UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 498—№ 541 (1975), 85—90.

513. ON A FUNCTIONAL EQUATION HAVING DETERMINANT FORM*

Radosav Ž. Đorđević and *Gradimir V. Milovanović*

In this paper we shall determine general continuous solutions of the functional equation

$$(1)\qquad \begin{vmatrix} F_0(x) & F_1(x) & \cdots & F_n(x) \\ a_0^{x_1} & a_1^{x_1} & & a_n^{x_1} \\ \vdots & & & \\ a_0^{x_n} & a_1^{x_n} & & a_n^{x_n} \end{vmatrix} = 0,$$

under the following assumptions:

1° Unknown functions $F_i: \mathbf{R} \to \mathbf{R}$,

2° $0 < a_0 < \cdots < a_n$,

3° $x = \sum_{i=1}^{n} x_i$.

If by $\Delta_i\,(i=0, 1, \ldots, n)$ we denote the determinant

$$\Delta_i(x_1, \ldots, x_n) = \begin{vmatrix} a_0^{x_1} & a_1^{x_1} & \cdots & a_{i-1}^{x_1} & a_{i+1}^{x_1} & \cdots & a_n^{x_1} \\ \vdots & & & & & & \\ a_0^{x_n} & a_1^{x_n} & & a_{i-1}^{x_n} & a_{i+1}^{x_n} & & a_n^{x_n} \end{vmatrix},$$

i.e.

$$(2)\qquad \Delta_i(x_1, \ldots, x_n) = \begin{vmatrix} e^{k_0 x_1} & e^{k_1 x_1} & \cdots & e^{k_{i-1} x_1} & e^{k_{i+1} x_1} & \cdots & e^{k_n x_1} \\ \vdots & & & & & & \\ e^{k_0 x_n} & e^{k_1 x_n} & & e^{k_{i-1} x_n} & e^{k_{i+1} x_n} & & e^{k_n x_n} \end{vmatrix},$$

where we have put $k_i = \log a_i\,(i=0, 1, \ldots, n)$, the functional equation (1) becomes

$$(3)\qquad \sum_{i=0}^{n} (-1)^i F_i(x)\, \Delta_i(x_1, \ldots, x_n) = 0.$$

Let $S = \{k_0, k_1, \ldots, k_n\}$ and T_i be an operator for which

$$T_i S = (k_0, k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \qquad (i=0, 1, \ldots, n).$$

---

* Presented January 6, 1975 by B. CRSTICI and P. M. VASIĆ.

Note that $T_i S$ is an arranged n-tuple. Let us by $(T_i S)_r$ $(r=1, \ldots, n!)$ denote the r-th permutation of $T_i S \equiv (T_i S)_1$. $(T_i S)_r$ is also an arranged n-tuple for each r.

If $(T_i S)_r$ and $X=(x_1, \ldots, x_n)$ are understood as vectors, the coordinates of which are given by the corresponding arranged n-tuple, we shall denote their scalar product by $(T_i S)_r X$.

With regard to the notations introduced, determinant (2) may be represented in the form

$$\Delta_i(x_1, \ldots, x_n) = \sum_{r=1}^{n!} (-1)^{s_r} \exp\{(T_i S)_r X\},$$

where s_r is the number of inversions in permutation $(T_i S)_r$ in relation to the original permutation $T_i S$.

Then equation (3) becomes

$$\sum_{i=0}^{n} (-1)^i F_i(x) \left\{ \sum_{r=1}^{n!} (-1)^{s_r} \exp\{(T_i S)_r X\} \right\} = 0,$$

i.e.

$$(4) \qquad \sum_{r=1}^{n!} (-1)^{s_r} \left\{ \sum_{i=0}^{n} (-1)^i F_i(x) \exp\{(T_i S)_r X\} \right\} = 0,$$

that is

$$\sum_{r=1}^{n!} (-1)^{s_r} \left\{ \sum_{i=0}^{n} (-1)^i F_i(x) \exp\{(T_i S)_r X - (T_j S)_1 X\} \right\} = 0 \qquad (j \in \{1, \ldots, n\})$$

or

$$(5) \qquad \sum_{r=1}^{n!} (-1)^{s_r} \left\{ \sum_{i=0}^{n} (-1)^i F(i, j, r, X) \right\} = 0 \qquad (j \in \{1, \ldots, n\}),$$

where the expressions $F(i, j, r, X)$ are determined by

$$F(i, j, r, X) \equiv F_i(x) \exp\{(T_i S)_r X - (T_j S)_1 X\}$$
$$\equiv F_i(x) \exp\left(\sum_{m=1}^{n} \lambda_m(i, j, r) x_m \right) \qquad (\lambda_m \in \mathbf{R}).$$

With regard to the values taken by the numbers λ_m we shall distinguish three kinds of the expression $F(i, j, r, X)$:

1° Expressions which are the functions of x. Those are the ones for which $\lambda_m = 0$ $(m=1, \ldots, n)$. Such a single expression is $F(j, j, 1, X)$.

