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514. ON A MALET-HAMMOND'S FUNCTIONAL EQUATION*

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1. J. C. MALET (see [**1**]) has stated a problem:

Prove that the function $f(x)=b^x-a^x$ statisfies the functional equation

$$(a+b)f(x)=abf(x-1)+f(x+1) \qquad (a\neq b).$$

Solving this problem, J. HAMMOND [**1**] has proved a more general result. Function f, defined by

$$\text{(1)} \qquad f(x_1, \ldots, x_n)=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_0^{x_1} & a_1^{x_1} & & a_n^{x_1} \\ \vdots & & & \\ a_0^{x_n} & a_1^{x_n} & & a_n^{x_n} \end{vmatrix} \qquad (a_i>0;\ a_i<a_j \Leftrightarrow i<j),$$

satisfies the equation

$$\text{(2)} \qquad \Big(\sum_{k=0}^{n} a_k\Big) f(x_1, \ldots, x_n)=\Big(\prod_{k=0}^{n} a_k\Big) f(x_1-1, \ldots, x_n-1) + \sum_{k=1}^{n} f(x_1, \ldots, x_{k-1}, x_k+1, x_{k+1}, \ldots, x_n).$$

In this paper we will consider the functional equation (2), where $f\colon \mathbf{R}^n \to \mathbf{R}$, $a_i>0$ $(i=0, 1, \ldots, n)$, $a_i<a_j \Leftrightarrow i<j$.

Function f, defined by (1), is a particular solution of equation (2).

It is easy to show that the function

$$f(x_1, \ldots, x_n)=\begin{vmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} \\ a_0^{x_1} & a_1^{x_1} & & a_n^{x_1} \\ \vdots & & & \\ a_0^{x_n} & a_1^{x_n} & & a_n^{x_n} \end{vmatrix},$$

where α_{0i} $(i=0, 1, \ldots, n)$ are arbitrary real constants, is also a solution of equation (2).

Let us introduce the following notations:

$$X=\sum_{k=1}^{n} x_k, \quad A=\sum_{i=0}^{n} a_i, \quad \Delta(X; F_0, F_1, \ldots, F_n)=\begin{vmatrix} F_0(X) & F_1(X) & \cdots & F_n(X) \\ a_0^{x_1} & a_1^{x_1} & & a_n^{x_1} \\ \vdots & & & \\ a_0^{x_n} & a_1^{x_n} & & a_n^{x_n} \end{vmatrix},$$

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2. We shall determine all functions f of the form

$$f(x_1, \ldots, x_n) = \Delta(X; G_0, \ldots, G_n),$$

satisfying the equation (2).

According to this result we shall prove the following lemma.

Lemma 1. *If* $t_1 + t_2 + \cdots + t_n = t$,

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{i1} & & a_{in} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{vmatrix} \quad \text{and} \quad D_i = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ t_1 a_{i1} & & t_n a_{in} \\ \vdots & & \cdot \\ a_{n1} & & a_{nn} \end{vmatrix},$$

then the equality

$$\sum_{i=1}^{n} D_i = tD$$

is valid.

Proof. Since

$$D = \sum_{i=1}^{n} a_{ij} A_{ij} \quad \text{and} \quad D_i = \sum_{j=1}^{n} t_j a_{ij} A_{ij},$$

where A_{ij} is the algebraic complement of a_{ij}, so that

$$\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} t_j a_{ij} A_{ij}\Big) = \sum_{j=1}^{n} t_j \Big(\sum_{i=1}^{n} a_{ij} A_{ij}\Big) = \sum_{j=1}^{n} t_j D = tD.$$

Theorem 1. *If functions* H_i $(i = 0, 1, \ldots, n)$ *are general continuous solutions of equation*

