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CONSTRUCTION OF GAUSSIAN QUADRATURE FORMULAS FOR EVEN WEIGHT FUNCTIONS

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Instead of a quadrature rule of Gaussian type with respect to an even weight function on $(-a, a)$ with n nodes, we construct the corresponding Gaussian formula on $(0, a^2)$ with only $[(n+1)/2]$ nodes. Especially, such a procedure is important in the cases of nonclassical weight functions, when the elements of the corresponding three-diagonal Jacobi matrix must be constructed numerically. In this manner, the influence of numerical instabilities in the process of construction can be significantly reduced, because the dimension of the Jacobi matrix is halved. We apply this approach to Pollaczek's type weight functions on $(-1, 1)$, to the weight functions on $\mathbb{R}$ which appear in the Abel-Plana summation processes, as well as to a class of weight functions with four free parameters, which covers the generalized ultraspherical and Hermite weights. Some numerical examples are also included.

1. INTRODUCTION

Let $\mathcal{P}$ be the set of all algebraic polynomials and $\mathcal{P}_n$ be its subset of degree at most n. In this paper, we consider the Gauss-Christoffel quadrature rules with respect to the even weight function $x \mapsto w(x) = w(-x)$ on a symmetric interval $(-a, a)$ for $a > 0$,

$$\int_{-a}^{a} f(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{n} w_k f(x_k) + R_n(f; w), \tag{1}$$

where $R_n(f; w) = 0$ for each $f \in \mathcal{P}_{2n-1}$ and they are automatically exact for all odd functions.

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Suppose that the moments $\mu_k = \int_{-a}^{a} x^k w(x)\,\mathrm{d}x$, exist and are finite for any $k = 0, 1, \ldots$, and also $\mu_0 = \int_{-a}^{a} w(x)\,\mathrm{d}x > 0$. Then the quadrature rules (1) exist for each $n \in \mathbb{N}$ as well as the corresponding orthogonal polynomials. It is well known that $\mu_{2k+1} = 0$ for any $k = 0, 1, \ldots$, and the monic symmetric polynomials $\pi_k(x)$ orthogonal with respect to the even weight w on $(-a, a)$ satisfy the three-term recurrence relation (cf. [**12**, p. 102])

$$\pi_{k+1}(x) = x\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, \ldots, \tag{2}$$

with $\pi_{-1}(x) = 0$, $\pi_0(x) = 1$ and $\pi_1(x) = x$.

The recurrence coefficients β_k in (2) can be computed from the moments in terms of Hankel determinants

$$\Delta_k = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{k-1} \\ \mu_1 & \mu_2 & & \mu_k \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-2} \end{vmatrix},$$

by

$$\beta_k = \frac{\Delta_{k-1}\Delta_{k+1}}{\Delta_k^2} \quad (k \geq 1) \qquad \text{with} \quad \Delta_0 = 1.$$

Although β_0 in (2) may be arbitrary, it is sometimes convenient to define it as $\beta_0 = \mu_0 = \int_{-a}^{a} w(x)\,\mathrm{d}x$. By noting the definition

$$(p, q) = \int_{-a}^{a} p(x)q(x)w(x)\,\mathrm{d}x \quad \text{and} \quad \|p\| = \sqrt{(p, p)},$$

one can prove that the norm of π_n equals to

$$\|\pi_n\| = \sqrt{\beta_0\beta_1\cdots\beta_n} = \sqrt{\frac{\Delta_{n+1}}{\Delta_n}}.$$

For instance, the first few monic symmetric polynomials π_k in terms of moments are as follows

$$\begin{aligned}
\pi_2(x) &= x^2 - \frac{\mu_2}{\mu_0}, \\
\pi_3(x) &= x^3 - \frac{\mu_4}{\mu_2}\,x, \\
\pi_4(x) &= x^4 - \frac{\mu_6\mu_0 - \mu_4\mu_2}{\mu_4\mu_0 - \mu_2^2}\,x^2 + \frac{\mu_6\mu_2 - \mu_4^2}{\mu_4\mu_0 - \mu_2^2}\,, \\
\pi_5(x) &= x^5 - \frac{\mu_8\mu_2 - \mu_6\mu_4}{\mu_6\mu_2 - \mu_4^2}\,x^3 + \frac{\mu_8\mu_4 - \mu_6^2}{\mu_6\mu_2 - \mu_4^2}\,x\,.
\end{aligned}$$

A standard method for calculating the nodes x_k and the weight coefficients (Christoffel numbers) w_k in the quadrature (1) is based on their characterization

via an eigenvalue problem for the Jacobi matrix of order n associated with the even weight function $x \mapsto w(x)$. Thus, the nodes x_k are the eigenvalues of the symmetric tridiagonal Jacobi matrix (cf. [**12**, pp. 325–328])

$$J_n(w) = \begin{bmatrix} 0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & 0 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & 0 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & 0 \end{bmatrix}, \tag{3}$$

and the weight coefficients w_k are given by $w_k = \beta_0 v_{k,1}^2$ $(k = 1, \ldots, n)$, where $v_{k,1}$ is the first component of the eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,n}]^{\mathrm{T}})$ corresponding to the eigenvalue x_k, normalized such that $\mathbf{v}_k^{\mathrm{T}} \mathbf{v}_k = 1$. This popular method is called the Golub-Welsch procedure [**8**].

Unfortunately, for many weight functions the coefficients β_k in (2) are not explicitly known. In such cases, the corresponding polynomials π_k are known as *strong non–classical* orthogonal polynomials, and their recursion coefficients must be constructed numerically from the moment information. Such problems are very sensitive with respect to small perturbations in the input data. Fortunately, in the eighties of the last century, Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, with effective algorithms for numerically generating the first n recursion coefficients (the method of (modified) moments, the discretized Stieltjes–Gautschi procedure, and the Lanczos algorithm), which allow us to compute all orthogonal polynomials of degree $\leq n$ by a straightforward application of the three-term recurrence relation. A detailed stability analysis of these algorithms as well as several new applications of orthogonal polynomials are also included in the previously mentioned theory. The basic references are [**6, 7, 15**].

Because of $w(-x) = w(x)$ on $(-a, a)$, the nodes in the quadrature sum

$$Q_n(f; w) := \sum_{k=1}^{n} w_k f(x_k)$$

in (1) are symmetrically distributed with respect to the origin, and their weight coefficients are mutually equal for symmetric nodes. Taking only positive nodes, denoted by $x_k^{(n)}$ and the corresponding weight coefficients by $A_k^{(n)}$ for $k = 1, \ldots, m$ $(= [n/2])$, the quadrature sum can be expressed as

$$Q_n(f; w) := \begin{cases} \displaystyle\sum_{k=1}^{m} A_k^{(n)} \big(f(x_k^{(n)}) + f(-x_k^{(n)})\big), & n = 2m, \\ \displaystyle A_0^{(n)} f(0) + \sum_{k=1}^{m} A_k^{(n)} \big(f(x_k^{(n)}) + f(-x_k^{(n)})\big), & n = 2m+1, \end{cases} \tag{4}$$

where, in the case of odd n, $A_0^{(n)}$ (>0) is the weight coefficient for the node 0. Here,

$$0 < x_1^{(n)} < \cdots < x_m^{(n)} < a \quad \text{and} \quad A_k^{(n)} > 0, \quad k = 1, \ldots, m.$$

This paper is organized as follows. In Section 2, we shortly describe a simple transformation from $(-a, a)$ to $(0, a^2)$ and give recurrence coefficients for the corresponding orthogonal polynomials. Section 3 is devoted to the construction of two quadratures on $(0, a^2)$ and their connection with symmetric Gaussian quadratures on $(-a, a)$. These sections are introductory and record material that is essentially known (cf. [**12**], [**13**]), but needed in subsequent sections. The numerical construction of Gaussian rules related to the Pollaczek-type weight functions on $(-1, 1)$ is presented in Section 4, together with some numerical examples. Symmetric Gaussian quadrature rules on $\mathbb{R}$, which appear in the Abel-Plana summation formulas, are considered in Section 5. Finally, a class of symmetric weight functions with four free parameters that covers many well-known weights on $(-1, 1)$ and $\mathbb{R}$ are considered in Section 6.

