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548. ON SOME GENERALIZATIONS OF ZMOROVIČ'S INEQUALITY*

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In [1] V. A. ZMOROVIČ has proved the following theorem:

Theorem A. *If the function* $f\colon [a-h,\ a+h]\to \mathbf{R}$ *is twice continuously-differentiable, then*

$$\int_{a-h}^{a+h} (f''(x))^2 \geqq \frac{3}{2h^3}[f(a+h)-2f(a)+f(a-h)]^2,$$

with equality if and only if f is given by

$$f(x)=\begin{cases} C_1\{(h-a+x)^3+6h^2(a-x)\}+C_2x+C_3 & (x\in[a-h,\ a]) \\ C_1(h+a-x)^3+C_2x+C_3 & (x\in[a,\ a+h]), \end{cases}$$

where C_1, C_2, C_3 *are arbitrary real constants.*

The mentioned ZMOROVIČ's result is an improvement of the inequality

$$\int_{a-h}^{a+h} (f''(x))^2\,\mathrm{d}x \geqq \frac{1}{2h^3}[f(a+h)-2f(a)+f(a-h)]^2,$$

which was, through geometric considerations, obtained by M. A. LAVRENT'EV (see [1]), under stronger conditions. Namely, LAVRENT'EV has proved the last inequality under the condition that f is a four times continuously-differentiable function on $[a-h,\ a+h]$.

A similar result may be found in [2] (Theorem 264):

Theorem B. *If*

$$f(-1)=-1,\quad f(1)=1,\quad f'(-1)=f'(1)=0$$

and k is a positive integer, then

$$\int_{-1}^{1} (f''(x))^{2k}\,\mathrm{d}x \geqq 2\left(\frac{4k-1}{2k-1}\right)^{2k-1},$$

with inequality unless

$$f(x)=\frac{4k-1}{2k}x-\frac{2k-1}{2k}x^{(4k-1)/(2k-1)}.$$

This paper will give some generalizations of Theorem A.

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Let us first introduce the notation

$$\Delta=\frac{1}{h^2}[f(a+h)-2f(a)+f(a-h)]$$

and let us note that

$$\int_0^h (h-t)\big(f''(a-t)+f''(a+t)\big)\,\mathrm{d}t=h^2\Delta. \tag{1}$$

Theorem 1. *Let the function $f:[a-h,\ a+h]\to\mathbf{R}$ be twice continuously-differentiable and $g:[a-h,\ a+h]\to\mathbf{R}^+$ continuous.*

Then

$$\int_{a-h}^{a+h} g(x)\,|f''(x)|^r\,\mathrm{d}x \geqq \frac{h^{2r}}{\delta(r)^{r-1}}\,|\Delta|^r \qquad (r>1), \tag{2}$$

where

$$\delta(r)=\int_0^h (h-t)^{\frac{r}{r-1}}\Big(g(a-t)^{\frac{1}{1-r}}+g(a+t)^{\frac{1}{1-r}}\Big)\,\mathrm{d}t. \tag{3}$$

Equality in (2) *holds if and only if the function f is given by*

$$f(x)=\begin{cases} A_1\displaystyle\int_a^x (x-t)\left|\frac{h-a+t}{g(t)}\right|^{\frac{1}{r-1}}\mathrm{d}t+A_2x+A_3 & (x\in[a-h,\ a])\\[2ex] A_1\displaystyle\int_a^x (x-t)\left|\frac{h+a-t}{g(t)}\right|^{\frac{1}{r-1}}\mathrm{d}t+A_2x+A_3 & (x\in[a,\ a+h]),\end{cases} \tag{4}$$

where A_1, A_2, A_3 are arbitrary real constants.

