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573. ON GENERALIZATION OF THE INEQUALITY OF A. OSTROWSKI AND SOME RELATED APPLICATIONS*

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A. OSTROWSKI ([1]) has proved the following result:

Theorem A. *Let f be a differentiable function on (a, b) and let, on (a, b), $|f'(x)| \leqq M$. Then, for every $x \in (a, b)$,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leqq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) M.$$

G. V. MILOVANOVIĆ ([2]) has generalized the Theorem A when f is function of several variables. In the same paper [2], by use of Theorem A, the following theorem, which generalize the result of G. W. MACKEY ([3], [4, p. 297]), is proved:

Theorem B. *Let $f: \mathbf{R} \to \mathbf{R}$ be a differentiable function defined on $[0, 1]$ and such that $|f'(x)| \leqq M$ for $0 < x < 1$. Then*

$$\left| \int_0^1 f(x)\, dx - \sum_{k=1}^{n} \lambda_k f(x_k) \right| \leqq \frac{M}{2} \sum_{k=1}^{n} \left((x_k - a_{k-1})^2 + (a_k - x_k)^2 \right),$$

where

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

and

$$\lambda_k = a_k - a_{k-1}, \quad a_{k-1} \leqq x_k \leqq a_k \qquad (k = 1, \ldots, n).$$

In this paper we shall give generalizations of these results when $|f^{(n)}(x)| \leqq M$ $(\forall x \in (a, b))$, and $n > 1$.

Theorem 1. *Let $f: \mathbf{R} \to \mathbf{R}$ be n (>1) times differentiable function such that $|f^{(n)}(x)| \leqq M$ $(\forall x \in (a, b))$. Then, for every $x \in [a, b]$*

$$\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y)\, dy \right| \leqq \frac{M}{n(n+1)!} \cdot \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a},$$

where F_k is defined by

$$(1) \qquad F_k \equiv F_k(f; n; x, a, b) \equiv \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$

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Proof. Integrating by use of TAYLOR's formula

$$f(x)=f(y)+\sum_{k=1}^{n-1}\frac{1}{k!}f^{(k)}(y)(x-y)^k+\frac{1}{n!}f^{(n)}(\xi)(x-y)^n,$$

where $\xi=y+\theta(x-y)$ $(0<\theta<1)$, we obtain

$$(2)\quad f(x)(b-a)-\int_a^b f(y)\,dy-\sum_{k=1}^{n-1}\frac{1}{k!}\int_a^b f^{(k)}(y)(x-y)^k\,dy=\frac{1}{n!}\int_a^b f^{(n)}(\xi)(x-y)^n\,dy.$$

Putting $I_k=\frac{1}{k!}\int_a^b f^{(k)}(y)(x-y)^k\,dy$ $(k\geqq 1)$ and $I_0=\int_a^b f(y)\,dy$, by means of partial integration on I_k, we have

$$I_k=\frac{1}{k!}\{f^{(k-1)}(b)(x-b)^k-f^{(k-1)}(a)(x-a)^k\}+I_{k-1}\qquad(1\leqq k\leqq n-1),$$

i.e.

$$(n-k)(I_k-I_{k-1})=-(b-a)F_k\qquad(1\leqq k\leqq n-1),$$

where F_k is defined by (1).

Since

$$\sum_{k=1}^{n-1}(n-k)(I_k-I_{k-1})=-(b-a)\sum_{k=1}^{n-1}F_k,$$

i.e.

$$I_0+\sum_{k=1}^{n-1}I_k=nI_0-(b-a)\sum_{k=1}^{n-1}F_k,$$

it follows, from (2),

$$\left|f(x)(b-a)+(b-a)\sum_{k=1}^{n-1}F_k-nI_0\right|=\frac{1}{n!}\left|\int_a^b f^{(n)}(\xi)(x-y)^n\,dy\right|\leqq\frac{M}{n!}\int_a^b|x-y|^n\,dy$$

$$=\frac{M}{(n+1)!}(x-a)^{n+1}+(b-x)^{n+1}\}.$$

Thus the proof is finished.

For $n=2$, the Theorem 1 reduces to the following corollary:

Corollary 1. *Let $f:\mathbf{R}\to\mathbf{R}$ be twice differentiable function such that $|f''(x)|\leqq M$ $(\forall x\in(a,b))$. Then*

$$\left|\frac{1}{2}\left(f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|\leqq\frac{M(b-a)^2}{4}\left(\frac{1}{12}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)$$

for every $x\in[a,b]$.

