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A STUDY OF GENERALIZED SUMMATION THEOREMS FOR THE SERIES $_2F_1$ WITH AN APPLICATIONS TO LAPLACE TRANSFORMS OF CONVOLUTION TYPE INTEGRALS INVOLVING KUMMER'S FUNCTIONS $_1F_1$

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Motivated by recent generalizations of classical theorems for the series $_2F_1$ [Integral Transform. Spec. Funct. **229**(11), (2011), 823–840] and interesting Laplace transforms of Kummer's confluent hypergeometric functions obtained by Kim *et al.* [Math. Comput. Modelling **55** (2012), 1068–1071], first we express generalized summations theorems in explicit forms and then by employing these, we derive various new and useful Laplace transforms of convolution type integrals by using product theorem of the Laplace transforms for a pair of Kummer's confluent hypergeometric function.

1. INTRODUCTION AND PRELIMINARIES

We begin with the definition of the *generalized hypergeometric function* with p numerator and q denominator parameters is defined by [**7**]

$$(1)\qquad {}_pF_q\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\,\middle|\, z\right] = {}_pF_q\left[\begin{matrix} (a) \\ (b) \end{matrix}\,\middle|\, z\right] := \sum_{m\ge 0} \frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\,\frac{z^m}{m!}\,,$$

where the Pochhammer symbol is defined by $(c)_0 = 1$, $(c)_n = c(c+1)\cdots(c+n-1)$, and $b_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $j = \overline{1,s} := 1,2,\dots,s$. The series converges for all $z \in \mathbb{C}$ if $p \le q$. It is divergent for all $z \neq 0$ when $p > q+1$, unless at least one numerator parameter

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is a negative integer in which case (1) is a polynomial. Finally, if $p = q + 1$, the series converges on the unit circle $|z| = 1$ when $\operatorname{Re}\left(\sum b_j - \sum a_j\right) > 0$. The importance of the generalized hypergeometric function lies in the fact that almost all elementary functions such as exponential, binomial, trigonometric, hyperbolic, logarithmic etc. are special case of this function. From the application point of view, it is very important to represent the hypergeometric function by the well-known gamma function. Thus the well known classical summation theorems such as those of Gauss second summation theorem, Bailey summation theorem and Kummer's summation theorem for the series ${}_2F_1$ which are given below, play an important role in the theory of hypergeometric functions.

Gauss's second summation theorem

$$(2) \qquad {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})},$$

Bailey's summation theorem

$$(3) \qquad {}_2F_1\left[\begin{matrix} a, 1-a \\ b \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}b+\frac{1}{2}a)\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})}$$

and Kummer's summation theorem

$$(4) \qquad {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix} \,\Big|\, -1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}.$$

During 1992–96, in a series of three research papers, Lavoie *et al.* [**3, 4, 5**] generalized various classical summation theorems such as the Gauss second, Bailey and Kummer ones for the ${}_2F_1$ series, as well as the Watson, Dixon and Whipple ones for the ${}_3F_2$ series.

In our present investigation, we are interested in the following generalizations of Gauss's second, Bailey and Kummer summation theorems, respectively, recorded in [**5**],

$$(5) \qquad {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix} \,\Big|\, \frac{1}{2}\right]$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2})}$$
$$\times\left\{\frac{A_i(a,b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{B_i(a,b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}i-\lfloor\frac{i}{2}\rfloor)}\right\}$$
$$= L_i(a,b),$$

$$(6)\qquad {}_2F_1\left[\begin{matrix} a, 1-a+i \\ b \end{matrix} \,\Big|\, \frac{1}{2}\right]$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma(1-a+\frac{1}{2}i+\frac{1}{2}|i|)} \left\{ \frac{C_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}a-\lfloor\frac{1+i}{2}\rfloor)} + \frac{D_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-\lfloor\frac{i}{2}\rfloor)} \right\}$$
$$= M_i(a,b),$$

and

$$(7)\qquad {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+i \end{matrix} \,\Big|\, -1\right]$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(1-b)\Gamma(1+a-b+i)}{2^{a}\Gamma(1-b+\frac{1}{2}i+\frac{1}{2}|i|)} \left\{ \frac{E_i(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{F_i(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}i-\lfloor\frac{i}{2}\rfloor)} \right\}$$
$$= N_i(a,b).$$

For $i=0$, these formulas respectively give classical Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem, given by (2), (3) and (4), respectively (cf. [**8**]).

Here, and in what follows, as usual, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to the real number x and its modulus is denoted by $|x|$. The coefficients which appears in (5), (6) and (7) were determined and listed for $|i|\le 5$, in the papers [**5**] and [**2**].

In 2011, Rakha and Rathie [**6**] generalized above mentioned classical theorems in the most genral form for any $i\in\mathbb{Z}$ and established six general summation theorems (two each). For details we refer their original paper [**6**]. The aim of this research paper is two fold. First we provide the explicit expressions for all pairs of coefficients $\{A_i(a,b), B_i(a,b)\}$, $\{C_i(a,b), D_i(a,b)\}$ and $\{E_i(a,b), F_i(a,b)\}$ in (5), (6) and (7), respectively, thereafter, we demonstrate how one can easily obtain Laplace transforms of convolution type integral by using product theorem for Laplace transforms for a pair of Kummer's confluent hypergeometric function by employing generalized summation formulas (5), (6) and (7) .

