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691. ON SOME MODIFICATIONS OF A THIRD ORDER METHOD FOR SOLVING EQUATIONS*

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Some iterative formulas for determination of a simple zero of a function are considered in this paper. These formulas have at most the first derivative f' of function f and their order of convergence is higher than two.

Let us mention some iterative formulas in which the highest derivate is eliminated.

Let $x=\xi$ be a simple zero of a differentiable function f, and assume that x_n is the approximate value of ξ which is sufficiently close to ξ. Substituting

$$f'(x_n) \simeq \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$$

in NEWTON-RAPHSON's formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \qquad (n=0,\ 1,\ldots), \tag{1}$$

we get the method of chords. Further, using (1) and approximation

$$f'(x_n) \simeq \frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)},$$

STEFFENSEN [1] defined the second order iterative method

$$x_{n+1}=x_n-\frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)} \qquad (n=0,\ 1,\ldots).$$

Assume that x_0 is sufficiently close to $x=\xi$. It is well known that the iterative process

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)^2 f''(x_n)}{2f'(x_n)^3} \qquad (n=0,\ 1,\ldots) \tag{2}$$

produces a sequence which converges cubically to zero $x=\xi$ of function f (see, e.g. [2]).

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Using HERMITE's interpolation formula for function f defined for $x=x_{n-1}$ and $x=x_n$, from (2) follows the iterative formula (see [2])

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{1}{2}\cdot\frac{f(x_n)^2}{f'(x_n)}\cdot\bar{f}''(x_n) \qquad (n=1,\ 2,\ldots), \tag{3}$$

where $\bar{f}''(x_n)=-\frac{6}{\varepsilon_n^2}[f(x_n)-f(x_{n-1})]+\frac{2}{\varepsilon_n}[2f'(x_n)+f'(x_{n-1})]$ and $\varepsilon_n=x_n-x_{n-1}$.

The iterative process formed in this way has the order of convergence $1+\sqrt{3}$.

We shall now consider some iterative formulas based on (2) and which also contain only the first derivative. The order of convergence of the iterative methods defined by these formulas is higher than two.

Substituting $f''(x_n)\simeq\frac{f'(x_n+\varepsilon_n)-f'(x_n)}{\varepsilon_n}$ in (2), where $\varepsilon_n\to 0$ as $n\to+\infty$, we obtain

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)^2}{2f'(x_n)^3}\cdot\frac{f'(x_n+\varepsilon_n)-f'(x_n)}{\varepsilon_n} \qquad (n=0,\ 1,\ldots). \tag{4}$$

Suppose that a simple root $x=\xi$ of the equation $f(x)=0$ is isolated in interval $[\alpha,\beta]$. Let $x_0\in[\alpha,\beta]$ be initial approximation for ξ, chosen so that the sequence (x_n), defined by (4), converges to ξ as $n\to+\infty$.

Theorem 1. *Let $f\in C_3[\alpha,\beta]$ and let be $\varepsilon_n=x_{n-1}-x_n$. Then the iterative process (4) has the order of convergence $r=1+\sqrt{2}$, i.e.*

$$|x_{n+1}-\xi|\sim A\,|x_n-\xi|^{1+\sqrt{2}} \quad (n\to+\infty),\ \textit{where}\quad A=\left|\frac{f'''(\xi)}{4f'(\xi)}\right|^{1+\sqrt{2}}.$$

Proof. For $\varepsilon_n=x_{n-1}-x_n$, the formula (4) becomes

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)^2}{2f'(x_n)^3}\cdot\frac{f'(x_n)-f'(x_{n-1})}{x_n-x_{n-1}} \qquad (n=1,\ 2,\ldots). \tag{5}$$

Introduce notation

$$h_n=-\frac{f(x_n)}{f'(x_n)},\qquad s_n=\frac{f''(x_n)}{f'(x_n)},\qquad r_n=\frac{f'''(x_n)}{f'(x_n)}.$$