2. TRANSFORMATION AND PRESERVATION OF ORTHOGONALITY

Suppose in (1) that $x \mapsto f(x)$ is an even function, so that

$$\int_{-a}^{a} f(x)w(x)\,\mathrm{d}x = 2\int_0^a f(x)w(x)\,\mathrm{d}x = \int_0^{a^2} f(\sqrt{t})\,\frac{w(\sqrt{t})}{\sqrt{t}}\,\mathrm{d}t. \tag{5}$$

On the other hand, according to (1), (4) and (5) we have

$$I(\varphi_1; w_1) = \int_0^{a^2} f(\sqrt{t})\,\frac{w(\sqrt{t})}{\sqrt{t}}\,\mathrm{d}t = Q_n(f; w) + R_n(f; w), \tag{6}$$

where two new functions are defined on $(0, a^2)$ as

$$w_1(t) := \frac{w(\sqrt{t})}{\sqrt{t}} \quad \text{and} \quad \varphi_1(t) := f(\sqrt{t}). \tag{7}$$

Similarly, we need to define

$$w_2(t) := \sqrt{t}\,w(\sqrt{t}) \quad \text{and} \quad \varphi_2(t) := \frac{f(\sqrt{t}) - f(0)}{t}. \tag{8}$$

The orthogonal polynomials with respect to the weight functions $w_1(t)$ and $w_2(t)$ defined on $(0, a^2)$ can be directly expressed in terms of the polynomials $\pi_k(x)$ which are orthogonal with respect to the symmetric weight w on $(-a, a)$. In fact, according to Theorem 2.2.11 of [**12**, p. 102] we have:

(i) $p_\nu(t) := \pi_{2\nu}(\sqrt{t})$ are orthogonal with respect to the weight function $w_1(t) = w(\sqrt{t})/\sqrt{t}$ on $(0, a^2)$, and

(ii) $q_n(t) := \pi_{2n+1}(\sqrt{t})/\sqrt{t}$ are orthogonal with respect to the weight function $w_2(t) = \sqrt{t}\, w(\sqrt{t})$ on $(0, a^2)$.

Also, these (monic) polynomials satisfy the three-term recurrence relations (Theorem 2.2.12 in [**12**, p. 102]),

$$p_{\nu+1}(t) = (t - a_\nu)p_\nu(t) - b_\nu p_{\nu-1}(t), \quad \nu = 0, 1, \dots, \tag{9}$$

and

$$q_{\nu+1}(t) = (t - c_\nu)q_\nu(t) - d_\nu q_{\nu-1}(t), \quad \nu = 0, 1, \dots, \tag{10}$$

with $p_0(t) = 1$, $p_{-1}(t) = 0$ and $q_0(t) = 1$, $q_{-1}(t) = 0$, respectively, where the coefficients in (9) and (10) are given by

$$a_0 = \beta_1, \quad a_\nu = \beta_{2\nu} + \beta_{2\nu+1}, \quad b_\nu = \beta_{2\nu-1}\beta_{2\nu},$$

and

$$c_0 = \beta_1 + \beta_2, \quad c_\nu = \beta_{2\nu+1} + \beta_{2\nu+2}, \quad d_\nu = \beta_{2\nu}\beta_{2\nu+1},$$

in which β_k are the same values as in (2). In addition, we can define

$$b_0 := \int_0^{a^2} w_1(t)\,\mathrm{d}t = \int_{-a}^{a} w(x)\,\mathrm{d}x = \mu_0$$

and

$$d_0 := \int_0^{a^2} w_2(t)\,\mathrm{d}t = \int_0^{a^2} t\, w_1(t)\,\mathrm{d}t = \int_{-a}^{a} x^2 w(x)\,\mathrm{d}x = \mu_2,$$

i.e., $b_0 := \beta_0$ and $d_0 := \beta_0\beta_1$.

In the case of strong nonclassical weights, the coefficients a_ν and b_ν in (9), as well as c_ν and d_ν in (10), must be constructed numerically (cf. [**6**], [**12**, pp. 160–166]).

The orthogonal polynomials $p_\nu(t)$ and their recurrence relation (9) are applied in constructing Gaussian quadratures with respect to the weight function $w_1(t) = w(\sqrt{t})/\sqrt{t}$ on $(0, a^2)$, while the polynomials $q_\nu(t)$ and their recurrence relation (10) are appropriate for constructing Gauss-Radau rules (cf. [**12**, p. 329]).

By noting these comments and (4), the construction of quadratures (1) will be significantly simplified. Namely, instead of constructing a quadrature formula on $(-a, a)$ with n nodes, we construct a quadrature formula on $(0, a^2)$ with only $[(n + 1)/2]$ nodes. In particular, it is very important in the cases of nonclassical weight functions, when the recurrence coefficients in the three-term relations for the corresponding orthogonal polynomials must be constructed numerically, before the procedure for constructing nodes and Christoffel numbers (by the Golub-Welsch procedure from the Jacobi matrices). In this manner, the influence of numerical instabilities in the process of construction can be significantly reduced. Also, in this way, the dimensions of the corresponding Jacobi matrices are halved.

3. CONSTRUCTION OF TWO RULES OF GAUSSIAN TYPE

We consider now two quadrature formulas for computing the integral $I(\varphi_1; w_1)$ given in (6).

3.1. Gauss-Christoffel quadrature formula with the weight $\mathbf{w_1(t)}$

The first formula is a m-point Gauss-Christoffel quadrature formula with respect to the weight function $t \mapsto w_1(t) = w(\sqrt{t})/\sqrt{t}$ on $(0, a^2)$,

$$I(g; w_1) = \int_0^{a^2} g(t)\, w_1(t)\, \mathrm{d}t = \sum_{k=1}^{m} B_k^{(m)} g(\tau_k^{(m)}) + R_m^{\mathrm{GC}}(g; w_1), \tag{11}$$

with the nodes $0 < \tau_1^{(m)} < \cdots < \tau_m^{(m)} < a^2$ and the corresponding weight coefficients $B_k^{(m)}$ $(k = 1, \ldots, m)$. The remainder term $R_m^{\mathrm{GC}}(g; w_1) = 0$ for each $g \in \mathcal{P}_{2m-1}$.

Proposition 3.1. *The nodes $\tau_k^{(m)}$ $(k = 1, \ldots, m)$ in the formula* (11)*, that is, the zeros of the polynomial $p_m(t)$ in* (9)*, are the eigenvalues of the Jacobi matrix*

$$J_m(w_1) = \begin{bmatrix} \beta_1 & \sqrt{\beta_1\beta_2} & & & \mathbf{O} \\ \sqrt{\beta_1\beta_2} & \beta_2 + \beta_3 & \sqrt{\beta_3\beta_4} & & \\ & \sqrt{\beta_3\beta_4} & \beta_4 + \beta_5 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{2m-3}\beta_{2m-2}} \\ \mathbf{O} & & & \sqrt{\beta_{2m-3}\beta_{2m-2}} & \beta_{2m-2} + \beta_{2m-1} \end{bmatrix},$$

where β_k are the same values as in (2)*. Also, the weight coefficients $B_k^{(m)}$ are given by $B_k^{(m)} = \beta_0 v_{k,1}^2$, where $v_{k,1}$ is the first component of the eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,m}]^{\mathrm{T}})$ corresponding to the eigenvalue $\tau_k^{(m)}$ and normalized such that $\mathbf{v}_k^{\mathrm{T}}\mathbf{v}_k = 1$.*

3.2. Gauss-Radau quadrature formula with the weight $\mathbf{w_1(t)}$

The $(m+1)$-point Gauss-Radau quadrature formula with respect to the same weight function $w_1(t)$ as before and the new nodes $0 = \theta_0^{(m)} < \theta_1^{(m)} < \cdots < \theta_m^{(m)} < a^2$ and weight coefficients $C_k^{(m)}$ are given by

$$I(g; w_1) = \int_0^{a^2} g(t)\, w_1(t)\, \mathrm{d}t = C_0^{(m)} g(0) + \sum_{k=1}^{m} C_k^{(m)} g(\theta_k^{(m)}) + R_{m+1}^{\mathrm{GR}}(g; w_1). \tag{12}$$

It is clear that $R_{m+1}^{\mathrm{GR}}(g; w_1) = 0$ for each $g \in \mathcal{P}_{2m}$.