The paper is organized as follows. In Section 2, we derive explicit expressions for these coefficients for each integer $i=0,\pm1,\pm2,\dots$ in the generalized summation formulas (5), (6) and (7). By using product theorem for Laplace transform for a pair of Kummer's confluent hypergeometric functions, presented shortly in Section 3, in Section 4 we employ the generalized summation formulas (5), (6) and (7) in order to derive new Laplace transforms of convolution type integrals involving ${}_1F_1(a;b;x)$. Some interesting special cases are also included. Finally, some concluding remarks and observations are presented in Section 5.

2. EXPLICIT EXPRESSIONS FOR COEFFICIENTS IN THE GENERALIZED SUMMATION FORMULAS

2.1. Generalized Gauss's second summation theorem

Theorem 2.1. *For each* $\nu \in \mathbb{N}_0$*, the coefficients* $A_i(a,b)$ *and* $B_i(a,b)$ *in* (5) *are given by*

$$\begin{aligned}
A_{-2\nu}(a,b) = A_{2\nu}(a,b) &= \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b}{2}\right)_j \left(\frac{a+1}{2} - (\nu - j)\right)_{\nu-j},\\
B_{-2\nu}(a,b) = -B_{2\nu}(a,b) &= \sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b+1}{2}\right)_j \left(\frac{a}{2} - (\nu - j - 1)\right)_{\nu-j-1},\\
A_{-(2\nu+1)}(a,b) = -A_{2\nu+1}(a,b) &= \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b+1}{2}\right)_j \left(\frac{a+1}{2} - (\nu - j)\right)_{\nu-j},\\
B_{-(2\nu+1)}(a,b) = B_{2\nu+1}(a,b) &= \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b}{2}\right)_j \left(\frac{a}{2} - (\nu - j)\right)_{\nu-j}.
\end{aligned}$$

Proof. We start with a recent result by Rakha and Rathie [**6**, Theorem 1],

$$(8) \qquad {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+i+1)\right)\Gamma\left(\frac{1}{2}(a-b-i+1)\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}(b+1)\right)\Gamma\left(\frac{1}{2}(a-b+i+1)\right)} S_i(a,b),$$

where

$$S_i(a,b) = \sum_{r=0}^{i} \binom{i}{r} \frac{(-1)^r \Gamma\left(\frac{1}{2}(b+r)\right)}{\Gamma\left(\frac{1}{2}(a-i+r+1)\right)}.$$

Taking even and odd indices r in these sums for $i = 2\nu$ and $i = 2\nu + 1$, we have

$$S_{2\nu}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu}{2j} \frac{\Gamma\left(\frac{1}{2}b + j\right)}{\Gamma\left(\frac{1}{2}(a+1) - (\nu - j)\right)} - \sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \frac{\Gamma\left(\frac{1}{2}(b+1) + j\right)}{\Gamma\left(\frac{1}{2}a - (\nu - j - 1)\right)}$$

and

$$S_{2\nu+1}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \frac{\Gamma\left(\frac{1}{2}b + j\right)}{\Gamma\left(\frac{1}{2}a - (\nu - j)\right)} - \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \frac{\Gamma\left(\frac{1}{2}(b+1) + j\right)}{\Gamma\left(\frac{1}{2}(a+1) - (\nu - j)\right)},$$

respectively. Since

$$\Gamma(z+j) = (z)_j \Gamma(z) \quad \text{and} \quad \frac{1}{\Gamma(z-j)} = \frac{(z-j)_j}{\Gamma(z)},$$

the previous sums reduce to

$$\begin{aligned} S_{2\nu}(a,b) &= \frac{\Gamma(\frac{1}{2}b)}{\Gamma(\frac{1}{2}(a+1))} \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b}{2}\right)_j \left(\frac{a+1}{2} - (\nu - j)\right)_{\nu-j} \\ &\quad - \frac{\Gamma(\frac{1}{2}(b+1))}{\Gamma(\frac{1}{2}a)} \sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b+1}{2}\right)_j \left(\frac{a}{2} - (\nu - j - 1)\right)_{\nu-j-1} \end{aligned}$$

and

$$\begin{aligned} S_{2\nu+1}(a,b) &= \frac{\Gamma(\frac{1}{2}b)}{\Gamma(\frac{1}{2}a)} \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b}{2}\right)_j \left(\frac{a}{2} - (\nu - j)\right)_{\nu-j} \\ &\quad - \frac{\Gamma(\frac{1}{2}(b+1))}{\Gamma(\frac{1}{2}(a+1))} \sum_{j=0}^{\nu-1} \binom{2\nu+1}{2j+1} \left(\frac{b+1}{2}\right)_j \left(\frac{a+1}{2} - (\nu - j)\right)_{\nu-j}. \end{aligned}$$

Now, comparing (8) with (5) for $i \geq 0$, i.e.,

$$(9) \quad {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+i+1)\right)\Gamma\left(\frac{1}{2}(a-b-i+1)\right)}{\Gamma\left(\frac{1}{2}(a-b+i+1)\right)} \times \left\{ \frac{A_i(a,b)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}b+s_{i+1}\right)} + \frac{B_i(a,b)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+s_i\right)} \right\},$$

where $s_i = i/2 - \lfloor i/2 \rfloor$ (i.e., 0 for even and 1/2 for odd i), we obtain the explicit expressions for the coefficients $A_i(a,b)$ and $B_i(a,b)$, for even and odd indices, given in the statement of this theorem.