Proof. Let us put $\gamma(t)=\Big(g(a-t)^{\frac{1}{1-r}}+g(a+t)^{\frac{1}{1-r}}\Big)^{\frac{1-r}{r}}$. Then (3) becomes

$$\delta(r)=\int_0^h\left(\frac{h-t}{\gamma(t)}\right)^{\frac{r}{r-1}}\mathrm{d}t. \tag{5}$$

Since

$$\int_{a-h}^{a+h} g(x)\,|f''(x)|^r\,\mathrm{d}x=\int_0^h\big(g(a-t)\,|f''(a-t)|^r+g(a+t)\,|f''(a+t)|^r\big)\,\mathrm{d}t,$$

putting

$$p_1=g(a-t),\ p_2=g(a+t),\ z_1=f''(a-t),\ z_2=f''(a+t),$$

and using inequality (see [3], [4])

$$|z_1+\cdots+z_n|^r\leqq\Big(\sum_{i=1}^n p_i^{\frac{1}{1-r}}\Big)^{r-1}(p_1|z_1|^r+\cdots+p_n|z_n|^r) \tag{6}$$

$$\big(z_i\in\mathbf{C},\ p_i>0\ (i=1,\ldots,n),\ r>1\big),$$

we have

$$\int_{a-h}^{a+h} g(x)\,|f''(x)|^r\,\mathrm{d}x \geqq \int_0^h \frac{|f''(a-t)+f''(a+t)|^r}{\left(g(a-t)^{\frac{1}{1-r}}+g(a+t)^{\frac{1}{1-r}}\right)^{r-1}}\,\mathrm{d}t$$

$$=\int_0^h \big(\gamma(t)\,|f''(a-t)+f''(a+t)|\big)^r\,\mathrm{d}t,$$

or, with regard to (5),

$$(7)\qquad \int_{a-h}^{a+h} g(x)|f''(x)|^r\,\mathrm{d}x$$

$$\geqq \frac{1}{\delta(t)^{r-1}}\Big(\int_0^h \big(\gamma(t)\,|f''(a-t)+f''(a+t)|\big)^r\,\mathrm{d}t\Big)\Big(\int_0^h \Big(\frac{h-t}{\gamma(t)}\Big)^{\frac{r}{r-1}}\mathrm{d}t\Big)^{r-1}.$$

Applying HÖLDER's inequality to the right side in (7), we obtain

$$\int_{a-h}^{a+h} g(x)\,|f''(x)|^r\,\mathrm{d}x \geqq \frac{1}{\delta(r)^{r-1}}\Big(\int_0^h (h-t)\,|f''(a-t)+f''(a+t)|\,\mathrm{d}t\Big)^r$$

$$\geqq \frac{1}{\delta(r)^{r-1}}\Big|\int_0^h (h-t)\big(f''(a-t)+f''(a+t)\big)\,\mathrm{d}t\Big|^r,$$

from where, with regard to (1), follows (2).

Since in (6) equality holds if and only if

$$p_1|z_1|^{r-1}=\cdots=p_n|z_n|^{r-1} \text{ and } z_k\bar{z}_j\geqq 0\ \ (k,\ j=1,\ldots,n),$$

and in HÖLDER's inequality

$$\int_\alpha^\beta |\Phi(x)\Psi(x)|\,\mathrm{d}x \leqq \Big(\int_\alpha^\beta |\Phi(x)|^p\,\mathrm{d}x\Big)^{1/p}\Big(\int_\alpha^\beta |\Psi(x)|^q\,\mathrm{d}x\Big)^{1/q}\quad \Big(\frac{1}{p}+\frac{1}{q}=1,\ p>1\Big)$$

if and only if $|\Phi(x)|^p=C|\Psi(x)|^q$, where C is a real constant (see [5], p. 54), we conclude that equality in (2) holds if and only if

$$(8)\qquad \begin{aligned} &g(a-t)^{\frac{1}{r-1}}f''(a-t)=g(a+t)^{\frac{1}{r-1}}f''(a+t),\\ &\big(\gamma(t)\big(f''(a-t)+f''(a+t)\big)\big)^r=C\Big(\frac{h-t}{\gamma(t)}\Big)^{\frac{r}{r-1}}\qquad (C\in\mathbf{R}). \end{aligned}$$

From (8), if we put $C=A_1^r$, follows

$$f''(a-t)=A_1\Big(\frac{h-t}{g(a-t)}\Big)^{\frac{1}{r-1}},\ f''(a+t)=A_1\Big(\frac{h-t}{g(a+t)}\Big)^{\frac{1}{r-1}}\quad (0\leqq t\leqq h),$$

from where, by integration, we obtain (4).