In order to construct the formula (12), we need to introduce a function h, $g(t) = g(0) + th(t)$, to get

$$I(g; w_1) = g(0) \int_0^{a^2} w_1(t)\,\mathrm{d}t + \int_0^{a^2} h(t)\,t\,w_1(t)\,\mathrm{d}t.$$

This means that

$$I(g; w_1) = \beta_0 g(0) + \int_0^{a^2} h(t) w_2(t)\,\mathrm{d}t = \beta_0 g(0) + I(h; w_2), \tag{13}$$

because $w_2(t) = tw_1(t)$ according to (7) and (8). To compute the integral $I(h; w_2)$ we can directly construct the Gauss-Christoffel rule with respect to the second weight function $w_2(t) = \sqrt{t}\,w(\sqrt{t})$ on $(0, a^2)$ as

$$I(h; w_2) = \int_0^{a^2} h(t)\,w_2(t)\,\mathrm{d}t = \sum_{k=1}^{m} D_k^{(m)} h(\theta_k^{(m)}) + R_m^{\mathrm{GC}}(h; w_2), \tag{14}$$

where $0 < \theta_1^{(m)} < \cdots < \theta_m^{(m)} < a^2$ and $D_k^{(m)}$ are the corresponding weight coefficients. Note in (14) that the remainder term $R_m^{\mathrm{GC}}(h; w_2) = 0$ for each $h \in \mathcal{P}_{2m-1}$. Thus, by noting (13) and (14) we first get

$$\begin{aligned} I(g; w_1) &= \beta_0 g(0) + \sum_{k=1}^{m} D_k^{(m)} h(\theta_k^{(m)}) + R_m^{\mathrm{GC}}(h; w_2) \\ &= \beta_0 g(0) + \sum_{k=1}^{m} D_k^{(m)} \frac{g(\theta_k^{(m)}) - g(0)}{\theta_k^{(m)}} + R_m^{\mathrm{GC}}(h; w_2) \\ &= \left(\beta_0 - \sum_{k=1}^{m} \frac{D_k^{(m)}}{\theta_k^{(m)}}\right) g(0) + \sum_{k=1}^{m} \frac{D_k^{(m)}}{\theta_k^{(m)}}\, g(\theta_k^{(m)}) + R_m^{\mathrm{GC}}(h; w_2), \end{aligned}$$

and then comparing this with (12), the weight coefficients of the Gauss-Radau quadrature (12) are

$$C_0^{(m)} = \beta_0 - \sum_{k=1}^{m} \frac{D_k^{(m)}}{\theta_k^{(m)}}, \quad C_k^{(m)} = \frac{D_k^{(m)}}{\theta_k^{(m)}} \quad (k = 1, \ldots, m), \tag{15}$$

and $R_{m+1}^{\mathrm{GR}}(g; w_1) = R_m^{\mathrm{GC}}(h; w_2)$ for $h(t) = (g(t) - g(0))/t$. This means that the nodes of the Gauss-Radau quadrature rule with respect to the weight function $w_1(t)$ are in fact the nodes of the Gauss-Christoffel formula with respect to the weight function $w_2(t) = t\,w_1(t)$ on $(0, a^2)$.

Proposition 3.2. *The nodes* $\theta_k^{(m)}$ $(k = 1, \ldots, m)$ *in the formula* (12)*, that is, the*

zeros of the polynomial $q_m(t)$ *in* (10)*, are the eigenvalues of the Jacobi matrix*

$$J_m(w_2) = \begin{bmatrix} \beta_1+\beta_2 & \sqrt{\beta_2\beta_3} & & & \mathbf{O} \\ \sqrt{\beta_2\beta_3} & \beta_3+\beta_4 & \sqrt{\beta_4\beta_5} & & \\ & \sqrt{\beta_4\beta_5} & \beta_5+\beta_6 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{2m-2}\beta_{2m-1}} \\ \mathbf{O} & & & \sqrt{\beta_{2m-2}\beta_{2m-1}} & \beta_{2m-1}+\beta_{2m} \end{bmatrix}.$$

where β_k *are the same values as in* (2) *and the weight coefficients* $C_k^{(m)}$ *are given by* (15)*, where* $D_k^{(m)}$ *is determined by the first component* $v_{k,1}$ *of the normalized eigenvector* $\mathbf{v}_k$ $(= [v_{k,1} \ \dots \ v_{k,m}]^{\mathrm{T}})$ *of the Jacobi matrix* $J_m(w_2)$ *corresponding to the eigenvalue* $\theta_k^{(m)}$*, i.e.,* $D_k^{(m)} = \beta_0\beta_1 v_{k,1}^2$*,* $k = 1, \dots, m$*.*

Remark 3.3. The quadratures (11) and (12) can be related to the basic quadrature (1), which allows a much simpler construction of these symmetric quadratures given in form

$$\int_{-a}^{a} f(x)w(x)\,\mathrm{d}x = Q_n(f;w) + R_n(f;w), \tag{16}$$

where $Q_n(f;w)$ is defined by (4). If we have the recursion coefficients β_k in the explicit form, in our construction we use Proposition 3.1 for even n and Proposition 3.2 for odd n. However, in the case of strong nonclassical weights, we first numerically construct the recursion coefficients a_ν and b_ν in (9), and c_ν and d_ν in (10), and then the Jacobi matrices $J_m(w_1)$ and $J_m(w_2)$ are given by

$$J_m(w_1) = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \mathbf{O} \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \sqrt{b_2} & a_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{b_{m-1}} \\ \mathbf{O} & & & \sqrt{b_{m-1}} & a_{m-1} \end{bmatrix}$$

and

$$J_m(w_2) = \begin{bmatrix} c_0 & \sqrt{d_1} & & & \mathbf{O} \\ \sqrt{d_1} & c_1 & \sqrt{d_2} & & \\ & \sqrt{d_2} & c_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{d_{m-1}} \\ \mathbf{O} & & & \sqrt{d_{m-1}} & c_{m-1} \end{bmatrix}.$$

Corollary 3.4. *The positive nodes* $x_k^{(n)}$ *of the symmetric quadrature rule* (16) *are given by* $x_k^{(n)} = \sqrt{\tau_k^{(m)}}$ $(n = 2m)$ *and* $x_k^{(n)} = \sqrt{\theta_k^{(m)}}$ $(n = 2m+1)$, *and the corresponding weight coefficients by*

$$1^\circ \ A_k^{(n)} = \frac{1}{2} B_k^{(m)} \quad (k = 1, \dots, m) \quad \textit{for even} \ \ n = 2m,$$

and

$$2^\circ \ A_0^{(n)} = C_0^{(m)},\ A_k^{(n)} = \frac{1}{2} C_k^{(m)} \quad (k = 1, \dots, m) \quad \textit{for odd} \ \ n = 2m+1,$$

where $\tau_k^{(m)}$ *and* $B_k^{(m)}$ *and* $\theta_k^{(m)}$ *and* $C_k^{(m)}$ *are defined in Propositions* 3.1 *and* 3.2, *respectively.*

Corollary 3.5. *Let* $f : (-a, a) \to \mathbb{R}$ *be an even function and* $\varphi_1 : (0, a^2) \to \mathbb{R}$ *and* $\varphi_2 : (0, a^2) \to \mathbb{R}$ *be defined by* (7) *and* (8) *respectively. The remainder term in* (16) *is given by*

$$R_n(f; w) = \begin{cases} R_m^{\mathrm{GC}}(\varphi_1; w_1) & n = 2m, \\ R_m^{\mathrm{GC}}(\varphi_2; w_2) & n = 2m+1, \end{cases}$$

where $R_m^{\mathrm{GC}}(\,\cdot\,; w_\nu)$ *is the remainder term of the Gauss-Christoffel rule with respect to the weight function* w_ν $(\nu = 1, 2)$ *on* $(0, a^2)$.

