad y(x_0) = y_0,$$

where $x_n = x_0 + nh \in (-1, 1)$.

In this case, relation (20) is reduced to

$$y_{n+1} = d(h, x_n) y_n + h \sum_{i=0}^{k-1} D_i^{(k)}(h, x_n) G_{n-i}, \tag{24}$$

where

$$d(h, x_n) = \frac{A(x_n)w(x_n)}{A(x_{n+1})w(x_{n+1})} = \frac{(1 - x_n)^{\alpha+1}(1 + x_n)^{\beta+1}}{(1 - x_{n+1})^{\alpha+1}(1 + x_{n+1})^{\beta+1}}$$

and

$$D_i^{(k)}(h, x_n) = \frac{1}{A(x_{n+1})w(x_{n+1})} \int_0^1 w(x_n + \lambda h) C_i^{(k)}(\lambda) d\lambda.$$

Putting

$$c(h,x_n)=\frac{(1-x_n)^{\alpha}(1+x_n)^{\beta}}{(1-x_{n+1})^{\alpha+1}(1+x_{n+1})^{\beta+1}}$$

and

$$\Phi_i^{(k)}(h,x_n)=\int_0^1(1-x_n-\lambda h)^{\alpha}(1+x_n+\lambda h)^{\beta}C_i^{(k)}(\lambda)\mathrm{d}\lambda, \tag{25}$$

rule (24) can be simplified as

$$y_{n+1}=c(h,x_n)\left((1-x_n^2)y_n+h\sum_{i=0}^{k-1}\Phi_i^{(k)}(h,x_n)G_{n-i}\right). \tag{26}$$

In the special (Legendre) case $\alpha=\beta=0$, (25) is reduced to

$$\Phi_i^{(k)}(h,x_n)=\int_0^1 C_i^{(k)}(\lambda)d\lambda=B_i^{(k)}, \tag{27}$$

where $B_i^{(k)}$ are the same coefficients as for standard (non-weighted) Adams-Bashforth formulas given by (10).

3.2. Case $(a,b)=(0,\infty)$

Now, consider the Cauchy problem of Laguerre type

$$xy'+(\gamma+1-x)y=G(x,y),\quad y(x_0)=y_0,$$

in which $x_n=nh$ for $n=0,1,\ldots$, and the main relation (20) is reduced to the corresponding equation (24) with

$$d(h,x_n)=\frac{A(x_n)w(x_n)}{A(x_{n+1})w(x_{n+1})}=\mathrm{e}^h\left(\frac{x_n}{x_{n+1}}\right)^{\gamma+1},$$

and

$$D_i^{(k)}(h,x_n)=\frac{B_i^{(k)}(h,x_n)}{A(x_{n+1})w(x_{n+1})}=\frac{\mathrm{e}^h}{x_{n+1}^{\gamma+1}}\int_0^1(x_n+\lambda h)^{\gamma}\mathrm{e}^{-\lambda h}C_i^{(k)}(\lambda)\mathrm{d}\lambda.$$

In other words, we have

$$D_i^{(k)}(h,x_n)=x_n^{-(\gamma+1)}d(h,x_n)\Phi_i^{(k)}(h,x_n),$$

such that

$$\Phi_i^{(k)}(h,x_n)=\int_0^1(x_n+\lambda h)^{\gamma}\mathrm{e}^{-\lambda h}C_i^{(k)}(\lambda)\mathrm{d}\lambda. \tag{28}$$

Hence, the relation corresponding to (20) takes the form

$$y_{n+1}=\mathrm{e}^h\left(\frac{x_n}{x_{n+1}}\right)^{\gamma+1}y_n+\frac{\mathrm{e}^h h}{x_{n+1}^{\gamma+1}}\sum_{i=0}^{k-1}\Phi_i^{(k)}(h,x_n)G_{n-i}. \tag{29}$$

Note that when $\gamma = 0$, the coefficients (28) are independent of x_n and

$$\Phi_i^{(k)}(h, x_n) = \Phi_i^{(k)}(h) = \int_0^1 \mathrm{e}^{-\lambda h} C_i^{(k)}(\lambda)\mathrm{d}\lambda. \tag{30}$$

