MULTI-PARAMETER MATHIEU, AND ALTERNATING MATHIEU SERIES

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Abstract. The main purpose of this paper is to present a multi–parameter study of the familiar Mathieu series and the alternating Mathieu series $\mathscr{S}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}(\boldsymbol{r})$. The computable series expansions of the their related integral representations are obtained in terms of higher transcendental hypergeometric functions like Lauricella's hypergeometric function $F_C^{(m)}[\boldsymbol{x}]$, Fox–Wright Ψ function, Srivastava–Daoust S generalized Lauricella function, Riemann Zeta Dirichlet Eta function, while the extensions concern products of Bessel and modified Bessel functions of the first kind, hyper–Bessel and Bessel–Clifford functions. Auxiliary Laplace–Mellin transforms, bounding inequalities for the hyper–Bessel and Bessel–Clifford functions are established- which are also of independent but considerable interest. A set of bounding inequalities are presented either for the hyper–Bessel and Bessel–Clifford functions which are to our best knowledge new, or also for all considered extended Mathieu–type series. Next, functional bounding inequalities, log–convexity porperties and Turán inequality results are presented for the investigated extensions of multi–parameter Mathieu–type series. We end the exposition by a thorough discussion closes the exposition including important details, bridges to occuring new questions like the similar kind multi–parameter treatment of the complete Butzer–Flocke–Hauss Ω function which is intimately connected with the Mathieu series family.

1. Introduction and preliminaries

The series representation of Riemann Zeta function $\zeta(s)$ is defined by [81, p. 164, Eq. (1)]

$$\zeta(s) = \sum_{n\geq 1} n^{-s}, \qquad \Re(s) > 1\,,$$

and its integral representation is given as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\mathrm{e}^x - 1}\, \mathrm{d}x, \qquad \Re(s) > 1. \tag{1.1}$$

The close relative of the Riemann Zeta function known as Dirichlet Eta function (or the alternating Riemann Zeta function) $\eta(s)$ and its integral representation is given by [81, p. 384, Eq. (35)]

$$\eta(s) = \sum_{n\geq 1} (-1)^{n-1} n^{-s}, \qquad \Re(s) > 0\,,$$

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and

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\mathrm{e}^x + 1}\,\mathrm{d}x, \qquad \Re(s) > 0, \tag{1.2}$$

respectively.

In the study of elasticity of solid bodies, Émile Leonard Mathieu (1835-1890) investigated the celebrated infinite series of the form [54]

$$S(r) = \sum_{n\geq 1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0. \tag{1.3}$$

A remarkable useful integral representation for $S(r)$ is given by Emersleben [21] in the following elegant form

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x\sin(rx)}{\mathrm{e}^x - 1}\,\mathrm{d}x. \tag{1.4}$$

The alternating Mathieu series

$$\widetilde{S}(r) = \sum_{n\geq 1} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0, \tag{1.5}$$

and one of its integral representation was established by Pogány *et al.* [74, p. 72, Eq. (2.8)]

$$\widetilde{S}(r) = \frac{1}{r}\int_0^\infty \frac{x\sin(rx)}{\mathrm{e}^x + 1}\,\mathrm{d}x; \tag{1.6}$$

another kind integral forms for Mathieu and alternaning Mathieu series were obtained by Milovanović and Pogány [60, pp. 185-186, Corollary 2.2], Tomovski and Pogány [89, p. 7, Theorem 3.2] derived Cauchy principal value integrals for these series, also see [11, 13, 20] in this integral forms respect. We notice that $S(0) = 2\,\zeta(3)$, while $\widetilde{S}(0) = 2\,\eta(3)$ but that are not the only connecting links between the Riemann ζ, Dirichlet η and the Mathieu series and its alternating variant. The Mathieu series (1.3) and alternating Mathieu series (1.5) can also be written in terms of the Riemann Zeta function $\zeta(s)$ and Dirichlet Eta function $\eta(s)$, respectively [13, p. 863, Eq. (2.3–4)]

$$S(r) = 2\sum_{n\geq 0} (-1)^n\,(n+1)\,\zeta(2n+3)\,r^{2n}, \qquad |r| < 1 \tag{1.7}$$

and

$$\widetilde{S}(r) = 2\sum_{n\geq 0} (-1)^n\,(n+1)\,\eta(2n+3)\,r^{2n}, \qquad |r| < 1. \tag{1.8}$$

The generalization of Mathieu series we can realize considering the related integral representation extending the integrand by a weight function. Namely, re-write (1.4) into the form

$$S(r) = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{3/2}}{\mathrm{e}^x - 1}\sqrt{\frac{2}{\pi r x}}\,\sin(rx)\,\mathrm{d}x = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{3/2}}{\mathrm{e}^x - 1}\,J_{\frac12}(rx)\,\mathrm{d}x\,, \tag{1.9}$$

where $J_{\frac12}$ stands for the Bessel function of the first kind of the order $\frac12$. Now, diverse points of view occur which we can develop in several possible directions. We mention few of them:

(i) The generalization methodology of the multiparameter unified and generalized Voigt function considered by Khan *et al.* [30] in which a finite product of different arguments sine function is inserted into integrand.

(ii) Replacing the product of trigonometric functions with a product of Bessel functions of the first kind was done in the article by Srivastava and Pogány [86, p. 195, Eq. (4)]. Motivated by (1.9) we can proceed using ONE general order Bessel function of the first kind $J_\nu(rx)$ instead of $J_{\frac{1}{2}}(rx)$ in (1.9) or, secondly, the product of sines from **(i)** we replace with a PRODUCT $\prod_{j=1}^m J_{\nu_j}(r_j x)$; $\min\{\nu_1, \cdots, \nu_m\} > -1$. By these methods we extend either the integrals (1.3) and (1.5) or *a fortiori* the series (1.7) and (1.8).

(iii) As a further extension of Bessel function the multiparameter Bessel–Clifford function [16, 26, 27], which is actually a normalized and simplified argument variant of the Delerue hyper-Bessel function [15, 16, 32, 33, 34, 36, 37], can be used in **(ii)**.

However, the hyper–Bessel function unification of the input Bessel and modified Bessel functions is of considerable interest.

(iv) A wide range of literature offered on the bounding inequalities for the Bessel functions family members (see [1, 2, 3, 5, 6, 17, 18, 31, 54, 60, 67, 69, 70, 72, 74, 79, 80, 86, 90]) suggests to include a special chapter for obtaining and discussing bounding inequalities for the multi–parameter extensions of the Mathieu–type series.

(v) Diverse mathematical models are connected with the here exposed theory of Mathieu series, particularly with the general concept of the so–called $(\boldsymbol{a}, \boldsymbol{\lambda})$–series, see [70]. So, the inserted 7. Application section is devoted to certain applications of different type Mathieu series and generalizations in quantum physics (Casimir effect's mathematics); the 2D clamped plates and membranes vibration model described by the fourth order homogeneous and non-homogeneous differential equation $\Delta^2 f = 0$ associated with Neumann boundary condition; the same type Neumann problem for the 3D prism (both investigated by Mathieu himself); ODE description and definition of the Butzer–Flocke–Hauss complete Ω and the related study devoted to bounds' magnitude which are derived by the celebrated Chaplygin Differential Comparison Theorem.

The main objectives of this work are to present a multi–parameter study of the familiar Mathieu series and alternating Mathieu series $\mathscr{S}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}(\boldsymbol{r})$. Series expansions are obtained in terms of Lauricella's hypergeometric functions by using generalized Weber-Schafheitlin integral. Relationships of the multi–parameter Mathieu series and alternating Mathieu series with the Riemann Zeta function and the Dirichlet Eta function are also considered.

Next, a set of bounding inequalities are presented either for the hyper–Bessel and Bessel–Clifford functions which are to our best knowledge new, or also for all considered extended Mathieu–type series, while a short section concerning log–convexity and Turán inequalities are given for all considered extended Mathieu series.

The Applications section covers the topics listed above in **(v)**, while a thorough discussion finishes the exposed results.

2. Preliminaries. Hypergeometric and Bessel type special functions

In the beginning part of this section we recall power series definitions of generalizations of hypergeometric function which take place in our considerations.

The Fox-Wright function, which is a generalization of hypergeometric function, is defined as follows [24], [92, p. 4, Eq. (2.4)]:

$$ {}_p\Psi_q\left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p) \\ (b_1, B_1), \cdots, (b_q, B_q) \end{array} \middle| z \right] = \sum_{k\geq 0} \frac{\prod\limits_{j=1}^{p} \Gamma(a_j + kA_j)}{\prod\limits_{j=1}^{q} \Gamma(b_j + kB_j)} \frac{z^k}{k!}, \tag{2.1} $$

where $A_j > 0,\ j = 1, \cdots, p; B_j > 0,\ j = 1, \cdots, q$. The convergence conditions for the series at the right-hand side of (2.1) follow from the known asymptotic of the Euler Gamma–function. The defining series in (2.1) converges in the whole complex z-plane when

$$ \Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0. $$

If $\Delta = 0$, then the series in (2.1) converges for $|z| < \rho$, and $|z| = \rho$ under the condition $\Re(\theta) > \frac{1}{2}$ where

$$ \rho = \prod_{j=1}^{p} A_j^{-A_j} \cdot \prod_{j=1}^{q} B_j^{B_j}, \qquad \theta = \sum_{j=1}^{q} b_j - \sum_{k=1}^{p} a_k + \frac{p-q}{2}. \tag{2.2} $$

The Fox–Wright function extends the generalized hypergeometric function ${}_pF_q[z]$ which power series form reads

$$ {}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \middle| z \right] = \sum_{k\geq 0} \frac{\prod\limits_{l=1}^{p} (a_l)_k}{\prod\limits_{l=1}^{q} (b_l)_k} \frac{z^k}{k!}, $$

where, as usual, we make use of the Pochhammer symbol (or raising factorial)

$$ (\tau)_0 = 1; \quad (\tau)_k = \tau(\tau+1)\cdots(\tau+k-1) = \frac{\Gamma(\tau+k)}{\Gamma(\tau)}, \qquad k \in \mathbb{N}. $$

In the special case $A_r = B_s = 1; r = 1, \cdots, p; s = 1, \cdots, q$, the Fox–Wright function ${}_p\Psi_q[z]$ reduces (up to the multiplicative constant) to the generalized hypergeometric function

$$ {}_p\Psi_q\left[\begin{array}{c} (a_1, 1), \cdots, (a_p, 1) \\ (b_1, 1), \cdots, (b_q, 1) \end{array} \middle| z \right] = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}\, {}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \middle| z \right]. $$

The Bessel function of the first kind of the order ν has the power series definition

$$ J_\nu(z) = \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)}, \qquad -z \notin \mathbb{N};\ \nu \in \mathbb{C}, \tag{2.3} $$

while the modified Bessel functions of the first kind of the order ν has the expansion

$$ I_\nu(z) = \sum_{k\geq 0} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)}, \qquad -z \notin \mathbb{N};\ \nu \in \mathbb{C}. $$

Their multi–parameter analogues are the hyper–Bessel function by Delerue has the power series definition (see the inventory article [15] and also [1, 12, 16, 37]):

$$J_{\boldsymbol{\nu}}^{(m)}(z) = \left(\frac{z}{m+1}\right)^{\sum\limits_{j=1}^m \nu_j} \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{m+1}\right)^{k(m+1)}}{\Gamma(\nu_1+k+1)\cdots\Gamma(\nu_m+k+1)\,k!}\,,$$

and its modified variant is[1]

$$I_{\boldsymbol{\nu}}^{(m)}(z) = \left(\frac{z}{m+1}\right)^{\sum\limits_{j=1}^m \nu_j} \sum_{k\geq 0} \frac{\left(\frac{z}{m+1}\right)^{k(m+1)}}{\Gamma(\nu_1+k+1)\cdots\Gamma(\nu_m+k+1)\,k!}\,.$$

For $m = 1$ we arrive at the classical Bessel and modified Bessel functions, while for $m \geq 2$ we deduce the so–called Bessel–Clifford (or normalized hyper–Bessel) functions [28, p. 11, Eq. (2.5)]

$$C_{\boldsymbol{\nu}}^{(m)}(\pm z) = \begin{cases} z^{-\sum\limits_{j=1}^m \nu_j/(m+1)} J_{\boldsymbol{\nu}}^{(m)}\big((m+1)\,z^{1/(m+1)}\big) \\ z^{-\sum\limits_{j=1}^m \nu_j/(m+1)} I_{\boldsymbol{\nu}}^{(m)}(z)\big((m+1)\,z^{1/(m+1)}\big) \end{cases} = \sum_{k\geq 0} \frac{(\mp 1)^k z^k}{\prod\limits_{j=1}^m \Gamma(\nu_j+k+1)\,k!}\,. \tag{2.4}$$

In the case $m = 1$ the Bessel–Clifford function is related with the standard Bessel function of the first kind of the order $\nu_1 \equiv \nu$ *via* the equality [28, p. 11, Eq. (2.2)]

$$C_\nu^{(1)}(z) = C_\nu(z) = z^{-\frac{\nu}{2}}\, J_\nu(2\sqrt{z})\,. \tag{2.5}$$

Another type generalization includes the four multivariable Lauricella generalized hypergeometric series of m variables. We will apply $F_C^{(m)}$ (see the original definitions in Lauricella's introductory memoir [44, p. 113]) to infer some our main results. Its definition reads [85, p. 33, Eq. (1)]

$$F_C^{(m)}[\alpha,\beta;\boldsymbol{\gamma};\boldsymbol{x}] = \sum_{\boldsymbol{k}\geq 0} (\alpha)_{k_1+\cdots+k_m}\,(\beta)_{k_1+\cdots+k_m} \prod_{j=1}^m \frac{x_j^{k_j}}{(\gamma_j)_{k_j}\,k_j!}\,.$$

The convergence domain is $\sqrt{|x_1|} + \cdots + \sqrt{|x_m|} < 1$ established in [44, p. 116].