Remark 3.4. Some fast variants of the Golub-Welsch algorithm for symmetric weight functions in Matlab have been considered in [**13**], including numerical experiments with Gegenbauer and Hermite weight functions.

4. GAUSSIAN RULES RELATED TO THE POLLACZEK WEIGHT

Recently De Bonis, Mastroianni, and Notarangelo [**5**] have considered Gaussian quadrature rules with respect to the Pollaczek-type weight $w(x; \lambda) = \mathrm{e}^{-(1-x^2)^{-\lambda}}$, $\lambda > 0$, on $(-1, 1)$ in order to evaluate integrals of the form

$$I(f; \lambda) = \int_{-1}^{1} f(x) \mathrm{e}^{-(1-x^2)^{-\lambda}} \, \mathrm{d}x, \tag{17}$$

where f is a Riemann integrable function, in particular, f can increase exponentially at the endpoints ± 1. Also, their rule is useful for approximating integrals of functions that decay exponentially at ± 1 (e.g., when f is bounded or has a slower growth than exponential at the endpoints).

In [**5**], the authors use the first $2n$ moments

$$\mu_k = \int_{-1}^{1} x^k w(x; \lambda) \, \mathrm{d}x, \quad k = 0, 1, \dots, 2n-1,$$

in order to construct the first n recursive coefficients and the corresponding Gaussian quadratures with $\leq n$ nodes, by the package `OrthogonalPolynomials` ([**1**]), which is downloadable from the web site `http://www.mi.sanu.ac.rs/~gvm/`.

Since $w(x;\lambda)$ is even on $(-1,1)$, in our construction, we can use the following weight functions on $(0,1)$

$$w_1(t;\lambda) = \frac{\mathrm{e}^{-(1-t)^{-\lambda}}}{\sqrt{t}} \quad \text{and} \quad w_2(t;\lambda) = \sqrt{t}\,\mathrm{e}^{-(1-t)^{-\lambda}}. \tag{18}$$

The weight functions $x \mapsto w(x;\lambda)$ on $(-1,1)$ and $x \mapsto w_1(x;\lambda)$ on $(0,1)$ for $\lambda = 1/10$, $\lambda = 1/2$ and $\lambda = 10$ are displayed in Fig. 1, left and right, respectively. Note

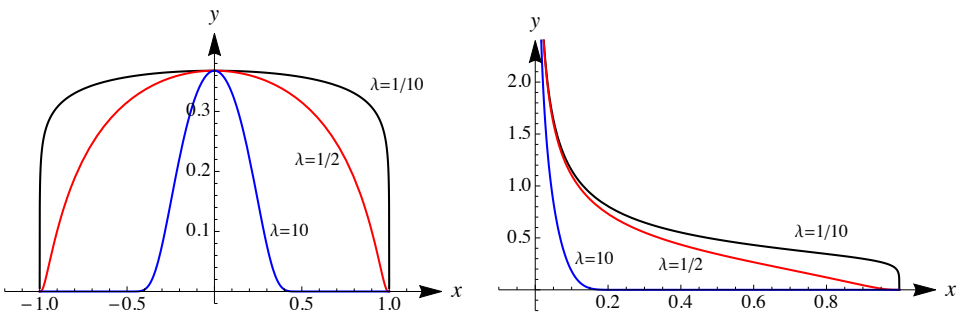

Figure 1: The weight functions $w(x;\lambda) = \mathrm{e}^{-(1-x^2)^{-\lambda}}$ (*left*) and $w_1(x;\lambda)$ (*right*) for three parameters $\lambda = 1/10$, $\lambda = 1/2$ and $\lambda = 10$.

that for a very small value of λ, the weight $x \mapsto w(x;\lambda)$ is very close to a constant value (Legendre weight) in $(-1,1)$ and tends exponentially to zero at the endponts ± 1.

According to results of Section 3, to construct Gaussian quadrature rules with respect to the weight w on $(-1,1)$, for n (or less) nodes, we need the corresponding Gaussian quadrature rules with respect to the weight function w_1 (and w_2) on $(0,1)$, but only for $[n/2]$ nodes. Thus, if we want to construct the quadrature sum $Q_n(f;w)$ for even number of nodes $\le n$ $(= 2m)$, we should first compute the first m coefficients a_ν and b_ν for $\nu = 0,1,\ldots,m-1$ (see Remark 3.3), starting with the first $2m$ moments with respect to the weight function w_1, i.e.,

$$\mu_k^{(1)}(\lambda) = \int_0^1 t^{k-1/2}\mathrm{e}^{-(1-t)^{-\lambda}}\,\mathrm{d}t, \quad k = 0,1,\ldots,2m-1.$$

As an illustration, we take $m = 25$ (i.e., $n = 50$) and $\lambda = 10$. In this case, with the moments $\mu_k^{(1)}(10)$, $k = 0,1,\ldots,49$, calculated with `WorkingPrecision -> 80`, using MATHEMATICA package `OrthogonalPolynomials` (see [**1, 17**]), we get the first 25 recursive coefficients a_ν and b_ν with maximal relative error less than 3.30×10^{-60}.

These coefficients enable us to establish the Gaussian quadrature rules (11) for each $m \le 25$, i.e., the symmetric quadratures (16) on $(-1,1)$ for each even $n = 2m \le 50$, according to Corollary 3.4.

EXAMPLE 4.1. For a given function f, defined on $(-1,1)$ by

$$f(x) = \frac{3\mathrm{e}^{-\frac{1}{\sqrt{1-x^2}}} - 2\sin(3x) - x^2}{(1-x^2)^2}, \tag{19}$$

we consider the integral $I(f;\lambda)$ (with respect to the Pollaczek weight function), given by (17). For $\lambda = 1/2$ and $\lambda = 10$, the corresponding values are

$$I\left(f;\tfrac{1}{2}\right) = -0.100853578447701253704966132370110608871510279078813023527 0\ldots$$

$$I(f;10) = 0.182895219233483199388012214330942401509423267232629315052 76\ldots,$$

obtained in MATHEMATICA with `WorkingPrecision -> 60`. Graphics of the function (19) and the corresponding integrands in (17) are presented in Fig. 2 and Fig. 3, respectively.

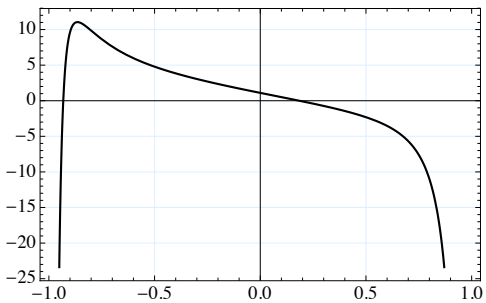

Figure 2: Graphic of the function $x \mapsto f(x)$ given by (19)

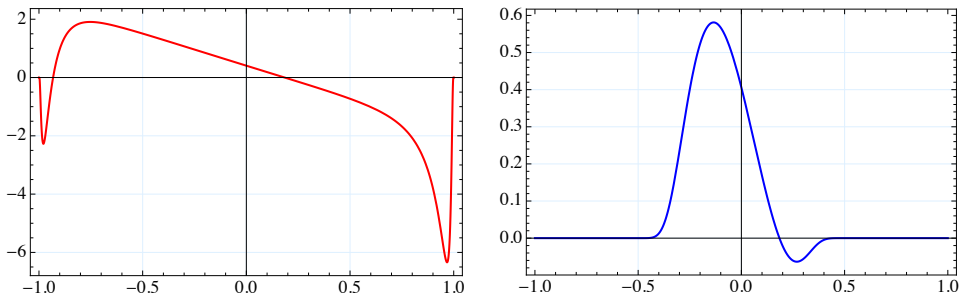

Figure 3: Integrand in $I(f;\lambda)$ for $\lambda = 1/2$ (*left*) and $\lambda = 10$ (*right*)