For instance, for $k = 1(1)5$ and $i = 0, 1, \ldots, k-1$, relation (30) gives

$\underline{k=1}$:

$$\Phi_0^{(1)}(h) = \frac{1-\mathrm{e}^{-h}}{h};$$

$\underline{k=2}$:

$$\Phi_0^{(2)}(h) = \frac{1+h-\mathrm{e}^{-h}(1+2h)}{h^2},$$

$$\Phi_1^{(2)}(h) = -\frac{1-\mathrm{e}^{-h}(1+h)}{h^2};$$

$\underline{k=3}$:

$$\Phi_0^{(3)}(h) = \frac{2+3h+2h^2-\mathrm{e}^{-h}(2+5h+6h^2)}{2h^3},$$

$$\Phi_1^{(3)}(h) = -\frac{2(1+h)-\mathrm{e}^{-h}(2+4h+3h^2)}{h^3},$$

$$\Phi_2^{(3)}(h) = \frac{2+h-\mathrm{e}^{-h}(2+3h+2h^2)}{2h^3};$$

$\underline{k=4}$:

$$\Phi_0^{(4)}(h) = \frac{6+12h+11h^2+6h^3-2\mathrm{e}^{-h}(3+9h+13h^2+12h^3)}{6h^4},$$

$$\Phi_1^{(4)}(h) = -\frac{2(3+5h+3h^2)-\mathrm{e}^{-h}(6+16h+19h^2+12h^3)}{2h^4},$$

$$\Phi_2^{(4)}(h) = \frac{6+8h+3h^2-2\mathrm{e}^{-h}(3+7h+7h^2+4h^3)}{2h^4},$$

$$\Phi_3^{(4)}(h) = -\frac{2(3+3h+h^2)-\mathrm{e}^{-h}(6+12h+11h^2+6h^3)}{6h^4};$$

$\underline{k=5}$:

$$\Phi_0^{(5)}(h) = \frac{12+30h+35h^2+25h^3+12h^4-\mathrm{e}^{-h}(12+42h+71h^2+77h^3+60h^4)}{12h^5},$$

$$\Phi_1^{(5)}(h) = -\frac{2(12+27h+26h^2+12h^3)-\mathrm{e}^{-h}(24+78h+118h^2+107h^3+60h^4)}{6h^5},$$

$$\Phi_2^{(5)}(h) = \frac{12+24h+19h^2+6h^3-\mathrm{e}^{-h}(12+36h+49h^2+39h^3+20h^4)}{2h^5},$$

$$\Phi_3^{(5)}(h) = -\frac{2(12+21h+14h^2+4h^3)-\mathrm{e}^{-h}(24+66h+82h^2+61h^3+30h^4)}{6h^5},$$

$$\Phi_4^{(5)}(h) = \frac{12+18h+11h^2+3h^3-\mathrm{e}^{-h}(12+30h+35h^2+25h^3+12h^4)}{12h^5}.$$

Remark 2. According to (30), it is clear that $\lim\limits_{h\to 0}\Phi_i^{(k)}(h) = B_i^{(k)}$, $i = 0, 1, \ldots, k-1$, where $B_i^{(k)}$ are the coefficients of standard (non-weighted) Adams-Bashforth formulas given by (10).

Remark 3. Consider the Cauchy-Laguerre problem $xy' + (1-x)y = y$ (i.e., $G(x,y) = y$), which is simplified as

$$y' = y, \quad y(0) = 1,$$

with the exact solution $y = \mathrm{e}^x$. Considering the simplest method (for $k = 1$) gives

$$y_{n+1} = \frac{\mathrm{e}^h}{1 + \frac{h}{x_n}}\left(y_n + \frac{h}{x_n}\Phi_0^{(1)}(h)G_n\right) = \frac{\mathrm{e}^h}{x_{n+1}}\left(x_n y_n + (1 - \mathrm{e}^{-h})G(x_n, y_n)\right).$$