If x_n is sufficiently close to root $x=\xi$, we can write SCHRÖDER's development (see [3])

$$\xi-x_n=h_n-\frac{h_n^2 s_n}{2}+\frac{h_n^3}{6}(3s_n^2-r_n)+O(h_n^4). \tag{6}$$

On the other hand, applying TAYLOR's formula, we have

$$f'(x_{n-1})=f'(x_n)+\varepsilon_n f''(x_n)+\frac{\varepsilon_n^2}{2}f'''(\zeta_n),\qquad \zeta_n=x_n+\theta\varepsilon_n \quad (0<\theta<1). \tag{7}$$

Since $\varepsilon_n = x_{n-1} - x_n = \frac{f(x_{n-1})}{f'(x_{n-1})} + O(h_{n-1}^2) = -h_{n-1} + O(h_{n-1}^2)$, from (5), (6) and (7), we obtain

$$x_{n+1} - \xi = \frac{h_n^2}{2f'(x_n)} \left[f''(x_n) + \frac{f'(x_n) - f'(x_{n-1})}{x_{n-1} - x_n} + O(h_n) \right]$$
$$= \frac{h_n^2}{4} \left[\frac{\varepsilon_n f'''(\zeta_n)}{f'(x_n)} + O(h_n) \right]$$
$$= \frac{h_n^2}{4} \left\{ \frac{f'''(\zeta_n)}{f'(x_n)} [-h_{n-1} + O(h_{n-1}^2)] + O(h_n) \right\}.$$

Introduce the errors

$$e_{n+1} = x_{n+1} - \xi, \quad e_n = x_n - \xi = -h_n + O(h_n^2), \quad e_{n-1} = x_{n-1} - \xi = -h_{n-1} + O(h_{n-1}^2).$$

Hence

$$h_n = -e_n + O(e_n^2), \qquad h_{n-1} = -e_{n-1} + O(e_{n-1}^2).$$

We have now

$$e_{n+1} = \frac{f'''(\zeta_n)}{4f'(x_n)} [e_n + O(e_n^2)]^2 [e_{n-1} + O(e_{n-1}^2) + O(e_n)]$$
$$= \frac{f'''(\zeta_n)}{4f'(x_n)} [e_n^2 e_{n-1} + O(e_n^2 e_{n-1}^2)] = \frac{f'''(\zeta_n)}{4f'(x_n)} e_n^2 e_{n-1} [1 + O(e_{n-1})].$$

Thus, x_n tends to ξ as $n \to +\infty$ and

(8) $$|e_{n+1}| \sim \left| \frac{f'''(\xi)}{4f'(\xi)} \right| |e_n|^2 |e_{n-1}|.$$

Assume that the order of the iterative process (5) is r, i.e.

(9) $$|e_{n+1}| \sim A |e_n|^r \quad (A > 0).$$

From (8) and (9) it follows

$$\left| \frac{f'''(\xi)}{4f'(\xi)} \right| |e_n|^2 |e_{n-1}| \sim A |e_n|^r,$$

hence

(10) $$|e_n| \sim \left| \frac{1}{A} \cdot \frac{f'''(\xi)}{4f'(\xi)} \right|^{\frac{1}{r-2}} |e_{n-1}|^{\frac{1}{r-2}}.$$

Comparing (9) and (10) we obtain $r = \frac{1}{r-2}$, $A = \left| \frac{f'''(\xi)}{4f'(\xi)} \right|^{\frac{1}{r-2}}$.

From equation $r^2 - 2r - 1 = 0$ we find that the order of convergence is $r = 1 + \sqrt{2}$. Now, the asymptotic constant A is given by $A = \left| \frac{f'''(\xi)}{4f'(\xi)} \right|^{1+\sqrt{2}}$. □

The order of convergence of the iterative process (3) is higher than the order of the method represented above. On the other hand, the iterative formula (5) is simpler and requires less numerical operations then (3).

