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518. ON SOME INTEGRAL INEQUALITIES*

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A. OSTROWSKI ([**1**]) has proved the following theorem:

Theorem A. *Let f be a differentiable function on (a, b) and let, on (a, b), $|f'(x)| \leqq M$. Then, for every $x \in (a, b)$,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x \right| \leqq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a)\, M. \tag{1}$$

REMARK. If $f: \mathbf{R} \to \mathbf{R}$ is a differentiable function defined on $[a, b]$ and such that $|f'(x)| \leqq M\, (\forall\, x \in (a, b))$, then the inequality (1) holds for every $x \in [a, b]$ (see [**2**]).

G. W. MACKEY in [3] has given the following result.

Theorem B. *Let f be a differentiable real-valued function defined on $[0, 1]$ and such that*

$$|f'(x)| \leqq M \quad \text{for} \quad 0 < x < 1.$$

Then

$$\left| \int_0^1 f(x)\, \mathrm{d}x - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leqq \frac{M}{n}. \tag{2}$$

These results are presented in the monograph [**4**, p. 297].

In this paper we shall give some generalisations of these results.

Theorem 1. *Let $f: \mathbf{R} \to \mathbf{R}$ be a differentiable function defined on $[0, 1]$ and such that*

$$|f'(x)| \leqq M \quad \text{for} \quad 0 < x < 1.$$

Then

$$\left| \int_0^1 f(x)\, \mathrm{d}x - \sum_{k=1}^{n} \lambda_k f(x_k) \right| \leqq \frac{M}{2} \sum_{k=1}^{n} \left((x_k - a_{k-1})^2 + (a_k - x_k)^2 \right), \tag{3}$$

where

$$0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$$

and

$$\lambda_k = a_k - a_{k-1}, \qquad a_{k-1} \leqq x_k \leqq a_k \ (k = 1, \ldots, n).$$

Proof. Let $E(f; k) = f(x_k) - \dfrac{1}{\lambda_k} \displaystyle\int_{a_{k-1}}^{a_k} f(x)\, \mathrm{d}x.$

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According to Theorem A and the Remark, we have

$$\left|E(f;k)\right| \leqq \left(\frac{1}{4}+\frac{\left(x_k-\frac{a_{k-1}+a_k}{2}\right)^2}{(a_k-a_{k-1})^2}\right)(a_k-a_{k-1})M,$$

i. e.

(4) $$\lambda_k\,|E(f;k)| \leqq \frac{M}{2}\left((x_k-a_{k-1})^2+(a_k-x_k)^2\right).$$

Since

$$\left|\int_0^1 f(x)\,dx-\sum_{k=1}^{n}\lambda_k f(x_k)\right| = \left|\sum_{k=1}^{n}\lambda_k\left(f(x_k)-\frac{1}{\lambda_k}\int_{a_{k-1}}^{a_k} f(x)\,dx\right)\right| \leqq \sum_{k=1}^{n}\lambda_k\,|E(f;k)|,$$

according to the inequality (4), we obtain (3).

Thus the theorem is proved.

Corollary 1. *If* $x_k=a_k$ *or* $x_k=a_{k-1}$, *from* (3) *it follows*

$$\left|\int_0^1 f(x)\,dx-\sum_{k=1}^{n}\lambda_k f(x_k)\right| \leqq \frac{M}{2}\sum_{k=1}^{n}\lambda_k^2.$$

Corollary 2. *If* $x_k=\frac{1}{2}(a_{k-1}+a_k)$, *from* (3) *we obtain*

$$\left|\int_0^1 f(x)\,dx-\sum_{k=1}^{n}\lambda_k f(x_k)\right| \leqq \frac{M}{4}\sum_{k=1}^{n}\lambda_k^2.$$

Corollary 3. *If* $a_k=\frac{k}{n}$ $(k=0, 1, \ldots, n)$, *from* (3) *it follows*

(5) $$\left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}f(x_k)\right| \leqq \frac{M}{2n}\sum_{k=1}^{n}\left\{\frac{1}{n}+2\left(x_k-\frac{k}{n}\right)+2n\left(x_k-\frac{k}{n}\right)^2\right\}.$$

If $x_k=a_k=\frac{k}{n}$ or $x_k=a_{k-1}=\frac{k-1}{n}$, from (5) we have

$$\left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}f(x_k)\right| \leqq \frac{M}{2n}.$$

This inequality is stronger than the inequality (2).

If $x_k=\frac{1}{2}(a_{k-1}+a_k)=\frac{2k-1}{2n}$, from (5) we obtain

(6) $$\left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}f(x_k)\right| \leqq \frac{M}{4n}.$$

Inequality (6) is given in [**5**, p. 151].