In a similar way, using [**6**, Theorem 2], we get the coefficients for negative indices. □

Corollary 2.1. *For each $\nu \in \mathbb{N}_0$ we have $B_{2\nu+1}(b,a) = -A_{2\nu+1}(a,b)$, i.e., the following identity*

$$(10) \quad \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b}{2}\right)_j \left(\frac{a}{2} - (\nu - j)\right)_{\nu-j} = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b+1}{2}\right)_j \left(\frac{a+1}{2} - (\nu - j)\right)_{\nu-j}$$

holds.

Proof. Note first that the interchange of parameters a and b does not change the value of the hypergeometric function in (9), and put, for $i = 2\nu + 1$,

$$K_{2\nu+1}(a,b) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b)+\nu+1\right)\Gamma\left(\frac{1}{2}(a-b)-\nu\right)}{\Gamma\left(\frac{1}{2}(a-b)+\nu+1\right)}.$$

This constant can be written in the form

$$K_{2\nu+1}(a,b)=\frac{\sqrt{\pi}\,\Gamma\left(\dfrac{a+b}{2}\right)\left(\dfrac{a+b}{2}\right)_\nu}{\left(\dfrac{a-b}{2}\right)_{\nu+1}\left(\dfrac{a-b}{2}-\nu\right)_\nu}=\frac{\sqrt{\pi}\,\Gamma\left(\dfrac{a+b}{2}\right)\left(\dfrac{a+b}{2}\right)_\nu}{\dfrac{a-b}{2}\displaystyle\prod_{j=1}^{\nu}\left[\left(\dfrac{a-b}{2}\right)^2-j^2\right]},$$

from which we conclude that $K_{2\nu+1}(b,a)=-K_{2\nu+1}(a,b)$.

Now, interchanging a and b in (9) for $i=2\nu+1$, we get that

$$B_{2\nu+1}(b,a)=-A_{2\nu+1}(a,b).$$

According to Theorem 2.1 we conclude that the identity (10) is valid for each $\nu\in\mathbb{N}_0$. □

Remark 2.1. Following Theorem 2.1 the coefficients in (5) are:

$$\begin{aligned}
&A_0(a,b)=1,\quad A_1(a,b)=-1,\quad A_2(a,b)=\frac{1}{2}(a+b-1),\\
&A_3(a,b)=-\frac{1}{2}(3a+b-2),\quad A_4(a,b)=\frac{1}{4}\left(a^2+6ab+b^2-4a-4b+3\right),\\
&A_5(a,b)=-\frac{1}{4}\left(5a^2+10ab+b^2-10a-6b+8\right),\\
&A_6(a,b)=\frac{1}{8}(a+b-1)\left(a^2+14ab-8a+b^2-8b+15\right),\\
&A_7(a,b)=-\frac{1}{8}(3a+b-2)\left[4(a+b-1)^2-(a-b)^2+16\right]\\
&\qquad\qquad+\frac{1}{4}(a+b-1)\left[(a-b)^2-4\right],\\
&A_8(a,b)=\frac{1}{16}(a-b)^2\left[(a-b)^2-26\right]\\
&\qquad\qquad+\frac{1}{2}(a+b-1)^2\left[(a+b-1)^2-(a-b)^2+9\right]+\frac{25}{16},\quad\ldots
\end{aligned}$$

and

$$\begin{aligned}
&B_0(a,b)=0,\quad B_1(a,b)=1,\quad B_2(a,b)=-2,\quad B_3(a,b)=\frac{1}{2}(a+3b-2),\\
&B_4(a,b)=-2(a+b-1),\quad B_5(a,b)=\frac{1}{4}\left(a^2+10ab+5b^2-6a-10b+8\right),\\
&B_6(a,b)=\frac{1}{2}\left[(a-b)^2-4(a+b-1)^2-9\right],\\
&B_7(a,b)=\frac{1}{8}(a+3b-2)\left[4(a+b-1)^2-(a-b)^2+16\right]\\
&\qquad\qquad-\frac{1}{4}(a+b-1)\left[(a-b)^2-4\right],\\
&B_8(a,b)=(a+b-1)\left[(a-b)^2-2(a+b-1)^2-17\right],\quad\ldots.
\end{aligned}$$

2.2. Generalized Bailey's summation theorem

Using a similar procedure as in the previous subsection, as well as Theorems 5 and 6 from [**6**], we can prove the following generalization of Bailey's summation theorem:

Theorem 2.2. *For each $\nu \in \mathbb{N}_0$, the coefficients $C_i(a,b)$ and $D_i(a,b)$ in (6) are given by*

$$C_{2\nu}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} - (2\nu - j)\right)_{\nu-j},$$

$$D_{2\nu}(a,b) = -\sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} - (2\nu - j)\right)_{\nu-j-1},$$

$$C_{2\nu+1}(a,b) = -\sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} - (2\nu - j + 1)\right)_{\nu-j},$$

$$D_{2\nu+1}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} - (2\nu - j + 1)\right)_{\nu-j},$$

$$C_{-2\nu}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} + j\right)_{\nu-j},$$

$$D_{-2\nu}(a,b) = \sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} + j\right)_{\nu-j-1},$$

$$C_{-(2\nu+1)}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2} + j\right)_{\nu-j},$$

$$D_{-(2\nu+1)}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2} + j\right)_{\nu-j}.$$

Remark 2.2. For $|i| \le 7$, the coefficients $C_i(a,b)$ and $D_i(a,b)$ are:

$$C_0(a,b) = 1, \quad C_1(a,b) = -1, \quad C_2(a,b) = b - 2, \quad C_3(a,b) = a - 2b + 3,$$
$$C_4(a,b) = -a^2 + 5a + 2\left(b^2 - 6b + 6\right),$$
$$C_5(a,b) = a^2 + a(2b - 13) - 4b^2 + 22b - 20,$$
$$C_6(a,b) = (b-4)\left(-3a^2 + 21a + 4b^2 - 32b + 30\right),$$
$$C_7(a,b) = -a^3 + a^2(4b - 6) + a\left(4b^2 - 68b + 181\right) - 8b^3 + 92b^2 - 288b + 210;$$

$$C_{-1}(a,b)=1,\quad C_{-2}(a,b)=b,\quad C_{-3}(a,b)=2b-a,$$
$$C_{-4}(a,b)=-a(a+3)+2b(b+2),$$
$$C_{-5}(a,b)=-a^2-a(2b+7)+4b(b+2),$$
$$C_{-6}(a,b)=(b+2)\left(-3a^2-15a+4b(b+4)\right),$$
$$C_{-7}(a,b)=a^3-a^2(4b+1)-2a\left(2b^2+22b+31\right)+8b\left(b^2+6b+8\right);$$

$$D_0(a,b)=0,\quad D_1(a,b)=1,\quad D_2(a,b)=-2,\quad D_3(a,b)=a+2b-7,$$
$$D_4(a,b)=-4(b-3),\quad D_5(a,b)=-a^2+a(2b-1)+4b^2-34b+62,$$
$$D_6(a,b)=2\left(a^2-7a-4b^2+32b-54\right),$$
$$D_7(a,b)=-a^3+a^2(30-4b)+a\left(4b^2-4b-107\right)+8b^3-124b^2+576b-762;$$

$$D_{-1}(a,b)=1,\quad D_{-2}(a,b)=2,\quad D_{-3}(a,b)=a+2b+2,\quad D_{-4}(a,b)=4(b+1),$$
$$D_{-5}(a,b)=-a^2+a(2b-1)+4(b+1)(b+3),$$
$$D_{-6}(a,b)=8(b+1)(b+3)-2a(a+5),$$
$$D_{-7}(a,b)=-a^3-a^2(4b+19)+a(4(b-1)b-58)+8(b+1)(b+3)(b+5).$$

2.3. Generalized Kummer's summation theorem

Also, in similar way as before and using Theorems 3 and 4 from [**6**], we can prove the following generalization of Kummer's summation theorem:

Theorem 2.3. *For each $\nu \in \mathbb{N}_0$, the coefficients $E_i(a,b)$ and $F_i(a,b)$ in (6) are given by*

$$E_{2\nu}(a,b) = \sum_{j=0}^{\nu}\binom{2\nu}{2j}\left(\frac{a-2b+1}{2}+\nu\right)_j\left(\frac{a+1}{2}-(\nu-j)\right)_{\nu-j},$$

$$F_{2\nu}(a,b) = -\sum_{j=0}^{\nu-1}\binom{2\nu}{2j+1}\left(\frac{a-2b}{2}+\nu+1\right)_j\left(\frac{a}{2}-(\nu-j-1)\right)_{\nu-j-1},$$

$$E_{2\nu+1}(a,b) = -\sum_{j=0}^{\nu}\binom{2\nu+1}{2j}\left(\frac{a-2b}{2}+\nu+1\right)_j\left(\frac{a}{2}-(\nu-j)\right)_{\nu-j},$$

$$F_{2\nu+1}(a,b) = \sum_{j=0}^{\nu}\binom{2\nu+1}{2j+1}\left(\frac{a-2b+1}{2}+\nu+1\right)_j\left(\frac{a+1}{2}-(\nu-j)\right)_{\nu-j},$$

$$E_{-2\nu}(a,b) = \sum_{j=0}^{\nu}\binom{2\nu}{2j}\left(\frac{a-2b+1}{2}-\nu\right)_j\left(\frac{a+1}{2}-(\nu-j)\right)_{\nu-j},$$