(3) $$\Delta(X; H_0, \ldots, H_n) = 0,$$

functional equation (2) *has the general solution given by*

(4) $$f(x_1, \ldots, x_n) = \Delta(X; G_0, \ldots, G_n),$$

if and only if functions G_i *satisfy equations*

(5) $$(A - a_i)\, G_i(X+1) - A G_i(X) + a_i G_i(X-n) = H_i(X).$$

Proof. If we substitute (4) in equation (2) we obtain

$$A\,\Delta(X; G_0, G_1, \ldots, G_n) = \Big(\prod_{i=0}^{n} a_i\Big) \begin{vmatrix} G_0(X-n) & G_1(X-n) & \cdots & G_n(X-n) \\ a_0^{x_1 - 1} & a_1^{x_1 - 1} & & a_n^{x_1 - 1} \\ \vdots & & & \\ a_0^{x_n - 1} & a_1^{x_n - 1} & & a_n^{x_n - 1} \end{vmatrix}$$

$$+ \sum_{k=1}^{n} \begin{vmatrix} G_0(X+1) & G_1(X+1) & \cdots & G_n(X+1) \\ a_0^{x_1} & a_1^{x_1} & & a_n^{x_1} \\ \vdots & & & \\ a_0^{x_k + 1} & a_1^{x_k + 1} & & a_n^{x_k + 1} \\ \vdots & & & \\ a_0^{x_n} & a_1^{x_n} & & a_n^{x_n} \end{vmatrix}.$$

Using the Lemma 1, the last equation becomes

$$A\,\Delta(X; G_0, \dots, G_n) = \Delta(X-n; a_0 G_0, \dots, a_n G_n)$$
$$+ A\,\Delta(X+1; G_0, \dots, G_n) - \Delta(X+1; a_0 G_0, \dots, a_n G_n);$$

hence, it follows

$$\Delta(X+1; AG_0, \dots, AG_n) - \Delta(X+1; a_0 G_0, \dots, a_n G_n)$$
$$- \Delta(X; AG_0, \dots, AG_n) + \Delta(X-n; a_0 G_0, \dots, a_n G_n) = 0,$$

$$\Delta\big(X+1; (A-a_0) G_0, \dots, (A-a_n) G_n\big) - \Delta(X; AG_0, \dots, AG_n)$$
$$+ \Delta(X-n; a_0 G_0, \dots, a_n G_n) = 0,$$

i. e.,

$$\Delta(X; H_0, \dots, H_n) = 0.$$

Thus, Theorem is proved.

The continuous solutions of equation (3) are (see [**2**])

(6) 1° $H_0(X) = H(X)$, $H_i(X) = 0$ $(i = 1, \dots, n-1)$, $H_n(X) = (-1)^{n-1} q^X H(X)$,

where H is an arbitrary continuous function with values in **R**, if $a_0, a_1, \dots, a_n$ make a geometric progression, where $a_i = q^i a_0$, or

2° $H_i(X) = 0$ $(i = 0, 1, \dots, n)$,

if $a_0, a_1, \dots, a_n$ do not make a geometric progression.

Basing on that it may be concluded that for defining functions G_i, as solutions of equations (5), one should recognize these two cases. We are about to show that there is no need for that, i.e., that it is enough to take only $H_i(X) = 0$ $(i = 0, 1, \dots, n)$.

Namely, equations (5), that is equations

$$(A - a_i)\, G_i(X+n+1) - AG_i(X+n) + a_i G_i(X) = H_i(X+n) \qquad (i = 0, 1, \dots, n),$$

to which, using the operator E, one may give a concise form

$$\Phi_i(E)\, G_i(X) = H_i(X+n) \qquad (i = 0, 1, \dots, n), \tag{7}$$

where

$$\Phi_i(E) = (A - a_i)\, E^{n+1} - AE^n + a_i \qquad (i = 0, 1, \dots, n),$$

have general solutions given by

$$G_i(X) = g_i(X) + \tilde{g}_i(X) \qquad (i = 0, 1, \dots, n),$$

where $\tilde{g}_i$ are particular solutions of equations (7) and g_i general solutions of the corresponding homogeneous equations

$$\Phi_i(E)\, G_i(X) = 0 \qquad (i = 0, 1, \dots, n).$$

Then

$$f(x_1, \dots, x_n) = \Delta(X; g_0, \dots, g_n) + \Delta(X; \tilde{g}_0, \dots, \tilde{g}_n).$$

We are going to show that $\Delta(X; \tilde{g}_0, \dots, \tilde{g}_n) = 0$.