Finally, the Srivastava-Daoust generalization of the Lauricella hypergeometric functions in m variables defined by [82, p. 454]

$$S_{C:D';\cdots;D^{(n)}}^{A:B';\cdots;B^{(m)}}\left(\begin{array}{c|c} [(a):\theta',\cdots,\theta^{(m)}]:[(b'):\varphi'];\cdots;[(b^{(n)}):\varphi^{(m)}] & x_1 \\ & \vdots \\ [(c):\psi';\cdots;\psi^{(m)}]:[(d'):\delta'];\cdots;[(d^{(m)}):\delta^{(m)}] & x_m \end{array}\right)$$

$$= \sum_{\boldsymbol{k}\geq 0} \frac{\prod\limits_{j=1}^A (a_j)_{k_1\theta_j'+\cdots+k_m\theta_j^{(m)}} \prod\limits_{j=1}^{B'} (b_j')_{k_1\varphi_j'} \cdots \prod\limits_{j=1}^{B^{(m)}} \big(b_j^{(m)}\big)_{k_m\varphi_j^{(m)}}}{\prod\limits_{j=1}^C (c_j)_{k_1\psi_j'+\cdots+k_m\psi_j^{(m)}} \prod\limits_{j=1}^{D'} (d_j')_{k_1\delta_j'} \cdots \prod\limits_{j=1}^{D^{(m)}} \big(d_j^{(m)}\big)_{k_m\delta_j^{(m)}}} \frac{x_1^{k_1}}{k_1!}\cdots\frac{x_m^{k_m}}{k_m!}, \tag{2.6}$$

where the parameters satisfy

$$\theta_1',\cdots,\theta_A',\cdots,\delta_1^{(m)},\cdots,\delta_{D^{(m)}}^{(m)} > 0.$$

[1] Here, and in what follows, we shall write the shorthand $\boldsymbol{a} = (a_1,\cdots,a_m)$.

For convenience, we write (a) to denote the sequence of A parameters $a_1, \cdots, a_A$, with similar interpretations for $(b'), \cdots, (d^{(m)})$. Empty products should be interpreted as unity. Srivastava and Daoust [83, pp. 157–158] reported that the series in (2.6) converges absolutely

(i) for all $\boldsymbol{x} = (x_1, \cdots, x_m) \in \mathbb{C}^m$ when

$$\Delta_\ell = 1 + \sum_{j=1}^{C} \psi_j^{(\ell)} + \sum_{j=1}^{D^{(\ell)}} \delta_j^{(\ell)} - \sum_{j=1}^{A} \theta_j^{(\ell)} - \sum_{j=1}^{B^{(\ell)}} \varphi_j^{(\ell)} > 0, \qquad \ell = \overline{1, m};$$

(ii) for $|x_\ell| < \eta_\ell$ when $\Delta_\ell = 0$, $\ell = 1, \cdots, m$, where

$$\eta_\ell := \min_{\mu_1, \cdots, \mu_m > 0} \left\{ \mu_\ell^{1 + \sum\limits_{j=1}^{D^{(\ell)}} \delta_j^{(\ell)} - \sum\limits_{j=1}^{B^{(\ell)}} \varphi_j^{(\ell)}} \frac{\prod\limits_{j=1}^{C} \left(\sum\limits_{\ell=1}^{m} \mu_\ell \psi_j^{(\ell)} \right)^{\psi_j^{(\ell)}} \prod\limits_{j=1}^{D^{(\ell)}} \left(\delta_j^{(\ell)} \right)^{\delta_j^{(\ell)}}}{\prod\limits_{j=1}^{A} \left(\sum\limits_{\ell=1}^{n} \mu_\ell \theta_j^{(\ell)} \right)^{\theta_j^{(\ell)}} \prod\limits_{j=1}^{B^{(\ell)}} \left(\varphi_j^{(\ell)} \right)^{\varphi_j^{(\ell)}}} \right\}.$$

When all $\Delta_\ell < 0$, $\mathscr{S}^{A:B';\cdots;B^{(m)}}_{C:D';\cdots;D^{(m)}}(\boldsymbol{x})$ diverges exclusively at the origin, that is, this series is formal.

3. Multi–parameter Mathieu series $\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$

The extended Mathieu series $S(r)$ and its alternating variant $\widetilde{S}(r)$ read

$$S_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1}\, J_\nu(rx)\, \mathrm{d}x, \qquad \mu + \nu \geq 1, \tag{3.1}$$

$$\widetilde{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x + 1}\, J_\nu(rx)\, \mathrm{d}x, \qquad \mu + \nu \geq 0, \tag{3.2}$$

where in both cases $r > 0$, $\mu > 0$. Clearly $S_{\frac{3}{2},\frac{1}{2}}(r) = S(r)$ and $\widetilde{S}_{\frac{3}{2},\frac{1}{2}}(r) = \widetilde{S}(r)$.

Now, we introduce the multi–parameter Mathieu series and alternating Mathieu series $\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$; $\boldsymbol{a} = (a_1, \cdots, a_m)$ as follows

$$\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^{m} J_{\nu_j}(r_j\, x) \cdot \mathrm{d}x, \qquad \mu + \sum_{j=1}^{m} \nu_j \geq 1; \tag{3.3}$$

$$\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x + 1} \prod_{j=1}^{m} J_{\nu_j}(r_j\, x) \cdot \mathrm{d}x; \qquad \mu + \sum_{j=1}^{m} \nu_j \geq 0 \tag{3.4}$$

$$K_m(\boldsymbol{r}) = \left(\frac{\pi}{2}\right)^{\frac{m}{2}} \prod_{j=1}^{m} r_j^{-\frac{1}{2}}, \qquad \boldsymbol{r} \in \mathbb{R}_+^m.$$

At this point we remark that the Mathieu series $S(r)$ (so does its recently introduced alternating counterpart $\widetilde{S}(r)$), was considered by É. L. Mathieu for $r > 0$. Obviously, the definition (1.3) allows $r \in \mathbb{R}$. Following Mathieu's approach in taking the domain $r > 0$, we consider the first orthant $\mathbb{R}_+^m$ as the domain for $\boldsymbol{r}$ for the multi–parameter Mathieu series, also since we remain by this choice in real domain for $\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$.

Remark 1. *It is worth to mention that by means of the relation*

$$\frac{1}{\mathrm{e}^x+1} = \frac{1}{\mathrm{e}^x-1} - \frac{2}{\mathrm{e}^{2x}-1},$$

in (3.4) *and simplifying, we get the simple inter-relation*

$$\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = \mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) - 2^{-\mu-\frac{m}{2}+1}\,\mathscr{S}_{\mu,\boldsymbol{\nu}}\big(\tfrac12\,\boldsymbol{r}\big).$$

Accordingly, the analysis of the alternating multi–parameter Mathieu-type series study can be treated considering exclusively the multi–parameter Mathieu's $\mathscr{S}_{\mu,\boldsymbol{\nu}}$.

Theorem 1. *For all* $\mu \ge 1$, $\boldsymbol{\nu}+1 \in \mathbb{R}_+^{m+1}$, $\mu + \sum\limits_{j=1}^m \nu_j > 0$ *and* $\boldsymbol{r} \in \mathbb{R}_+^m$ *such that* $\sum\limits_{j=1}^m r_j < \mu$ *we have*

$$\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = \kappa_m(\mu,\boldsymbol{\nu}) \sum_{n\ge1} \frac{1}{n^{\mu+\sum\limits_{j=1}^m \nu_j}}\, F_C^{(m)}\Big[\frac12\Big(\mu+\sum_{j=1}^m \nu_j\Big), \frac12\Big(\mu+\sum_{j=1}^m \nu_j+1\Big); \boldsymbol{\nu}+1; -\frac{\boldsymbol{r}^2}{n^2}\Big]\,,$$

$$\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = \kappa_m(\mu,\boldsymbol{\nu}) \sum_{n\ge1} \frac{(-1)^{n-1}}{n^{\mu+\sum\limits_{j=1}^m \nu_j}}\, F_C^{(m)}\Big[\frac12\Big(\mu+\sum_{j=1}^m \nu_j\Big), \frac12\Big(\mu+\sum_{j=1}^m \nu_j+1\Big); \boldsymbol{\nu}+1; -\frac{\boldsymbol{r}^2}{n^2}\Big]\,,$$

where

$$\kappa_m(\mu,\boldsymbol{\nu}) = \frac{\pi^{\frac{m}{2}}\,\Gamma\Big(\mu+\sum\limits_{j=1}^m \nu_j\Big)}{2^{\frac{m}{2}+\sum_{j=1}^m \nu_j}} \prod_{j=1}^m \frac{r_j^{\nu_j-\frac12}}{\Gamma(\nu_j+1)}\,.$$

Proof. Insert the binomial series expansion of the kernel $(\mathrm{e}^x-1)^{-1} = \sum_{n\ge1}\mathrm{e}^{-nx}$, into (3.3), valid for the whole integration domain $x>0$. The legitimate integral–sum interchange, which can be proved e.g. by the dominated convergence theorem results in

$$\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \sum_{n\ge1} \int_0^\infty x^{\mu-1}\,\mathrm{e}^{-nx} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\cdot \mathrm{d}x\,. \tag{3.5}$$

Making use of the generalized Weber–Schafheitlin integral [84, p. 2, Eq.(2.2)]

$$\int_0^\infty x^{\mu-1}\mathrm{e}^{-\alpha x} \prod_{j=1}^m J_{\nu_j}(\beta_j x)\,\mathrm{d}x = \frac{\prod\limits_{j=1}^m \big(\frac{\beta_j}{2}\big)^{\nu_j}\Gamma\Big(\mu+\sum\limits_{j=1}^m \nu_j\Big)}{\alpha^{\mu+\sum\limits_{j=1}^m \nu_j}\,\Gamma(\nu_1+1)\cdots\Gamma(\nu_m+1)}$$
$$\times\, F_C^{(m)}\Big[\frac12\Big(\mu+\sum_{j=1}^m \nu_j\Big), \frac12\Big(\mu+\sum_{j=1}^m \nu_j+1\Big); \boldsymbol{\nu}+1; -\frac{\boldsymbol{\beta}^2}{\alpha^2}\Big], \tag{3.6}$$

which parameter space consists from $\Re(\mu)>0$, $\Re\big(\mu+\sum_{j=1}^m \nu_j\big) > 0$ and $|\beta_1|+\cdots+|\beta_m| < |\mu|$, specifying in (3.5) $\alpha = n$ and $\beta_j = r_j$; $j=1,\cdots,m$, we conclude the first asserted formula.

The derivation procedure of the series expansion result for $\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ applies the binomial series $(1+\mathrm{e}^x)^{-1} = \sum_{n\ge1}(-1)^{n-1}\mathrm{e}^{-nx}$, $x>0$. Now, the path to the final formula is obvious. □

The next result gives an insight how our series expansion formulae are related to the one-parameter initial Mathieu series (3.3) and(3.4).