This completes the proof.

Corollary 1. *If the function* $f:[a-h,\ a+h]\to \mathbf{R}$ *is twice continuously-differentiable, then*

$$\int_{a-h}^{a+h} |f''(x)|^r\,dx \geq \left(\frac{2r-1}{2r-2}\right)^{r-1} h\,|\Delta|^r \qquad (r>1). \tag{9}$$

Equality in (9) *holds if and only if the function* f *is given by*

$$f(x)=\begin{cases} C_1\left\{(h-a+x)^{\frac{2r-1}{r-1}}+\frac{4r-2}{r-1}h^{\frac{r}{r-1}}(a-x)\right\}+C_2x+C_3 & (x\in[a-h,\ a]) \\ C_1(h+a-x)^{\frac{2r-1}{r-1}}+C_2x+C_3 & (x\in[a,\ a+h]), \end{cases}$$

where $C_i\,(i=1,\ 2,\ 3)$ *are arbitrary real constants.*

Putting $g(x)=1$, the statement of the Corollary 1 follows from Theorem 1.

REMARK. For $r=2$, the Corollary 1 reduces to Theorem A.

Observing the left side of inequality (2) in the form

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx,$$

where $t\mapsto\Phi(t)=|t|^r$ $(r>1)$, we conclude that Φ is a convex function. That gave us the idea to generalize Theorem 1, for a more general function Φ.

First, we give the following definition.

Definition. *Continuous function* $\Phi:\mathbf{R}\to\mathbf{R}^+$ *belongs to the class* M *if there is a convex function* $F:\mathbf{R}\to\mathbf{R}^+$ *and real numbers* λ *and* m *so that for each* $x\in\mathbf{R}$ *the inequalities*

$$F(x)\leq\Phi(x)^{1/m}\leq\lambda F(x) \qquad (\lambda\geq 1;\ m>1)$$

are valid.

In this paper we have taken convex functions to mean continuous and JENSEN convex functions, as defined in [5].

One of the possible generalizations of Theorem 1 is:

Theorem 2. *Let the functions* $\Phi:\mathbf{R}\to\mathbf{R}^+$, $f:[a-h,a+h]\to\mathbf{R}$, $g:[a-h,a+h]\to\mathbf{R}^+$ *satisfy the conditions:*

1° $\Phi\in M$;

2° f *twice continuously-differentiable;*

3° g *continuous.*

Then

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx \geq \frac{h^{2m}}{\lambda^m\,\delta(m)^{m-1}}\,\Phi(\Delta),$$

where δ *is given by* (3).

Proof. Since $\Phi\in M$, i.e. for each $x\in\mathbf{R}$ the inequalities

(10) $$F(x)\leqq\Phi(x)^{1/m}\leqq\lambda F(x)\qquad(\lambda\geqq 1;\ m>1)$$

hold, where F is a convex function, we have

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx=\int_0^h\{g(a-t)\,\Phi(f''(a-t))+g(a+t)\,\Phi(f''(a+t))\}\,dt$$

$$=\int_0^h\{g(a-t)\,F(f''(a-t))^m+g(a+t)\,F(f''(a+t))^m\}\,dt.$$

Applying (6) to the right side of the last inequality, we obtain

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx\geqq\int_0^h\{\gamma(t)(F(f''(a-t)+F(f''(a+t)))\}^m\,dt,$$

where, now $\gamma(t)=\Big(g(a-t)^{\frac{1}{1-m}}+g(a+t)^{\frac{1}{1-m}}\Big)^{\frac{1-m}{m}}$.

According to JENSEN's inequality we have

(11) $$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx\geqq 2^m\int_0^h\big(\gamma(t)\,F(u(t))\big)^m\,dt,$$

where, with regard to the assumption for function f, function $t\mapsto u(t)=$ $=\frac{1}{2}(f''(a-t)+f''(a+t))$ is continuous on $[0,h]$.