Now we shall give an application of Theorem 1. Namely, in the quadrature formula

$$\int_0^1 f(x)\,dx=\frac{1}{n}\sum_{j=1}^{m}\lambda_j f(x_j)+\frac{1}{n}\sum_{k=1}^{n-1}\sum_{j=1}^{m}F_k^{(j)}+R_m(f), \tag{3}$$

where

$$0=a_0<a_1<\cdots<a_m=1,$$

$$\lambda_j=a_j-a_{j-1},\quad a_{j-1}\leqq x_j\leqq a_j \qquad (j=1,\ldots,m),$$

$$F_k^{(j)}\equiv\lambda_j F_j(f;\, n;\, x_j,\, a_{j-1},\, a_j)$$

$$\equiv\frac{n-k}{k!}\{(x_j-a_{j-1})^k f^{(k-1)}(a_{j-1})-(x_j-a_j)^k f^{(k-1)}(a_j)\}$$

we shall estimate $R_m(f)$.

Theorem 2. *Let $f:\mathbf{R}\to\mathbf{R}$ be n (>1) times differentiable function such that $|f^{(n)}(x)|\leqq M$ $(\forall x\in(0,1))$. For the error $R_m(f)$ in the quadrature formula* (3), *holds*

$$|R_m(f)|\leqq\frac{M}{n(n+1)!}\sum_{j=1}^{m}\{(x_j-a_{j-1})^{n+1}+(a_j-x_j)^{n+1}\}. \tag{4}$$

Proof. Let

$$E_j=\frac{1}{n}\left(f(x_j)+\sum_{k=1}^{n-1}\frac{1}{\lambda_j}F_k^{(j)}\right)-\frac{1}{\lambda_j}\int_{a_{j-1}}^{a_j}f(x)\,dx.$$

Since

$$|R_m(f)|=\left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{j=1}^{m}\lambda_i f(x_j)-\frac{1}{n}\sum_{k=1}^{n-1}\sum_{j=1}^{m}F_k^{(j)}\right|=\left|\sum_{j=1}^{m}\lambda_j E_j\right|\leqq\sum_{j=1}^{m}\lambda_j|E_j|,$$

on the basis of Theorem 1, we have

$$|R_m(f)|\leqq\frac{M}{n(n+1)!}\sum_{j=1}^{m}\{(x_j-a_{j-1})^{n+1}+(a_j-x_j)^{n+1}\}.$$

Thus, the Theorem 2 is proved.

We now present some corollaries of the Theorem 2.

Corollary 2. *If $x_j=a_j$ $(j=1,\ldots,m)$ then*

$$|R_m(f)|=\left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{j=1}^{m}\lambda_j f(a_j)-\frac{1}{n}\sum_{k=1}^{n-1}\frac{n-k}{k!}\sum_{j=1}^{m}\lambda_j^k f^{(k-1)}(a_{j-1})\right|$$

$$\leqq\frac{M}{n(n+1)!}\sum_{j=1}^{m}\lambda_j^{n+1}.$$

If, moreover, $a_j=\frac{j}{m}$ $(j=1,\ldots,m)$ then

$$|R_m(f)|\leqq\frac{M}{n(n+1)!\,m^n}. \tag{5}$$

Note that for $n=2$, (5) reduces to

$$|R_m(f)| = \left| \int_0^1 f(x)\,dx - \frac{1}{2m} \sum_{j=1}^{m} \left(f\left(\frac{j-1}{m}\right) + f\left(\frac{j}{m}\right) \right) \right| \leq \frac{M}{12\,m^2},$$

what represents the error of generalized trapezoidal rule (see [5], p. 95).

Corollary 3. *If* $x_j = \frac{1}{2}(a_{j-1} + a_j)$, (4) *is reduced to*

$$|R_m(f)| \leq \frac{M}{2^n\, n\,(n+1)!} \sum_{j=1}^{m} \lambda_j^{n+1}.$$

If, moreover, $a_j = \frac{j}{m}$ $(j = 1, \ldots, m)$ *then*

$$|R_m(f)| \leq \frac{M}{2^n n m^n (n+1)!}.$$

In this case, formula (3) is reduced to

$$\int_0^1 f(x)\,dx = \frac{1}{nm} \sum_{j=1}^{m} f\left(\frac{2j-1}{2m}\right) + \frac{1}{n} \sum_{k=1}^{n-1} \frac{n-k}{(2m)^k\, k!} \sum_{j=1}^{m} \left(f^{(k-1)}\left(\frac{j-1}{m}\right) + (-1)^{k-1} f^{(k-1)}\left(\frac{j}{m}\right) \right) + R_m(f). \tag{6}$$

In special case for $n=3$, formula (6) becomes

$$\int_0^1 f(x)\,dx = \frac{1}{3m} \sum_{j=1}^{m} \left(f\left(\frac{j-1}{m}\right) + f\left(\frac{2j-1}{2m}\right) + f\left(\frac{j}{m}\right) \right) - \frac{1}{24\,m^2} (f'(1) - f'(0)) + R_m(f),$$

where

$$|R_m(f)| \leq \frac{1}{576\,\mathrm{m}^3} \max_{x \in (0,1)} |f'''(x)|.$$

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