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515. A GENERALIZATION OF E. LANDAU'S THEOREM*

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1. E. LANDAU ([1]) has proved the following theorem.

Theorem A. *Let $x \mapsto f(x)$ be a real function which on an interval I, of length not less than 2 satisfies the conditions $|f(x)| \leqq 1$ and $|f''(x)| \leqq 1$. Then*

$$|f'(x)| \leqq 2$$

for all $x \in I$, where 2 is the best possible constant.

There are several generalizations of this result in many senses. We shall state some of these generalizations related to this paper.

1° J. D. KEČKIĆ (see [2, pp. 381—382]) has given the following result.

Theorem B. *Let $x \mapsto f(x)$ be a real function which on an interval I, of length not less than $a\,(a>0)$ satisfies the conditions $|f(x)| \leqq 1$ and $|f''(x)| \leqq 1$. Then*

$$|f'(x)| \leqq \frac{2}{a} + \frac{a}{2} \qquad (\forall\, x \in I).$$

Theorem B represents a generalization of Theorem A and reduces to in for $a = 2$.

2° V. G. AVAKUMOVIĆ and S. ALJANČIĆ have proved the theorem in [3].

Theorem C. *The condition $|\varphi''(x)| \leqq 1$ $(0 \leqq x \leqq 1)$ implies*

$$|\varphi'(x) - \varphi(1) + \varphi(0)| \leqq \frac{1}{2} - x + x^2 \qquad (0 \leqq x \leqq 1).$$

The polynomial $x \mapsto \frac{1}{2} - x + x^2$ is the best possible.

3° I. B. LACKOVIĆ and M. S. STANKOVIĆ have proved the following theorem in [4].

Theorem D. *Let the function $f: \mathbf{R}^n \to \mathbf{R}$ be defined on the set*

$$K_n = \{(x_1, \ldots, x_n) \mid 0 \leqq x_i \leqq a,\ a>0,\ i=1, \ldots, n\}$$

and let $|f(x_1, \ldots, x_n)| \leqq 1$ for all $(x_1, \ldots, x_n) \in K_n$. Furthermore, let us suppose that all the first derivatives of f are continuous in K_n. If all the derivatives of the second order of the function f are continuous, and if $\left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| \leqq 1$ $(i, j = 1, \ldots, n)$ for all $(x_1, \ldots, x_n) \in \mathring{K}_n$, where

$$\mathring{K}_n = \{(x_1, \ldots, x_n) \mid 0 < x_i < a,\ i = 1, \ldots, n\},$$

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then

$$\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\right|\leqq\frac{2}{a}+n^2\frac{a}{2}$$

for all $(x_1, \ldots, x_n)\in K_n$.

The same paper [4] gives D. D. ADAMOVIĆ's remark without proofs which represent a generalization of Theorem D. Stated as a theorem the remark is as follows.

Theorem E. *Let the function* $f:\mathbf{R}^n\to\mathbf{R}$ *be defined on the set*

$$L_n=\{(x_1, \ldots, x_n)\mid a_i\leqq x_i\leqq b_i,\ a_i<b_i\ (1\leqq i\leqq n)\},$$

and let $|f(x_1, \ldots, x_n)|\leqq 1$ *for all* $(x_1, \ldots, x_n)\in L_n$. *If all the first order derivatives of* f *are continuous on* L_n *and differentiable on the set*

$$\mathring{L}_n=\{(x_1, \ldots, x_n)\mid a_i<x_i<b_i,\ 1\leqq i\leqq n\}$$

and if $\left|\frac{\partial^2 f}{\partial x_i\partial x_j}\right|\leqq 1\ (i, j=1, \ldots, n)$ *for all* $(x_1, \ldots, x_n)\in\mathring{L}_n$, *then*

$$\left|\sum_{i=1}^{n}(b_i-a_i)\frac{\partial f}{\partial x_i}\right|\leqq 2+\frac{1}{2}\left(\sum_{i=1}^{n}b_i-\sum_{i=1}^{n}a_i\right)^2$$

for all $(x_1, \ldots, x_n)\in L_n$.