Now, substituting $x_n = nh$ in the above relation yields

$$y_{n+1} = \frac{(1+nh)\mathrm{e}^h - 1}{(n+1)h} y_n, \quad \text{with} \quad y_0 = 1.$$

For example, we have

$$y_1 = \frac{\mathrm{e}^h - 1}{h}, \quad y_2 = \frac{(1+h)\mathrm{e}^h - 1}{2h}\cdot\frac{\mathrm{e}^h - 1}{h},$$
$$y_3 = \frac{(1+2h)\mathrm{e}^h - 1}{3h}\cdot\frac{(1+h)\mathrm{e}^h - 1}{2h}\cdot\frac{\mathrm{e}^h - 1}{h},$$

or, in general,

$$y_n = \prod_{\nu=1}^{n}\frac{[1 + (\nu-1)h]\mathrm{e}^h - 1}{\nu h}.$$

The method is convergent, i.e.,

$$\lim_{\substack{n\to+\infty \\ (nh = x = \mathrm{const})}} y_n = \lim_{n\to+\infty}\prod_{\nu=1}^{n}\frac{\left[1 + (\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n} - 1}{\dfrac{\nu x}{n}} = \mathrm{e}^x. \tag{31}$$

In order to prove (31) we define a sequence $\{a_n\}_{n\in\mathbb{N}}$ by

$$a_n = \sum_{\nu=1}^{n}\log\left\{\frac{\left[1 + (\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n} - 1}{\dfrac{\nu x}{n}}\right\}^n$$

and apply the well known Stolz-Cesàro theorem. Namely, if we prove the convergence

$$\lim_{n\to+\infty}\frac{a_n - a_{n-1}}{n-(n-1)} = \lim_{n\to+\infty}\log\left\{\frac{\left[1 + \left(1 - \dfrac{1}{n}\right)x\right]\mathrm{e}^{x/n} - 1}{x}\right\}^n = L,$$

then the limit $\lim\limits_{n\to+\infty}\dfrac{a_n}{n}$ also exists and it is equal to L. Since

$$\frac{1}{x}\left\{\left[1+\left(1-\frac{1}{n}\right)x\right]\mathrm{e}^{x/n}-1\right\}=1+\frac{x}{n}+O(n^{-2}),$$

we conclude that $L=\log\mathrm{e}^x=x$. Therefore, we obtain (31), because

$$\begin{aligned}\lim_{n\to+\infty}\frac{a_n}{n}&=\lim_{n\to+\infty}\sum_{\nu=1}^{n}\log\left\{\frac{\left[1+(\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n}-1}{\dfrac{\nu x}{n}}\right\}\\&=\lim_{n\to+\infty}\log\left\{\prod_{\nu=1}^{n}\frac{\left[1+(\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n}-1}{\dfrac{\nu x}{n}}\right\}=x.\end{aligned}$$

3.3. Case $(a,b)=(-\infty,\infty)$

Now, consider the Cauchy problem of Hermite type

$$y'-2xy=G(x,y),\quad y(x_0)=y_0,$$

in which $x_n=x_0+nh\in(-\infty,\infty)$ and the main relation (20) is reduced to the corresponding equation (24) with

$$d(h,x_n)=\frac{\mathrm{e}^{-x_n^2}}{\mathrm{e}^{-x_{n+1}^2}}=\mathrm{e}^{x_{n+1}^2-x_n^2}=\mathrm{e}^{h(2x_n+h)},$$

and

$$D_i^{(k)}(h,x_n)=\frac{B_i^{(k)}(h,x_n)}{\mathrm{e}^{-x_{n+1}^2}}=\mathrm{e}^{x_{n+1}^2}\int_0^1\mathrm{e}^{-(x_n+\lambda h)^2}C_i^{(k)}(\lambda)\mathrm{d}\lambda.$$

In other words, we have $D_i^{(k)}(h,x_n)=d(h,x_n)\Phi_i^{(k)}(h,x_n)$, such that

$$\Phi_i^{(k)}(h,x_n)=\int_0^1\mathrm{e}^{-(2x_n\lambda h+\lambda^2h^2)}C_i^{(k)}(\lambda)\mathrm{d}\lambda.$$