Corollary 1.1. *For all* μ, $\nu+1>0$, $\mu+\nu>0$ *we have*

$$\begin{aligned}\mathscr{S}_{\mu,\nu}(r) &= \kappa_1(\mu,\nu)\sum_{n\geq 1}\frac{1}{n^{\mu+\nu}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(\mu+\nu+1)\\ \nu+1\end{matrix}\Big|-\frac{r^2}{n^2}\Big]\\ &= \kappa_1(\mu,\nu)\sum_{n\geq 1}\frac{1}{(n^2+r^2)^{\frac{\mu+\nu}{2}}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(\nu-\mu+1)\\ \nu+1\end{matrix}\Big|\frac{r^2}{n^2+r^2}\Big].\end{aligned}$$

Moreover, when $\mu+\nu+1>0$ *there holds*

$$\begin{aligned}\widetilde{\mathscr{S}}_{\mu,\nu}(r) &= \kappa_1(\mu,\nu)\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^{\mu+\nu}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(\mu+\nu+1)\\ \nu+1\end{matrix}\Big|-\frac{r^2}{n^2}\Big]\\ &= \kappa_1(\mu,\nu)\sum_{n\geq 1}\frac{(-1)^{n-1}}{(n^2+r^2)^{\frac{\mu+\nu}{2}}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(\nu-\mu+1)\\ \nu+1\end{matrix}\Big|\frac{r^2}{n^2+r^2}\Big],\end{aligned}$$

where

$$\kappa_1(\mu,\nu)=\frac{\sqrt{\pi}\,r^{\nu-\frac12}\,\Gamma(\mu+\nu)}{2^{\nu+\frac12}\,\Gamma(\nu+1)}.$$

Proof. Putting $m=1$ in Theorem 1, then having in mind that now $\boldsymbol{\nu}=\nu_1\equiv\nu$, $\boldsymbol{r}=r_1\equiv r$ and the Lauricella $F_C^{(1)}$ reduces to the Gaussian ${}_2F_1$ function we have instead of (3.6) the integral (see Watson's historically developmental comments [91, p. 384 *et seq.*, Eqs. (2) and (3)] concerning this result who attributed the first formula to Hankel and Gegenbauer):

$$\begin{aligned}\int_0^\infty x^{\mu-1}\mathrm{e}^{-ax}\,J_\nu(bx)\,\mathrm{d}x &= \frac{\left(\frac{b}{2}\right)^\nu\Gamma(\mu+\nu)}{a^{\mu+\nu}\,\Gamma(\nu+1)}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(\mu+\nu+1)\\ \nu+1\end{matrix}\Big|-\frac{b^2}{a^2}\Big]\\ &= \frac{\left(\frac{b}{2}\right)^\nu\Gamma(\mu+\nu)}{(a^2+b^2)^{\frac12(\mu+\nu)}\,\Gamma(\nu+1)}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(1-\mu+\nu)\\ \nu+1\end{matrix}\Big|\frac{b^2}{a^2+b^2}\Big],\end{aligned}$$

where the second equality follows by the Pfaff linear transform of the argument in Gaussian hypergeometric function. Following the steps in the proof of Theorem 1, we conclude the statements specifying $a=n; b=r$. □

The reconstruction of the sine functions product in multi–parameter Mathieu series' integrand in (3.3) and (3.4), respectively, the Bessel function's order should be specified as $\nu_j=\frac12$, $j=1,\cdots,m$, in our setting $\boldsymbol{\nu}=\boldsymbol{\frac12}$. The associated integrals read

$$\mathscr{S}_{\mu,\frac12}(\boldsymbol{r})=\frac{1}{\prod\limits_{j=1}^m r_j}\int_0^\infty\frac{x^{\mu-\frac{m}{2}-1}}{\mathrm{e}^x-1}\prod_{j=1}^m\sin(r_j\,x)\cdot\mathrm{d}x, \tag{3.7}$$

$$\widetilde{\mathscr{S}}_{\mu,\frac12}(\boldsymbol{r})=\frac{1}{\prod\limits_{j=1}^m r_j}\int_0^\infty\frac{x^{\mu-\frac{m}{2}-1}}{\mathrm{e}^x+1}\prod_{j=1}^m\sin(r_j\,x)\cdot\mathrm{d}x. \tag{3.8}$$

To restore $S(r)$ and $\widetilde{S}(r)$ in (1.4) and (1.6) it is enough to specify $m=1$, $\mu=\frac52$ in the previous couple of formulae. Hence, we have

Corollary 1.2. *For all* $\mu \geq 1$ *and* $\boldsymbol{r} \in \mathbb{R}_+^m$ *such that* $\sum\limits_{j=1}^m r_j < \mu$, *there holds*

$$\mathscr{S}_{\mu,\frac{\boldsymbol{1}}{\boldsymbol{2}}}(\boldsymbol{r}) = \Gamma\left(\mu + \frac{m}{2}\right) \sum_{n\geq 1} \frac{1}{n^{\mu+\frac{m}{2}}}\, F_C^{(m)}\left[\frac{1}{2}\left(\mu + \frac{m}{2}\right), \frac{1}{2}\left(\mu + \frac{m}{2} + 1\right); \frac{\boldsymbol{3}}{\boldsymbol{2}}; -\frac{\boldsymbol{r}^2}{n^2}\right],$$

$$\widetilde{\mathscr{S}}_{\mu,\frac{\boldsymbol{1}}{\boldsymbol{2}}}(\boldsymbol{r}) = \Gamma\left(\mu + \frac{m}{2}\right) \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{\mu+\frac{m}{2}}}\, F_C^{(m)}\left[\frac{1}{2}\left(\mu + \frac{m}{2}\right), \frac{1}{2}\left(\mu + \frac{m}{2} + 1\right); \frac{\boldsymbol{3}}{\boldsymbol{2}}; -\frac{\boldsymbol{r}^2}{n^2}\right].$$

Theorem 2. *For all* $\mu > 0, \boldsymbol{\nu} + 1 \in \mathbb{R}_+^m$ *such that* $\mu + \sum\limits_{j=1}^m \nu_j > 1$, *and for all* $\boldsymbol{r} \in \mathbb{R}_+^m$, *we have*

$$\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = L_m(\boldsymbol{r}) \sum_{\boldsymbol{k}\geq 0} \Gamma\Big(\mu + \sum_{j=1}^m (\nu_j + 2k_j)\Big)\, \zeta\Big(\mu + \sum_{j=1}^m (\nu_j + 2k_j)\Big) \prod_{j=1}^m \frac{(-1)^{k_j}\, \big(\frac{r_j}{2}\big)^{2k_j}}{\Gamma(k_j + \nu_j + 1)\, k_j!}.$$

Moreover, for the same parameter space, when $\mu + \sum\limits_{j=1}^m \nu_j > 0$,

$$\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = L_m(\boldsymbol{r}) \sum_{\boldsymbol{k}\geq 0} \Gamma\Big(\mu + \sum_{j=1}^m (\nu_j + 2k_j)\Big)\, \eta\Big(\mu + \sum_{j=1}^m (\nu_j + 2k_j)\Big) \prod_{j=1}^m \frac{(-1)^{k_j}\, \big(\frac{r_j}{2}\big)^{2k_j}}{\Gamma(k_j + \nu_j + 1)\, k_j!},$$

where the constant

$$L_m(\boldsymbol{r}) = \pi^{\frac{m}{2}}\, 2^{-\frac{m}{2} - \sum\limits_{j=1}^m \nu_j} \prod_{j=1}^m r_j^{\nu_j - \frac{1}{2}}.$$

Proof. Using the Bessel function series form (2.3) in (3.3), we have

$$\begin{aligned}
\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) &= K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m J_{\nu_j}(r_j\, x) \cdot \mathrm{d}x \\
&= K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m \sum_{k_j\geq 0} \frac{(-1)^{k_j}}{\Gamma(k_j + \nu_j + 1)\, k_j!} \Big(\frac{r_j x}{2}\Big)^{\nu_j + 2k_j} \cdot \mathrm{d}x \\
&= K_m(\boldsymbol{r}) \sum_{\boldsymbol{k}\geq 0} \int_0^\infty \frac{x^{\mu + \sum_{j=1}^m (\nu_j + 2k_j) - 1}}{\mathrm{e}^x - 1}\, \mathrm{d}x \cdot \prod_{j=1}^m \frac{(-1)^{k_j}}{\Gamma(k_j + \nu_j + 1)\, k_j!} \Big(\frac{r_j}{2}\Big)^{\nu_j + 2k_j} \\
&= K_m(\boldsymbol{r}) \sum_{\boldsymbol{k}\geq 0} \Gamma\Big(\mu + \sum_{j=1}^m (\nu_j + 2k_j)\Big)\, \zeta\Big(\mu + \sum_{j=1}^m (\nu_j + 2k_j)\Big) \prod_{j=1}^m \frac{(-1)^{k_j}\, \big(\frac{r_j}{2}\big)^{\nu_j + 2k_j}}{\Gamma(k_j + \nu_j + 1)\, k_j!},
\end{aligned}$$

which is equivalent to the first statement of the theorem. In the derivation procedure we apply the integral representation (1.1) of the Riemann Zeta function.

Similarly, if we start with the expression (3.4) we obtain the second formula with the aid of Dirichlet Eta function's integral form (1.2). In both cases the parameter constraints are controlled by (1.1) and (1.2) convergence conditions, respectively. □

Now we re-consider the integral expressions (3.7) and (3.8) of the multi–parameter Mathieu series $\mathscr{S}_{\mu,\frac{\boldsymbol{1}}{\boldsymbol{2}}}(\boldsymbol{r})$ and its alternating counterpart $\widetilde{\mathscr{S}}_{\mu,\frac{\boldsymbol{1}}{\boldsymbol{2}}}(\boldsymbol{r})$. The related multiple series expansions in terms of Riemann ζ and Dirichlet η functions give the following results.

Corollary 2.1. *For all $\mu > 0$ such that $\mu + \frac{m}{2} > 1$, and for all $\boldsymbol{r} \in \mathbb{R}_+^m$, we have*

$$\mathscr{S}_{\mu,\frac{1}{2}}(\boldsymbol{r}) = \sum_{\boldsymbol{k}\geq 0} \Gamma\Big(\mu + \frac{m}{2} + 2\sum_{j=1}^m k_j\Big)\zeta\Big(\mu + \frac{m}{2} + 2\sum_{j=1}^m k_j\Big) \prod_{j=1}^m \frac{(-1)^{k_j} r_j^{2k_j}}{(2k_j+1)!}\,, \tag{3.9}$$

while $\mu + \frac{m}{2} > 0$ implies

$$\mathscr{S}_{\mu,\frac{1}{2}}(\boldsymbol{r}) = \sum_{\boldsymbol{k}\geq 0} \Gamma\Big(\mu + \frac{m}{2} + 2\sum_{j=1}^m k_j\Big)\eta\Big(\mu + \frac{m}{2} + 2\sum_{j=1}^m k_j\Big) \prod_{j=1}^m \frac{(-1)^{k_j} r_j^{2k_j}}{(2k_j+1)!}\,. \tag{3.10}$$

Proof. The Maclaurin series expansion of the sine function in the integrand of (3.7) implies

$$\begin{aligned}\mathscr{S}_{\mu,\frac{1}{2}}(\boldsymbol{r}) &= \frac{1}{\prod\limits_{j=1}^m r_j} \int_0^\infty \frac{x^{\mu-\frac{m}{2}-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m \sum_{k_j\geq 0} \frac{(-1)^{k_j}(r_j x)^{2k_j+1}}{(2k_j+1)!} \cdot \mathrm{d}x \\ &= \sum_{\boldsymbol{k}\geq 0} (-1)^{\sum_{j=1}^m k_j} \prod_{j=1}^m \frac{r_j^{2k_j}}{(2k_j+1)!} \int_0^\infty \frac{x^{\mu+\frac{m}{2}+2\sum_{j=1}^m k_j - 1}}{\mathrm{e}^x - 1}\,\mathrm{d}x\,.\end{aligned}$$

By virtue of the integral (1.1) with the parameter $s = \mu + \frac{m}{2} + 2\sum_{j=1}^m k_j > 1$ we conclude the first result (3.9). Similar steps lead to (3.10). □

4. Extending Mathieu series *via* hyper–Bessel functions

The next step in generalization of the Mathieu series' integrals includes the product of several hyper–Bessel functions which replaces the product of Bessel functions of the first kind in (3.3) and (3.4). This extension includes the Bessel–Clifford functions case by virtue of the inter–relation $C_\nu(z) = z^{-\nu/2}\, J_\nu(2\sqrt{z})$, see in the context of (2.5). Thus, the hyper–Bessel generalized Mathieu–type integral expression and its alternating version read

$$\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\, x)\cdot \mathrm{d}x, \qquad \mu + \sum_{j=1}^m \sum_{k=1}^{\ell_j} \nu_{jk} > 1; \tag{4.1}$$

$$\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x + 1} \prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\, x)\cdot \mathrm{d}x; \qquad \mu + \sum_{j=1}^m \sum_{k=1}^{\ell_j} \nu_{jk} > 0 \tag{4.2}$$

$$K_m(\boldsymbol{r}) = \Big(\frac{\pi}{2}\Big)^{\frac{m}{2}} \prod_{j=1}^m r_j^{-\frac{1}{2}}, \qquad \boldsymbol{r} \in \mathbb{R}_+^m,$$

where the multi–index $\bar{\boldsymbol{\nu}} := (\boldsymbol{\nu_1}, \cdots, \boldsymbol{\nu_m})$ and $\boldsymbol{\nu_j} = (\nu_{j1}, \cdots, \nu_{j\ell_j})$.