For $(a_1, \ldots, a_n)=(0, \ldots, 0)$, $(b_1, \ldots, b_n)=(a, \ldots, a)$ Theorem E is reduced to Theorem D.

4° In [5] A. OSTROWSKI has proved the following result.

Theorem F. *Let* $x\mapsto f(x)$ *be a differentiable function on* (a, b) *and let, on* (a, b), $|f'(x)|\leqq N$. *Then, for every* $x\in(a, b)$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\leqq\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)N.$$

REMARK. For $\varphi(t)=(b-a)^{-1}N^{-1}\int_0^t f(a+(b-a)s)\,\mathrm{d}s$ Theorem C reduces to Theorem F.

2. This paper also gives a generalization of E. LANDAU's theorem and it relates to the operators in BANACH space.

Let X and Y be BANACH spaces. If $a, b\in X$ $(a\neq b)$, let us define the functional $g:X\to\mathbf{R}^+$ as follows

$$g(x)=\|x-a\|^2+\|b-x\|^2\qquad(x\in X).$$

Let $D\subset\{x\mid g(x)\leqq\|b-a\|^2,\ x\in X\}$ be a convex set such that $a, b\in\bar{D}$, where $\bar{D}$ is the closure of D.

If $F:X\to Y$ is an operator which is twice FRÉCHET-differentiable on $\bar{D}$, the following Theorem holds.

Theorem 1. *If*

(1) $$\|F(x)\| \leqq M \qquad (\forall x \in \bar{D})$$

and

$$\|F''_{(\alpha)}(h, h)\| \leqq N \|h\|^2 \qquad (\forall h \in X \wedge \forall \alpha \in D),$$

then

$$\|F'_{(x)}(b-a)\| \leqq 2M + \frac{N}{2} g(x) \leqq 2M + \frac{N}{2} \|b-a\|^2 \qquad (\forall x \in \bar{D}).$$

Proof. Let $x \in \bar{D}$ and $x + th \in D\,(0 < t < 1)$. TAYLOR's formula, namely

$$F(x+h) = F(x) + F'_{(x)}(h) + W(x, h) \quad \left(W(x, h) = \frac{1}{2} F''_{(x+th)}(h, h)\right),$$

where

(2) $$\|W(x, h)\| = \frac{1}{2} \|F''_{(x+th)}(h, h)\| \leqq \frac{N}{2} \|h\|^2,$$

for $h = a - x$ and $h = b - x$, becomes in turn

$$F(a) = F(x) + F'_{(x)}(a-x) + W(x, a-x),$$
$$F(b) = F(x) + F'_{(x)}(b-x) + W(x, b-x).$$

From these equations it follows that

$$F(b) - F(a) = F'_{(x)}(b-x) - F'_{(x)}(a-x) + W(x, b-x) - W(x, a-x),$$

or, with regard to the linearity of operator $F'_{(x)}$,

(3) $$F'_{(x)}(b-a) = F(b) - F(a) + W(x, a-x) - W(x, b-x).$$

From (3) it immediately follows that

$$\|F'_{(x)}(b-a)\| \leqq \|F(b)\| + \|F(a)\| + \|W(x, a-x)\| + \|W(x, b-x)\|,$$

hence, using (1) and (2), we obtain

$$\|F'_{(x)}(b-a)\| \leqq 2M + \frac{N}{2}\{\|x-a\|^2 + \|b-x\|^2\} = 2M + \frac{N}{2} g(x).$$

Since $x \in \bar{D}$, i. e., $g(x) \leqq \|b-a\|^2$, we have

$$\|F'_{(x)}(b-a)\| \leqq 2M + \frac{N}{2} g(x) \leqq 2M + \frac{N}{2} \|b-a\|^2,$$

which proves the theorem.

REMARK. Theorem 1 holds if condition (1) is substituted by the weaker condition

$$\|F(b) - F(a)\| \leqq 2M.$$

Corollary 1. *If* $X = Y = \mathbf{R}$, $\|x-y\| = |x-y|\ (x, y \in \mathbf{R})$, $F = f$, $D = \{x \mid x \in (\alpha, \beta),\ 0 < a \leqq \beta - \alpha\}$, $M = 1$, $N = 1$, *then Theorem* B *follows from Theorem* 1.