Hence, the relation corresponding to (20) takes the form

$$y_{n+1}=\mathrm{e}^{h(2x_n+h)}\left(y_n+\sum_{i=0}^{k-1}\Phi_i^{(k)}(h,x_n)G_{n-i}\right).\tag{32}$$

Remark 4. As in the case of non-weighted methods, in applications of these methods for $k>1$, we need the additional starting values $y_i=s_i(h)$, $i=1,\ldots,k-1$, such that $\lim\limits_{h\to0}=y_0$ (cf. [11, pp. 32–36]).

4. Numerical examples

In order to illustrate the efficiency of our method, in this section we give two numerical examples for singular Cauchy problems on $(0, \infty)$ and $(-1, 1)$. In particular, the weighted Adams-Bashforth methods with respect to the standard Laguerre weight given in Subsection 3.2 have the simplest form and they can find adequate application in solving weighted singular Cauchy problems. The third case when $(a, b) = (-\infty, \infty)$ is not interesting for applications because equation (15) is not singular in a finite domain.

Example 1. We first consider a singular Cauchy problem

$$(1-x^2)y' - 3xy = \frac{y^2\left((1-x^2)\tan(x) + 4x + 1\right)\sec(x)}{x-1}, \quad y(-1) = 2\cos 1.$$

Here,

$$G(x,y) = \frac{y^2\left((1-x^2)\tan(x) + 4x + 1\right)\sec(x)}{x-1} + xy$$

and the exact solution of this problem is given by $y(x) = (1-x)\cos x$.

In order to solve the problem for $x \in [-1, 0]$, we apply the k-step method (26), with $\alpha = \beta = 0$ (Legendre case) and $\Phi_i^{(k)}(h, x_n)$ given by (27). For the sake of simplicity, in the case $k > 1$, for starting values we use the exact values. Otherwise, some other ways must be applied (see Remark 4).

Relative errors obtained by this k-step method when $k = 1, 2, \ldots, 5$, for $h = 0.02$ and $h = 0.01$, are displayed in log-scale in Figures 1 and 2, respectively.

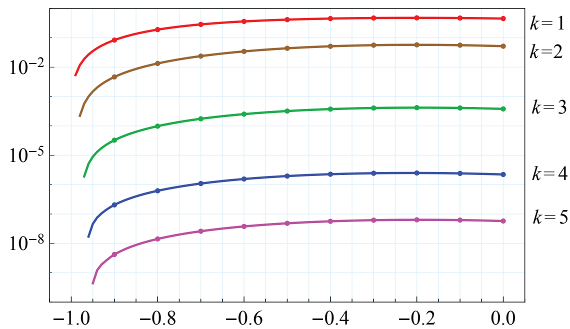

Figure 1: *Relative errors for methods (26) in Example 1 for $h = 0.02$ and $k = 1, 2, \ldots, 5$*

We consider now the actual errors, $|y_{n+k} - y(x_{n+k})|$ at a fixed point $x = x_{n+k}$ obtained by using the k-step method (26) for different stepsize h and different k. We take $x = -0.5$ and $h = 0.05$, 0.02 and 0.01. The corresponding errors are presented in Table 3. Numbers in parentheses indicate decimal exponents.

Note that the effect of reducing the step-size h to the accuracy of the solution is greater if k is higher. Assuming an asymptotic relation in the form

$$e(h,k,x) = |y_{n+k}^h - y(x_{n+k})| \approx C_k h^{\alpha_k}, \tag{33}$$

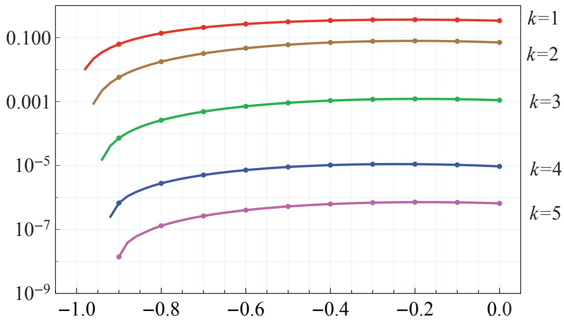

Figure 2: *Relative errors for methods (26) in Example 1 for* $h = 0.01$ *and* $k = 1, 2, \ldots, 5$

