Filomat 35:9 (2021), 3193–3202
https://doi.org/10.2298/FIL2109193M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inequalities for the Maximum Modulus of Univariate Constrained Polynomials

Gradimir V. Milovanović[a], Abdullah Mir[b]

[a]*Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia*
and
University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia
[b]*Department of Mathematics, University of Kashmir, Srinagar, 190006, India*

Abstract. This paper deals with the problem of finding some upper bound estimates for the maximum modulus of a univariate complex polynomial on a disk under certain constraints on the zeros and on the functions involved. A variety of interesting results follow as special cases from our general results.

1. Introduction

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n and $P'(z)$ is the derivative of $P(z)$. For brevity, we introduce the following notations:

$$\phi_k(r,R,\alpha) := \left(\frac{R+k}{r+k}\right)^n - |\alpha|, \qquad \psi_k(r,R,\alpha) := \left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|,$$

$$\phi_k(r,R) := \left(\frac{R+k}{r+k}\right)^n, \qquad \psi_k(r,R) := \left(\frac{Rk+1}{rk+1}\right)^n,$$

where $\alpha \in \mathbb{C}$ is such that $|\alpha| \le 1$. Note that

$$\left(\frac{R+k}{r+k}\right)^n - |\alpha| > 0 \quad \text{and} \quad \left(\frac{Rk+1}{rk+1}\right)^n - |\alpha| > 0 \quad \text{for } R > r.$$

The study of comparison inequalities that relate the norm between polynomials on the disk $|z| = R,\ R > 0$, is a classical topic in analysis. The extremal problems of analytic functions and the results where some approaches to obtaining the classical inequalities are developed on using various methods of the geometric function theory are known for various norms and for many classes of functions such as polynomials with various constraints, and on various regions of the complex plane. A classical result due to Bernstein [3] is that, for two polynomials $f(z)$ and $F(z)$ with degree of $f(z)$ not exceeding that of $F(z)$ and $F(z) \neq 0$ for $|z| > 1$,

2020 *Mathematics Subject Classification*. Primary 30C10; Secondary 30A10, 30C15
Keywords. Complex polynomial; Maximum modulus; Rouché's theorem; Zeros.
Received: 30 May 2021; Accepted: 03 June 2021
Communicated by Vladimir Rakočević
Research of the first author was partly supported by the Serbian Academy of Sciences and Arts (Project Φ-96).
Email addresses: gvm@mi.sanu.ac.rs (Gradimir V. Milovanović), drabmir@yahoo.com (Abdullah Mir)

the inequality $|f(z)| \le |F(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|f'(z)| \le |F'(z)|$ on $|z| = 1$. In particular, this result allows one to establish the famous Bernstein inequality [2] for the sup-norm on the unit circle: namely, if $P(z)$ is a polynomial of degree n, it is true that

$$\max_{|z|=1} \left|P'(z)\right| \le n \max_{|z|=1} \left|P(z)\right|. \tag{1}$$

On the other hand, concerning the maximum modulus of $P(z)$ on the circle $|z| = R \ge 1$, we have another classical result known as Bernstein-Walsh lemma ([20], Corollary 12.1.3), which states that, if $f(z)$ and $F(z)$ are two polynomials with degree of $f(z)$ not exceeding that of $F(z)$ and $F(z) \neq 0$ for $|z| > 1$, the inequality $|f(z)| \le |F(z)|$ on the unit circle $|z| = 1$ implies that $|f(z)| < |F(z)|$ for $|z| > 1$, unless $f(z) = e^{i\theta}F(z)$, $\theta \in \mathbb{R}$. From this, one can deduce that if $P \in \mathcal{P}_n$, then for $R \ge 1$,

$$\max_{|z|=R} \left|P(z)\right| \le R^n \max_{|z|=1} \left|P(z)\right|. \tag{2}$$

The inequalities (1) and (2) are related with each other and it was observed by Bernstein [3] that (1) can also be deduced from (2) by making use of Gauss-Lucas theorem and the proof of this fact was given by Govil et al. [4].