Now, let us apply Gauss-Pollaczek quadrature rule with $n = 10(10)50$ nodes to the integral $I(f;\lambda)$ and compare the results by ones obtained by the standard

Gauss-Legendre rules. Here, $Q_n(f;w)$ denotes the Gauss-Pollaczek quadrature sum defined by (4), and $r_n^P(f;\lambda)$ shows their relative errors,

$$r_n^P(f;\lambda) = \left| \frac{Q_n(f;w) - I(f;\lambda)}{I(f;\lambda)} \right|.$$

Relative errors for $\lambda = 1/2$ and $\lambda = 10$ are given in Table 1. Numbers in paren-

Table 1: Relative errors in quadrature sums when $n = 10(10)50$

n	$r_n^P(f;1/2)$	$r_n^L(f;1/2)$	$r_n^P(f;10)$	$r_n^L(f;10)$
10	1.66	1.01	4.32(−13)	3.52(−2)
20	2.38(−1)	1.43(−1)	2.94(−24)	1.21(−3)
30	4.54(−2)	1.12(−2)	5.27(−35)	1.57(−5)
40	1.04(−2)	4.87(−3)	1.86(−45)	2.93(−6)
50	2.71(−3)	7.09(−4)	1.09(−55)	1.82(−7)

theses indicate decimal exponents. The corresponding relative errors in Gauss-Legendre sums are denoted by $r_n^L(f;\lambda)$. As we can see, for $\lambda = 1/2$ both quadratures are slow and have similar behaviour, while for a larger λ $(= 10)$ the advantage of the Gauss-Pollaczek quadrature is clearly evident.

5. A CLASS OF SYMMETRIC WEIGHTS ON $\mathbb{R}$

In this section, we consider symmetric quadrature rules on $\mathbb{R}$ which play an important role in summation formulas of Abel-Plana type, which were intensively studied by Germund Dahlquist [**2, 3, 4**] (also see Milovanović [**14, 16**]). Such rules can be constructed in a simpler way if the corresponding formulas on $\mathbb{R}^+$ are first constructed. By noting the results of Sections 2 and 3, instead of the polynomials π_n orthogonal with respect to $x \mapsto w(x)$ on $\mathbb{R}$, we need the polynomials p_ν and q_ν, given by the recurrence relations (9) and (10), respectively. In other words, the recursive coefficients $\{a_\nu\}$ and $\{b_\nu\}$ for polynomials orthogonal with respect to the weight function $t \mapsto w(\sqrt{t})/\sqrt{t}$ on $\mathbb{R}^+$, as well as the coefficients $\{c_\nu\}$ and $\{d_\nu\}$ for polynomials orthogonal with respect to the weight function $t \mapsto \sqrt{t}\, w(\sqrt{t})$ on $\mathbb{R}^+$ must be computed.

In the sequel, let us mention some important cases of the symmetric weight $x \mapsto w(x)$ on $\mathbb{R}$.

1° In [**12**, p. 159] three interesting even weight functions on $\mathbb{R}$ are given, for which the recurrence coefficients β_k in (2) are known explicitly. They are respectively known as the *Abel weight*

$$w(x) = w^A(x) = \frac{x}{\mathrm{e}^{\pi x} - \mathrm{e}^{-\pi x}} = \frac{x}{2\sinh(\pi x)},$$

the *Lindelöf weight*

$$w(x) = w^L(x) = \frac{1}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}} = \frac{1}{2\cosh(\pi x)},$$

and the *logistic weight*

$$w(x) = w^{\log}(x) = \frac{\mathrm{e}^{-\pi x}}{(1+\mathrm{e}^{-\pi x})^2}.$$

The corresponding recurrence coefficients are

$$\beta_k^A = \frac{k(k+1)}{4}, \quad \beta_k^L = \frac{k^2}{4} \quad \text{and} \quad \beta_k^{\log} = \frac{k^4}{4k^2-1} \quad (k = 1, 2, \ldots),$$

with $\beta_0^A = 1/4$, $\beta_0^L = 1/2$ and $\beta_0^{\log} = 1/\pi$.

We mention also that $w^{\log}(x) = [w^L(x/2)]^2$.

For these weight functions, in the sequel we give the recurrence coefficients in (9) and (10) for polynomials orthogonal on $(0, \infty)$ with respect to $t \mapsto w(\sqrt{t})/\sqrt{t}$ and $t \mapsto w(\sqrt{t})\sqrt{t}$, respectively.

(i) In the Abel case we compute these coefficients as

$$a_\nu^A = \frac{(2\nu+1)^2}{2} \quad (\nu \in \mathbb{N}_0), \quad b_0^A = \frac{1}{4}, \quad b_\nu^A = \frac{\nu^2(4\nu^2-1)}{4} \quad (\nu \in \mathbb{N});$$
$$c_\nu^A = 2(\nu+1)^2 \quad (\nu \in \mathbb{N}_0), \quad d_0^A = \frac{1}{8}, \quad d_\nu^A = \frac{\nu(\nu+1)(2\nu+1)^2}{4} \quad (\nu \in \mathbb{N}).$$

(ii) Similarly in the Lindelöf case the corresponding coefficients are

$$a_\nu^L = \frac{8\nu^2+4\nu+1}{4} \quad (\nu \in \mathbb{N}_0), \quad b_0^L = \frac{1}{2}, \quad b_\nu^A = \frac{\nu^2(4\nu^2-1)}{4} \quad (\nu \in \mathbb{N});$$
$$c_\nu^L = \frac{8\nu^2+12\nu+5}{4} \quad (\nu \in \mathbb{N}_0), \quad d_0^L = \frac{1}{8}, \quad d_\nu^L = \frac{\nu^2(2\nu+1)^2}{4} \quad (\nu \in \mathbb{N}).$$

(iii) Finally, in the case of the logistic weight the recurrence coefficients in (9) are

$$a_\nu^{\log} = \frac{32\nu^4+32\nu^3+8\nu^2-1}{(4\nu-1)(4\nu+3)} \quad (\nu \in \mathbb{N}_0),$$
$$b_0^{\log} = \frac{1}{\pi}, \quad b_\nu^{\log} = \frac{16\nu^4(2\nu-1)^4}{(4\nu-3)(4\nu-1)^2(4\nu+1)} \quad (\nu \in \mathbb{N}),$$

and in (10) they are

$$c_\nu^{\log} = \frac{32\nu^4+96\nu^3+104\nu^2+48\nu+7}{16\nu^2+24\nu+5} \quad (\nu \in \mathbb{N}_0),$$
$$d_0^{\log} = \frac{1}{3\pi}, \quad d_\nu^{\log} = \frac{16\nu^4(2\nu+1)^4}{(4\nu-1)(4\nu+1)^2(4\nu+3)} \quad (\nu \in \mathbb{N}).$$

The first two weight functions appear in the so-called Abel-Plana summation formulas (cf. [**16**]). For example, under certain conditions for an analytic function

f in the complex plane, the finite sum $S_{n,m}(f) = \sum_{k=m}^{n}(-1)^k f(k)$ can be obtained from the *Abel summation formula*

$$S_{n,m}(f) = \frac{1}{2}\big((-1)^m f(m) + (-1)^n f(n+1)\big) - \int_{\mathbb{R}} h(x;m,n)w^A(x)\,\mathrm{d}x,$$

where

$$h(x;m,n) = (-1)^m \frac{f(m+\mathrm{i}x) - f(m-\mathrm{i}x)}{2\mathrm{i}x} + (-1)^n \frac{f(n+1+\mathrm{i}x) - f(n+1-\mathrm{i}x)}{2\mathrm{i}x}.$$

2° For other weight functions which also appear in summation formulas, since the explicit expressions of the coefficients β_k are not known, using the MATHEMATICA package `OrthogonalPolynomials` (see [**1, 17**]) enables us to obtain β_k in rational forms.