Corollary 2. *Let* $a < b$, $h > 0$ *and let* $x \mapsto f(x)$ *be a differentiable function on* $[a, b+h]$ *such that* $|f(x)| \leqq 1\ (\forall x \in [a, b+h])$ *and* $|\Delta_h f'(x)| \leqq 1\ (\forall x \in (a, b))$, *where* $\Delta_h g(x) = \frac{g(x+h) - g(x)}{h}$. *Then*

$$|\Delta_h f(x)| \leqq \frac{2}{b-a} + \frac{b-a}{2} \qquad (\forall x \in [a, b]).$$

To prove this, take $X=Y=\mathbf{R}$, $\|x-y\|=|x-y|$, $F(x)=\frac{1}{h}\int_x^{x+h} f(t)\,dt$ $(a\leqq x\leqq b)$, in Theorem 1. Note that $D=\{x\,|\,a<x<b\}$.

Corollary 3. *Let*

$x=(x_1, \ldots, x_n)$, $a=(a_1, \ldots, a_n)$, $b=(b_1, \ldots, b_n)$ $(a_i<b_i;\ i=1, \ldots, n)$

and let

$$D\subset\left\{(x_1, \ldots, x_n)\,\middle|\,\left(\sum_{i=1}^{n}|x_i-a_i|\right)^2+\left(\sum_{i=1}^{n}|b_i-x_i|\right)^2<\left(\sum_{i=1}^{n}|b_i-a_i|\right)^2\right\}$$

be a convex set such that $a, b\in\bar{D}$.

If $f:\mathbf{R}^n\to\mathbf{R}$ *is twice differentiable function on* $\bar{D}$ *and satisfy the conditions*

$$|f(x_1, \ldots, x_n)|\leqq M \quad (\forall(x_1, \ldots, x_n)\in\bar{D})$$

and

$$\left|\frac{\partial^2 f}{\partial x_i\partial x_j}\right|\leqq N \qquad (\forall(x_1, \ldots, x_n)\in D;\ i, j=1, \ldots, n), \tag{4}$$

then

$$\left|\sum_{i=1}^{n}(b_i-a_i)\frac{\partial f}{\partial x_i}\right|\leqq 2M+\frac{N}{2}\left\{\left(\sum_{i=1}^{n}|x_i-a_i|\right)^2+\left(\sum_{i=1}^{n}|b_i-x_i|\right)^2\right\}$$
$$\leqq 2M+\frac{N}{2}\left(\sum_{i=1}^{n}b_i-\sum_{i=1}^{n}a_i\right)^2$$

for every $(x_1, \ldots, x_n)\in\bar{D}$.

To prove this, in Theorem 1, take $X=\mathbf{R}^n$, $Y=\mathbf{R}$, $F(x)=f(x_1, \ldots, x_n)$ and

$$\|x-\bar{x}\|=\sum_{i=1}^{n}|x_i-\bar{x}_i| \quad (x, \bar{x}\in X), \qquad \|y-\bar{y}\|=|y-\bar{y}| \quad (y, \bar{y}\in Y).$$

Note that in this event from (4) follows

$$\|F''_{(x)}(h, h)\|=\left|\sum_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}h_i h_j\right|$$
$$\leqq\sum_{i,j}\left|\frac{\partial^2 f}{\partial x_i\partial x_j}\right|\cdot|h_i|\cdot|h_j|$$
$$\leqq N\left(\sum_i|h_i|\right)^2=N\|h\|^2$$

for every $x\in D$ and every $h\in X$.

Remark. For $M=1$, $N=1$, $D=\overset{\circ}{L}_n=\{x\,|\,a_i<x_i<b_i\ (i=1, \ldots, n)\}$ Corollary 3 reduces to Theorem E.

Theorem 2. *If*

(5) $$\| F''_{(\alpha)}(h, h) \| \leqq N \| h \|^2 \qquad (\forall h \in X \wedge \forall \alpha \in D),$$

then

(6) $$\| F'_{(x)}(b-a) - F(b) + F(a) \| \leqq \frac{N}{2} \{ \| x-a \|^2 + \| b-x \|^2 \}$$

for every $x \in \bar{D}$.