2° Expressions which may be the functions of x. Those are the ones for which $\lambda_m > 0$ $(m=1, \ldots, n)$.

Such a single expression is $F(0, n, 1, X)$. It is the function of x if and only if all λ_m are mutually equal, i.e. if and only if

$$k_1 - k_0 = k_2 - k_1 = \cdots = k_n - k_{n-1}.$$

3° Other expressions.

Theorem 1. *If $a_0, a_1, \ldots, a_n$ do not make a geometric progression the functional equation* (1) *in the set of continuous functions has only the trivial solutions $F_i(x)=0$ $(i=0, 1, \ldots, n)$.*

Proof. Instead of equation (1), let us observe its equivalent form (5).

Let $j \in \{1, \ldots, n-1\}$. In that case from all the expressions $F(i, j, r, X)$ only the expression $F(j, j, 1, X)$, given by

$$F(j, j, 1, X)=F_j(x),$$

is the function of x. With regard that each of the remaining expressions contains the exponentional factor which is not the function of x, we deduce that $F_j(x)=0$. Since j is an arbitrary element of the set $\{1, \ldots, n-1\}$, it follows

$$F_j(x)=0 \qquad (j=1, \ldots, n-1). \tag{6}$$

Using (6), equation (5) for $j=n$, becomes

$$\sum_{r=1}^{n!}(-1)^{s_r}\{F(0, n, r, X)+(-1)^n F(n, n, r, X)\}=0.$$

$F(n, n, 1, X)=F_n(x)$ is the function of x.

Expressions $F(0, n, r, X)$ and $F(n, n, r, X)$ for $r=2, 3, \ldots, n!$ are not the functions of x.

Since a_i $(i=0, 1, \ldots, n)$ by assumption do not make a geometric progression, all $\lambda_m(=k_m-k_{m-1})$ are not mutually equal, which means that neither

$$F(0, n, 1, X)=F_0(x)\exp\left\{\sum_{m=1}^{n}(k_m-k_{m-1})\,x_m\right\}$$

is the function of x.

From here, similarly as for $j \neq n$, we deduce that $F_n(x)=0$.

Therefore, equation (5), and (3) respectively, become

$$F_0(x)\,\Delta_0(x_1, \ldots, x_n)=0.$$

Since $\Delta_0 \neq 0$ and F_0, as assumed, is a continuous function, it follows $F_0(x)=0$.

Thus, the Theorem 1 is proved.

Theorem 2. *If $a_0, a_1, \ldots, a_n$ make a geometric progression with quotient q, the functional equation* (1) *has the general solution determined by*

$$F_0(x)=F(x),$$

$$F_j(x)=0 \qquad (j=1, \ldots, n-1),$$

$$F_n(x)=(-1)^{n-1}q^x F(x),$$

where F is an arbitrary continuous function with values in $\mathbf{R}$.

Proof. Let us observe equation (1) in form (5). If $j=1,\ldots,n-1$, as in the proof of Theorem 1 it follows $F_j(x)=0$. Using this, equation (4) becomes

$$\sum_{r=1}^{n!}(-1)^{s_r}\{F_0(x)\exp\{(T_0 S)_r X\}+(-1)^n F_n(x)\exp\{(T_n S)_r X\}\}=0,$$

i.e.

$$\sum_{r=1}^{n!}(-1)^{s_r}\exp\{(T_n S)_r X\}\left(F_0(x)\exp\{(T_0 S)_r X-(T_n S)_r X\}+(-1)^n F_n(x)\right)=0. \tag{7}$$

With regard that $\exp\{(T_0 S)_r X-(T_n S)_r X\}=\exp\{x\log q\}=q^x$, equation (7) becomes

$$\left(F_0(x)\,q^x+(-1)^n F_n(x)\right)\sum_{r=1}^{n!}(-1)^{s_r}\exp\{(T_n S)_r X\}=0,$$

i.e.

$$\left(F_0(x)\,q^x+(-1)^n F_n(x)\right)\Delta_n(x_1,,\ldots,x_n)=0.$$

Since $\Delta_n\neq 0$ and F_0 and F_n are continuous functions, it follows

$$F_0(x)\,q^x+(-1)^n F_n(x)=0,$$

wherefrom we obtain

$$F_0(x)=F(x),\quad F_n(x)=(-1)^{n-1}q^x F(x),$$

where $F:\mathbf{R}\to\mathbf{R}$ is an arbitrary continuous function.