If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ with $P(z) \neq 0$ in $|z| < 1$, then (1) and (2) can be respectively replaced by

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{2} \max_{|z|=1} \left|P(z)\right|, \tag{3}$$

and

$$\max_{|z|=R\ge 1} \left|P(z)\right| \le \frac{R^n+1}{2} \max_{|z|=1} \left|P(z)\right|. \tag{4}$$

Inequality (3) was conjectured by Erdős and later proved by Lax [9], where as inequality (4) was proved by Ankeny and Rivlin [1], for which they made use of (3). Jain [6] generalized both the inequalities (3) and (4) and proved that if $P \in \mathcal{P}_n$ with $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \le 1$ and $|z| = 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \le \frac{n}{2}\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \max_{|z|=1}\left|P(z)\right|, \tag{5}$$

and $R \ge 1$,

$$\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \le \frac{1}{2}\left[\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right] \max_{|z|=1}\left|P(z)\right|. \tag{6}$$

As refinements of (5) and (6), Jain [7] also established that if $P \in \mathcal{P}_n$ with $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \le 1$ and $|z| = 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \le \frac{n}{2}\left[\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \max_{|z|=1}\left|P(z)\right| - \left\{\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\} \min_{|z|=1}\left|P(z)\right|\right], \tag{7}$$

and

$$\begin{aligned}\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \le \frac{1}{2}\Bigg[&\left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\} \max_{|z|=1}\left|P(z)\right| \\ &- \left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| - \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\} \min_{|z|=1}\left|P(z)\right|\Bigg].\end{aligned} \tag{8}$$

Other classical majorization results for polynomials and related functions can be found in the comprehensive books of Milovanović et al. [14] and Rahman and Schmeisser [20]. Mezerji et al. [10] besides proving other results also obtained a result concerning minimum modulus of polynomials. In fact, they proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le k, k \le 1$, then for every $|\beta| \le 1$ and $R \ge 1$,

$$\min_{|z|=1} \left|P(Rz) + \beta\phi_k(1,R)P(z)\right| \ge \frac{1}{k^n}\left|R^n + \beta\phi_k(1,R)\right| \min_{|z|=k}\left|P(z)\right|. \tag{9}$$

In the same paper, Mezerji et al. generalized the inequality (8) by proving that if $P \in \mathcal{P}_n$ and $P(z) \ne 0$ in $|z| < k$, $k \ge 1$, then for every $|\beta| \le 1$ and $R \ge 1$,

$$\max_{|z|=1} \left|P(Rk^2z) + \beta\psi_k(1,R)P(k^2z)\right| \le \frac{1}{2}\left[\left\{k^n\left|R^n + \beta\psi_k(1,R)\right| + \left|1 + \beta\psi_k(1,R)\right|\right\}\min_{|z|=k}\left|P(z)\right| \right.$$
$$\left. - \left\{k^n\left|R^n + \beta\psi_k(1,R)\right| - \left|1 + \beta\psi_k(1,R)\right|\right\}\min_{|z|=k}\left|P(z)\right|\right]. \tag{10}$$

Although the literature on polynomial inequalities is vast and growing and over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities, including inequalities for polar derivatives. One can see in the literature (for example, refer [8], [10]–[19]), the latest research where some approaches to obtaining polynomial inequalities are developed on applying the methods and results of the geometric function theory. Recently, Kumar [8] found that there is a room for the generalization of the condition $R \ge 1$ in the above inequalities to $R \ge r > 0$, and proved the following results.

Theorem A. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its in $|z| \le k$, $k > 0$, then for every $|\beta| \le 1$, $|z| \ge 1$, and $R \ge r$, $Rr \ge k^2$,*

$$\min_{|z|=1} \left|P(Rz) + \beta\phi_k(r,R)P(rz)\right| \ge \frac{1}{k^n}\left|R^n + \beta\phi_k(r,R)\right| \min_{|z|=k}\left|P(z)\right|. \tag{11}$$