Lemma 2. *If $\Delta(X; H_0, \ldots, H_n)=0$ and if $\tilde{g}_0, \ldots, \tilde{g}_n$ are the particular solutions of equations (7), then*

$$\Delta(X; \tilde{g}_0, \ldots, \tilde{g}_n)=0. \tag{8}$$

Proof. If $H_i(X)=0$ $(i=0, 1, \ldots, n)$, the claim is correct, since equations (7) are reduced to the homogeneous ones.

Let now functions H_i be defined by (6). Then $a_i=q^i a_0$ $(i=0, 1, \ldots, n)$, so equations (7) become

$$\Phi_0(E)\, G_0(X)=H(X+n), \tag{9}$$

$$\Phi_i(E)\, G_i(X)=0 \qquad (i=1, \ldots, n-1), \tag{10}$$

$$\Phi_n(E)\, G_n(X)=(-1)^{n-1} q^{X+n} H(X+n). \tag{11}$$

From (10) it immediately follows $\tilde{g}_i(X)=0$ $(i=1, \ldots, n-1)$.

As $(A-a_n)\,q=A-a_0$, one has

$$\Phi_n(qE)=(A-a_n)\,q^{n+1}E^{n+1}-Aq^nE^n+a_n=q^n\,\Phi_0(E),$$

so

$$\begin{aligned}\Phi_n(E)\,G_n(X)&=\Phi_n(E)\big(q^Xq^{-X}G_n(X)\big)\\&=q^X\,\Phi_n(qE)\big(q^{-X}G_n(X)\big)\\&=q^{X+n}\,\Phi_0(E)\big(q^{-X}G_n(X)\big).\end{aligned}$$

If $\tilde{g}_0$ is a particular solution of equation (9), follows that equation (11) has a particular solution $\tilde{g}_n$ defined by

$$\tilde{g}_n(X)=(-1)^{n-1}q^X\tilde{g}_0(X).$$

As for the system of functions

$$\tilde{g}_0(X), \quad \tilde{g}_i(X)\,(=0) \quad (i=1, 2, \ldots, n-1), \quad \tilde{g}_n(X)\big(=(-1)^{n-1}q^X\tilde{g}_0(X)\big)$$

equality (8) holds true (see [**2**]), the Lemma is proved.

Theorem 2. *If $\alpha_{0i}(X)$ and $\alpha_{ki}(X)$ are arbitrary periodic constants and λ_{ki} $(k=1, 2, \ldots, n)$ roots of equations*

$$\frac{A-a_i}{a_i}\lambda^n-\lambda^{n-1}-\cdots-\lambda-1=0 \qquad (i=0, 1, \ldots, n),$$

the general solution of the form (4) of equation (2) is

$$f(x_1, \ldots, x_n)=\Delta(X; G_0, \ldots, G_n),$$

where functions G_i are defined by

$$G_i(X)=\alpha_{0i}(X)+\sum_{k=1}^{n}\alpha_{ki}(X)\,\lambda_{ki}{}^X \qquad (i=0, 1, \ldots, n).$$

Proof. Based on Theorem 1 and Lemma 2 each function G_i satisfies the equation

$$(A-a_i)\,G_i(X+n+1)-A\,G_i(X+n)+a_i\,G_i(X)=0. \tag{12}$$

Its characteristic equation is

$$(A-a_i)\,\lambda^{n+1}-A\,\lambda^n+a_i=0,$$

i. e.

$$(\lambda-1)\,P_i(\lambda)=0,$$

where

$$P_i(\lambda)=\frac{A-a_i}{a_i}\lambda^n-\lambda^{n-1}-\cdots-\lambda-1.$$

If λ_{ki} $(k=1,\ldots,n)$ denote the roots of equations $P_i(\lambda)=0$ $(i=0,1,\ldots,n)$, the general solution of equation (12) is defined by

$$G_i(X)=\alpha_{0i}(X)+\sum_{k=1}^{n}\alpha_{ki}(X)\,\lambda_{ki}^{X} \qquad (i=0,1,\ldots,n),$$

where $\alpha_{0i}(X)$ and $\alpha_{ki}(X)$ are arbitrary periodic constants.