Thus the proof is finished.

***Another proof of Theorem* 2.** Putting $a_i=q^i a_0\ (i=1,\ldots,n)$, equation (1) becomes

$$a_0{}^x\begin{vmatrix} F_0(x) & F_1(x) & \cdots & F_n(x)\\ 1 & q^{x_1} & & q^{nx_1}\\ \vdots & & & \\ 1 & q^{x_n} & & q^{nx_n}\end{vmatrix}=0,$$

i.e. it is reduced to equation

$$\sum_{i=1}^{n}(-1)^i F_i(x)\,D_i=0, \tag{8}$$

where

$$D_i=\begin{vmatrix} 1 & q^{x_1} & q^{2x_1} & \ldots & q^{(i-1)x_1} & q^{(i+1)x_1} & \ldots & q^{nx_1}\\ \vdots & & & & & & & \\ 1 & q^{x_n} & q^{2x_n} & & q^{(i-1)x_n} & q^{(i+1)x_n} & & q^{nx_n}\end{vmatrix}.$$

Since (see [2], Problem 2.50.)

$$D_i=\sigma_{n-i}(q^{x_1},\ldots,q^{x_n})\,V_n(q^{x_1},\ldots,q^{x_n}),$$

where σ_p represents a symmetrical function of the order p of the elements $q^{x_1}, \ldots, q^{x_n}$, and V_n the corresponding VANDERMONDE's determinant, equation (8) becomes

$$\sum_{i=0}^{n} (-1)^i F_i(x)\, \sigma_{n-i}\, V_n = 0,$$

i.e. it reduces to the equation

$$\sum_{i=0}^{n} (-1)^i F_i(x)\, \sigma_{n-i} = 0, \tag{9}$$

because $V_n \neq 0$, functions F_i being continuous, as it was supposed.

If we put

$$G_i(x) = (-1)^i F_i(x) \quad (i = 1, \ldots, n-1), \quad G_n(x) = F_0(x)\, q^x + (-1)^n F_n(x), \tag{10}$$

equation (9) becomes

$$\sum_{i=1}^{n} G_i(x)\, \sigma_{n-i} = 0. \tag{11}$$

For $x_2 = x_3 = \cdots = x_n = t$, $x_1 = x - (n-1)\, t$, we have

$$\sigma_i(q^{x_1}, \ldots, q^{x_n}) = \sigma_i(q^{x_1}, q^t, \ldots, q^t) = \binom{n-1}{i} q^{it} + \binom{n-1}{i-1} q^{x_1}\, q^{(i-1)t}\ (i = 1, \ldots, n-1),$$

so equation (11) becomes

$$\sum_{i=1}^{n} G_i(x) \left\{ \binom{n-1}{i-1} q^{(n-i)t} + \binom{n-1}{i} q^x q^{-it} \right\} = 0,$$

or

$$\sum_{j=0}^{2n-1} H_j(x)\, q^{jt} = 0, \tag{12}$$

where

$$H_j(x) = \begin{cases} \binom{n-1}{j-1} q^x\, G_{n-j}(x) & (j = 0, 1, \ldots, n-1) \\ \binom{n-1}{j-n} G_{2n-j}(x) & (j = n, \ldots, 2n-1). \end{cases} \tag{13}$$

Let now $x = \alpha = \text{const}\, (\alpha \in \mathbf{R})$. Then (12) is reduced to

$$\sum_{j=0}^{2n-1} H_j(\alpha)\, q^{jt} = 0. \tag{14}$$

Since $q \neq 1$, system of functions $1, q^t, q^{2t}, \ldots, q^{(2n-1)t}$ is linearly independent, thus from (14) it follows

$$H_j(\alpha) = 0 \qquad (j = 0, 1, \ldots, 2n-1)$$

for each real α, i.e. $H_j(x) \equiv 0$.

On the basis of this result and equality (13) we deduce that

$$G_i(x) \equiv 0 \qquad (i = 1, \ldots, n),$$

hence, according to (10), it follows

$$F_i(x) = 0 \qquad (i = 1, \ldots, n-1), \qquad F_0(x)\, q^x + (-1)^n F_n(x) = 0.$$

Here immediately follows that Theorem 2 holds.

REFERENCES

1. J. ACZEL: *Lectures on Functional Equations and Their Applications.* New York — London 1966.
2. D. S. MITRINOVIĆ: *Matrice i determinante — Zbornik zadataka i problema.* (Novi zbornik matematičkih problema, II). Beograd 1972.

Elektronski fakultet
18 000 Niš, Jugoslavija