Equality holds in (11) *for $P(z) = \alpha z^n$, $\alpha \ne 0$.*

Theorem B. *If $P \in \mathcal{P}_n$ and $P(z) \ne 0$, $|z| < k$, $k > 0$, then for every $|\beta| \le 1$ and $R \ge r$, $Rr \ge 1/k^2$, $|z| = 1$,*

$$\left|P(Rk^2z) + \beta\psi_k(r,R)P(rk^2z)\right| \le \frac{1}{2}\left[\left\{k^n\left|R^n + \beta\psi_k(r,R)\right| + \left|1 + \beta\psi_k(r,R)\right|\right\}\max_{|z|=k}\left|P(z)\right| \right.$$
$$\left. - \left\{k^n\left|R^n + \beta\psi_k(r,R)\right| - \left|1 + \beta\psi_k(r,R)\right|\right\}\min_{|z|=k}\left|P(z)\right|\right]. \tag{12}$$

Equality holds in (12) *for $P(z) = az^n + bk^n$, $|b| \ge |a|$.*

While thinking for the generalization of the above inequalities, we consider a more general problem of investigating the dependence of $\left|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\right|$ on the maximum of $\left|P(z)\right|$ on $|z| = k$ for every $|\alpha| \le 1$, $|\beta| \le 1$, and develop a unified method for arriving at these results.

2. Main results

We first prove the following general result which as a special case provides a generalization of Theorem A.

Theorem 2.1. *If $F \in \mathcal{P}_n$ and $F(z)$ has all its in $|z| \le k$, $k > 0$ and $P(z)$ is a polynomial of degree at most n such that*

$$\left|P(z)\right| \le \left|F(z)\right| \quad \text{for} \quad |z| = k.$$

Then for $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge k^2$ and $|z| \ge 1$, we have

$$\left|P(Rz) - \alpha P(rz) + \beta\phi_k(r,R,\alpha)P(rz)\right| \le \left|F(Rz) - \alpha F(rz) + \beta\phi_k(r,R,\alpha)F(rz)\right|. \tag{13}$$

Equality holds in (13) *for $P(z) = e^{i\gamma}F(z)$, γ real and $F(z)$ has all its zeros in $|z| \le k$.*

We now present and discuss some consequences of Theorem 2.1. If we apply this theorem to polynomials $P(z)$ and $(z^n/k^n)\min_{|z|=k}|P(z)|$, we get the following generalization of (11).

Corollary 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its in $|z| \le k$, $k > 0$, then for $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge k^2$ and $|z| \ge 1$, we have*

$$\left|P(Rz) - \alpha P(rz) + \beta\phi_k(r,R,\alpha)P(rz)\right| \ge \frac{|z|^n}{k^n}\left|R^n - \alpha r^n + \beta r^n\phi_k(r,R,\alpha)\right|\min_{|z|=k}\left|P(z)\right|. \tag{14}$$

Equality holds in (14) *for $P(z) = \gamma z^n$, $\gamma \ne 0$.*

Remark 2.1. For $\alpha = 0$ Theorem 2.1 reduces to Theorem A. If we take $r = k$, then inequality (14) takes the form

$$\left|P(Rz) - \alpha P(kz) + \beta\phi_k(k,R,\alpha)P(kz)\right| \ge |z|^n\left|\frac{R^n}{k^n} - \alpha + \beta\phi_k(k,R,\alpha)\right|\min_{|z|=k}\left|P(z)\right|.$$

Several other interesting results easily follow from Theorem 2.1 and here, we mention a few of them. Setting $F(z) = z^n M/k^n$, where $M = \max_{|z|=k}\left|P(z)\right|$ in Theorem 2.1, we get the following result.

Corollary 2.2. *If $P \in \mathcal{P}_n$, then for every α, β with $|\alpha| \le 1$, $|\beta| \le 1$, and $k > 0$ with $R \ge r$, $rR \ge k^2$, $|z| \ge 1$, we have*

$$\left|P(Rz) - \alpha P(rz) + \beta\phi_k(r,R,\alpha)P(rz)\right| \le \frac{|z|^n}{k^n}\left|R^n - \alpha r^n + \beta r^n\phi_k(r,R,\alpha)\right|\max_{|z|=k}\left|P(z)\right|. \tag{15}$$

Equality holds in (15) *for $P(z) = \gamma z^n$, $\gamma \ne 0$.*

Again, if we choose $\alpha = r = 1$, $k \le 1$, in Corollary 2.2, divide the two sides of (15) by $R-1$, making $R \to 1$ and noting that

$$\frac{\phi_k(1,R,1)}{R-1} \to \frac{n}{1+k},$$

we get the following result.

