UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 602 — № 633 (1978), 129—132.

619. A GENERALIZATION OF A PROBLEM GIVEN BY D. S. MITRINOVIĆ

G. V. Milovanović and *I. Ž. Milovanović*

D. S. MITRINOVIĆ (see [**1**]) has stated the problem:

Let a, b, c, d be positive numbers and m a natural number. Then

$$(1)\qquad \left|\sqrt[m]{a^m+b^m}-\sqrt[m]{c^m+d^m}\right|\leqq \min\left(|a-c|+|b-d|,\quad |a-d|+|b-c|\right).$$

Prove (1) and determine the corresponding estimation for

$$\left|\sqrt[m]{\sum_{k=1}^{n} a_k{}^m}-\sqrt[m]{\sum_{k=1}^{n} b_k{}^m}\right|,$$

where a_k and b_k $(k=1, \ldots, n)$ are positive numbers and m a natural number.

Solving this problem, D. D. ADAMOVIĆ ([**2**], [**3**]), proved a more general result:

Theorem A. 1° *If* $p\geqq 1$ *is a real number and* $a_k>0,\ b_k>0$ $(k=1, \ldots, n)$, *then*

$$(2)\qquad \left|\left(\sum_{k=1}^{n} a_k{}^p\right)^{1/p}-\left(\sum_{k=1}^{n} b_k{}^p\right)^{1/p}\right|\leqq \min\left\{\sum_{k=1}^{n}|a_k-b_{t(k)}|: t\in\mathcal{P}\right\},$$

where $\mathcal{P}$ *denotes the set of all permutations of the set* $\{1, \ldots, n\}$.

2° *Excluding trivial cases* $p=1$ *and* $n=1$, *the inequality* (2) *turns in equality if and only if its right side is equal to zero.*

3° *With additional conditions*

$$\sum_{k=1}^{+\infty} a_k{}^p<+\infty,\quad \sum_{k=1}^{+\infty} b_k{}^p<+\infty,$$

the statements 1° *and* 2° *are true also for* $n=+\infty$.

In this paper we will prove a more general result than the statement 1° in Theorem A. Namely, under certain conditions for function f and sequences $a=(a_1, \ldots, a_n)$, $b=(b_1, \ldots, b_n)$, $p=(p_1, \ldots, p_n)$ we will find a bound for

$$\left|f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right)-f^{-1}\left(\sum_{k=1}^{n} p_k f(b_k)\right)\right|.$$

First, we give the following definition.

Definition. *A convex function* $f: [0, +\infty] \to [0, +\infty]$ *belongs to the class* M *if* $f(0)=0$ *and if the function* F*, defined by* $F(t)=\log f(e^t)$*, is convex for all real* t*.*

The following auxiliary result generalizes a result of H. P. MULHOLLAND (see [4]):

Theorem 1. *Let* $f\in M$ *and let the sequences of nonnegative numbers* $a=(a_1, \dots, a_n)$, $b=(b_1, \dots, b_n)$, $p=(p_1, \dots, p_n)$, $p_n \geqq 1$ $(k=1, \dots, n)$ *be given; then the inequality*

$$f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k+b_k)\right) \leqq f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n} p_k f(b_k)\right) \tag{3}$$

is valid.

Proof. Similarly as in proof of Theorem 1 from [4], now we shall introduce the function g and the sequence $s=(s_1, \dots, s_n)$, by means of $f(x)=xg(x)$ and $s_k=a_k+b_k$ $(k=1, \dots, n)$, respectively. Put

$$G_f(a, p)=f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right). \tag{4}$$

In order to prove the inequality (3) it is enough to prove that the inequality

$$\frac{\sum_{k=1}^{n} p_k x_k g(s_k)}{G_f(x, p)} \leqq \frac{\sum_{k=1}^{n} p_k s_k g(s_k)}{G_f(s, p)} \tag{5}$$

holds for all sequences $x=(x_1, \dots, x_n)$ for which $0 \leqq x_k \leqq s_k$ $(k=1, \dots, n)$.

If we assume that the left hand side of the inequality (5) is a functional $U(x)$, then (5) can be represented in the form

$$U(x) \leqq U(s). \tag{6}$$

The last inequality, in case $n=2$, can be easily proved by the same method which is applied in proof of Theorem 1 in the paper [4].

Combining the inequalities, which are obtained from (6) for $x=a$ and $x=b$, we prove that the inequality (3) holds for $n=2$.