For instance, consider the Plana weight function

$$w(x) = w^P(x) = \frac{|x|}{\mathrm{e}^{|2\pi x|} - 1},$$

which appears in the so-called *Plana summation formula* (cf. [**19**], [**14**])

$$T_{m,n}(f) - \int_m^n f(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x;m,n)w^P(x)\,\mathrm{d}x$$

for the composite trapezoidal sum

$$T_{m,n}(f) = \sum_{k=m}^{n}{}'' f(k) = \frac{1}{2}f(m) + \sum_{k=m+1}^{n-1} f(k) + \frac{1}{2}f(n),$$

where

$$g(x;m,n) = \frac{f(n+\mathrm{i}x) - f(n-\mathrm{i}x)}{2\mathrm{i}x} - \frac{f(m+\mathrm{i}x) - f(m-\mathrm{i}x)}{2\mathrm{i}x}. \tag{20}$$

This formula holds for analytic functions in the strip $\Omega_{m,n} = \big\{z \in \mathbb{C} : m \le \operatorname{Re} z \le n\big\}$, such that

$$\int_0^{+\infty} |f(x+\mathrm{i}y) - f(x-\mathrm{i}y)|\mathrm{e}^{-|2\pi y|}\,\mathrm{d}y$$

exists, and $\lim_{|y|\to+\infty} \mathrm{e}^{-|2\pi y|}|f(x \pm \mathrm{i}y)| = 0$ uniformly in x, for every $m \le x \le n$ $(m, n \in \mathbb{N},\ m < n)$.

Using the package `OrthogonalPolynomials`, we can obtain the sequence of coefficients $\{\beta_k^P\}_{k\ge0}$ in rational forms as

$$\beta_0^P = \frac{1}{12},\ \ \beta_1^P = \frac{1}{10},\ \ \beta_2^P = \frac{79}{210},\ \ \beta_3^P = \frac{1205}{1659},\ \ \beta_4^P = \frac{262445}{209429},\ \ \beta_5^P = \frac{33461119209}{18089284070},$$

$$\beta_6^P = \frac{361969913862291}{137627660760070},\ \ \beta_7^P = \frac{85170013927511392430}{24523312685049374477},\ \ \text{etc.}$$

When k increases, these values are becoming more complicated (see [**16**]).

The corresponding weights for polynomials p_ν and q_ν on $\mathbb{R}^+$ are

$$w_1(t) = w_1^P(t) = \frac{1}{\mathrm{e}^{2\pi\sqrt{t}} - 1} \quad \text{and} \quad w_2(t) = w_2^P(t) = \frac{t}{\mathrm{e}^{2\pi\sqrt{t}} - 1}, \tag{21}$$

respectively. It is interesting to mention that at the Helsinki International Congress of Mathematicians (1978), Nikishin [**18**] pointed out the importance of some classes of orthogonal polynomials different from classical ones. In particular, he proposed obtaining explicit forms of polynomials (if possible) orthogonal with respect to the weight function w_1^P.

Taking the moments

$$\mu_{1,\nu}^P = \int_0^{+\infty} t^\nu w_1^P(t)\,\mathrm{d}t = \frac{(2\nu+1)!\zeta(2\nu+2)}{2^{2\nu+1}\pi^{2\nu+2}}, \quad \nu = 0, 1, \ldots, 2m-1,$$

and

$$\mu_{2,\nu}^P = \int_0^{+\infty} t^\nu w_2^P(t)\,\mathrm{d}t = \mu_{1,\nu+1}^P \frac{(2\nu+3)!\zeta(2\nu+4)}{2^{2\nu+3}\pi^{2\nu+4}}, \quad \nu = 0, 1, \ldots, 2m-1,$$

we can obtain the corresponding coefficients in (9) as

$$a_0^P = \frac{1}{10}, \quad a_1^P = \frac{871}{790}, \quad a_2^P = \frac{1672667011}{539062030}, \quad a_3^P = \frac{50634486717810987107}{8296534235776787390},$$

$$a_4^P = \frac{3241115879498605269828015564949609681}{3208013247516243608013276319334150 50}, \quad \text{etc.};$$

$$b_0^P = \frac{1}{12}, \quad b_1^P = \frac{79}{2100}, \quad b_2^P = \frac{1312225}{1441671}, \quad b_3^P = \frac{2491734801234609}{512172182993900},$$

$$b_4^P = \frac{276980623805265435471536707 00}{1769555822315229089057426013}, \quad \text{etc.},$$

as well as in (10),

$$c_0^P = \frac{10}{21}, \quad c_1^P = \frac{110200}{55671}, \quad c_2^P = \frac{239533652610}{53469214601}, \quad c_3^P = \frac{3126116063270299247432720 0}{3917478728549923835709789},$$

$$c_4^P = \frac{2032299617271932287823786429182679246048749956869 0}{1628454245165190286597605307125063916376617814289}, \quad \text{etc.};$$

$$d_0^P = \frac{1}{120}, \quad d_1^P = \frac{241}{882}, \quad d_2^P = \frac{423558471}{182722826}, \quad d_3^P = \frac{821210997517832607}{89904292554749621},$$

$$d_4^P = \frac{8087641966063021053585391796858341525 7}{3206594662841751899714894730399285285}, \quad \text{etc.},$$

but their explicit expressions (for each index) remains a mystery!

Another interesting summation formula is

$$\sum_{k=m}^{n} f(k) - \int_{m-1/2}^{n+1/2} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} g\big(x; m - \tfrac{1}{2}, n + \tfrac{1}{2}\big) w^M(x)\,\mathrm{d}x,$$

where the so-called *midpoint weight function* is defined by

$$w(x) = w^M(x) = \frac{|x|}{\mathrm{e}^{|2\pi x|}+1},$$

and $g\big(x; m-\frac{1}{2}, n+\frac{1}{2}\big)$ by (20). As before, one can consider the polynomials p_ν and q_ν orthogonal on $\mathbb{R}^+$ with respect to the weight functions

$$w_1(t) = w_1^M(t) = \frac{1}{\mathrm{e}^{2\pi\sqrt{t}}+1} \quad \text{and} \quad w_2(t) = w_2^M(t) = \frac{t}{\mathrm{e}^{2\pi\sqrt{t}}+1}, \tag{22}$$

and in a similar way, one can obtain the corresponding coefficients in (9) and (10) in the following the rational forms

$$a_0^M = \frac{7}{40},\ a_1^M = \frac{97153}{82840},\ a_2^M = \frac{2143300949275717}{675664735216120},\ a_3^M = \frac{22095355709373634969176841705426 1}{358005012158232650133557971060 40},$$

$$a_4^M = \frac{1308613469253930258531717464011751570501805639936024249720 7}{1286538803151559855777866179684631498656991773534847212200},\ \text{etc.};$$

$$b_0^M = \frac{1}{24},\quad b_1^M = \frac{2071}{33600},\quad b_2^M = \frac{15685119025}{15852295536},\quad b_3^M = \frac{5895324568676150049511881}{1170833101982789404702400},$$

$$b_4^M = \frac{919480999258696959661346213448241024976800075}{576540802597908800437584051097300398601002 12},\quad \text{etc.};$$

$$c_0^M = \frac{155}{294},\quad c_1^M = \frac{654837850}{323155833},\quad c_2^M = \frac{49647154589257771035}{10966854047350313398},$$

$$c_3^M = \frac{5430885812228074267126755757400276732980 0}{6765310743275018623908418926036774608781},$$

$$c_4^M = \frac{2383807210883859864106057473176692820172732110851477347996900634325131 8055}{1902789007849170506061772395575191790930210358707162334205873293472321134},\ \text{etc.};$$

$$d_0^M = \frac{7}{960},\quad d_1^M = \frac{199849}{691488},\quad d_2^M = \frac{366669459296427}{154646219485472},$$

$$d_3^M = \frac{2644652549156041551189819109731}{2860028859159418199911261554 08},$$

$$d_4^M = \frac{7071951106108162652736604339756545328619345537100911995491 1}{2782343550785232136311735142019287634629029202932721468080},\ \text{etc..}$$

Unfortunately, we were unable to discover their explicit forms!