Corollary 2.3. *If $P \in \mathcal{P}_n$, then for every $|\beta| \le 1$, $k \le 1$, $R > 1$ and $|z| \ge 1$,*

$$\left|zP'(z) + \frac{n\beta}{1+k}P(z)\right| \le \frac{n|z|^n}{k^n}\left|1 + \frac{\beta}{1+k}\right|\max_{|z|=k}\left|P(z)\right|. \tag{16}$$

Equality holds in (16) *for $P(z) = \gamma z^n$, $\gamma \ne 0$.*

Suppose $P \in \mathcal{P}_n$ and $P(z) \ne 0$ in $|z| < k$, the polynomial $Q(z) = z^n\overline{P(1/\bar{z})} \in \mathcal{P}_n$ and $Q(z)$ has all its zeros in $|z| \le 1/k$. Note that

$$|Q(z)| = \frac{1}{k^n}\left|P(k^2 z)\right| \quad \text{for} \quad |z| = \frac{1}{k}.$$

Applying Theorem 2.1, with $F(z)$ replaced by $k^nQ(z)$, we get the following result.

Corollary 2.4. *If $P \in \mathcal{P}_n$ and $P(z) \ne 0$ in $|z| < k$, $k > 0$, then for every $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge 1/k^2$, $|z| \ge 1$, we have*

$$\left|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\right| \le k^n\left|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\right|, \tag{17}$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$.

Next, we establish the following generalization of inequalities (5) and (6).

Theorem 2.2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r$, $rR \geq 1/k^2$ and $|z| = 1$, we have*

$$\begin{aligned}\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| \leq \frac{1}{2}\Big\{k^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| \\ + \Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|\Big\}\max_{|z|=k}\Big|P(z)\Big|. \qquad (18)\end{aligned}$$

Equality holds in (18) *for $P(z) = z^n + k^n$.*

Remark 2.2. If we take $\alpha = k = r = 1$ in Theorem 2.2 and divide both sides of (18) by $R-1$ and make $R \to 1$, we get (5). Also, if we take $\alpha = 0$ and $r = k = 1$ in Theorem 2.2, we get (6).

Instead of proving Theorem 2.2, we prove the following result which not only strengthens inequality (18) but also provides a generalization of Theorem B.

Theorem 2.3. *If $P \in \mathcal{P}_n$ and $P(z)$ has no zeros in $|z| < k$, $k > 0$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r$, $rR \geq 1/k^2$ and $|z| = 1$, we have*

$$\begin{aligned}&\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| \\ &\leq \frac{1}{2}\Big\{k^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| + \Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|\Big\}\max_{|z|=k}\Big|P(z)\Big| \\ &- \frac{1}{2}\Big\{k^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| - \Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|\Big\}\min_{|z|=k}\Big|P(z)\Big|. \qquad (19)\end{aligned}$$

Equality holds in (19) *for $P(z) = z^n + k^n$.*

Remark 2.3. If we take $\alpha = 0$ in Theorem 2.3, we get Theorem B. Similarly as above, if we take $\alpha = k = r = 1$ in Theorem 2.3 and divide both sides of (19) by $R-1$ and make $R \to 1$, we get (7), where as inequality (8) follows by taking $\alpha = 0$ and $r = k = 1$ in inequality (19).

3. Auxiliary results

We need the following lemmas to prove our theorems.

Lemma 3.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has no zeros in $|z| < k$, $k > 0$, then for $R \geq r$, $rR \geq k^2$, we have*

$$\Big|P(Rz)\Big| \geq \left(\frac{R+k}{r+k}\right)^n \Big|P(rz)\Big| \quad \textit{for} \quad |z| = 1.$$

The above lemma is due to Govil et al. [5].