Theorem 2 is thus proved.

EXAMPLE. Let $f:\mathbf{R}^2\to\mathbf{R}$ and let a, b, c be mutually different positive numbers. General solution of the form (3) of the functional equation

$$(a+b+c)\,f(x,y)=abc\,f(x-1,y-1)+f(x+1,y)+f(x,y+1)$$

is

$$f(x,y)=\begin{vmatrix} G_1(x+y) & G_2(x+y) & G_3(x+y)\\ a^x & b^x & c^x\\ a^y & b^y & c^y \end{vmatrix},$$

where functions G_i $(i=1,2,3)$, with values in $\mathbf{R}$, are given by

$$G_1(x)=\alpha_1(x)+\beta_1(x)\left(\frac{\sqrt{a^2+4a(b+c)}+a}{2(b+c)}\right)^x+\gamma_1(x)\left(\frac{\sqrt{a^2+4a(b+c)}-a}{2(b+c)}\right)^x\cos\pi x,$$

$$G_2(x)=\alpha_2(x)+\beta_2(x)\left(\frac{\sqrt{b^2+4b(c+a)}+b}{2(c+a)}\right)^x+\gamma_2(x)\left(\frac{\sqrt{b^2+4b(c+a)}-b}{2(c+a)}\right)^x\cos\pi x,$$

$$G_3(x)=\alpha_3(x)+\beta_3(x)\left(\frac{\sqrt{c^2+4c(a+b)}+c}{2(a+b)}\right)^x+\gamma_3(x)\left(\frac{\sqrt{c^2+4c(a+b)}-c}{2(a+b)}\right)^x\cos\pi x$$

and α_i, β_i, γ_i $(i=1,2,3)$ are real periodic constants.

3. Now we are going to point out to some generalizations.

If $a\in\mathbf{R}$, $m, r\in\mathbf{N}$, $0<a_i<a_j$ $(i<j)$, $A_m=a_0^m+a_1^m+\cdots+a_n^m$ and if function $f:\mathbf{R}^n\to\mathbf{R}$, for functional equations

$$af(x_1,\ldots,x_n)=\Big(\prod_{k=0}^{n}a_k\Big)f(x_1-1,\ldots,x_n-1)+\sum_{k=1}^{n}f(x_1,\ldots,x_{k-1},x_k+m,x_{k+1},\ldots,x_n) \tag{13}$$

and

(14) $$af(x_1, \dots, x_n) = (\prod_{k=0}^{n} a_k)^r f(x_1 - r, \dots, x_n - r)$$

$$+ \sum_{k=1}^{n} f(x_1, \dots, x_{k-1}, x_k + m, x_{k+1}, \dots, x_n),$$

the following results hold.

Theorem 3. *If* $\alpha_{ki}(X)$ *are the arbitrary periodic constants and* λ_{ki} *roots of equations*

$$(A_m - a_i^m) \lambda^{m+n} - a \lambda^n + a_i = 0 \qquad (i = 0, 1, \dots, n),$$

the general solution of equation (13) *is*

$$f(x_1, \dots, x_n) = \Delta(X; G_0, \dots, G_n),$$

where functions G_i *are defined by*

$$G_i(X) = \sum_{k=1}^{n+m} \alpha_{ki}(X) \lambda_{ki}^X \qquad (i = 0, 1, \dots, n).$$

Theorem 4. *If functions* H_i $(i = 0, 1, \dots, n)$ *are general continuous solutions of equation*

$$\Delta(X; H_0, \dots, H_n) = 0,$$

functional equation (14) *has the general solution given by*

$$f(x_1, \dots, x_n) = \Delta(X; G_0, \dots, G_n),$$

if and only if the functions G_i *satisfy equations*

$$(A_m - a_i^m) G_i(X + m) - aG_i(X) + a_i^r G_i(X - nr) = H_i(X).$$

Since the proofs of these theorems are similar to those of Theorem 1 and 2, they will not be given here.

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