Proof of the inequality (3) for arbitrary n, we give by mathematical induction. Suppose that (3) holds for any $n>2$. Then

$$f^{-1}\left(\sum_{k=1}^{n+1} p_k f(a_k+b_k)\right) = f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k+b_k) + p_{n+1} f(a_{n+1}+b_{n+1})\right)$$

$$\leqq f^{-1}\left(1\cdot f\left(f^{-1}\left(\sum_{k=1}^{n} p_k f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n} p_k f(b_k)\right)\right) + p_{n+1} f(a_{n+1}+b_{n+1})\right)$$

$$\leqq f^{-1}\left(\sum_{k=1}^{n+1} p_k f(a_k)\right) + f^{-1}\left(\sum_{k=1}^{n+1} p_k f(b_k)\right).$$

Thus the proof is finished.

Theorem 2. *If the conditions of Theorem* 1 *are fulfilled, the inequality*

(7) $$| G_f(a, p) - G_f(b, p) | \leqq \sum_{k=1}^{n} p_k | a_k - b_k |$$

holds, where G_f *is defined by* (4).

Proof. Since f is increasing function for $x \geqq 0$, then

$$f(a_k) = f((a_k - b_k) + b_k) \leqq f(| a_k - b_k | + b_k).$$

Applying Theorem 1 to the sequences $c = (| a_1 - b_1 |, \ldots, | a_n - b_n |)$ and b and using the property that f^{-1} is a increasing function we obtain

$$G_f(a, p) \leqq G_f(c + b, p) \leqq G_f(c, p) + G_f(b, p),$$

i. e.

(8) $$G_f(a, p) - G_f(b, p) \leqq G_f(c, p).$$

Permuting series a and b, the inequality (8) becomes

(9) $$G_f(b, p) - G_f(a, p) \leqq G_f(c, p).$$

Using the inequalities of P. M. Vasić (see [5], [3, pp. 368—369]) for convex functions we obtain

(10) $$\sum_{k=1}^{n} p_k f(c_k) \leqq f\left(\sum_{k=1}^{n} p_k c_k\right),$$

as $f(0) = 0$. From (8), (9), (10) we obtain

$$| G_f(a, p) - G_f(b, p) | \leqq \sum_{k=1}^{n} p_k | a_k - b_k |,$$

which completes the proof.

Remark 1. Inequality (7) holds if the function f is defined by

$$f(x) = x \exp\{H(\log x)\},$$

where $t \mapsto H(t)$ is a nondecreasing convex function in $(-\infty, +\infty)$.

For instance, the inequality (7) holds if

$$f(x) = x^p e^{qx} \qquad (p \geqq 1, \ q \geqq 0).$$

For $q = 0$, the inequality (7) can be reduced to

(11) $$\left| \left(\sum_{k=1}^{n} p_k a_k^p\right)^{\frac{1}{p}} - \left(\sum_{k=1}^{n} p_k b_k^p\right)^{\frac{1}{p}} \right| \leqq \sum_{k=1}^{n} p_k | a_k - b_k |.$$

If $p_k = 1$ $(k = 1, \ldots, n)$, from the inequality (11) follows

(12) $$\left| \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} - \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \right| \leqq \sum_{k=1}^{n} | a_k - b_k |.$$

wherefrom (2) is easily obtainable.

REMARK 2. If in (12) a_k and b_k are substituted with $p_k^{1/p}a_k$ and $p_k^{1/p}b_k$ respectively ($p_k \geqq 0$) we obtain

$$\left| \left(\sum_{k=1}^{n} p_k a_k^p \right)^{\frac{1}{p}} - \left(\sum_{k=1}^{n} p_k b_k^p \right)^{\frac{1}{p}} \right| \leqq \sum_{k=1}^{n} p_k^{\frac{1}{p}} | a_k - b_k |. \tag{13}$$

It is easy to show that for $p_k \geqq 1$ $(k = 1, \ldots, n)$ inequalities (11) and (13) are equivalent.

REFERENCES

1. D. S. MITRINOVIĆ: *Problem 109.* Mat. Vesnik **4** (19) (1967), 453.
2. D. D. ADAMOVIĆ: *Rešenje problema 109.* Mat. Vesnik **5** (20) (1968), 548—549.
3. D. S. MITRINOVIĆ (in cooperation with P. M. VASIĆ): *Analytic inequalities.* Berlin--Haidelberg-New York, 1970.
4. H. P. MULHOLLAND: *On generalizations of Minkowski's inequality in the form of a triangle inequality.* Proc. London. Math. Soc. (2) **51** (1950), 294—307.
5. P. M. VASIĆ: *O jednoj nejednakosti M. Petrovića za konveksne funkcije.* Matematička biblioteka, № 38, Beograd 1968, str. 101—104.

Elektronski fakultet
18 000 Niš, Yugoslavia