Lemma 3.2. *If $P \in \mathcal{P}_n$ and $Q(z) = z^n\overline{P(1/\bar{z})}$, then for $|\alpha| \leq 1$, $|\beta| \leq 1$, $k > 0$, $R \geq r$, $rR \geq 1/k^2$ and $|z| = 1$, we have*

$$\begin{aligned}&\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| + k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| \\ &\leq \Big\{k^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| + \Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|\Big\}\max_{|z|=k}|P(z)|. \qquad (20)\end{aligned}$$

Proof. Let $M = \max_{|z|=k}|P(z)|$. Then $|P(z)| \leq M$ for $|z| = k$. Therefore, for a given complex number μ with $|\mu| > 1$, it follows by Rouché's theorem that the polynomial $F(z) = P(z) - \mu M$ does not vanish in $|z| < k$. Applying Corollary 2.4 to the polynomial $F(z)$, we get for $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r$, $rR \geq 1/k^2$ and $|z| \geq 1$,

$$\Big|F(Rk^2z) - \alpha F(rk^2z) + \beta\psi_k(r,R,\alpha)F(rk^2z)\Big| \leq k^n\Big|H(Rz) - \alpha H(rz) + \beta\psi_k(r,R,\alpha)H(rz)\Big|,$$

where $H(z) = z^n\overline{F(1/\bar{z})} = Q(z) - \bar{\mu}Mz^n$.

The above inequality gives for $|\alpha| \le 1$, $|\beta| \le 1$ and $|z| \ge 1$,

$$\begin{aligned}\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z) - \mu\Big(1 - \alpha + \beta\psi_k(r,R,\alpha)\Big)M\Big| \\ \le k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz) - \bar{\mu}\Big(R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big)Mz^n\Big|.\end{aligned} \tag{21}$$

Now choosing the argument of μ on the right hand side of (21) such that

$$\begin{aligned}k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz) - \bar{\mu}\Big(R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big)Mz^n\Big| \\ = |\mu|Mk^n|z|^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,k)\Big| - k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big|,\end{aligned}$$

which is possible by Corollary 2.2, we get from (21) for $|\alpha| \le 1$, $|\beta| \le 1$ and $|z| \ge 1$,

$$\begin{aligned}\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| - |\mu|\Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|M \\ \le |\mu|Mk^n|z|^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| - k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big|,\end{aligned}$$

implying for $|z| = 1$,

$$\begin{aligned}\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| + k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| \\ \le |\mu|Mk^n|z|^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| + |\mu|\Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|M.\end{aligned}$$

Letting $|\mu| \to 1$ in the last inequality, we get the desired inequality (20), and this completes the proof of Lemma 3.2. □

4. Proofs of main results

Proof of Theorem 2.1. In case $R = r$, we have nothing to prove. Hence forth, we suppose $R > r$. By hypothesis $F(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \le |F(z)| \quad \text{for} \quad |z| = k, \tag{22}$$

therefore, if $F(z)$ has a zero of multiplicity ν at $z = ke^{i\theta_0}$, then $P(z)$ must also have a zero of multiplicity at least ν at $z = ke^{i\theta_0}$. We assume that $P(z)/F(z)$ is not a constant, otherwise, the inequality (13) is obvious. It follows by the maximum modulus principle that

$$|P(z)| < |F(z)| \quad \text{for} \quad |z| > k.$$

Suppose $F(z)$ has m zeros on $|z| = k$, where $0 \le m < n$, so that we can write

$$F(z) = F_1(z)F_2(z),$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree $n - m$ whose all zeros lie in $|z| < k$. This gives with the help of (22) that

$$P(z) = P_1(z)F_1(z),$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (22), we get

$$|P_1(z)| \le |F_2(z)| \quad \text{for} \quad |z| = k,$$

and $F_2(z) \ne 0$ for $|z| = k$. Therefore, for a given complex number λ with $|\lambda| > 1$, it follows from Rouché's theorem that the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \ge 1$ has all its zeros in $|z| < k$. Hence the polynomial

$$f(z) = F_1(z)(P_1(z) - \lambda F_2(z)) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$f(z) = (z - \eta e^{i\gamma})H(z),$$