Proof. Let $x \in \bar{D}$ and $x + th \in D\,(0 < t < 1)$. As in the proof of Theorem 1, the inequality (3) holds, i.e.,

$$F'_{(x)}(b-a) - F(b) + F(a) = W(x, a-x) - W(x, b-x),$$

from which follows

(7) $$\| F'_{(x)}(b-a) - F(b) + F(a) \| \leqq \| W(x, a-x) \| + \| W(x, b-x) \|.$$

From (7), (2) and using the made assumption (5), immediately follows (6).

Corollary 4. *Let*

$$x = (x_1, \dots, x_n), \quad a = (a_1, \dots, a_n), \quad b = (b_1, \dots, b_n) \; (a_i < b_i; \; i = 1, \dots, n),$$

and let

$$D \subset \left\{ (x_1, \dots, x_n) \,\middle|\, \left(\sum_{i=1}^{n} |x_i - a_i| \right)^2 + \left(\sum_{i=1}^{n} |b_i - x_i| \right)^2 < \left(\sum_{i=1}^{n} |b_i - a_i| \right)^2 \right\}$$

be a convex set such that $a, b \in \bar{D}$.

If $f\colon \mathbf{R}^n \to \mathbf{R}$ *is twice differentiable function on* $\bar{D}$ *and*

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \leqq N \qquad (\forall (x_1, \dots, x_n) \in D; \;\; i, j = 1, \dots, n),$$

then for all $(x_1, \dots, x_n) \in \bar{D}$

(8) $$\left| \sum_{i=1}^{n} (b_i - a_i) \frac{\partial f}{\partial x_i} - f(b_1, \dots, b_n) + f(a_1, \dots, a_n) \right| \leqq \frac{N}{2} \left\{ \left(\sum_{i=1}^{n} |x_i - a_i| \right)^2 + \left(\sum_{i=1}^{n} |b_i - x_i| \right)^2 \right\}.$$

Putting in Theorem 2 that $X = \mathbf{R}^n$, $Y = \mathbf{R}$, $F(x) = f(x_1, \dots, x_n)$ and

$$\| x - \bar{x} \| = \sum_{i=1}^{n} |x_i - \bar{x}_i| \;\; (x, \bar{x} \in X), \quad \| y - \bar{y} \| = |y - \bar{y}| \;\; (y, \bar{y} \in Y)$$

it is obtained Corollary 4.

REMARK. For $N = 1$ and $D = \mathring{L}_n = \{x \mid a_i < x_i < b_i (i = 1, \dots, n)\}$ (8) reduces to

$$\left| \sum_{i=1}^{n} (b_i - a_i) \frac{\partial f}{\partial x_i} - f(b_1, \dots, b_n) + f(a_1, \dots, a_n) \right| \leqq \frac{1}{2} \{(X-A)^2 + (B-X)^2\} = \frac{1}{2}(A^2 + B^2) - (A+B)X + X^2,$$

where $A = \sum_{i=1}^{n} a_i$, $B = \sum_{i=1}^{n} b_i$, $X = \sum_{i=1}^{n} x_i$.

This result represents a generalization of Theorem C.

Corollary 5. *If*

$$X=Y=\mathbf{R},\ \|x-y\|=|x-y|\ (x,\ y\in\mathbf{R}),\ D=\{x\mid a<x<b;\ a,\ b\in\mathbf{R}\},$$

and $F(x)=\int_a^x f(t)\,\mathrm{d}t$, *where* f *is a differentiable function defined on* $[a, b]$ *and* $|f'(x)|\leqq N$, *for every* $x\in D$, *then Theorem F follows from Theorem 2.*

On reading this paper in manuscript Prof. P. R. Beesack pointed out the possibility of the generalization of Theorem 1 as follows:

Theorem 3. *Let* D *be a convex set such that* $a, b\in\bar{D}$ *and let* $F: X\to Y$ *be twice Fréchet-differentiable on* $\bar{D}$ *and satisfy the condition*

$$\|F''_{(\alpha)}(h, h)\|\leqq H(\|h\|) \qquad (\forall h\in X\wedge\forall\alpha\in D),$$

where H *is a function from* $\mathbf{R}^+$ *to* $\mathbf{R}^+$. *Then for all* $x\in\bar{D}$

$$\|F'_{(x)}(b-a)\|\leqq\|F(b)-F(a)\|+\frac{1}{2}\{H(\|x-a\|)+H(\|b-x\|)\}.$$

Proof. Since

$$\|W(x, h)\|=\frac{1}{2}\|F''_{(x+th)}(h, h)\|\leqq\frac{1}{2}H(\|h\|),$$

from (3) it immediately follows that

$$\|F'_{(x)}(b-a)\|\leqq\|F(b)-F(a)\|+\|W(x, a-x)\|+\|W(x, b-x)\|$$
$$\leqq\|F(b)-F(a)\|+\frac{1}{2}\{H(\|x-a\|)+H(\|b-x\|)\},$$

which proves the Theorem.

Corollary 6. *If the convex set* D *and the function* $H:\mathbf{R}^+\to\mathbf{R}^+$ *satisfy*

$$H(\|x-a\|)+H(\|b-x\|)\leqq H(\|b-a\|) \qquad (\forall x\in\bar{D}),$$

then

$$\|F'_{(x)}(b-a)\|\leqq\|F(b)-F(a)\|+\frac{1}{2}H(\|b-a\|) \qquad (\forall x\in\bar{D}).$$

3. In [6] (p. 606) the following result is given.

Theorem G. *Let the function* $(s, t, u)\mapsto K(s, t; u)$ *be continuous and twice continuously-differentiable with respect to* u, *and*

$$(9)\qquad |K''_{uu}(s, t; u)|\leqq M|u|^{p-2}+N \quad (s, t\in[0, 1],\ |u|<+\infty;\ M, N>0, p\in\mathbf{R}).$$

If $p\geqq 2$, *operator* F, *defined by*

$$F(f)=\int_0^1 K\big(s, t; f(t)\big)\,\mathrm{d}t \qquad (f\in L^p),$$

maps the space L^p into the space $L^q\,(1\leqq q<+\infty)$ and is twice differentiable, when

$$F'_{(f)}(h)=\int_0^1 K'_u(s,\,t;f(t))\,h(t)\,\mathrm{d}t,$$

$$F''_{(f)}(h,\,k)=\int_0^1 K''_{uu}(s,\,t;f(t))\,h(t)\,k(t)\,\mathrm{d}t$$

for every $f\in L^p$.

Using this Theorem, we shall point out some corollaries of Theorems 1 and 2.

Let $X=L^p\,(p\geqq 2)$, $Y=L^q\,(1\leqq q<+\infty)$. Let us in space L^p notice the functions $t\mapsto a(t)\equiv 0$ and $t\mapsto b(t)>0$ $(t\in[0,\,1])$.

Let $D\subset\{f\mid \|f\|^2_{L^p}+\|b-f\|^2_{L^p}<\|b\|^2_{L^p},\ f\in L^p\}$ be a convex set such that $a(t),\,b(t)\in\bar{D}$.

First, we shall prove the following lemma.

Lemma. *If the conditions of Theorem G are fulfilled, holds the inequality*

$$\|F''_{(f)}(h,\,h)\|_{L^q}\leqq\Phi(f)\,\|h\|^2_{L^p}, \tag{10}$$

where functional $\Phi\colon L^p\to\mathbf{R}^+$ and is defined by

$$\Phi(f)=\begin{cases}2^{2/p}\big(M\,\|f\|^{p-2}_{L^p}+N\big) & p>2,\\ M+N & p=2.\end{cases}$$

Proof. We shall distinguish two cases.