$$F_{-2\nu}(a,b) = \sum_{j=0}^{\nu-1}\binom{2\nu}{2j+1}\left(\frac{a-2b}{2}-(\nu-1)\right)_j\left(\frac{a}{2}-(\nu-j-1)\right)_{\nu-j-1},$$

$$E_{-(2\nu+1)}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{a-2b}{2} - \nu\right)_j \left(\frac{a}{2} - (\nu - j)\right)_{\nu-j},$$

$$F_{-(2\nu+1)}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{a-2b+1}{2} - \nu\right)_j \left(\frac{a+1}{2} - (\nu - j)\right)_{\nu-j}.$$

Remark 2.3. For $|i| \leq 7$, the coefficients $E_i(a,b)$ and $F_i(a,b)$ are:

$$\begin{aligned}
&E_0(a,b) = 1, \;\; E_1(a,b) = -1, \;\; E_2(a,b) = a - b + 1, \;\; E_3(a,b) = -2a + 3b - 5,\\
&E_4(a,b) = 2a^2 - 4a(b-2) + b^2 - 3b + 2,\\
&E_5(a,b) = -4a^2 + 2a(5b-13) - 5b^2 + 25b - 32,\\
&E_6(a,b) = 4a^3 - 12a^2(b-3) + a\left(9b^2 - 51b + 74\right) - b^3 + 6b^2 - 11b + 6,\\
&E_7(a,b) = -8a^3 + 4a^2(7b-25) - 4a\left(7b^2 - 49b + 88\right)\\
&\qquad\qquad +7b^3 - 70b^2 + 245b - 302;
\end{aligned}$$

$$\begin{aligned}
&E_{-1}(a,b) = 1, \;\; E_{-2}(a,b) = a - b - 1, \;\; E_{-3}(a,b) = 2a - 3b - 4,\\
&E_{-4}(a,b) = 2a^2 - 4a(b+2) + b^2 + 5b + 6,\\
&E_{-5}(a,b) = 4a^2 - 2a(5b+12) + 5b^2 + 25b + 32,\\
&E_{-6}(a,b) = 4a^3 - 12a^2(b+3) + a\left(9b^2 + 57b + 92\right) - b^3 - 12b^2 - 47b - 60,\\
&E_{-7}(a,b) = 8a^3 - 4a^2(7b+24) + 4a\left(7b^2 + 49b + 88\right)\\
&\qquad\qquad -7b^3 - 77b^2 - 294b - 384;
\end{aligned}$$

$$\begin{aligned}
&F_0(a,b) = 0, \;\; F_1(a,b) = 1, \;\; F_2(a,b) = -2, \;\; F_3(a,b) = 2a - b + 1,\\
&F_4(a,b) = -4(a - b + 2), \;\; F_5(a,b) = 4a^2 + a(14 - 6b) + b^2 - 3b + 2,\\
&F_6(a,b) = -8a^2 + 16a(b-3) - 6b^2 + 34b - 52,\\
&F_7(a,b) = 8a^3 + a^2(68 - 20b) + 4a\left(3b^2 - 19b + 32\right) - b^3 + 6b^2 - 11b + 6;
\end{aligned}$$

$$\begin{aligned}
&F_{-1}(a,b) = 1, \;\; F_{-2}(a,b) = 2, \;\; F_{-3}(a,b) = 2a - b - 2,\\
&F_{-4}(a,b) = 4(a - b - 2), \;\; F_{-5}(a,b) = 4a^2 - 2a(3b+8) + b^2 + 7b + 12,\\
&F_{-6}(a,b) = 8a^2 - 16a(b+3) + 6b^2 + 38b + 64,\\
&F_{-7}(a,b) = 8a^3 - 4a^2(5b+18) + 4a\left(3b^2 + 23b + 46\right) - b^3\\
&\qquad\qquad -15b^2 - 74b - 120.
\end{aligned}$$

3. LAPLACE TRANSFORM OF PRODUCT OF ${}_1F_1(a;b;x)$

The Laplace transform (cf. [**10**]) of the function $f(t)$ is defined, as usual, by

$$L\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t.$$

Here, we recall the following general product theorem for Laplace transform [**9**, p. 43, Eq. 3.2.28]:

$$g_1(t)\,g_2(t) = \int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t f_1(\tau)\,f_2(t-\tau)\,\mathrm{d}\tau\right\}\mathrm{d}t,$$

which further applied to a pair of generalized hypergeometric functions to obtain the formula [**9**, p. 43, Eq. 3.2.29]:

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{\mu-1}\,(t-\tau)^{\nu-1}\,{}_AF_B\left[\begin{array}{c}(a)\\(b)\end{array}\Big|k\tau\right]\,{}_{A'}F_{B'}\left[\begin{array}{c}(a')\\(b')\end{array}\Big|k'(t-\tau)\right]\mathrm{d}\tau\right\}\mathrm{d}t$$
$$= \Gamma(\mu)\Gamma(\nu)\,s^{-\mu-\nu}\,{}_{A+1}F_B\left[\begin{array}{c}(a),\mu\\(b)\end{array}\Big|\frac{k}{s}\right]\,{}_{A'+1}F_{B'}\left[\begin{array}{c}(a'),\nu\\(b')\end{array}\Big|\frac{k'}{s}\right],$$

for $A < B$, $A' < B'$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(s) > 0$ or for $A = B$, $A' = B'$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(k)$, $\mathrm{Re}(s) > \mathrm{Re}(k')$.