where $\eta < k$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leq k$. Applying Lemma 3.1 to $H(z)$, we obtain for $R > r$, $rR \geq k^2$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} \left|f(Re^{i\theta})\right| &= \left|Re^{i\theta} - \eta e^{i\gamma}\right| \left|H(Re^{i\theta})\right| \\ &\geq \left|Re^{i\theta} - \eta e^{i\gamma}\right| \left(\frac{R+k}{r+k}\right)^{n-1} \left|H(re^{i\theta})\right| \\ &= \left(\frac{R+k}{r+k}\right)^{n-1} \frac{\left|Re^{i\theta} - \eta e^{i\gamma}\right|}{\left|re^{i\theta} - \eta e^{i\gamma}\right|} \left|re^{i\theta} - \eta e^{i\gamma}\right| \left|H(re^{i\theta})\right|. \end{aligned} \tag{23}$$

Now for $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} \left|\frac{Re^{i\theta} - \eta e^{i\gamma}}{re^{i\theta} - \eta e^{i\gamma}}\right|^2 &= \frac{R^2 + \eta^2 - 2R\eta cos(\theta - \gamma)}{r^2 + \eta^2 - 2r\eta cos(\theta - \gamma)} \\ &\geq \left(\frac{R+\eta}{r+\eta}\right)^2, \quad \text{for } R > r \text{ and } rR \geq k^2, \\ &> \left(\frac{R+k}{r+k}\right)^2, \quad \text{since } \eta < k. \end{aligned}$$

This implies

$$\left|\frac{Re^{i\theta} - \eta e^{i\gamma}}{re^{i\theta} - \eta e^{i\gamma}}\right| > \frac{R+k}{r+k},$$

which on using in (23) gives for $R > r$, $rR \geq k^2$ and $0 \leq \theta < 2\pi$,

$$\left|f(Re^{i\theta})\right| > \left(\frac{R+k}{r+k}\right)^n \left|f(re^{i\theta})\right|.$$

Equivalently,

$$\left|f(Rz)\right| > \left(\frac{R+k}{r+k}\right)^n \left|f(rz)\right|, \tag{24}$$

for $R > r$, $rR \geq k^2$ and $|z| = 1$. This implies for every $|\alpha| \leq 1$, $R > r$, $rR \geq k^2$ and $|z| = 1$,

$$\left|f(Rz) - \alpha f(rz)\right| \geq \left|f(Rz)\right| - |\alpha| \left|f(rz)\right| > \left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\} \left|f(rz)\right|. \tag{25}$$

Again, since $r < R$, it follows that $((r+k)/(R+k))^n < 1$, inequality (25) implies that

$$|f(rz)| < |f(Rz)| \quad \text{for } |z| = 1.$$

Also, all the zeros of $f(Rz)$ lie in $|z| \leq k/R$, and $R^2 > rR \geq k^2$, we have $k/R < 1$. A direct application of Rouché's theorem shows that the polynomial $f(Rz) - \alpha f(rz)$ has all its zeros in $|z| < 1$, for every $|\alpha| \leq 1$.

Applying Rouché's theorem again, it follows from (25) that for every $|\beta| \le 1$, $|\alpha| \le 1$, $R > r$, $rR \ge k^2$, all the zeros of the polynomial

$$g(z) = f(Rz) - \alpha f(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\} f(rz)$$
$$= P(Rz) - \alpha f(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\} P(rz) - \lambda\left[F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\} F(rz)\right] \tag{26}$$

lie in $|z| < 1$, for every λ with $|\lambda| > 1$, and this clearly implies that for $|z| \ge 1$ and $R > r$ with $rR \ge k^2$,

$$\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}P(rz)\right| \le \left|F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz)\right|. \tag{27}$$

To see that the inequality (27) holds, note that if the inequality (27) is not true, then there is a point $z = z_0$ with $|z_0| \ge 1$, such that

$$\left|P(Rz_0) - \alpha P(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}P(rz_0)\right| > \left|F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz_0)\right|. \tag{28}$$