The counterpart definitions of modified hyper–Bessel extensions of the Mathieu–type series' integral expressions we introduce when in the (4.1) and (4.2) we replace the hyper–Bessel function $J^{(m)}_{\boldsymbol{\nu_j}}(x)$ with the modified hyper–Bessel function $I^{(m)}_{\boldsymbol{\nu_j}}(x)$. Hence,

$$\mathscr{S}^I_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m I^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\, x)\cdot \mathrm{d}x,$$

$$\widetilde{\mathscr{S}}^{I}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x+1} \prod_{j=1}^m I^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\, x)\cdot \mathrm{d}x\,.$$

The parameter space remains $\boldsymbol{r} \in \mathbb{R}^m_+$, while the constant $K_m(\boldsymbol{r})$ coincides with the one in the above quoted relations (4.1) and (4.2).

Here, and in what follows, under Laplace–Mellin transform of a suitable input function $h(x)$ we mean the improper integral

$$\mathcal{LM}[h](p,s) = \int_0^\infty \mathrm{e}^{-px}x^{s-1}h(x)\,\mathrm{d}x\,,$$

where the parameter space of p and s depend on the nature and behavior of h.

In inferring the series form of the hyper–Bessel extensions of the Mathieu–type integrals we need two lemmata which precise the Laplace–Mellin transforms of the product of finitely man m, say, hyper–Bessel or modified hyper–Bessel functions of not necessarily equal number of order parameters $\ell_j, j=1,\cdots,m$. We could not find any traces in the literature of these Laplace–Mellin transformation result which are of particular interest by themselves.

Proposition 1. *For all $\Re(p), \mu, \nu_{jk}-1>0$, such that $\mu+\sum_{j=1}^m\sum_{k=1}^{\ell_j}\nu_{jk}>0$; $\ell_j\in\mathbb{N}$, when $j=1,\cdots,m; k=1,\cdots,\ell_j$, and $\boldsymbol{r}\in\mathbb{R}^m_+$, the following Laplace–Mellin transform holds*

$$\begin{aligned}\mathcal{LM}\Big[\prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\Big](p,\mu) &= \int_0^\infty \mathrm{e}^{-p\,x}\,x^{\mu-1}\prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\cdot\mathrm{d}x\\ &= B_p(\mu,\boldsymbol{r})\cdot S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix}\big[(\mu+\boldsymbol{\nu}^\star):\boldsymbol{\ell}+1\big]:\,[-:-];\cdots;[-:-]\\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big]\end{matrix}\;\Bigg|\,\boldsymbol{x}\right),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathcal{LM}\Big[\prod_{j=1}^m I^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\Big](p,\mu) &= \int_0^\infty \mathrm{e}^{-p\,x}\,x^{\mu-1}\prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\cdot\mathrm{d}x\\ &= B_p(\mu,\boldsymbol{r})\cdot S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix}\big[(\mu+\boldsymbol{\nu}^\star):\boldsymbol{\ell}+1\big]:\,[-:-];\cdots;[-:-]\\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big]\end{matrix}\;\Bigg|\,-\boldsymbol{x}\right),\end{aligned} \tag{4.4}$$

where $\boldsymbol{x}=(x_1,\cdots,x_m)^T$ with

$$x_j = -\Big(\frac{r_j}{p(\ell_j+1)}\Big)^{\ell_j+1},\quad j=1,\cdots,m;\qquad \text{and}\qquad \boldsymbol{\nu}^\star=\sum_{j=1}^m\sum_{k=1}^{\ell_j}\nu_{jk}\,.$$

The constant

$$B_p(\mu,\boldsymbol{r}) = \frac{\Gamma(\mu+\boldsymbol{\nu}^\star)}{p^{\,\mu+\boldsymbol{\nu}^\star}}\prod_{j=1}^m\prod_{k=1}^{\ell_j}\frac{\Big(\dfrac{r_j}{\ell_j+1}\Big)^{\nu_{jk}}}{\Gamma(\nu_{jk}+1)}\,.$$

Proof. In the integrand we express the hyper–Bessel functions by their Maclaurin series. After some routine calculations we get

$$\mathscr{I}_p(\mu, \bar{\boldsymbol{\nu}}, \boldsymbol{r}) = \prod_{j=1}^m \Big(\frac{r_j}{\ell_j+1}\Big)^{\sum\limits_{k=1}^{\ell} \nu_{jk}} \cdot \sum_{\boldsymbol{k}\geq 0} \frac{(-1)^{\sum\limits_{j=1}^m k_j} \prod\limits_{j=1}^m \big(\frac{r_j}{\ell_j+1}\big)^{k_j(\ell_j+1)}}{\prod\limits_{j=1}^m \Big(k_j! \prod\limits_{k=1}^{\ell_j} \Gamma(\nu_{jk}+k_j+1)\Big)} \int_0^\infty \mathrm{e}^{-p\,x} x^{q-1}\, \mathrm{d}x\,,$$

where

$$q = \mu + \sum_{j=1}^m \sum_{k=1}^{\ell_j} \nu_{jk} + \sum_{j=1}^m k_j(\ell_j+1)\,.$$

Inserting the Gamma integral's solution we have

$$\mathscr{I}_p(\mu, \bar{\boldsymbol{\nu}}, \boldsymbol{r}) = \frac{1}{p^\mu} \prod_{j=1}^m \Big(\frac{r_j}{p(\ell_j+1)}\Big)^{\sum\limits_{k=1}^{\ell_j} \nu_{jk}} \sum_{\boldsymbol{k}\geq 0} \frac{(-1)^{\sum\limits_{j=1}^m k_j} \prod\limits_{j=1}^m \big(\frac{r_j}{\ell_j+1}\big)^{k_j(\ell_j+1)}}{\prod\limits_{j=1}^m \Big(k_j! \prod\limits_{j=1}^{\ell_j} \Gamma(\nu_{jk}+k_j+1)\Big)}$$

$$\times\, \Gamma\Big(\mu + \sum_{j=1}^m \sum_{k=1}^{\ell_j} \nu_{jk} + \sum_{j=1}^m k_j(\ell_j+1)\Big)\,. \tag{4.5}$$

Now using the Pochhamer symbol notation there holds

$$\mathscr{I}_p(\mu, \bar{\boldsymbol{\nu}}, \boldsymbol{r}) = B_p(\mu, \bar{\boldsymbol{\nu}}, \boldsymbol{r}) \sum_{\boldsymbol{k}\geq 0} \frac{\Big(\mu + \sum\limits_{j=1}^m \sum\limits_{k=1}^{\ell_j} \nu_{jk}\Big)_{\sum\limits_{j=1}^m (\ell_j+1)k_j}}{\prod\limits_{j=1}^m \prod\limits_{k=1}^{\ell_j} (\nu_{jk}+1)_{k_j}} \prod_{j=1}^m \frac{(-1)^{k_j}}{k_j!} \Big(\frac{r_j}{p(\ell_j+1)}\Big)^{(\ell_j+1)k_j}\,.$$

In turn, comparing this expression with (2.6), we clearly get the Srivastava–Daoust S function as the stated result. The transform (4.4) we derive in the same manner, however the sign change of $\boldsymbol{x}$ in the S function implies the final expression. $\square$

The sub–case $\ell_1 = \cdots = \ell_m = \ell$ of Proposition 1 we can specify to a greater extent.

Proposition 2. *For all* $\Re(p), \mu, \nu_{jk} - 1 > 0; \ell \in \mathbb{N}$ *when* $j = 1, \cdots, m; k = 1, \cdots, \ell$ *such that* $\mu + \sum\limits_{j=1}^m \sum\limits_{k=1}^{\ell} \nu_{jk} > 0$ *and* $\boldsymbol{r} \in \mathbb{R}_+^m$ *the following Laplace–Mellin transforms hold:*

$$\mathcal{LM}\Big[\prod_{j=1}^m J_{\boldsymbol{\nu}_j}^{(\ell)}(r_j\, x)\Big](p,\mu) = \int_0^\infty \mathrm{e}^{-p\,x}\, x^{\mu-1} \prod_{j=1}^m J_{\boldsymbol{\nu}_j}^{(\ell)}(r_j\, x)\cdot \mathrm{d}x$$

$$= A_p(\mu, \boldsymbol{r}) \cdot S^{\boldsymbol{\ell}+1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{array}{c} \Big[\frac{1}{\ell+1}\Big(\mu+\boldsymbol{\nu}_\star + r\Big) : \boldsymbol{\ell}+1\Big]_{r=0}^{\ell} : [-:-]; \cdots ; [-:-] \\ - : \Big[\prod\limits_{k=1}^{\ell}(\nu_{1k}+1):1\Big]; \cdots ; \Big[\prod\limits_{k=1}^{\ell}(\nu_{mk}+1):1\Big] \end{array} \Bigg| \boldsymbol{x} \right),$$

and

$$\mathcal{LM}\Big[\prod_{j=1}^m I_{\boldsymbol{\nu}_j}^{(\ell)}(r_j\,x)\Big](p,\mu) = \int_0^\infty \mathrm{e}^{-p\,x}\,x^{\mu-1}\,\prod_{j=1}^m I_{\boldsymbol{\nu}_j}^{(\ell)}(r_j\,x)\cdot \mathrm{d}x$$

$$= A_p(\mu,\boldsymbol{r})\cdot S^{\boldsymbol{\ell}+1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{array}{c} \Big[\frac{1}{\ell+1}\Big(\mu+\boldsymbol{\nu}_\star+r\Big):\boldsymbol{\ell}+1\Big]_{r=0}^{\ell}:[-:-];\cdots;[-:-] \\ -:\Big[\prod\limits_{k=1}^{\ell}(\nu_{1k}+1:1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell}(\nu_{mk}+1):1\Big]\end{array}\Bigg|\,-\boldsymbol{x}\right),$$

where $\boldsymbol{x}=(x_1,\cdots,x_m)^T$ *with*

$$x_j = -\Big(\frac{r_j}{p(\ell+1)}\Big)^{\ell+1}, \quad j=1,\cdots,m\,; \qquad \text{and} \qquad \boldsymbol{\nu}_\star=\sum_{j=1}^m\sum_{k=1}^{\ell}\nu_{jk}\,.$$

The constant

$$A_p(\mu,\boldsymbol{r}) = \frac{(\ell+1)^{\mu-\frac12}}{(2\pi)^{\frac{\ell}{2}}\,p^{\mu+\boldsymbol{\nu}_\star}}\prod_{j=1}^m \frac{r_j^{\sum_{k=1}^{\ell}\nu_{jk}}\prod\limits_{r=1}^{\ell+1}\Gamma\Big[\frac{1}{(\ell+1)}\Big(\mu+\boldsymbol{\nu}_\star+r-1\Big)\Big]}{\prod\limits_{j=1}^{\ell}\Gamma(\nu_{jk}+1)}\,.$$