6. A CLASS OF SYMMETRIC WEIGHTS WITH FOUR FREE PARAMETERS

In this section, we consider a special case of symmetric weight functions on $(-a, a)$ with four free parameters that covers many well-known classical weights such as Legendre, first and second kind Chebyshev, ultraspherical, generalized ultraspherical, Hermite and generalized Hermite weight, i.e.,

$$w(x) = \exp\left(\int_c^x \frac{rt^2+s}{t(pt^2+q)}\,\mathrm{d}t\right) = w(-x),$$

where p, q, r, s are real parameters and c is a constant in $(-a, a)$.

It is shown in [**11**] that a special solution of the differential equation

$$x^2(px^2+q)\Phi_n''(x)+x(rx^2+s)\Phi_n'(x)-\left(n(r+(n-1)p)x^2+\frac{1-(-1)^n}{2}\,s\right)\Phi_n(x)=0,$$

is the symmetric polynomial in the form

$$\Phi_n(x)=S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) \tag{23}$$

$$=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{(2i-(-1)^n+2\,[n/2])\,p+r}{(2i-(-1)^n+2)\,q+s}\right)x^{n-2k},$$

whose monic form is given by

$$\widehat{S}_n(x)=\widehat{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right)=\prod_{i=0}^{[n/2]-1}\frac{\big(2i-(-1)^n+2\big)q+s}{\big(2i-(-1)^n+2[n/2]\big)p+r}\,S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right).$$

For instance, we have

$$\begin{aligned}
\widehat{S}_2\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) &= x^2+\frac{q+s}{p+r}\,,\\
\widehat{S}_3\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) &= x^3+\frac{3q+s}{3p+r}x\,,\\
\widehat{S}_4\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) &= x^4+2\frac{3q+s}{5p+r}x^2+\frac{(3q+s)(q+s)}{(5p+r)(3p+r)},\\
\widehat{S}_5\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) &= x^5+2\frac{5q+s}{7p+r}x^3+\frac{(5q+s)(3q+s)}{(7p+r)(5p+r)}x\,.
\end{aligned}$$

According to [**11**], the monic form of these polynomials satisfies the three-term recurrence relation

$$\widehat{S}_{k+1}(x)=x\widehat{S}_k(x)-\beta_k\begin{pmatrix} r & s \\ p & q \end{pmatrix}\widehat{S}_{k-1}(x)\qquad (k\geq 1), \tag{24}$$

where $\widehat{S}_0(x)=1$, $\widehat{S}_1(x)=x$, and

$$\beta_k\begin{pmatrix} r & s \\ p & q \end{pmatrix}=-\frac{pq\,k^2+\big((r-2p)q-(-1)^k ps\big)k+(r-2p)s(1-(-1)^k)/2}{(2pk+r-p)(2pk+r-3p)}.$$

This means that for the monic polynomials $\pi_k(x)=\widehat{S}_k(x)$, the coefficients

$$\beta_k=\beta_k(p,q,r,s)=\beta_k\begin{pmatrix} r & s \\ p & q \end{pmatrix}\qquad (k\geq 1), \tag{25}$$

depend on four parameters p, q, r, s. Moreover, if $\beta_k(p,q,r,s) > 0$, the generic form of the orthogonality relation is as

$$\int_{-a}^{a} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) \widehat{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) \widehat{S}_k\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) \mathrm{d}x = \big(\beta_0\beta_1\cdots\beta_n\big)\delta_{n,k}, \tag{26}$$

where

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) = \exp\left(\int_c^x \frac{(r-2p)t^2+s}{t(pt^2+q)}\,\mathrm{d}t\right) \tag{27}$$

and

$$\beta_0 = \int_{-a}^{a} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right) \mathrm{d}x.$$

Without loss of generality, we can assume only $a = 1$ for finite intervals and $a = \infty$ for the infinite interval.

Regarding [**11**], the function $(px^2+q)\,W\left(\begin{matrix} r & s \\ p & q \end{matrix}\,\middle|\, x\right)$ must vanish at $x = a$ in order to hold the orthogonality relation (26).

In general, there are four main sub-classes of distribution families (27) whose probability density functions are as follows (see [**11**])

$$K_1 W\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\, x\right) = \frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\beta+1)}|x|^{2\alpha}(1-x^2)^{\beta} \tag{28}$$

for $-1 \le x \le 1$, and

$$K_2 W\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\, x\right) = \frac{1}{\Gamma\left(\alpha+\frac{1}{2}\right)}|x|^{2\alpha}\mathrm{e}^{-x^2}, \tag{29}$$

$$K_3 W\left(\begin{matrix} -2\alpha-2\beta+2 & -2\alpha \\ 1 & 1 \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(\beta)}{\Gamma\left(\beta+\alpha-\frac{1}{2}\right)\Gamma\left(-\alpha+\frac{1}{2}\right)}\frac{|x|^{-2\alpha}}{(1+x^2)^{\beta}}, \tag{30}$$

$$K_4 W\left(\begin{matrix} -2\alpha+2 & 2 \\ 1 & 0 \end{matrix}\,\middle|\, x\right) = \frac{1}{\Gamma\left(\alpha-\frac{1}{2}\right)}|x|^{-2\alpha}\mathrm{e}^{-1/x^2} \tag{31}$$

for $-\infty < x < \infty$, where the values $\{K_i\}_{i=1}^4$ play the normalizing constant role in relations (28) to (31). Consequently, there are four sub-sequences of symmetric orthogonal polynomials (23).

According to (28), if $(p,q,r,s) = (-1, 1, -2\alpha-2\beta-2, 2\alpha)$ is substituted into (23), then

$$\begin{aligned} &S_n\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\, x\right) \\ &\quad = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \prod_{i=0}^{[n/2]-(k+1)} \frac{-2i-(2\beta+2\alpha+2-(-1)^n+2[n/2])}{2i+2\alpha+2-(-1)^n} x^{n-2k}, \end{aligned}$$

represents the explicit form of generalized ultraspherical polynomials (GUP). By noting (24) and (25), the recurrence relation of monic GUP takes the form

$$\widehat{S}_{k+1}(x) = x\,\widehat{S}_k(x) - \beta_k\begin{pmatrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{pmatrix}\widehat{S}_{k-1}(x),$$

in which

$$\beta_k\begin{pmatrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{pmatrix} = \frac{\big(k+(1-(-1)^k)\alpha\big)\,\big(k+(1-(-1)^k)\alpha+2\beta\big)}{(2k+2\alpha+2\beta-1)(2k+2\alpha+2\beta+1)}. \tag{32}$$

Hence, its orthogonality relation reads as

$$\begin{aligned}\int_{-1}^{1}|x|^{2\alpha}(1-x^2)^{\beta}\,\widehat{S}_n\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\,x\right)\widehat{S}_m\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix}\,\middle|\,x\right)\mathrm{d}x \\ = \int_{-1}^{1}|x|^{2\alpha}(1-x^2)^{\beta}\,\mathrm{d}x\prod_{i=1}^{n}\beta_i\begin{pmatrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{pmatrix}\delta_{n,m},\end{aligned}$$

where

$$\int_{-1}^{1}|x|^{2\alpha}(1-x^2)^{\beta}\,\mathrm{d}x = B\Big(\alpha+\frac{1}{2},\beta+1\Big) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\beta+1)}{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}.$$

The above relation shows that the constraints on the parameters α and β should be $\alpha+1/2>0$ and $\beta+1>0$.