Now, because by hypothesis all the zeros of $F(z)$ lie in $|z| \le k$, all the zeros of $F(Rz)$ lie in $|z| \le k/R < 1$, and therefore if we use Rouché's theorem and arguments similar to the above, we will get that all the zeros of

$$F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz)$$

lie in $|z| < 1$ for every $R > r,\ rR \ge k^2$, that is,

$$F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz_0) \ne 0$$

for every z_0 with $|z_0| \ge 1$. Therefore, if we take

$$\lambda = \frac{P(Rz_0) - \alpha P(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}P(rz_0)}{F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz_0)},$$

then λ is a well defined real or complex number and in view of (28) we also have $|\lambda| > 1$. Therefore, with this choice of λ, we get from (26) that $g(z_0) = 0$ for some z_0, satisfying $|z_0| \ge 1$, which is clearly a contradiction to the fact that all the zeros of $g(z)$ lie in $|z| < 1$. Thus, for $R > r, rR \ge k^2$, inequality (27) holds for $|z| \ge 1$, and this completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. Let $m = \min_{|z|=k} |P(z)|$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Theorem 2.2 in this case. Henceforth, we assume that $P(z)$ has all its zeros in $|z| > k$, so that $m > 0$. It follows by Rouché's theorem that $h(z) = P(z) - \lambda m$ does not vanish in $|z| < k$, for every λ with $|\lambda| < 1$. Applying Corollary 2.4 to the polynomial $h(z)$, we get for every $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge 1/k^2$, and $|z| \ge 1$,

$$\left|h(Rk^2z) - \alpha h(rk^2z) + \beta\psi_k(r,R,\alpha)h(rk^2z)\right| \le k^n\left|T(Rz) - \alpha T(rz) + \beta\psi_k(r,R,\alpha)T(rz)\right|,$$

where $T(z) = z^n\overline{h(1/\bar{z})} = Q(z) - \bar{\lambda}mz^n$.

Equivalently,

$$\left|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z) - \lambda\Big(1 - \alpha + \beta\psi_k(r,R,\alpha)\Big)m\right|$$
$$\le k^n\left|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz) - \bar{\lambda}\Big(R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big)mz^n\right|, \tag{29}$$

for every $|\alpha| \le 1,\ |\beta| \le 1,\ R \ge r,\ rR \ge 1/k^2$ and $|z| \ge 1$. Since all the zeros of $Q(z)$ lie in $|z| \le 1/k$, we may apply Corollary 2.1 to $Q(z)$ and get for every $|\alpha| \le 1,\ |\beta| \le 1,\ rR \ge 1/k^2$ and $|z| \ge 1$,

$$\begin{aligned}\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| &\ge |z|^n k^n \Big|R^n - \alpha r^n + \beta\psi_k(r,R,\alpha)\Big| \min_{|z|=1/k} |Q(z)| \\ &= |z|^n \Big|R^n - \alpha r^n + \beta\psi_k(r,R,\alpha)\Big| \min_{|z|=k} |P(z)|. \end{aligned} \tag{30}$$

Now, choosing the argument of λ on the right hand side of (29) such that

$$\begin{aligned}&k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz) - \bar{\lambda}\Big(R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big)mz^n\Big| \\ &\qquad = k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| - k^n\Big|\lambda\Big|\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big|m|z|^n,\end{aligned}$$

which is possible by (30), we get from (29) for every $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge 1/k^2$ and $|z| = 1$,

$$\begin{aligned}&\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)| - |\lambda|\,\Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|m \\ &\qquad \le k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| - k^n|\lambda|\,\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big|m.\end{aligned} \tag{31}$$

Letting $|\lambda| \to 1$ in (31), we obtain for every $|\alpha| \le 1,\ |\beta| \le 1,\ R \ge r,\ rR \ge 1/k^2$ and $|z| = 1$,

$$\begin{aligned}&\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| - k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| \\ &\qquad \le -\Big\{k^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| - \Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|\Big\}m.\end{aligned} \tag{32}$$