h	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
0.05	2.27(−1)	1.14(−1)	4.76(−3)	6.68(−5)	1.45(−5)
0.02	4.10(−1)	7.87(−2)	1.20(−3)	1.18(−5)	6.89(−7)
0.01	5.46(−1)	5.77(−2)	4.21(−4)	2.57(−6)	6.34(−8)

Table 3: *Absolute errors in the obtained sequences* $\{y_{n+k}\}_n$ *at the point* $x = x_{n+k} = -0.5$*, using the* k*-step method (26) for* $k = 1, 2, \ldots, 5$ *and* $h = 0.05,\ 0.02$ *and* 0.01

where $x_{n+k} = -1 + (n+k)h = x =$ const, and C_k and α_k are some constants, we can calculate the following quotient for two different steps h_1 and h_2,

$$\frac{e(h_1, k, x)}{e(h_2, k, x)} \approx \left(\frac{h_1}{h_2}\right)^{\alpha_k}.$$

Therefore,

$$\alpha_k = \frac{\log(e(h_1, k, x)/e(h_2, k, x))}{\log(h_1/h_2)}. \tag{34}$$

These values are presented in Table 4 for $h_1/h_2 = 2$ and $h_1/h_2 = 5$. As we can see, the obtained values for the exponents α_k are very similar in these two cases.

h_1/h_2	α_1	α_2	α_3	α_4	α_5
2	−0.41	0.45	1.51	2.20	3.44
5	−0.54	0.42	1.51	2.02	3.38

Table 4: *The parameters* α_k *obtained from (34) for* $k = 1, 2, \ldots, 5$ *and different stepsizes*

As we can see, the actual order of the method is reduced approximately for one order of magnitude. This effect is mentioned in Section 3 after Proposition 1.

Example 2. Here we consider again the Cauchy problem of Jacobi type

$$(1 - x^2)y' - 2xy = 1 - x - 4x^2 - 5x^3 + xy, \quad y(-1) = 1,$$

whose exact solution is $y(x) = x^2 + x + 1$. According to Proposition 1, in this simple case, we have

$$g(x) = G(x, y(x)) = -4x^3 - 3x^2 + 1,$$

as well as

$$g'(x) = -12x^2 - 6x, \quad g''(x) = -24x - 6, \quad g'''(x) = -24, \quad g^{(iv)}(x) = 0.$$

Now, we apply the k-step method (26) for $k = 1(1)4$, $h = 0.05$ and $h = 0.01$ (see Tables 5 and 6), and for starting values (when $k > 1$) we use the exact values of the solution. In these tables, we only give the relative errors of the obtained values for $x = -0.9, -0.8, -0.7, -0.6, -0.5$ (m.p. is machine precision).

x	$k=1$	$k=2$	$k=3$	$k=4$
-0.9	$5.62(-2)$	$5.09(-3)$		
-0.8	$4.88(-2)$	$6.37(-3)$	$3.17(-4)$	m.p.
-0.7	$4.37(-2)$	$7.03(-3)$	$4.46(-4)$	m.p.
-0.6	$3.80(-2)$	$7.29(-3)$	$5.62(-4)$	m.p.
-0.5	$3.18(-2)$	$7.23(-3)$	$6.42(-4)$	m.p.

Table 5: *Relative errors in the obtained approximate sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (26) *for* $k = 1(1)4$ *and* $h = 0.05$

x	$k=1$	$k=2$	$k=3$	$k=4$
-0.9	$9.81(-3)$	$2.61(-4)$	$3.19(-6)$	m.p.
-0.8	$9.19(-3)$	$2.82(-4)$	$3.92(-6)$	m.p.
-0.7	$8.37(-3)$	$2.94(-4)$	$4.53(-6)$	m.p.
-0.6	$7.33(-3)$	$2.96(-4)$	$5.11(-6)$	m.p.
-0.5	$6.15(-3)$	$2.89(-4)$	$5.62(-6)$	m.p.