Now let $D=\{(x_1,\dots,x_m)\mid a_i<x_i<b_i\ (i=1,\dots,m)\}$ and let $\bar{D}$ be the closure of D.

We now propose the following generalisation of Theorem A.

Theorem 2. *Let $f:\mathbf{R}^m\to\mathbf{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial x_i}\right|\le M_i\ (M_i>0;\ i=1,\dots,m)$ in D. Then, for every $X=(x_1,\dots,x_m)\in\bar{D}$,*

$$\left| f(x_1,\dots,x_m)-\frac{1}{\prod_{i=1}^m (b_i-a_i)}\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} f(y_1,\dots,y_m)\,dy_1\cdots dy_m\right| \le \sum_{i=1}^m\left(\frac{1}{4}+\frac{\left(x_i-\frac{a_i+b_i}{2}\right)^2}{(b_i-a_i)^2}\right)(b_i-a_i)\,M_i. \tag{7}$$

Proof. Let $X=(x_1,\dots,x_m)$ and $Y=(y_1,\dots,y_m)$ $(X\in\bar{D},\ Y\in D)$. According to TAYLOR's formula, we have

$$f(X)-f(Y)=\sum_{i=1}^m \frac{\partial f(C)}{\partial x_i}(x_i-y_i), \tag{8}$$

where $C=(y_1+\theta(x_1-y_1),\dots,y_m+\theta(x_m-y_m))$ $(0<\theta<1)$.

Integrating (8), we obtain

$$f(X)\operatorname{mes} D-\int\cdots\int_D f(Y)\,dY=\sum_{i=1}^m\int\cdots\int_D \frac{\partial f(C)}{\partial x_i}(x_i-y_i)\,dY, \tag{9}$$

where $dY=dy_1\cdots dy_m$ and $\operatorname{mes} D=\prod_{i=1}^m (b_i-a_i)$.

From (7) it follows

$$\left| f(X)\operatorname{mes} D-\int\cdots\int_D f(Y)\,dY\right|\le\sum_{i=1}^m\left|\int\cdots\int_D \frac{\partial f(C)}{\partial x_i}(x_i-y_i)\,dY\right|$$

$$\le\sum_{i=1}^m\int\cdots\int_D\left|\frac{\partial f(C)}{\partial x_i}\right|\cdot|x_i-y_i|\,dY,$$

and

$$\left| f(X)\operatorname{mes} D-\int\cdots\int_D f(Y)\,dY\right|\le\sum_{i=1}^m M_i\int\cdots\int_D |x_i-y_i|\,dY, \tag{10}$$

respectively, owing to the assumption $\left|\frac{\partial f}{\partial x_i}\right|\le M_i$ $(M_i>0;\ i=1,\dots,m)$.

Since

$$\int_{a_i}^{b_i}|x_i-y_i|\,dy_i=\frac{1}{4}(b_i-a_i)^2+\left(x_i-\frac{a_i+b_i}{2}\right)^2,$$

we have

$$\int_D \cdots \int |x_i - y_i| \, dY = \frac{\text{mes}\, D}{b_i - a_i} \int_{a_i}^{b_i} |x_i - y_i| \, dy_i$$

$$= (\text{mes}\, D)\,(b_i - a_i) \left(\frac{1}{4} + \frac{\left(x_i - \frac{a_i + b_i}{2}\right)^2}{(b_i - a_i)^2} \right).$$

Since mes $D > 0$, inequality (10) becomes

$$\left| f(X) - \frac{1}{\prod_{i=1}^{m} (b_i - a_i)} \int_D \cdots \int f(Y) \, dY \right| \leq \sum_{i=1}^{m} \left(\frac{1}{4} + \frac{\left(x_i - \frac{a_i + b_i}{2}\right)^2}{(b_i - a_i)^2} \right) (b_i - a_i) M_i.$$

Thus the proof is finished.

The Theorem 2 can be generalised as follows.

Theorem 3. *Let $f: \mathbf{R}^m \to \mathbf{R}$ be a differentiable function defined on $\bar{D}$ and let $\left| \frac{\partial f}{\partial x_i} \right| \leq M_i$ ($M_i > 0$; $i = 1, \ldots, m$) in D. Furthermore, let function $X \mapsto p(X)$ be defined, integrable and $p(X) > 0$ for every $X \in \bar{D}$. Then, for every $X \in \bar{D}$,*

$$\left| f(X) - \frac{\int_D \cdots \int p(Y) f(Y) \, dY}{\int_D \cdots \int p(Y) \, dY} \right| \leq \frac{\sum_{i=1}^{m} M_i \int_D \cdots \int p(Y) \, |x_i - y_i| \, dY}{\int_D \cdots \int p(Y) \, dY}.$$

This Theorem can be prove similarly to Theorem 2.