If $A = B = 1 = A' = B'$, we get known formula [**9**, p. 43, Eq. 3.2.30]:

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{\mu-1}\,(t-\tau)^{\nu-1}\,{}_1F_1\left[\begin{array}{c}a\\c\end{array}\Big|k\tau\right]\,{}_1F_1\left[\begin{array}{c}a'\\c'\end{array}\Big|k'(t-\tau)\right]\mathrm{d}\tau\right\}\mathrm{d}t \tag{11}$$
$$= \Gamma(\mu)\Gamma(\nu)\,s^{-\mu-\nu}{}_2F_1\left[\begin{array}{c}a,\mu\\c\end{array}\Big|\frac{k}{s}\right]\,{}_2F_1\left[\,a',\nu;c'\,\Big|\frac{k'}{s}\right]$$

for $\mathrm{Re}(c) > 0$, $\mathrm{Re}(c') > 0$, $\mathrm{Re}(s) > \mathrm{Re}(k)$, $\mathrm{Re}(s) > \mathrm{Re}(k')$, $|s| > |k|$ and $|s| > |k'|$.

In the next section we present various new Laplace type integrals by using product theorem for Laplace transform for a pair of Kummer's confluent hypergeometric functions by employing generalized Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem, (5), (6) and (7), with coefficients given by Theorems 2.1, 2.2 and 2.3, respectively.

4. NEW LAPLACE TRANSFORMS OF CONVOLUTION TYPE INTEGRALS INVOLVING ${}_1F_1(a;b;x)$

Theorem 4.1. *For* $\mathrm{Re}(b) > 0$*,* $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$*, the following general result*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{b-1}\,(t-\tau)^{b'-1}{}_1F_1\left[\begin{array}{c}a\\\frac{1}{2}(a+b+i+1)\end{array}\Big|\frac{1}{2}\tau s\right]\right. \tag{12}$$
$$\left.\times\,{}_1F_1\left[\begin{array}{c}a'\\\frac{1}{2}(a'+b'+j+1)\end{array}\Big|\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t$$
$$= \frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,L_i(a,b)\,L_j(a',b'),$$

hold, where $L_i(a,b)$ *is given in* (5).

Proof. The proof of this theorem is quite straight forward. In order to prove this result, setting $k = \frac{1}{2}s$, $k' = \frac{1}{2}s$, $\mu = b$, $\nu = b'$, $c = \frac{1}{2}(a+b+i+1)$ and $c' = \frac{1}{2}(a'+b'+j+1)$ for $i, j \in \mathbb{Z}$ in (11), we have

$$\begin{aligned}(13)\quad \int_0^\infty \mathrm{e}^{-st}\Bigg\{\int_0^t \tau^{b-1}\,(t-\tau)^{b'-1}{}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+i+1)\end{matrix}\,\Big|\,\frac{1}{2}\tau s\right] \\ \times {}_1F_1\left[\begin{matrix} a' \\ \frac{1}{2}(a'+b'+j+1)\end{matrix}\,\Big|\,\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\Bigg\}\,\mathrm{d}t \\ = \frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1)\end{matrix}\,\Big|\,\frac{1}{2}\right]{}_2F_1\left[\begin{matrix} a', b' \\ \frac{1}{2}(a'+b'+j+1)\end{matrix}\,\Big|\,\frac{1}{2}\right].\end{aligned}$$

We, now observe that the two ${}_2F_1$ appearing on the right-hand side of (13) can be evaluated with the help of generalized Gauss's second summation theorem (5), which yields at once the desired formula (12). □

The following results can also be proven in a similar lines by applying appropriate summation theorems (5), (6) and (7).

Theorem 4.2. *For* $\mathrm{Re}(1-a+i) > 0$, $\mathrm{Re}(1-a'+j) > 0$ $(i, j \in \mathbb{Z})$ *and* $\mathrm{Re}(s) > 0$, *the following general result holds true:*

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\Bigg\{\int_0^t \tau^{-a+i}\,(t-\tau)^{-a'+j}\,{}_1F_1\left[\begin{matrix} a \\ b\end{matrix}\,\Big|\,\frac{1}{2}\tau s\right]{}_1F_1\left[\begin{matrix} a' \\ b'\end{matrix}\,\Big|\,\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\Bigg\}\,\mathrm{d}t \\ = \frac{\Gamma(1-a+i)\,\Gamma(1-a'+j)}{s^{i+j+2-a-a'}}\,M_i(a,b)\,M_j(a',b'),\end{aligned}$$

where $M_i(a,b)$ *is given in* (6).