Proof. Firstly, we equalize all parameter sizes $\ell_j=\ell, j=1,\cdots,m$ in (4.5). Then, by the Gauss multiplication theorem [85, p. 16, Eq. (16)]

$$\Gamma(nz) = (2\pi)^{\frac12(1-n)}n^{n\,z-\frac12}\,\prod_{j=1}^n\Gamma\Big(z+\frac{j-1}{n}\Big), \qquad z\neq 0,-\frac1n,-\frac2n,\cdots; \quad n\in\mathbb{N}\,,$$

transform (4.5) and rewrite the resulting expression in Pochhammer symbols notation. Hence,

$$\begin{aligned}\mathscr{I}_p(\mu,\bar{\boldsymbol{\nu}},\boldsymbol{r}) &= \frac{(\ell+1)^{\mu-\frac12}}{(2\pi)^{\frac{\ell}{2}}\,p^{\mu}}\prod_{j=1}^m\Big(\frac{r_j}{p}\Big)^{\sum_{k=1}^{\ell}\nu_{jk}}\cdot\sum_{\boldsymbol{k}\geq 0}\frac{(-1)^{\sum_{j=1}^m k_j}\prod\limits_{j=1}^m\big(\frac{r_j}{p}\big)^{(\ell+1)k_j}}{\prod\limits_{j=1}^m\Big(k_j!\prod\limits_{j=1}^{\ell}\Gamma(\nu_{jk}+k_j+1)\Big)} \\ &\quad\times\prod_{r=1}^{\ell+1}\Gamma\Big[\frac{1}{(\ell+1)}\Big(\mu+\sum_{j=1}^m\sum_{k=1}^{\ell}\nu_{jk}+r-1\Big)+\sum_{j=1}^m k_j\Big] \\ &= A_p(\mu,\boldsymbol{r})\sum_{\boldsymbol{k}\geq 0}\frac{\prod\limits_{r=1}^{\ell+1}\Big(\frac{1}{(\ell+1)}\Big[\mu+\sum\limits_{j=1}^m\sum\limits_{k=1}^{\ell}\nu_{jk}+r-1\Big]\Big)_{\sum_{j=1}^m k_j}}{\prod\limits_{j=1}^m\prod\limits_{j=1}^{\ell}(\nu_{jk}+1)_{k_j}}\prod_{j=1}^m\frac{\big[-\big(\frac{r_j}{p}\big)^{\ell+1}\big]^{k_j}}{k_j!},\end{aligned}$$

where

$$A_p(\mu,\boldsymbol{r}) = \frac{(\ell+1)^{\mu-\frac12}}{(2\pi)^{\frac{\ell}{2}}\,p^{\mu+\sum_{j=1}^m\sum_{k=1}^{\ell}\nu_{jk}}}\prod_{j=1}^m\frac{r_j^{\sum_{k=1}^{\ell}\nu_{jk}}\prod\limits_{r=1}^{\ell+1}\Gamma\Big[\frac{1}{(\ell+1)}\Big(\mu+\sum\limits_{j=1}^m\sum\limits_{k=1}^{\ell}\nu_{jk}+r-1\Big)\Big]}{\prod\limits_{k=1}^{\ell}\Gamma(\nu_{jk}+1)}\,.$$

Now, we recognize the multiple series as the Srivastava–Daoust S function of m variables. □

Theorem 3. *Let the parameter space and the domain be the same as in* Proposition 1. *Then*

$$\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = C_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{1}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \big[(\mu+\boldsymbol{\nu}^\star):\boldsymbol{\ell}+1\big]:[-:-];\cdots;[-:-] \\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, \boldsymbol{x}\right),$$

$$\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = C_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \big[(\mu+\boldsymbol{\nu}^\star):\boldsymbol{\ell}+1\big]:[-:-];\cdots;[-:-] \\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, \boldsymbol{x}\right),$$

$$\mathscr{S}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = C_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{1}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \big[(\mu+\boldsymbol{\nu}^\star):\boldsymbol{\ell}+1\big]:[-:-];\cdots;[-:-] \\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, -\boldsymbol{x}\right),$$

$$\widetilde{\mathscr{S}}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = C_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \big[(\mu+\boldsymbol{\nu}^\star):\boldsymbol{\ell}+1\big]:[-:-];\cdots;[-:-] \\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, -\boldsymbol{x}\right),$$

where $\boldsymbol{x} = (x_1, \cdots, x_m)^T$ *with coordinates*

$$x_j = -\Big(\frac{r_j}{n(\ell_j+1)}\Big)^{\ell_j+1}, \quad j = 1, \cdots, m; \qquad \text{and} \qquad \boldsymbol{\nu}^\star = \sum_{j=1}^m \sum_{k=1}^{\ell_j} \nu_{jk}\,,$$

while

$$C_m(\boldsymbol{r}) = \Big(\frac{\pi}{2}\Big)^{\frac{m}{2}}\, \Gamma(\mu+\boldsymbol{\nu}^\star) \prod_{j=1}^m \frac{\big[r_j/(\ell_j+1)\big]^{\sum\limits_{k=1}^{\ell_j}\nu_{jk}}}{\sqrt{r_j}\,\prod\limits_{k=1}^{\ell_j}\Gamma(\nu_{jk}+1)}\,.$$

Proof. The derivation follows the method from Theorem 1. The binomial series of $(\mathrm{e}^x-1)^{-1}$ gives

$$\begin{aligned} \mathscr{S}_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) &= K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1} \prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\cdot \mathrm{d}x \\ &= K_m(\boldsymbol{r}) \sum_{n\geq 1} \int_0^\infty x^{\mu-1}\,\mathrm{e}^{-nx} \prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\cdot \mathrm{d}x\,. \end{aligned}$$

Now, it is enough to take the Laplace–Mellin transform result (4.3) to earn the first statement, setting $p = n$. As to the computation series expansion of the remaining three Mathieu–type series' integrals $\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$, $\mathscr{S}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$ we apply the same method, this time starting with the corresponding series of $(\mathrm{e}^x \pm 1)^{-1}$. □

The synthesis of Proposition 2 and Theorem 3 can be formulate as

Corollary 3.1. *Let the parameter space and the domain be the same as in* Proposition 2. *Then we have*

$$\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = D_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{1}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{\boldsymbol{\ell}+1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \Big[\frac{1}{\ell+1}\Big(\mu+\boldsymbol{\nu}_\star+r\Big):\boldsymbol{\ell}+1\Big]_{r=0}^{\ell}:[-:-];\cdots \\ -:\Big[\prod\limits_{k=1}^{\ell}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, \boldsymbol{x}\right),$$

$$\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = D_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{\boldsymbol{\ell}+1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \Big[\frac{1}{\ell+1}\Big(\mu+\boldsymbol{\nu}_\star+r\Big):\boldsymbol{\ell}+1\Big]_{r=0}^{\ell} : [-:-];\cdots \\ -:\Big[\prod\limits_{k=1}^{\ell}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, \boldsymbol{x}\right),$$

$$\mathscr{S}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = D_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{1}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{\boldsymbol{\ell}+1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \Big[\frac{1}{\ell+1}\Big(\mu+\boldsymbol{\nu}_\star+r\Big):\boldsymbol{\ell}+1\Big]_{r=0}^{\ell} : [-:-];\cdots \\ -:\Big[\prod\limits_{k=1}^{\ell}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, -\boldsymbol{x}\right),$$

$$\widetilde{\mathscr{S}}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}) = D_m(\boldsymbol{r}) \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{\mu+\boldsymbol{\nu}^\star}}\, S^{\boldsymbol{\ell}+1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix} \Big[\frac{1}{\ell+1}\Big(\mu+\boldsymbol{\nu}_\star+r\Big):\boldsymbol{\ell}+1\Big]_{r=0}^{\ell} : [-:-];\cdots \\ -:\Big[\prod\limits_{k=1}^{\ell}(\nu_{1k}+1):1\Big];\cdots;\Big[\prod\limits_{k=1}^{\ell}(\nu_{mk}+1):1\Big] \end{matrix} \,\Bigg|\, -\boldsymbol{x}\right),$$

where $\boldsymbol{x} = (x_1,\cdots,x_m)^T$ *with coordinates*

$$x_j = -\Big(\frac{r_j}{n(\ell+1)}\Big)^{\ell+1}, \quad j=1,\cdots,m; \qquad \text{and} \qquad \boldsymbol{\nu}_\star = \sum_{j=1}^m\sum_{k=1}^{\ell}\nu_{jk}\,.$$

The constant

$$D_m(\boldsymbol{r}) = \frac{\pi^{\frac{m-\ell}{2}}(\ell+1)^{\mu-\frac12}}{2^{\frac{m+\ell}{2}}} \prod_{j=1}^m r_j^{\sum\limits_{k=1}^{\ell}\nu_{jk}-\frac12} \prod_{k=1}^{\ell}\frac{\Gamma\big[\frac{1}{\ell+1}\big(\mu+\boldsymbol{\nu}_\star+k\big)\big]}{\Gamma(\nu_{jk}+1)}\,.$$

Remark 2. *The most general results are presented in the ongoing section. Several specifications in the parameter range for* $\bar{\boldsymbol{\nu}}$ *in* Propositions 1 *and* 2, *in* Theorem 3 *and in* Corollary 3.1 *imply the achievements of the previous research chapters.*

So, for instance putting either $\ell_j = 1, j = 1,\cdots,m$ *in* Theorem 3, *or, which gives the same result,* $\ell = 1$ *in* Corollary 3.1, *we arrive at* Theorem 1. *If we additionally reduce* $m = 1$, *we arrive at* Corollary 1.1 *for the computational power series expansions of the Bessel–function extensions of the Mathieu–type series integral representations.*

5. Extending Mathieu series *via* Bessel–Clifford function

In this section we exploit the inter–relations (2.5) between the Bessel–function of the first kind and the related Bessel–Clifford function:

$$C^{(1)}_\nu(z) = C_\nu(z) = z^{-\nu/2}\,J_\nu(2\sqrt{z}) = \sum_{k\geq 0}\frac{(-1)^k\,z^k}{\Gamma(\nu+k+1)\,k!}\,,$$

which is extended to the multi–parameter Bessel–Clifford and modified Bessel–Clifford functions (2.4). The straightforward consequences of that display (also see [28, p. 11, Eq. (2.5)]) are the reverse formulae

$$J^{(m)}_{\boldsymbol{\nu}}(x) = \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m\nu_j} C^{(m)}_{\boldsymbol{\nu}}\Big(\frac{x^{m+1}}{(m+1)^{m+1}}\Big)\,, \tag{5.1}$$

$$I^{(m)}_{\boldsymbol{\nu}}(x) = \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m\nu_j} C^{(m)}_{\boldsymbol{\nu}}\Big(-\frac{x^{m+1}}{(m+1)^{m+1}}\Big)\,, \tag{5.2}$$

Let us introduce the extension of the initial Mathieu series' integral form by means of the product of m Bessel–Clifford functions (2.4) by which we replace the product of hyper–Bessel functions in definitions (4.1) and (4.2). Therefore the Bessel-Clifford extended Mathieu series integrals read:

$$\mathscr{S}^C_{\mu,\bar{\boldsymbol{\nu}}}(s,\boldsymbol{r}) = K_m(\boldsymbol{r})\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\,\prod_{j=1}^m C^{(\ell_j)}_{\boldsymbol{\nu}_j}(r_jx^s)\,\mathrm{d}x, \tag{5.3}$$

$$\widetilde{\mathscr{S}}^C_{\mu,\bar{\boldsymbol{\nu}}}(s,\boldsymbol{r}) = K_m(\boldsymbol{r})\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x+1}\,\prod_{j=1}^m C^{(\ell_j)}_{\boldsymbol{\nu}_j}(r_jx^s)\,\mathrm{d}x\,, \tag{5.4}$$

where in both cases $s, \boldsymbol{r} > 0,\ \mu > 0, \bar{\boldsymbol{\nu}} + 1 > 0$. However, the shorthand notations $\boldsymbol{\nu}^\star, \boldsymbol{\nu}_\star$ used in the previous section 4 remain the same. The auxiliary parameter $s > 0$ we put for the sake of completeness of unification suggested *via* the formulae (5.1) and (5.2).

Theorem 4. *For all $s, \mu, \bar{\boldsymbol{\nu}} > -1$ such that $\mu + \boldsymbol{\nu}^\star > 0$, and $\boldsymbol{r} \in \mathbb{R}^m_+$ we have*

$$\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(s,\boldsymbol{r}) = \sum_{n\ge1}\frac{H_m(\boldsymbol{r})}{n^\mu}\,S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix}[(\mu):s,\cdots,s]:-;\cdots;-\\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big],\cdots,\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big]\end{matrix}\,\middle|\,\boldsymbol{x}\right),$$

$$\widetilde{\mathscr{S}}^C_{\mu,\boldsymbol{\nu}}(s,\boldsymbol{r}) = H_m(\boldsymbol{r})\sum_{n\ge1}\frac{(-1)^{n-1}}{n^\mu}\,S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix}[(\mu):s,\cdots,s]:-;\cdots;-\\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big],\cdots,\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big]\end{matrix}\,\middle|\,\boldsymbol{x}\right),$$

where

$$\boldsymbol{x} = -\Big(\frac{\boldsymbol{r}}{n}\Big)^s, \quad \textit{and} \quad H_m(\boldsymbol{r}) = \Gamma(\mu)\prod_{j=1}^m\sqrt{\frac{\pi}{2r_j}}\,\Big(\prod_{k=1}^{\ell_j}\Gamma(\nu_{jk}+1)\Big)^{-1}.$$

Proof. Consider the integral $\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ in (5.3). The legitimate integration–summation order interchange which follows the binomial series expansion $(\mathrm{e}^x-1)^{-1}$ and the Bessel–Clifford functions power series' multiplication results in

$$\begin{aligned}\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(s,\boldsymbol{r}) &= K_m(\boldsymbol{r})\sum_{n\ge1}\sum_{\boldsymbol{k}\ge0}\prod_{j=1}^m\frac{(-1)^{k_j}r_j^{sk_j}}{k_j!\,\prod\limits_{k=1}^{\ell_j}\Gamma(\nu_{jk}+k_j+1)}\cdot\int_0^\infty x^{\mu+s\sum\limits_{j=1}^m k_j-1}\,\mathrm{e}^{-nx}\,\mathrm{d}x\\ &= H_m(\boldsymbol{r})\,\sum_{n\ge1}\frac{1}{n^\mu}\sum_{\boldsymbol{k}\ge0}\frac{(\mu)_{sk_1+\cdots+sk_m}}{\prod\limits_{j=1}^m\prod\limits_{k=1}^{\ell_j}(\nu_{jk}+1)_{k_j}}\cdot\prod_{j=1}^m\frac{(-1)^{k_j}}{k_j!}\,\Big(\frac{r_j}{n}\Big)^{sk_j}\\ &= H_m(\boldsymbol{r})\,\sum_{n\ge1}\frac{1}{n^\mu}\,S^{1:-;\cdots;-}_{-:1;\cdots;1}\left(\begin{matrix}[(\mu):s,\cdots,s]:-;\cdots;-\\ -:\Big[\prod\limits_{k=1}^{\ell_1}(\nu_{1k}+1):1\Big],\cdots,\Big[\prod\limits_{k=1}^{\ell_m}(\nu_{mk}+1):1\Big]\end{matrix}\,\middle|\,-\Big(\frac{\boldsymbol{r}}{n}\Big)^s\right),\end{aligned}$$

which finishes the proof.