Theorem 2. *Let* $f \in C_3[\alpha, \beta]$ *and let* $\varepsilon_n = f(x_n)$. *Then, the iterative process* (4) *has the order of convergence three, i.e.*

$$|x_{n+1} - \xi| \sim B\,|x_n - \xi|^3 \qquad (n \to +\infty),$$

where

$$B = \frac{1}{12 f'(\xi)^2}\,|3 f'(\xi)^2 f'''(\xi) + 2 f'(\xi) f'''(\xi) - 6 f''(\xi)^2|.$$

Proof. For $\varepsilon_n = f(x_n)$ the formula (4) becomes

$$(11) \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f'(x_n + f(x_n)) - f'(x_n)}{2 f'(x_n)^2}\right] \qquad (n = 0,\ 1, \ldots).$$

Since $f'(x_n + \varepsilon_n) = f'(x_n) + \varepsilon_n f''(x_n) + \frac{1}{2}\varepsilon_n^2 f'''(\zeta_n)$, where $\zeta_n = x_n + \varepsilon_n \theta$ $(0 < \theta < 1)$ and $\varepsilon_n = f(x_n)$, the formula (11) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2 f''(x_n)}{2 f'(x_n)^3} - \frac{f(x_n)^3 f'''(\zeta_n)}{4 f'(x_n)^3}.$$

Using (6) we obtain

$$(12) \qquad x_{n+1} - \xi = h_n^3\left(\frac{f'''(\zeta_n)}{4} - \frac{s_n^2}{2} + \frac{r_n}{6}\right) + O(h_n^4).$$

Since $h_n = \xi - x_n + O((\xi - x_n)^2)$, from (12) it follows

$$|x_{n+1} - \xi| \sim \frac{1}{12}\left|\frac{3 f'(\xi)^2 f'''(\xi) + 2 f'(\xi) f'''(\xi) - 6 f''(\xi)^2}{f'(\xi)^2}\right| |x_n - \xi|^3 \quad (n \to +\infty). \ \square$$

In the following examples we shall use iterative formulas (5) and (11).

Example 1. Applying the iterative formula (5) we shall determine a simple zero of polynomial

$$P(x) = x^6 - 4x^5 + x^4 + 5x^3 + 4x^2 - x - 6,$$

starting from approximate value $x_0 = 1.8$. Since iterative process (5) requires two initial approximations, using Newton's formula, we find

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = 2.088633519.$$

The result of application of iterative formula (5) is given in Table 1.

n	x_{n-1}	x_n	$\varepsilon_n = x_{n-1} - x_n$	x_{n+1}
1	1.8	2.088633519	−0.288633519	1.999758772
2	2.088633519	1.999758772	0.088874747	2.000000006

Table 1

Remark 1. The real roots of equation $P(x) = 0$ are −1, 1, 2 and 3.

EXAMPLE 2. By the graphical method it is found that $x_0 = 0.7$ is the approximate value of the smallest root (among infinite number of roots) of transcendental equation

$$f(x) \equiv e^{-x} - 2 \sin x + 1 = 0.$$

In order to get a better approximation of this root, we shall apply iterative formula (11). The results are given in Table 2. As the criterion for stopping iterative process, it is taken the absolute value $|f(x_n)|$.

n	x_n	$\varepsilon_n = f(x_n)$
0	0.7	0.2081499293
1	0.8076369413	0.0005988781
2	0.8079645521	$1.49 \cdot 10^{-10}$

Table 2

REMARK 2. The correct value of the smallest root to 11 decimal places is $\xi = 0.80796455218$.

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O NEKIM MODIFIKACIJAMA JEDNOG METODA TREĆEG REDA ZA REŠAVANJE JEDNAČINA

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U radu se razmatraju iterativne formule oblika

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2}{2f'(x_n)^3} \cdot \frac{f'(x_n + \varepsilon_n) - f'(x_n)}{\varepsilon_n} \qquad (n = 0, 1, \ldots).$$

za određivanje prostog korena jednačine $f(x) = 0$, uzimajući: 1° $\varepsilon_n = x_{n-1} - x_n$; 2° $\varepsilon_n = f(x_n)$. Red konvergencije ovih formula je $1 + \sqrt{2}$ i 3 respektivno. Primena formula ilustrovana je na primerima.