The second sub-class is the generalized Hermite polynomials

$$S_n\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\,x\right) = \sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\prod_{i=0}^{[n/2]-(k+1)}\frac{-2}{2i+(-1)^{n+1}+2+2\alpha}\,x^{n-2k},$$

satisfying the monic recurrence relation

$$\widehat{S}_{k+1}(x) = x\,\widehat{S}_k(x) - \beta_k\begin{pmatrix} -2 & 2\alpha \\ 0 & 1 \end{pmatrix}\widehat{S}_{k-1}(x),$$

with

$$\beta_k\begin{pmatrix} -2 & 2\alpha \\ 0 & 1 \end{pmatrix} = \frac{k}{2}+\frac{1-(-1)^k}{2}\,\alpha, \tag{33}$$

and the orthogonality relation

$$\begin{aligned}\int_{-\infty}^{\infty}|x|^{2\alpha}\mathrm{e}^{-x^2}\widehat{S}_n\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\,x\right)\widehat{S}_m\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix}\,\middle|\,x\right)\mathrm{d}x \\ = \left(\frac{1}{2^n}\prod_{i=1}^{n}\big((1-(-1)^i)\alpha+i\big)\right)\Gamma\Big(\alpha+\frac{1}{2}\Big)\delta_{n,m},\end{aligned}$$

provided that $\alpha+1/2>0$. According to Favard's theorem [**7, 12**], if $\beta_n(p,q,r,s)>0$ holds only for a finite number of positive integers, i.e., $n=1,\dots,N$, then the related polynomials are finitely orthogonal. In this sense, there are two kinds of classical symmetric finite orthogonal polynomials.

The first finite class is orthogonal with respect to the weight function $x\mapsto |x|^{-2\alpha}(1+x^2)^{-\beta}$ on $(-\infty,\infty)$ with the initial vector $(p,q,r,s)=(1,1,-2\alpha-2\beta+2,-2\alpha)$, whose explicit form is as

$$\widehat{S}_n\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\,\middle|\, x\right)$$
$$=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\prod_{i=0}^{[n/2]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2\alpha-2b}{2i+(-1)^{n+1}+2-2\alpha}\,x^{n-2k},$$

and satisfies the recurrence relation (24) with

$$\beta_k\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\right)=-\frac{\left[k-\alpha+(-1)^k\alpha\right]\left(k-(1-(-1)^k)\alpha-2\beta\right)}{(2k-2\alpha-2\beta+1)(2k-2\alpha-2\beta-1)}. \tag{34}$$

Hence, its orthogonality relation takes the form

$$\int_{-\infty}^{\infty}\frac{|x|^{-2\alpha}}{(1+x^2)^{\beta}}\,\widehat{S}_n\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\,\middle|\, x\right)\widehat{S}_m\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\,\middle|\, x\right)\mathrm{d}x$$
$$=\prod_{i=1}^{n}\beta_i\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\right)\frac{\Gamma\left(\beta+\alpha-\frac{1}{2}\right)\Gamma\left(-\alpha+\frac{1}{2}\right)}{\Gamma(\beta)}\,\delta_{n,m},$$

if and only if

$$\beta_n\left(\begin{matrix}-2\alpha-2\beta+2 & -2\alpha\\ 1 & 1\end{matrix}\right)>0;\quad \beta+\alpha>\frac{1}{2},\ \alpha<\frac{1}{2}\quad\text{and}\quad \beta>0.$$

In other words, the finite polynomial set $\{S_n(1,1,-2\alpha-2\beta+2,-2\alpha\,;x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $|x|^{-2\alpha}(1+x^2)^{-\beta}$ on $(-\infty,\infty)$ if and only if $N\le\alpha+\beta-1/2$, $\alpha<1/2$ and $\beta>0$.

Similarly, the second finite class is orthogonal with respect to the weight $x\mapsto|x|^{-2\alpha}\mathrm{e}^{-1/x^2}$ on $(-\infty,\infty)$ with the initial vector $(p,q,r,s)=(1,0,-2\alpha+2,2)$, whose explicit form is as

$$\widehat{S}_n\left(\begin{matrix}-2\alpha+2 & 2\\ 1 & 0\end{matrix}\,\middle|\, x\right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\prod_{i=0}^{[n/2]-(k+1)}\left(i+\left[\frac{n}{2}\right]-\frac{(-1)^n}{2}+1-\alpha\right)x^{n-2k},$$

and satisfies the recurrence relation (24) with

$$\beta_k\left(\begin{matrix}-2\alpha+2 & 2\\ 1 & 0\end{matrix}\right)=\frac{2(-1)^k(k-\alpha)+2\alpha}{(2k-2\alpha+1)(2k-2\alpha-1)}, \tag{35}$$

and finally has the orthogonality relation

$$\int_{-\infty}^{\infty} |x|^{-2\alpha} \mathrm{e}^{-1/x^2} \widehat{S}_n\left(\begin{matrix} -2\alpha+2 & 2 \\ 1 & 0 \end{matrix} \,\middle|\, x\right) \widehat{S}_m\left(\begin{matrix} -2\alpha+2 & 2 \\ 1 & 0 \end{matrix} \,\middle|\, x\right) \mathrm{d}x$$

$$= \prod_{i=1}^{n} \beta_i\left(\begin{matrix} -2\alpha+2 & 2 \\ 1 & 0 \end{matrix}\right) \Gamma\Big(\alpha - \frac{1}{2}\Big)\, \delta_{n,m},$$

if and only if $N = \max\{m, n\} \leq \alpha - 1/2$. This means that the finite polynomial set $\{S_n(1, 0, -2\alpha+2, 2\,; x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $|x|^{-2\alpha}\mathrm{e}^{-1/x^2}$ on $(-\infty, \infty)$ if $N \leq \alpha - 1/2$. The following table summarizes the main characteristics of the four introduced sub-classes. For other symmetric orthogonal polynomials see e.g. [**9, 10**].

Table 2: Four special cases of $S_n(p, q, r, s; x)$

Definition	Weight	β_k
$S_n\left(\begin{matrix} -2\alpha-2\beta-2 & 2\alpha \\ -1 & 1 \end{matrix} \,\middle|\, x\right)$	$\lvert x\rvert^{2\alpha}(1-x^2)^{\beta}$	$\frac{\left(k+\alpha-(-1)^k\alpha\right)\left(k+(1-(-1)^k)\alpha+2\beta\right)}{(2k+2\alpha+2\beta-1)(2k+2\alpha+2\beta+1)}$
$S_n\left(\begin{matrix} -2 & 2\alpha \\ 0 & 1 \end{matrix} \,\middle|\, x\right)$	$\lvert x\rvert^{2\alpha}\mathrm{e}^{-x^2}$	$\frac{k}{2} + \frac{1-(-1)^k}{2}\,\alpha$
$S_n\left(\begin{matrix} -2\alpha-2\beta+2 & -2\alpha \\ 1 & 1 \end{matrix} \,\middle|\, x\right)$	$\dfrac{\lvert x\rvert^{-2\alpha}}{(1+x^2)^{\beta}}$	$-\frac{\left(k-\alpha+(-1)^k\alpha\right)\left(k-(1-(-1)^k)\alpha-2\beta\right)}{(2k-2\alpha-2\beta+1)(2k-2\alpha-2\beta-1)}$
$S_n\left(\begin{matrix} -2\alpha+2 & 2 \\ 1 & 0 \end{matrix} \,\middle|\, x\right)$	$\lvert x\rvert^{-2\alpha}\mathrm{e}^{-1/x^2}$	$\frac{2(-1)^k(k-\alpha)+2\alpha}{(2k-2\alpha+1)(2k-2\alpha-1)}$

In the last column of this table we give the explicit expressions for the recursion coefficients β_k in the three-term recurrence relation. We use them in the construction of the corresponding Jacobi matrices.

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