The second formula's derivation follows the same lines. □

The specification $m = 1$ gives a more familiar form since the Srivastava–Daoust S function in Theorem 4 reduces to the generalized hypergeometric function ${}_1F_\ell$. Indeed,

Corollary 4.1. *For all* $\mu, \boldsymbol{\nu}+1>0; r>0$ *we have*

$$\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(r) = \frac{\sqrt{\pi}\,\Gamma(\mu)}{\sqrt{2r}\,\prod\limits_{j=1}^{\ell}\Gamma(\nu_j+1)} \sum_{n\geq 0} \frac{1}{n^\mu}\, {}_1F_\ell\Big[\begin{array}{c}\mu\\ \boldsymbol{\nu}+1\end{array}\Big| -\frac{r}{n}\Big], \tag{5.5}$$

$$\widetilde{\mathscr{S}}^C_{\mu,\boldsymbol{\nu}}(r) = \frac{\sqrt{\pi}\,\Gamma(\mu)}{\sqrt{2r}\,\prod\limits_{j=1}^{\ell}\Gamma(\nu_j+1)} \sum_{n\geq 0} \frac{(-1)^{n-1}}{n^\mu}\, {}_1F_\ell\Big[\begin{array}{c}\mu\\ \boldsymbol{\nu}+1\end{array}\Big| -\frac{r}{n}\Big]. \tag{5.6}$$

Proof. The proof follows the steps of the proving procedure in getting the results of Theorem 1. The inserted binomial series, the power series form of Bessel–Clifford function and the Gamma function integral transform the integral under consideration to

$$\begin{aligned}\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(r) &= \sqrt{\frac{\pi}{2r}} \sum_{n\geq 1}\int_0^\infty x^{\mu-1}\mathrm{e}^{-nx}\, C^{(m)}_{\boldsymbol{\nu}}(rx)\,\mathrm{d}x \\ &= \sqrt{\frac{\pi}{2r}} \sum_{n\geq 1}\sum_{k\geq 0} \frac{(-1)^k r^k}{k!\,\prod\limits_{j=1}^{m}\Gamma(\nu_j+1)} \int_0^\infty x^{\mu+k-1}\,\mathrm{e}^{-nx}\,\mathrm{d}x \\ &= \sqrt{\frac{\pi}{2r}} \frac{\Gamma(\mu)}{\prod\limits_{j=1}^{m}\Gamma(\nu_j+1)} \sum_{n\geq 1}\frac{1}{n^\mu}\sum_{k\geq 0}\frac{(\mu)_k}{\prod\limits_{j=1}^{m}(\nu_j)_k}\frac{\left(-\frac{r}{n}\right)^k}{k!},\end{aligned}$$

which is exactly the stated formula (5.5).

Mimicking this procedure we easily conclude (5.6) as well. □

Another approach in studying (5.5) gives a series expansion of $\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(r)$ in terms of Riemann Zeta, and $\widetilde{\mathscr{S}}^C_{\mu,\boldsymbol{\nu}}(r)$ expressed *via* Dirichlet Eta function.

Corollary 4.2. *For all* $\mu-1, \boldsymbol{\nu}+1>0; r>0$ *we have*

$$\begin{aligned}\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(r) &= \sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{\Gamma(\mu+k)\,\zeta(\mu+k)}{\prod\limits_{j=1}^{\ell}\Gamma(\nu_j+1)}\,\frac{(-r)^k}{k!},\\ \widetilde{\mathscr{S}}^C_{\mu,\boldsymbol{\nu}}(r) &= \sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{\Gamma(\mu+k)\,\eta(\mu+k)}{\prod\limits_{j=1}^{\ell}\Gamma(\nu_j+1)}\,\frac{r^k}{k!}.\end{aligned}$$

Proof. We have by direct calculation:

$$\mathscr{S}^C_{\mu,\boldsymbol{\nu}}(r) = \sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k r^k}{k!\,\prod\limits_{j=1}^{\ell}\Gamma(\nu_j+1)}\int_0^\infty \frac{x^{\mu+k-1}}{\mathrm{e}^x-1}\,\mathrm{d}x.$$

The integral (1.1) confirms the firstly stated formula. Applying the integral (1.2) in same fashion evaluation for the alternating Bessel–Clifford extension $\widetilde{\mathscr{S}}^C_{\mu,\boldsymbol{\nu}}(r)$ we get the second relation. □

6. Functional bounding inequalities

The inclusion of this chapter in the manuscript has a dual purpose. Firstly, giving a set of bounding inequalities of the extended Mathieu–type series integral forms, secondly, obtaining their computable series representations inferred to this section in the same way we can bound these series, considering that these are special kind summation formulae.

For estimation purposes we recall certain bounding inequalities for J_ν, I_ν on the positive real half–axis. First we mention Lommel's results [48], [49, pp. 548–549] (see also [91, p. 406])

$$|J_\nu(x)| \le 1, \qquad |J_{\nu+1}(x)| \le \frac{1}{\sqrt{2}}, \qquad \nu > 0,\, x \in \mathbb{R}, \tag{6.1}$$

and the bound by Minakshisundaram and Szász [62, p. 37]

$$|J_\nu(x)| \le \frac{1}{\Gamma(\nu+1)} \left(\frac{|x|}{2}\right)^\nu, \qquad x \in \mathbb{R}. \tag{6.2}$$

Another bounds were derived by Landau [43], who gave in a sense best possible bounds for J_ν with respect to ν and t. These bounds read as follows

$$|J_\nu(x)| \le b_L\, \nu^{-1/3}, \qquad b_L = \sqrt[3]{2} \sup_{x\ge0} \mathrm{Ai}(x), \tag{6.3}$$

$$|J_\nu(x)| \le c_L\, |x|^{-1/3}, \qquad c_L = \sup_{x\ge0} x^{1/3} J_0(t)\,, \tag{6.4}$$

where $\mathrm{Ai}(\cdot)$ stands for the Airy function.

Krasikov established uniform bounds for $|J_\nu|$. Let $\nu > -1/2$, then

$$J_\nu^2(x) \le \frac{4\big(4x^2 - (2\nu+1)(2\nu+5)\big)}{\pi\big((4x^2-\lambda)^{3/2} - \lambda\big)} =: \mathfrak{K}_\nu(t),$$

for all $t > \frac12\sqrt{\lambda + \lambda^{2/3}}, \lambda := (2\nu+1)(2\nu+3)$. The estimate is sharp in certain sense, see [39, Theorem 2]. In turn, Krasikov's recently published a set of more precise and simpler bounds [40, 41]. Precisely, for all $\nu \ge 1/2$ and for all $t \ge 0$ there holds [40, p. 210, Theorem 3]

$$\left| x^2 - \left|\nu^2 - \frac14\right|\right|^{1/4} |J_\nu(x)| \le \sqrt{\frac{2}{\pi}}, \tag{6.5}$$

where the right–hand–side constant is sharp. Next, his result [40, p. 210, Theorem 4] imply

$$|J_\nu(x)| \le \sqrt{\frac{2}{\pi x}} + \rho\, c \left|\nu^2 - \frac14\right| x^{-3/2}\,, \qquad x > 0,\, |\rho| < 1\,, \tag{6.6}$$

where

$$c = \begin{cases} \left(\frac{2}{\pi}\right)^{3/2} & x \ge 0,\, |\nu| \le 1/2 \\ \frac45 & 0 < x < \sqrt{|\nu^2 - 1/4|},\, \nu > 1/2 \\ \frac{2}{\pi} & x \ge \sqrt{|\nu^2 - 1/4|},\, \nu > 1/2 \end{cases}.$$

Here c cannot be less then $1/\sqrt{2\pi}$. For another kind bounds upon $J_\nu(t)$ consult [40, Theorems 2, 5, 6] and [41, Theorems 2, 4].

It is worth to mention that Olenko [65, Theorem 1] established the upper bound

$$\sup_{x\geq 0}\sqrt{t}|J_\nu(x)| \leq b_L\sqrt{\nu^{1/3}+\frac{\alpha_1}{\nu^{1/3}}+\frac{3\alpha_1^2}{10\nu}} = d_O, \qquad \nu>0\,, \tag{6.7}$$

where α_1 is the smallest positive zero of the Airy–function Ai and b_L is the Landau's constant from above. In this respect we point out Krasikov's result [40, p. 211, Eq. (7)].

Further considerable upper bounds are listed e.g. in [2, 3, 71, 80, 86]. We also draw the attention to the work by Paris [67] (see also [79]), who reported on the inequality

$$1 \leq \frac{J_\nu(\nu x)}{x^\nu\, J_\nu(\nu)} \leq \mathrm{e}^{\nu(1-x)}, \qquad \nu>0;\, x\in(0,1].$$

A different approach was used by Srivastava and Pogány in [86]. Let us denote $\chi_S(x)$ the characteristic (or indicator) function of a set S, that is $\chi_S(x)=1$ for all $x\in S$, else $\chi_S(x)=0$. Here the integration interval is the positive real half-axis, therefore we need an efficient bound for $|J_\mu(t)|$ on $(0,A]$, $A>\sqrt{\lambda+\lambda^{2/3}}/2$. So, the bounding function

$$|J_\mu(x)| \leq \mathscr{V}_\mu(x) := \frac{d_O}{\sqrt{x}}\,\chi_{(0,A_\lambda]}(x) + \sqrt{\mathfrak{K}_\mu(x)}\,\big(1-\chi_{(0,A_\lambda]}(x)\big)\,, \tag{6.8}$$

where, by simplicity reasons, our choice would be

$$A_\lambda = \tfrac12\left(\lambda+(\lambda+1)^{2/3}\right),$$

because $\mathfrak{K}_\nu(x)$ is positive and monotonous decreasing for $x\in\frac12\big((\lambda+\lambda^{2/3}),\infty\big)$, compare [86, §3]. Moreover, we point out that for A_λ we can take any $\frac12\big(\lambda+(\lambda+\eta)^{2/3}\big)$, $\eta>0$. (The interested reader is referred to [2] too). Obviously, combining (6.5), (6.6) in $\mathscr{V}_\mu(x)$ replacing Olenko's result and/or $\mathfrak{K}_\nu(x)$ in (6.8), we can define further bounding functions for $|J_\nu|$.