Theorem 4.3. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following general result holds true:*

$$\begin{aligned}(14)\quad \int_0^\infty \mathrm{e}^{-st}\Bigg\{\int_0^t \tau^{b-1}\,(t-\tau)^{b'-1}\,{}_1F_1\left[\begin{matrix} a \\ 1+a-b+i\end{matrix}\,\Big|\,-\tau s\right] \\ \times {}_1F_1\left[\begin{matrix} a' \\ 1+a'-b'+j\end{matrix}\,\Big|\,-(t-\tau)s\right]\mathrm{d}\tau\Bigg\}\,\mathrm{d}t \\ = \frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,N_i(a,b)\,N_j(a',b'),\end{aligned}$$

where $N_i(a,b)$ *is given in* (7).

Theorem 4.4. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(1-a'+j) > 0$ $(i,j \in \mathbb{Z})$ *and* $\mathrm{Re}(s) > 0$, *the following general result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{b-1}\,(t-\tau)^{-a'+j}\,{}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+i+1)\end{matrix}\,\Big|\,\frac{1}{2}\tau s\right] \times {}_1F_1\left[\begin{matrix} a' \\ b'\end{matrix}\,\Big|\,\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t = \frac{\Gamma(b)\,\Gamma(1-a'+j)}{s^{j+1-a'+b}}\,L_i(a,b)\,M_j(a',b'),$$

where $L_i(a,b)$ *and* $M_i(a,b)$ *are given in* (5) *and* (6), *respectively.*

Theorem 4.5. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following general result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{b-1}\,(t-\tau)^{b'-1}\,{}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+i+1)\end{matrix}\,\Big|\,\frac{1}{2}\tau s\right] \times {}_1F_1\left[\begin{matrix} a' \\ 1+a'-b'+j\end{matrix}\,\Big|\,-(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t = \frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,L_i(a,b)\,N_j(a',b'), \tag{15}$$

where $L_i(a,b)$ *and* $N_i(a,b)$ *are given in* (5) *and* (7), *respectively.*

Theorem 4.6. *For* $\mathrm{Re}(1-a+i) > 0$ $(i \in \mathbb{Z})$ $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following general result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{-a+i}\,(t-\tau)^{b'-1}\,{}_1F_1\left[\begin{matrix} a \\ b\end{matrix}\,\Big|\,\frac{1}{2}\tau s\right] \times {}_1F_1\left[\begin{matrix} a' \\ 1+a'-b'+j\end{matrix}\,\Big|\,-(t-\tau)s\right]\mathrm{d}t = \frac{\Gamma(1-a+i)\,\Gamma(b')}{s^{i+1-a+b'}}\,M_i(a,b)\,N_j(a',b'), \tag{16}$$

where $M_i(a,b)$ *and* $N_i(a,b)$ *are given in* (6) *and* (7), *respectively.*

5. SPECIAL CASES

For $i = 0$, Theorems 4.1 to 4.6 yield the following interesting results for classical ones asserted in the following corollaries:

Corollary 5.1. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{b-1}(t-\tau)^{b'-1}\,{}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+1)\end{matrix}\,\middle|\,\frac{1}{2}\tau s\right]\right.$$
$$\left.\times\,{}_1F_1\left[\begin{matrix} a' \\ \frac{1}{2}(a'+b'+1)\end{matrix}\,\middle|\,\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t$$
$$=\frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a'+b'+1))}{\Gamma(\frac{1}{2}a'+\frac{1}{2})\Gamma(\frac{1}{2}b'+\frac{1}{2})}.$$

Corollary 5.2. *For* $\mathrm{Re}(1-a) > 0$, $\mathrm{Re}(1-a') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{-a}(t-\tau)^{-a'}\,{}_1F_1\left[\begin{matrix} a \\ b\end{matrix}\,\middle|\,\frac{1}{2}\tau s\right]{}_1F_1\left[\begin{matrix} a' \\ b'\end{matrix}\,\middle|\,\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t$$
$$=\frac{\Gamma(1-a)\,\Gamma(1-a')}{s^{2-a-a'}}\,\frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}b+\frac{1}{2}a)\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})}\,\frac{\Gamma(\frac{1}{2}b')\Gamma(\frac{1}{2}b'+\frac{1}{2})}{\Gamma(\frac{1}{2}b'+\frac{1}{2}a')\Gamma(\frac{1}{2}b'-\frac{1}{2}a'+\frac{1}{2})}.$$

Corollary 5.3. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{b-1}(t-\tau)^{b'-1}\,{}_1F_1\left[\begin{matrix} a \\ 1+a-b\end{matrix}\,\middle|\,-\tau s\right]\right.$$
$$\left.\times\,{}_1F_1\left[\begin{matrix} a' \\ 1+a'-b'\end{matrix}\,\middle|\,-(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t$$
$$=\frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}\,\frac{\Gamma(1+\frac{1}{2}a')\Gamma(1+a'-b')}{\Gamma(1+a')\Gamma(1+\frac{1}{2}a'-b')}.$$