Case $1: p>2$. Based on Theorem G, applying Hölder's inequality, we have

$$\begin{aligned}|F''_{(f)}(h,\,h)|&=\Big|\int_0^1 K''_{uu}\big(s,\,t;f(t)\big)\,h(t)^2\,\mathrm{d}t\Big|\\ &\leqq\int_0^1 \big|K''_{uu}\big(s,\,t;f(t)\big)\big|\cdot|h(t)|^2\,\mathrm{d}t\\ &\leqq\Big(\int_0^1 \big|K''_{uu}(s,\,t;f(t))\big|^{\frac{p}{p-2}}\,\mathrm{d}t\Big)^{\frac{p-2}{p}}\Big(\int_0^1 |h(t)|^p\,\mathrm{d}t\Big)^{\frac{2}{p}}\\ &=P(s,f)\,\|h\|^2_{L^p},\end{aligned}$$

i.e.,

$$\Big(\int_0^1 |F''_{(f)}(h,\,h)|^q\,\mathrm{d}s\Big)^{\frac{1}{q}}\leqq\Big(\int_0^1 P(s,f)^q\,\mathrm{d}s\Big)^{\frac{1}{q}}\cdot\|h\|^2_{L^p},$$

or

$$\|F''_{(f)}(h,\,h)\|_{L^q}\leqq\Big(\int_0^1 P(s,f)^q\,\mathrm{d}s\Big)^{\frac{1}{q}}\cdot\|h\|^2_{L^p}, \tag{11}$$

where $P(s,f)=\Big(\int_0^1 |K''_{uu}(s,\,t;f(t))|^{\frac{p}{p-2}}\,dt\Big)^{\frac{p-2}{p}}$.

According to (9) and the inequality (see, for example [**2**, pp. 338—339, inequality 3.9.7])

$$|z_1+z_2|^r \leqq c_r(|z_1|^r+|z_2|^r) \qquad (z_1,\,z_2\in\mathbf{C};\ r\geqq 0),$$

where $c_r=1\,(0\leqq r\leqq 1)$ and $c_r=2^{r-1}\,(r>1)$, we see that $P(s,f)$ satisfies

$$P(s,f)=\Big(\int_0^1 |K''_{uu}(s,\,t;f(t))|^{\frac{p}{p-2}}\,dt\Big)^{\frac{p-2}{p}} \leqq \Big(\int_0^1 (M|f(t)|^{p-2}+N)^{\frac{p}{p-2}}\,dt\Big)^{\frac{p-2}{p}}$$

$$\leqq \Big(\int_0^1 2^{\frac{2}{p-2}}\Big(M^{\frac{p}{p-2}}|f(t)|^p+N^{\frac{p}{p-2}}\Big)\,dt\Big)^{\frac{p-2}{p}} = 2^{\frac{2}{p}}\Big(M^{\frac{p}{p-2}}\int_0^1 |f(t)|^p\,dt+N^{\frac{p}{p-2}}\Big)^{\frac{p-2}{p}}$$

$$\leqq 2^{\frac{2}{p}}\Big(M\Big(\int_0^1 |f(t)|^p\,dt\Big)^{\frac{p-2}{p}}+N\Big)=2^{\frac{2}{p}}\big(M\|f\|_{L^p}^{p-2}+N\big)=\Phi(f) \qquad (p>2).$$

On the basis of this, from (11) it follows that

$$\|F''_{(f)}(h,\,h)\|_{L^q}\leqq \Phi(f)\cdot\|h\|^2_{L^p} \qquad (p>2). \tag{12}$$

Case $2: p=2$. Then

$$|F''_{(f)}(h,\,h)|\leqq \int_0^1 |K''_{uu}(s,\,t;f(t))|\cdot|h(t)|^2\,dt,$$

from which, using (9) we obtain

$$|F''_{(f)}(h,\,h)|\leqq (M+N)\int_0^1 |h(t)|^2\,dt,$$

i.e.

$$\|F''_{(f)}(h,\,h)\|_{L^q}\leqq (M+N)\|h\|^2_{L^2}=\Phi(f)\,\|h\|^2_{L^2} \qquad (p=2). \tag{13}$$

From (12) and (13) follows (10), which proves the lemma.

Notice that $\Phi(f)\leqq\Phi(b)$ $(\forall f\in\bar{D})$.