Since some Mathieu–type integrals are expressed in terms of modified Bessel I_ν too, we apply an estimate by Luke [50, p. 399, Eq. (3)]

$$I_\mu(x) < \frac{\left(\frac{x}{2}\right)^\mu}{\Gamma(\mu+1)}\cosh x, \qquad x>0, \mu+1>0\,. \tag{6.9}$$

Finally, the functional bounds for the hyper–Bessel function and modified hyper–Bessel function when $\boldsymbol{\nu}>-1$, for all $x\in\big(0,j_{\nu_m,1}^{m+1}\big]$ [1, p. 284, Corollary 1]

$$J_{\boldsymbol{\nu}}^{(m)}(x) \leq \frac{1}{A(\boldsymbol{\nu})}\Big(\frac{x}{m+1}\Big)^{\sum_{j=1}^m \nu_j}\exp\Big\{-\frac{x^{m+1}}{(m+1)^{m+1}A(\boldsymbol{\nu})}\Big\}, \qquad A(\boldsymbol{\nu}) = \prod_{j=1}^m(\nu_j+2). \tag{6.10}$$

Here $j_{\nu_m,1}$ denotes the first positive zero of the normalized hyper–Bessel function

$$\mathscr{I}_{\boldsymbol{\nu}}^{(m)}(x) = \Big(\frac{x}{m+1}\Big)^{-\sum_{j=1}^m\nu_j}\prod_{j=1}^m\Gamma(\nu_j+1)\cdot J_{\boldsymbol{\nu}}^{(m)}(x) = \sum_{k\geq 0}\frac{(-1)^k}{k!\,\prod_{j=1}^m(\nu_j+1)_k}\Big(\frac{x}{m+1}\Big)^{k(m+1)},$$

A Redheffer–type inequality was reported by Aktaş *et al.* [1, p. 284, Theorem 7]:

$$\Big(\frac{j_{\nu_m,1}^{m+1}-x^{m+1}}{j_{\nu_m,1}^{m+1}}\Big)^{\alpha_{\nu_m}} \leq \mathscr{I}_{\boldsymbol{\nu}}^{(m)}(x) \leq \Big(\frac{j_{\nu_m,1}^{m+1}-x^{m+1}}{j_{\nu_m,1}^{m+1}}\Big)^{\beta_{\nu_m}}, \qquad x\in\big(0,j_{\nu_m,1}\big);$$

where the best constants are

$$\alpha_{\nu_m} = \frac{j_{\nu_m,1}^{m+1}}{(m+1)^{m+1}\, A(\boldsymbol{\nu})}, \quad \text{and} \quad \beta_{\nu_m} = 1\,.$$

In turn, this implies for the same domain of the argument the estimate

$$\frac{\big(\frac{x}{m+1}\big)^{\sum\limits_{j=1}^m \nu_j}}{\prod\limits_{j=1}^m \Gamma(\nu_j+1)} \Big(\frac{j_{\nu_m,1}^{m+1} - x^{m+1}}{j_{\nu_m,1}^{m+1}}\Big)^{\alpha_{\nu_m}} \le J_{\boldsymbol{\nu}}^{(m)}(x) \le \frac{\big(\frac{x}{m+1}\big)^{\sum\limits_{j=1}^m \nu_j}}{\prod\limits_{j=1}^m \Gamma(\nu_j+1)} \Big(\frac{j_{\nu_m,1}^{m+1} - x^{m+1}}{j_{\nu_m,1}^{m+1}}\Big)^{\beta_{\nu_m}}\,. \tag{6.11}$$

Also, we need the definition of the Hurwitz–Lerch Zeta function[87, p. 488, Eq. (1.1)]

$$\zeta(s,a) = \Phi(z,s,a) = \sum_{k\ge 0} \frac{z^k}{(k+a)^s}\,,$$

where $-a \notin \mathbb{N}_0; s \in \mathbb{C}$ when $|z|<1$ and $\Re(s)>1$ when $|z|=1$. Special cases of interest for our exposure are the Riemann Zeta $\zeta(s) = \Phi(1,s,1)$, the Dirichlet Eta $\eta(s) = \Phi(-1,s,1); \Re(s)>0$.

Finally, we draw the attention to the Remark 1 by which there holds for some suitable positive function $\mathbf{A}(r)$ that

$$|\mathscr{S}_{\mu,\nu}(r)| \le \mathbf{A}(r) \quad \Rightarrow \quad |\widetilde{\mathscr{S}_{\mu,\nu}}(r)| \le \mathbf{A}(r) + 2^{-\mu+\frac12}\, \mathbf{A}\Big(\frac{r}{2}\Big)\,. \tag{6.12}$$

Theorem 5. *For all $\mu,\nu>1$ and for all $r>0$ we have*

$$\begin{aligned} |S_{\mu,\nu}(r)| &\le \sqrt{\frac{\pi}{2r}}\Big(\frac{r}{\nu}\Big)^\nu \Big[J_\nu(\nu)\, \mathrm{e}^\nu\, \Gamma(\mu+\nu)\, \zeta(\mu+\nu, r+1) - \frac{b_L}{\mu\sqrt[3]{\nu}}\Big(\mathrm{e}^{\frac{\mu\nu}{r(\mu+1)}}-1\Big)^{-1}\Big)\Big] \\ &\quad + \sqrt{\frac{\pi}{2r}}\, \frac{b_L}{\sqrt[3]{\nu}}\, \Gamma(\mu)\zeta(\mu) =: H_{\mu,\nu}(r), \\ |\widetilde{S}_{\mu,\nu}(r)| &\le H_{\mu,\nu}(r) + 2^{-\mu+\frac12} H_{\mu,\nu}\Big(\frac{r}{2}\Big). \end{aligned}$$

Proof. Observe the transformations of the Bessel function in the integrand as follows

$$J_\nu(rx) = J_\nu\Big(\nu\cdot\frac{r}{\nu}x\Big)\,;$$

apply now the Paris upper bound when $\nu>0$ and $x \in \big(0, \frac{\nu}{r}\big]$. This result in

$$J_\nu(rx) \le \Big(\frac{r}{\nu}\Big)^\nu J_\nu(\nu)\, \mathrm{e}^\nu\, x^\nu\, \mathrm{e}^{-rx}, \qquad 0 < x \le \frac{\nu}{r}\,. \tag{6.13}$$

Splitting the integration domain into $\mathbb{R}_+ = \big(0, \frac{\nu}{r}\big] \cup \big(\frac{\nu}{r}, \infty\big)$, we conclude

$$\begin{aligned} |S_{\mu,\nu}(r)| &\le \sqrt{\frac{\pi}{2r}}\Big(\frac{r}{\nu}\Big)^\nu J_\nu(\nu)\, \mathrm{e}^\nu \int_0^{\frac{\nu}{r}} \frac{x^{\mu+\nu-1}\mathrm{e}^{-rx}}{\mathrm{e}^x-1}\,\mathrm{d}x + \sqrt{\frac{\pi}{2r}} \int_{\frac{\nu}{r}}^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\,|J_\nu(rx)|\,\mathrm{d}x \\ &\le \sqrt{\frac{\pi}{2r}}\Big(\frac{r}{\nu}\Big)^\nu J_\nu(\nu)\, \mathrm{e}^\nu \int_0^{\frac{\nu}{r}} \frac{x^{\mu+\nu-1}\mathrm{e}^{-rx}}{\mathrm{e}^x-1}\,\mathrm{d}x + \sqrt{\frac{\pi}{2r}}\, \frac{b_L}{\sqrt[3]{\nu}} \int_{\frac{\nu}{r}}^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\,\mathrm{d}x\,, \end{aligned} \tag{6.14}$$

where Landau's first upper bound (6.3) was employed. The first integral we handle in the following way:

$$\int_0^q \frac{x^{\mu+\nu-1}\mathrm{e}^{-rx}}{\mathrm{e}^x-1}\,\mathrm{d}x \le \sum_{n\ge 1} \int_0^\infty x^{\mu+\nu-1}\, \mathrm{e}^{-(r+n)x}\,\mathrm{d}x$$

$$= \sum_{n\geq 1} \frac{\Gamma(\mu+\nu)}{(r+n)^{\mu+\nu}} = \Gamma(\mu+\nu)\,\zeta(\mu+\nu, r+1)\,.$$

As to the second integral in (6.14), consider the integral

$$I(p,q) = \int_q^\infty \frac{x^{p-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x = \sum_{n\geq 1} \frac{\Gamma(p, nq)}{n^p}\,,$$

where $\Gamma(t,z) = \int_z^\infty x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x$ denotes the upper incomplete Gamma function and $p-1, q > 0$. Obviously $I(p,q) \leq \Gamma(p)\,\zeta(p)$, but we can derive a more sophisticated bound with the bilateral bound for the lower incomplete Gamma function $\gamma(t,z) = \Gamma(t) - \Gamma(t,z) = \int_0^z x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x$, reported by Neuman [64, p. 1213, Theorem 4.1]

$$\exp\Big\{-\frac{ax}{a+1}\Big\} \leq a\,x^{-a}\,\gamma(a,x) \leq \frac{1}{a+1}\big(1 + a\,\mathrm{e}^{-x}\big)\,, \qquad \min\{a,x\} > 0.$$

Rewriting this Neuman's formula into the equivalent form:

$$\Gamma(a) - \frac{x^a}{a(a+1)}\big(1 + a\,\mathrm{e}^{-x}\big) \leq \Gamma(a,x) \leq \Gamma(a) - \frac{x^a}{a}\,\exp\Big\{-\frac{ax}{a+1}\Big\}\,, \tag{6.15}$$

we conclude

$$I(p,q) \leq \Gamma(p)\,\zeta(p) - \frac{q^p}{p}\Big(\exp\Big\{\frac{pq}{p+1}\Big\} - 1\Big)^{-1}.$$

Collecting the bounds derived, we finish the proof.

As to the bound (6.14), it is enough to recall (6.12) for $m = 1$. □

Remark 3. *In estimating the modulus of the Bessel function $J_\nu(x)$, we may choose any of the above listed bounds like the ones by Lommel* (6.1)*, Landau's second* (6.4)*, Krasikov* (6.6)*, Olenko* (6.7)*, Minakshisundaram and Szász* (6.2) *or Srivastava and Pogány* (6.8)*.*

However, the sharpness of the bounds depends on the range and constraints of the involved parameters.

Now we switch to the kernel consisting from the product of m Bessel functions of the first kind.

Theorem 6. *For all $\mu, \boldsymbol{\nu}+1 > 0$ such that $\mu + \sum_{j=1}^m \nu_j > 1$, and for all $\boldsymbol{r} > 0$ we have*

$$\begin{aligned}|\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})| \leq K_m(\boldsymbol{r})\Bigg\{&\mathrm{e}^{\sum_{j=1}^m \nu_j} \prod_{j=1}^m \left(\frac{r_j}{\nu_j}\right)^{\nu_j} J_{\nu_j}(\nu_j)\cdot\Gamma\Big(\mu + \sum_{j=1}^m \nu_j\Big)\,\zeta\Big(\mu + \sum_{j=1}^m \nu_j, 1 + \sum_{j=1}^m r_j\Big)\\ &+ \prod_{j=1}^m \frac{b_L}{\sqrt{r_j}\,\sqrt[3]{\nu_j}}\left(\Gamma(\mu)\zeta(\mu) - \frac{(\nu_\star(r))^\mu}{\mu}\Big(\exp\Big\{\frac{\mu\,\nu_\star(r)}{\mu+1}\Big\} - 1\Big)^{-1}\right)\Bigg\} =: \mathbf{H}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})\,,\end{aligned}$$

where

$$r^\star(\nu) = \max_{1\leq j\leq m}\left(\frac{r_j}{\nu_j}\right) \quad \text{and} \quad \nu_\star(r) = \min_{1\leq j\leq m}\left(\frac{\nu_j}{r_j}\right). \tag{6.16}$$

Moreover, for all $\mu + \sum_{j=1}^m \nu_j > 0$ there holds

$$|\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})| \leq \mathbf{H}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) + 2^{-\mu-\frac{m}{2}+1}\mathbf{H}_{\mu,\boldsymbol{\nu}}\Big(\frac{\boldsymbol{r}}{2}\Big)\,.$$

Proof. We follow the methodology used in proof of the previous theorem. By means of Paris' upper bound (6.13) for the interval $x \in (0, \nu_\star(r)\big]$, and simultaneously by Neuman estimate (6.15) for $x > \nu_\star(r)$, we deduce

$$\begin{aligned}
|\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})| &= K_m(\boldsymbol{r}) \int_0^{\nu_\star(r)} \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m J_{\nu_j}(r_j\, x) \cdot \mathrm{d}x \\
&\quad + K_m(\boldsymbol{r}) \int_{\nu_\star(r)}^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1} \prod_{j=1}^m J_{\nu_j}(r_j\, x) \cdot \mathrm{d}x \\
&\le G_m(\boldsymbol{r})\, \Gamma\Big(\mu + \sum_{j=1}^m \nu_j\Big)\, \zeta\Big(\mu + \sum_{j=1}^m \nu_j, 1 + \sum_{j=1}^m r_j\Big) \\
&\quad + K_m(\boldsymbol{r}) \frac{b_L^m}{\prod\limits_{j=1}^m \sqrt[3]{\nu_j}} \left(\Gamma(\mu)\zeta(\mu) - \frac{(\nu_\star(r))^\mu}{\mu} \Big(\exp\left\{ \frac{\mu}{(m+1)\, r^\star(\nu)} \right\} - 1 \Big)^{-1} \right).
\end{aligned}$$