Corollary 5.4. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(1-a') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{b-1}(t-\tau)^{-a'}\,{}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+1)\end{matrix}\,\middle|\,\frac{1}{2}\tau s\right]{}_1F_1\left[\begin{matrix} a' \\ b'\end{matrix}\,\middle|\,\frac{1}{2}(t-\tau)s\right]\mathrm{d}\tau\right\}\mathrm{d}t$$
$$=\frac{\Gamma(b)\,\Gamma(1-a')}{s^{1-a'+b}}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}\,\frac{\Gamma(1+\frac{1}{2}a')\Gamma(1+a'-b')}{\Gamma(1+a')\Gamma(1+\frac{1}{2}a'-b')}.$$

Corollary 5.5. *For* $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\bigg\{\int_0^t \tau^{b-1}\,(t-\tau)^{b'-1}\,{}_1F_1\Big[\begin{matrix} a \\ \frac{1}{2}(a+b+1)\end{matrix}\ \Big|\frac{1}{2}\tau s\Big] \times {}_1F_1\Big[\begin{matrix} a' \\ 1+a'-b'\end{matrix}\ \Big| -(t-\tau)s\Big]\,\mathrm{d}\tau\bigg\}\,\mathrm{d}t$$
$$= \frac{\Gamma(b)\,\Gamma(b')}{s^{b+b'}}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}\,\frac{\Gamma(1+\frac{1}{2}a')\Gamma(1+a'-b')}{\Gamma(1+a')\Gamma(1+\frac{1}{2}a'-b')}.$$

Corollary 5.6. *For* $\mathrm{Re}(1-a) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\bigg\{\int_0^t \tau^{-a}\,(t-\tau)^{b'-1}\,{}_1F_1\Big[\begin{matrix} a \\ b\end{matrix}\ \Big|\frac{1}{2}\tau s\Big] \times {}_1F_1\Big[\begin{matrix} a' \\ 1+a'-b'\end{matrix}\ \Big| -(t-\tau)s\Big]\,\mathrm{d}\tau\bigg\}\,\mathrm{d}t$$
$$= \frac{\Gamma(1-a)\,\Gamma(b')}{s^{1-a+b'}}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}\,\frac{\Gamma(1+\frac{1}{2}a')\Gamma(1+a'-b')}{\Gamma(1+a')\Gamma(1+\frac{1}{2}a'-b')}.$$

Similarly, for $i \in \mathbb{Z}$ other results can also be obtained.

6. CONCLUDING REMARKS AND OBSERVATIONS

It is interesting to mention here that the following three results which are closely related to (14), (15) and (16) can also be obtained on similar lines. These are given here without proof.

Theorem 6.1. *For* $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$, *the following general result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\bigg\{\int_0^t \tau^{a-1}\,(t-\tau)^{b'-1}\,{}_1F_1\Big[\begin{matrix} b \\ 1+a-b+i\end{matrix}\ \Big| -\tau s\Big] \times {}_1F_1\Big[\begin{matrix} a' \\ 1+a'-b'+j\end{matrix}\ \Big|(\tau-t)s\Big]\,\mathrm{d}\tau\bigg\}\,\mathrm{d}t = \frac{\Gamma(a)\,\Gamma(b')}{s^{a+b'}}\,N_i(a,b)\,N_j(a',b'),$$

where $N_i(a,b)$ *is given in* (7).

Theorem 6.2. *For* $\mathrm{Re}(a) > 0$*,* $\mathrm{Re}(b') > 0$ *and* $\mathrm{Re}(s) > 0$*, the following general result holds true:*

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{a-1}\,(t-\tau)^{b'-1}{}_1F_1\left[\begin{matrix} b \\ \frac{1}{2}(a+b+i+1)\end{matrix}\;\Big|\,\frac{1}{2}\tau s\right]\right.$$

$$\left.\times\,{}_1F_1\left[\begin{matrix} a' \\ 1+a'-b'+j\end{matrix}\;\Big|(\tau-t)s\right]\mathrm{d}\tau\right\}\mathrm{d}t = \frac{\Gamma(a)\,\Gamma(b')}{s^{a+b'}}\,L_i(a,b)\,N_j(a',b'),$$

where $L_i(a,b)$ *and* $N_i(a,b)$ *are given in* (5) *and* (7)*, respectively.*

Theorem 6.3. *For* $\mathrm{Re}(1-a+i) > 0$ $(i \in \mathbb{Z})$

$$\int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t \tau^{-a+i}\,(t-\tau)^{a'-1}\,{}_1F_1\left[\begin{matrix} a \\ b\end{matrix}\;\Big|\,\frac{1}{2}\tau s\right]{}_1F_1\left[\begin{matrix} b' \\ 1+a'-b'+j\end{matrix}\;\Big|(\tau-t)s\right]\mathrm{d}t\right.$$

$$= \frac{\Gamma(1-a+i)\,\Gamma(a')}{s^{i+1-a+a'}}\,M_i(a,b)\,N_j(a',b'),$$

where $M_i(a,b)$ *and* $N_i(a,b)$ *are given in* (6) *and* (7)*, respectively.*

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