On the other hand, since $x\mapsto F(x)$ is a convex function, using JENSEN's integral inequality (see for example [6], Theorem 6, p. 228), we have

(12) $$\int_0^h(h-t)\,F(u(t))\,dt\geqq\int_0^h(h-t)\,dt\cdot F\left\{\frac{\int_0^h(h-t)\,u(t)\,dt}{\int_0^h(h-t)\,dt}\right\}=\frac{h^2}{2}F(\Delta),$$

because $\dfrac{\int_0^h(h-t)\,u(t)\,dt}{\int_0^h(h-t)\,dt}=\dfrac{1}{h^2}\big(f(a+h)-2f(a)+f(a-h)\big)=\Delta.$

Similarly to the proof of Theorem 1, using HÖLDER's inequality, from (11) follows

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dt\geqq\frac{2^m}{\delta(m)^{m-1}}\Big(\int_0^h(h-t)\,F(u(t))\,dt\Big)^m,$$

from where, with regard to (12), we have

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx \geqq \frac{2m}{\delta(m)^{m-1}}\left(\frac{h^2}{2}F(\Delta)\right)^m.$$

Since, according to (10)

$$F(x) \geqq \frac{1}{\lambda}\Phi(x)^{1/m} \qquad (m>1),$$

we finally obtain

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx \geqq \frac{h^{2m}}{\lambda^m\,\delta(m)^{m-1}}\Phi(\Delta),$$

which completes the proof.

REMARK. If $\Phi(x)^{1/m}=F(x)$ $(m>1)$, where F is a convex function, the inequality

$$\int_{a-h}^{a+h} g(x)\,\Phi(f''(x))\,dx \geqq \frac{h^{2m}}{\delta(m)^{m-1}}\Phi(\Delta), \tag{13}$$

holds.

The last inequality reduces to (2) for $\Phi(x)=|x|^r$ $(r>1)$ and $m=r$.

Corollary 2. *If functions* $\Phi:\mathbf{R}\to\mathbf{R}^+$ *and* $f:[a-h,\ a+h]\to\mathbf{R}$ *satisfy the conditions:*

1° $\Phi(x)^{1/m}=F(x)$ $(m>1$; *F convex function*);

2° *f twice continuously-differentiable.*

Then

$$\int_{a-h}^{a+h} \Phi(f''(x))\,dx \geqq \left(\frac{2m-1}{2m-2}\right)^{m-1} h\,\Phi(\Delta). \tag{14}$$

The inequality (14) is stronger if m is greater.

If we put $g(x)=1$ $(\forall x\in[a-h,\ a+h])$ in (13), we obtain (14). On the other, since $m\mapsto C_m=\left(\frac{2m-1}{2m-2}\right)^{m-1}$ $(m>1)$ is an increasing function, inequality (14) is stronger if m is greater. The possible maximum value for C_m is $C_\infty = \lim_{m\to+\infty} C_m=\sqrt{e}$ and it exists, for example for the function $x\mapsto\Phi(x)=e^{\lambda x}$ $(\lambda\in\mathbf{R})$.

REFERENCES

1. V. A. ZMOROVIČ: *On some inequalities* (Russian). Izv. Polytehn. Inst. Kiev **19** (1956), 92—107.
2. G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA: *Inequalities.* Cambridge 1934.
3. P. M. VASIĆ and J. D. KEČKIĆ: *Some inequalities for complex numbers.* Math. Balkanica **1** (1971), 282—286.
4. P. S. BULLEN: *A note on a recent paper of P. M. Vasić and J. D. Kečkić.* Math. Balkanica **2** (1972), 1—2.
5. D. S. MITRINOVIĆ (In cooperation with P. M. VASIĆ): *Analytic inequalities.* Berlin — Heidelberg — New York 1970.
6. И. П. НАТАНСОН: *Теория функций вещественной переменной.* Москва 1974.