We use the following notations:

$$m, n_i \in \mathbf{N} \quad (i = 1, \ldots, m);$$

$$0 = a_{i0} < a_{i1} < \cdots < a_{in_i} = 1 \quad (i = 1, \ldots, m);$$

$$a_{i, k_i - 1} \leq x_{ik_i} \leq a_{ik_i}, \quad \lambda_{ik_i} = a_{ik_i} - a_{i, k_i - 1} \quad (k_i = 1, \ldots, n_i; \; i = 1, \ldots, m);$$

$$k = (k_1, \ldots, k_m), \quad X = (x_1, \ldots, x_m), \quad X_k = (x_{1k_1}, \ldots, x_{mk_m});$$

$$D = \{X \mid 0 < x_i < 1; \; i = 1, \ldots, m\};$$

$$D(k) = \{X_k \mid a_{i, k_i - 1} < x_{ik_i} < a_{ik_i} \; (k_i = 1, \ldots, n_i; \; i = 1, \ldots, m)\};$$

$$dX = dx_1 \cdots dx_m;$$

$$E(f; k) = f(X_k) - \frac{1}{\prod_{i=1}^{m} \lambda_{ik_i}} \int_{D(k)} \cdots \int f(X) \, dX.$$

Theorem 4. *Let $f:\mathbf{R}^m\to\mathbf{R}$ be a differentiable function defined on $\overline{D}$ and let* $\left|\frac{\partial f}{\partial x_i}\right|\leqq M_i$ $(M_i>0;\ i=1,\dots,m)$ *in D.*

Then

$$\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| \leqq\frac{1}{2}\sum_{i=1}^{m}M_i\left(\sum_{k_i=1}^{n_i}H(x_{ik_i};k_i)\right), \tag{11}$$

where

$$H(t;k_i)=(t-a_{i,k_i-1})^2+(a_{ik_i}-t)^2.$$

Proof. According to Theorem 2, we have

$$|E(f;k)|\leqq\frac{1}{2}\sum_{i=1}^{m}\frac{M_i}{\lambda_{ik_i}}H(x_{ik_i};k_i). \tag{12}$$

Since $\bigcup_{k_1=1}^{n_1}\cdots\bigcup_{k_m=1}^{n_m}\overline{D}(k)=\overline{D}$, we have

$$\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| =\left|\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}E(f;k)\right| \leqq\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}|E(f;k)|.$$

Using (12), the last inequality becomes

$$\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right| \leqq\frac{1}{2}\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}\left(\sum_{i=1}^{m}\frac{M_i}{\lambda_{ik_i}}H(x_{ik_i};k_i)\right) =\frac{1}{2}\sum_{i=1}^{m}M_i\left(\sum_{k_i=1}^{n_i}H(x_{ik_i};k_i)\right).$$

The proof is finished.

Corollary 1. *If $x_{ik_i}=a_{ik_i}$ or $x_{ik_i}=a_{i,k_i-1}$, from (9) it follows*

$$\left|\int_D\cdots\int f(X)\,dX-\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right|\leqq\frac{1}{2}\sum_{=1}^{m}M_i\left(\sum_{k_i=1}^{n_i}\lambda_{ik_i}^2\right).$$

Furthermore, if $\lambda_{ik_i}=\frac{1}{n_i}$, *holds*

$$\left|\int_D\cdots\int f(X)\,\mathrm{d}X-\frac{1}{n_1\cdots n_m}\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}f(X_k)\right|\leqq\frac{1}{2}\sum_{i=1}^{m}\frac{M_i}{n_i}.$$

Corollary 2. *If* $x_{ik_i}=\frac{1}{2}(a_{i,k_i-1}+a_{ik_i})$, *holds*

$$\left|\int_D\cdots\int f(X)\,\mathrm{d}X-\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}\lambda_{1k_1}\cdots\lambda_{mk_m}f(X_k)\right|\leqq\frac{1}{4}\sum_{i=1}^{m}M_i\left(\sum_{k_i=1}^{n_i}\lambda_{ik_i}^2\right).$$

Furthermore, if $\lambda_{ik_i}=\frac{1}{n_i}$, *we have*

$$\left|\int_D\cdots\int f(X)\,\mathrm{d}X-\frac{1}{n_1\cdots n_m}\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m}f(X_k)\right|\leqq\frac{1}{4}\sum_{i=1}^{m}\frac{M_i}{n_i}.$$

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