Table 6: *Relative errors in the obtained approximate sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (26) *for* $k = 1(1)4$ *and* $h = 0.01$

As in Example 1, we consider asymptotic relation (33) at $x = x_{n+k} = -0.5$, when $k = 1, 2, 3$ and $h_1 = 0.05$ and $h_2 = 0.01$. The values of the corresponding exponent (34) are presented in Table 7.

k	$k=1$	$k=2$	$k=3$
$h_1/h_2 = 5$	$\alpha_1 = 1.00$	$\alpha_2 = 2.00$	$\alpha_3 = 2.94$

Table 7: *The parameters* α_k *obtained from (34) for* $k = 1, 2, 3$ *and* $h_1/h_2 = 5$

As we can see, in this polynomial case, there is not previously mentioned defect in the order. Note that the local truncation error (22) is equal to zero for each $k \geq 4$, because of $g^{(k)}(x) = 0$. Also, we see that the actual errors in Tables 5 and 6 for $k = 4$ are on the level of machine precision.

Example 3. Now we consider the Cauchy problem of Laguerre type

$$xy' + (1-x)y = \frac{3x^2+1}{(x^2+1)^2}\mathrm{e}^{-x}y^2, \quad y(0) = 1,$$

whose exact solution is $y = (x^2+1)\mathrm{e}^x$. We apply the k-step method (29) for $k = 1(1)6$. The corresponding relative errors for $h = 0.05$ are given in Table 8, and for $h = 0.01$ in Table 9. In these tables, we only give relative errors of the obtained values for $x = 0(0.1)1$. As in Example 1, for starting values we use the exact values of solution.

x	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.1	5.66(−2)	3.74(−3)				
0.2	1.18(−1)	1.43(−2)	7.17(−4)	1.89(−5)		
0.3	1.81(−1)	2.62(−2)	1.62(−3)	7.55(−5)	2.89(−6)	6.33(−8)
0.4	2.45(−1)	3.94(−2)	2.62(−3)	1.32(−4)	5.83(−6)	2.39(−7)
0.5	3.08(−1)	5.42(−2)	3.76(−3)	1.98(−4)	9.39(−6)	4.16(−7)
0.6	3.70(−1)	7.10(−2)	5.10(−3)	2.74(−4)	1.33(−5)	6.11(−7)
0.7	4.30(−1)	9.01(−2)	6.66(−3)	3.63(−4)	1.79(−5)	8.48(−7)
0.8	4.88(−1)	1.11(−1)	8.49(−3)	4.67(−4)	2.33(−5)	1.11(−6)
0.9	5.42(−1)	1.35(−1)	1.06(−2)	5.88(−4)	2.95(−5)	1.42(−6)
1.0	5.93(−1)	1.62(−1)	1.31(−2)	7.29(−4)	3.67(−5)	1.77(−6)

Table 8: *Relative errors in the obtained sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (29) *for* $k = 1, 2, \ldots, 6$ *and* $h = 0.05$

x	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.1	5.26(−2)	1.86(−3)	2.63(−5)	2.92(−7)	2.97(−9)	2.85(−11)
0.2	1.05(−1)	4.17(−3)	6.20(−5)	7.37(−7)	8.17(−9)	8.70(−11)
0.3	1.58(−1)	6.73(−3)	1.02(−4)	1.23(−6)	1.39(−8)	1.51(−10)
0.4	2.12(−1)	9.66(−3)	1.47(−4)	1.80(−6)	2.04(−8)	2.24(−10)
0.5	2.67(−1)	1.31(−2)	2.00(−4)	2.45(−6)	2.80(−8)	3.09(−10)
0.6	3.23(−1)	1.71(−2)	2.63(−4)	3.23(−6)	3.70(−8)	4.10(−10)
0.7	3.80(−1)	2.18(−2)	3.38(−4)	4.16(−6)	4.76(−8)	5.28(−10)
0.8	4.35(−1)	2.73(−2)	4.26(−4)	5.25(−6)	6.02(−8)	6.67(−10)
0.9	4.89(−1)	3.38(−2)	5.30(−4)	6.53(−6)	7.49(−8)	8.32(−10)
1.0	5.41(−1)	4.12(−2)	6.51(−4)	8.03(−6)	9.22(−8)	1.02(−9)