Also, by Lemma 3.2, we have for $|z| = 1$,

$$\begin{aligned}&\Big|P(Rk^2z) - \alpha P(rk^2z) + \beta\psi_k(r,R,\alpha)P(rk^2z)\Big| + k^n\Big|Q(Rz) - \alpha Q(rz) + \beta\psi_k(r,R,\alpha)Q(rz)\Big| \\ &\qquad \le \Big\{k^n\Big|R^n - \alpha r^n + \beta r^n\psi_k(r,R,\alpha)\Big| + \Big|1 - \alpha + \beta\psi_k(r,R,\alpha)\Big|\Big\} \max_{|z|=k} \Big|P(z)\Big|.\end{aligned} \tag{33}$$

Finally, adding (32) and (33), we get (19) and this completes the proof of Theorem 2.3. □

Proof of Theorem 2.2. The proof of this theorem follows by combining Corollary 2.4 and Lemma 3.2. We omit the details. □

References

[1] N. C. Ankeny, T. J. Rivlin, On a theorem of S. Bernstein, Pacific J. Math. **5** (1955), 849–852.

[2] S. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné, Mem. Acad. R. Belg. **4** (1912), 1–103.

[3] S. Bernstein, Sur la limitation des dérivées des polynômes, Comptes Rendus de l' Academic des Sciences (Paris), **190** (1930), 338–340.

[4] N. K. Govil, M. A. Qazi, Q. I. Rahman, Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius, Math. Ineq. Appl. **6** (2003), 453–466.

[5] N. K. Govil, A. Liman, W. M. Shah, Some inequalities concerning derivative and maximum modulus of polynomials, Austr. J. Math. Anal. Appl. **8** (2011), 1–8.

[6] V. K. Jain, Generalization of certain well known inequalities for polynomials, Glas. Mat. **32** (1997), 45–52.

[7] V. K. Jain, Inequalities for a polynomial and its derivative, Proc. Indian Acad. Sci. Math. Sci. **110** (2000), 137–146.

[8] P. N. Kumar, On the generalization of polynomial inequalities in the complex domain, J. Contem. Math. Anal. **50** (2015), 14–21.

[9] P. D. Lax, Proof of a conjecture of P. Erdős on the derivative of a polynomial, Bull. Amer. Math. Soc. **50** (1944), 509–513.

[10] H. A. S. Mezerji, M. A. Baseri, M. Bidhkam, A. Zireh, Generalization of certain inequalities for a polynomial and its derivative, Lob. J. Math. **33** (2012), 68–74.

[11] G. V. Milovanović, A. Mir, Generalizations of Zygmund-type integral inequalities for the polar derivative of a complex polynomial, J. Inequal. Appl. **2020**, Paper No. 136, 12 pp.

[12] G. V. Milovanović, A. Mir, On the Erdős-Lax inequality concerning polynomials, Math. Inequal. Appl. **23** (2020), 1499–1508.

[13] G. V. Milovanović, A. Mir, A. Ahmad, Certain estimates of Turán's-type for the maximum modulus of the polar derivative of a polynomial, Publ. Inst. Math. (Beograd) (N.S.) **108** (122) (2020), 119–128.
[14] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, Topics in Polynomials, Extremal problems, Inequalities, Zeros, World Scientific, Singapore, 1994.
[15] A. Mir, On an operator preserving inequalities between polynomials, Ukrainian Math. J. **69** (2018), 1234–1247.
[16] A. Mir, Bernstein type integral inequalities for a certain class of polynomials, Mediterranean J. Math. **16** (2019), (Art. 143) pp. 1–11.
[17] A. Mir, Generalizations of Bernstein and Turán-type inequalities for the polar derivative of a complex polynomial, Mediterranean J. Math. **17** (2020), (Art. 14) pp. 1–12.
[18] A. Mir, I. Hussain, On the Erdős-Lax inequality concerning polynomials, C. R. Acad. Sci. Paris Ser. I, **355** (2017), 1055–1062.
[19] A. Mir, M. I. Sheikh, Some upper bound estimates for the maximal modulus of the polar derivative of a polynomial, J. Contemporary Math. Anal. **55** (2020), 189–195.
[20] Q. I. Rahman, G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, 2002.