Corollary 7. *If the conditions of Theorem G are fulfilled, then using Theorem* 2 *and the proved Lemma, the inequality*

$$\Big\|\int_0^1 K'_u(s,\,t;f(t))\,b(t)\,dt-\int_0^1 K(s,\,t;\,b(t))\,dt+\int_0^1 K(s,\,t;\,0)\,dt\Big\|_{L^q}$$

$$\leqq \frac{1}{2}\,\Phi(b)\big(\|f\|^2_{L^p}+\|b-f\|^2_{L^p}\big)$$

holds for every $f\in\bar{D}$.

Corollary 8. *If*

$$\sup_{f\in\bar{D}} \|F(f)\|_{L^q} = \sup_{f\in\bar{D}} \Big(\int_0^1 \Big|\int_0^1 K\big(s, t; f(t)\big)\,dt\Big|^q ds\Big)^{1/q} \leqq M$$

and if the conditions of Theorem G are fulfilled, using Theorem 1 *and the Lemma, the inequality*

$$\Big\|\int_0^1 K'_u\big(s, t; f(t)\big)b(t)\,dt\Big\|_{L^q} \leqq 2M + \frac{1}{2}\Phi(b)\|b\|^2_{L^p}$$

holds for every $x\in\bar{D}$.

Let now:

1° $X=C^2[\alpha, \beta]$, $Y=\mathbf{R}$;

2° $\|f-g\| = \max_{x\in[\alpha,\beta]} |f(x)-g(x)| + \max_{x\in[\alpha,\beta]} |f'(x)-g'(x)|$ $(f, g\in X)$;

3° $\|y'-y''\| = |y'-y''|$ $(y', y''\in Y)$;

4° Function $(x, u, v)\mapsto G(x, u, v)$ is twice continuously-differentiable for $x\in[\alpha, \beta]$ and $u, v\in\mathbf{R}$;

5° Functional $F: C^2[\alpha, \beta]\to\mathbf{R}$ is defined by $F(f)=\int_\alpha^\beta G\big(x, f(x), f'(x)\big)\,dx$;

6° $a(x)=0$, $b(x)>0$ $(a, b\in C^2[\alpha, \beta])$;

7° $D\subset\{f \mid \|f\|^2+\|b-f\|^2<\|b\|^2,\ f\in C^2[\alpha, \beta]\}$ is a convex set such that $a, b\in\bar{D}$.

Let us introduce a notation $(u, v)=\int_\alpha^\beta G''_{uv}\big(x, f(x), f'(x)\big)\,dx$.

If

$$\max\{\sup_{f\in D}(f, f),\ \sup_{f\in D}(f, f'),\ \sup_{f\in D}(f', f')\}\leqq N, \tag{14}$$

then from

$$F''_{(f)}(h, k)=\int_\alpha^\beta \big\{G''_{ff}\,h(x)k(x)+G''_{ff'}\big(h(x)k(x)\big)'+G''_{f'f'}\,h'(x)k'(x)\big\}\,dx$$

follows the inequality

$$|F''_{(f)}(h, h)|\leqq N\big(\max_{x\in[\alpha,\beta]}|h(x)|+\max_{x\in[\alpha,\beta]}|h'(x)|\big)^2=N\|h\|^2 \qquad (\forall h\in X).$$

Corollary 9. *If the inequality* (14) *holds, from Theorem* 2 *it follows that*

$$\Big|\int_\alpha^\beta \big(G'_f(x, f, f')b(x)+G'_{f'}(x, f, f')b'(x)\big)\,dx-\int_\alpha^\beta G(x, b, b')\,dx$$
$$+\int_\alpha^\beta G(x, 0, 0)\,dx\Big|\leqq\frac{N}{2}(\|f\|^2+\|b-f\|^2)$$

for every $x\in\bar{D}$.

Corollary 10. *If*

$$\sup_{f \in \overline{D}} \left| \int_{\alpha}^{\beta} G(x, f(x), f'(x))\, dx \right| \leqq M,$$

then using Theorem 1 *and the inequality* (14) *it follows that*

$$\left| \int_{\alpha}^{\beta} (G'_f(x, f, f')\, b(x) + G'_{f'}(x, f, f')\, b'(x))\, dx \right| \leqq 2M + \frac{N}{2} \| b \|^2$$

for every $f \in \overline{D}$.

*

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