Table 9: *Relative errors in the obtained sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (29) *for* $k = 1, 2, \ldots, 6$ *and* $h = 0.01$

Using Proposition 1 we determine the main term of the truncation error at the point $x_{n+k} = x$, for example, when $h = 0.01$ and $x = 0.5$.

Since

$$\int_0^1 \binom{\lambda+k-1}{k} \mathrm{e}^{-(x-h+\lambda h)}\mathrm{d}\lambda = \mathrm{e}^{-x}Q_k, \quad Q_k = \frac{1}{k!}\int_0^1 (\lambda)_k \mathrm{e}^{-(\lambda-1)h}\mathrm{d}\lambda,$$

we first calculate the values: $Q_1 = 0.501671$, $Q_2 = 0.41792$, $Q_3 = 0.376058$, $Q_4 = 0.34955$, $Q_5 = 0.330639$, and $Q_6 = 0.316389$. The corresponding derivatives

$$g^{(k)}(x) = xy^{(k+1)} + (k+1-x)y^{(k)} - ky^{(k-1)}$$

are

$$\begin{aligned} g'(x) &= \mathrm{e}^x\left(3x^2+6x+1\right), & g''(x) &= \mathrm{e}^x\left(3x^2+12x+7\right),\\ g'''(x) &= \mathrm{e}^x\left(3x^2+18x+19\right), & g^{(iv)}(x) &= \mathrm{e}^x\left(3x^2+24x+37\right),\\ g^{(v)}(x) &= \mathrm{e}^x\left(3x^2+30x+61\right), & g^{(vi)}(x) &= \mathrm{e}^x\left(3x^2+36x+91\right), \end{aligned}$$

where $\xi_k \in \big(x-(k+1)h, x-h\big)$, $k = 1, 2, \ldots, 6$.

Now, taking $x = x_{n+k} = (n+k)h = 0.5$, $h = 0.01$, and $\xi_k = x - h = 0.49$ in (22), we obtain an approximation of the main term $(\widehat{T}_h^w)_{n+k}$ in the form

$$(\widehat{T}_h^w)_{n+k} \approx \frac{h^k g^{(k)}(x-h)Q_k}{x}, \qquad k = 1, 2, \ldots, 6,$$

whose numerical values for $k = 1, 2, \ldots, 6$, after dividing by $y(0.5) = 2.0609$, are: $3.70(-2)$, $9.00(-4)$, $1.70(-5)$, $2.74(-7)$, $4.00(-9)$, $5.48(-11)$, respectively. As expected, the actual global errors from Table 9 (the row referring to $x = 0.5$) are larger compared to the corresponding local truncation errors.

x	α_1	α_2	α_3	α_4	α_5	α_6
0.5	0.08	0.95	1.92	2.87	3.81	4.75
1.0	0.05	0.93	1.94	2.90	3.86	4.80

Table 10: *The parameters α_k obtained from (34) for $k = 1, 2, \ldots, 6$ and $h1/h2 = 2$ at two points $x = 0.5$ and $x = 1.0$.*

Finally, as in Example 1, we assume the behavior of the actual errors in the form (33), where $x_{n+k} = (n+k)h = x = \text{const}$. We compare actual errors obtained for $h_1 = 0.02$ and $h_2 = 0.01$ at two points $x = 0.5$ and $x = 1.0$. The results for α_k, $k = 0, 1, \ldots, 6$ are presented in Table 10. As we can see, the obtained values of the exponents α_k at these points are very close, but again with a defect of one order in its magnitude!

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