The first assertion is proved. The second one follows by (6.12). □

For the next functional inequalities we need upper bound for the modulus of the hyper–Bessel and Bessel–Clifford functions. In this goal, according to (6.16), we introduce the convention

$$\nu_\star(1) = \nu_1 \le \cdots \le \nu_m\,.$$

Theorem 7. *For all $x > 0$ and $\nu_\star(1) = \nu_1 \ge 1$, $m \in \mathbb{N}$ we have*

$$\big|C_{\boldsymbol{\nu}}^{(m)}(\pm x)\big| \le \sum_{j=1}^m x^{-\frac{\nu_j}{2}} I_{\nu_j}(2\sqrt{x}) \le \sum_{j=1}^m \frac{\cosh(2\sqrt{x})}{\Gamma(\nu_j+1)} \tag{6.17}$$

$$\left\{ \begin{array}{c} \big|J_{\boldsymbol{\nu}}^{(m)}(x)\big| \\ I_{\boldsymbol{\nu}}^{(m)}(x) \end{array} \right\} \le \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m \nu_j} \sum_{j=1}^m \Big(\frac{x}{m+1}\Big)^{-\frac{\nu_j}{2}(m+1)} I_{\nu_j}\Big\{ 2\Big(\frac{x}{m+1}\Big)^{\frac{1}{2(m+1)}} \Big\} \tag{6.18}$$

$$\le \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m \nu_j} \sum_{j=1}^m \frac{1}{\Gamma(\nu_j+1)} \cosh\Big\{ 2\Big(\frac{x}{m+1}\Big)^{\frac{1}{2(m+1)}} \Big\}. \tag{6.19}$$

Proof. It is well–known that any positive m–tuple $\boldsymbol{a} = (a_1, \cdots, a_m)$ is endowed by the property

$$a_1 \cdots a_m \le a_1 + \cdots + a_m; \qquad a_j \in [0,1], j = 1, \cdots, m\,, \tag{6.20}$$

where the equality is achieved either for $m = 1$, or all $a_j = 0$. On the other side $\Gamma(\nu_\star(1)+k+1) \ge 1$ when $\nu_\star(1) \ge 1$ and $k \in \mathbb{N}_0$. Choosing $a_j = \Gamma(\nu_j + k + 1)$ we conclude

$$\frac{1}{\prod\limits_{j=1}^m \Gamma(\nu_j + k + 1)} \le \sum_{j=1}^m \frac{1}{\Gamma(\nu_j + k + 1)}.$$

Multiply this inequality with $(\pm x)^k, x > 0$ and summ up with respect to $k \in \mathbb{N}_0$. Now, by virtue of definition (2.4) of the Bessel–Clifford function, relation (2.5) between the one–dimensional Bessel–Clifford function and the Bessel function od the first kind and the property $|J_\nu(z)| \le I_\nu(z)$, we conclude for positive argument $x>0$ that

$$\big|C_{\boldsymbol\nu}^{(m)}(\pm x)\big| \le C_{\boldsymbol\nu}^{(m)}(-x) = \sum_{k\ge 0}\frac{x^k}{k!\,\prod\limits_{j=1}^m \Gamma(\nu_j+k+1)}$$
$$\le \sum_{j=1}^m\sum_{k\ge 0}\frac{x^k}{k!\,\Gamma(\nu_j+k+1)} = \sum_{j=1}^m x^{-\frac{\nu_j}{2}}\, I_{\nu_j}(2\sqrt{x})\,.$$

Applying Luke's functional upper bound (6.9), we arrive at the inequality (6.18) for the m–order parameter Bessel–Clifford function.

As to the bound (6.18) for hyper–Bessel function, the reversed relation (5.1) for (2.4) and the previous estimate imply

$$\Big|J_{\boldsymbol\nu}^{(m)}\big((m+1)\,x^{1/(m+1)}\big)\Big| \le x^{\frac{1}{m+1}\sum\limits_{j=1}^m \nu_j}\big|C_{\boldsymbol\nu}^{(m)}(\pm x)\big| \le x^{\frac{1}{m+1}\sum\limits_{j=1}^m \nu_j}\sum_{j=1}^m x^{-\frac{\nu_j}{2}}\, I_{\nu_j}(2\sqrt{x})\,.$$

The argument change $(m+1)x^{1/(m+1)}\mapsto x$ gives

$$\Big|J_{\boldsymbol\nu}^{(m)}(x)\Big| \le \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m \nu_j}\sum_{j=1}^m\Big(\frac{x}{m+1}\Big)^{-\frac{\nu_j}{2}\,(m+1)} I_{\nu_j}\Big\{2\,\Big(\frac{x}{m+1}\Big)^{\frac{1}{2(m+1)}}\Big\}\,.$$

and Luke's upper bound (6.9) establishes the final result. The identical bound for the modified hyper–Bessel function holds now in a straightforward way for all $x>0$. □

Now, we consider $\mathscr{S}_{\mu,\bar{\boldsymbol\nu}}^C(s,\boldsymbol r)$, $\widetilde{\mathscr{S}}_{\mu,\bar{\boldsymbol\nu}}^C(s,\boldsymbol r)$ expressed by (5.3) and (5.4), respectively.

Theorem 8. *For all $\mu-1>0$, $\bar{\boldsymbol\nu}+1>0$, $s\in(0,2)$ and $\boldsymbol r>0$ we have that*

$$\big|\mathscr{S}_{\mu,\bar{\boldsymbol\nu}}^C(s,\boldsymbol r)\big| \le \mathfrak{K}_m^C(s,\boldsymbol r)\prod_{j=1}^m\sum_{k=1}^{\ell_j}\frac{r_j^{-\frac{\nu_{jk}}{2}}}{\Gamma(\nu_{jk}+1)}\sum_{n\ge1}\frac{1}{n^\mu}\,{}_2\Psi_1\left[\begin{matrix}(\frac{\mu}{2},\frac{s}{2}),\,(\frac{\mu}{2}+\frac12,\frac{s}{2})\\ (\frac12,1)\end{matrix}\,\middle|\,\frac{2^s\,r_j}{n^s}\right]. \tag{6.21}$$

Moreover, when $s\in(0,2),\mu>0$, $\bar{\boldsymbol\nu}+1>0$ and for all $\boldsymbol r>0$ there holds

$$\big|\widetilde{\mathscr{S}}_{\mu,\bar{\boldsymbol\nu}}^C(s,\boldsymbol r)\big| \le \mathfrak{K}_m^C(s,\boldsymbol r)\prod_{j=1}^m\sum_{k=1}^{\ell_j}\frac{r_j^{-\frac{\nu_{jk}}{2}}}{\Gamma(\nu_{jk}+1)}\sum_{n\ge1}\frac{(-1)^{n-1}}{n^\mu}\,{}_2\Psi_1\left[\begin{matrix}(\frac{\mu}{2},\frac{s}{2}),\,(\frac{\mu}{2}+\frac12,\frac{s}{2})\\ (\frac12,1)\end{matrix}\,\middle|\,\frac{2^s\,r_j}{n^s}\right]. \tag{6.22}$$

For $s=2$ both formulae are valid when $\boldsymbol r\in(0,\frac14)^m$. Here the constant

$$\mathfrak{K}_m^C(s,\boldsymbol r) = \frac{\pi^{\frac{m}{2}}\,\Gamma(\mu+1)}{2^{\frac{m}{2}+1}}\prod_{j=1}^m r_j^{-\frac12}.$$

Proof. Starting with (5.3) the following subsequent majorizations follow:

$$\big|\mathscr{S}_{\mu,\bar{\boldsymbol\nu}}^C(s,\boldsymbol r)\big| \le K_m(\boldsymbol r)\int_0^\infty\frac{x^{\mu-1}}{{\rm e}^x-1}\,\prod_{j=1}^m\big|C_{\boldsymbol\nu_j}^{(\ell_j)}(\pm r_jx^s)\big|\,{\rm d}x \le K_m(\boldsymbol r)\int_0^\infty\frac{x^{\mu-1}}{{\rm e}^x-1}\,\prod_{j=1}^m C_{\boldsymbol\nu_j}^{(\ell_j)}(-r_jx^s)\,{\rm d}x$$
$$\le K_m(\boldsymbol r)\int_0^\infty\frac{x^{\mu-1}}{{\rm e}^x-1}\,\prod_{j=1}^m\sum_{k=1}^{\ell_j} r_j^{-\frac{\nu_{jk}}{2}}\,x^{-\frac{s}{2}\nu_{jk}}\,I_{\nu_{jk}}\big(2\sqrt{r_j}\,x^{\frac{s}{2}}\big)\,{\rm d}x\,,$$

where the last estimate there follows by the auxiliary bound (6.17). Hence, routine transformations by the second appropriate bound in (6.17) imply

$$\big|\mathscr{S}^C_{\mu,\bar{\boldsymbol\nu}}(s,\boldsymbol r)\big| \le K_m(\boldsymbol r)\sum_{n\ge1}\prod_{j=1}^m\sum_{k=1}^{\ell_j}\frac{r_j^{-\frac{\nu_{jk}}{2}}}{\Gamma(\nu_{jk}+1)}\int_0^\infty x^{\mu-1}\,\mathrm e^{-nx}\,\cosh\big(2\sqrt{r_j}\;x^{\frac s2}\big)\,\mathrm dx\,.$$

The inner integral is in fact the Laplace–Mellin transform of the hyperbolic cosine function of power function argument. Its general closed form becomes

$$\mathscr{LM}\big[\cosh\big(ax^q\big)\big](p,\mu) = \frac{\Gamma(\mu+1)}{2\,p^\mu}\,{}_2\Psi_1\left[\begin{array}{c}(\frac\mu2,q),\,(\frac\mu2+\frac12,q)\\ (\frac12,1)\end{array}\middle|\frac{4^{q-1}\,a^2}{p^{2q}}\right], \tag{6.23}$$

where $\mu+1>0$, $\Re(p)>0$, $q\in(0,1)$ and for $q=1$, $|a|<\Re(p)$. Indeed, since the series form of the cosine hyperbolic and the reversing of the order of summation and integration ensure the set of transformations by the Legendre duplication formula, the formula (6.23) is evident:

$$\begin{aligned}\mathscr{LM}\big[\cosh\big(ax^q\big)\big](p,\mu) &= \sum_{n\ge0}\frac{a^{2n}}{(2n)!}\int_0^\infty \mathrm e^{-px}\,x^{\mu+2nq-1}\,\mathrm dx\\ &= \frac{1}{p^\mu}\sum_{n\ge0}\frac{a^{2n}}{\Gamma(2n+1)}\,\frac{\Gamma(\mu+2nq)}{p^{2nq}}\\ &= \frac{\Gamma(\mu+1)}{2\,p^\mu}\sum_{n\ge0}\frac{(\frac\mu2)_{qn}\,(\frac\mu2+\frac12)_{qn}}{(\frac12)_n\,n!}\Big(\frac{4^{q-1}a^2}{p^{2q}}\Big)^n\,.\end{aligned}$$

Consequently, specifying $p=n, q=s/2$, $a=2\sqrt{r_j}$ we arrive at the stated bound (6.21). As to the convergence of the Fox–Wright function term, consulting (2.2), $\Delta=2-s>0$ is satisfied by the conditions of the theorem. For $s=2$, the radius $\rho=4r_j<1, j=1,\cdots,m$ confirm the convergence region $0<\boldsymbol r<\frac14$, while for $s>2$ the series becomes formal, that is, converges exclusively at $\boldsymbol r=0$.

The formula (6.22) it follows by a similar approach - here we expand the binomial $(\mathrm e^x+1)^{-1}$ and integrate term-wise like in earlier exposed alternating series cases. □

An important special case of Theorem 8 we present in terms of the Hurwitz–Lerch Zeta function.

Corollary 8.1. *For all $\mu-1,\bar{\boldsymbol\nu}+1>0$ and for all $\boldsymbol r\in\mathbb R_+^m\setminus\{1,4,9,\cdots\}^m$ we have*

$$\big|\mathscr{S}^C_{\mu,\bar{\boldsymbol\nu}}(2,\boldsymbol r)\big| \le \mathfrak K^C_m(2,\boldsymbol r)\,\prod_{j=1}^m\sum_{k=1}^{\ell_j}\frac{r_j^{-\frac{\nu_{jk}}{2}}}{\Gamma(\nu_{jk}+1)}\,\big[\Phi(1,\mu,1-2\sqrt{r_j})+\Phi(1,\mu,1+2\sqrt{r_j})\big]\,.$$

Moreover, for the same parameter space and $\mu>0$, there holds

$$\big|\widetilde{\mathscr{S}}^C_{\mu,\bar{\boldsymbol\nu}}(2,\boldsymbol r)\big| \le \mathfrak K^C_m(2,\boldsymbol r)\,\prod_{j=1}^m\sum_{k=1}^{\ell_j}\frac{r_j^{-\frac{\nu_{jk}}{2}}}{\Gamma(\nu_{jk}+1)}\,\big[\Phi\big(-1,\mu,1-2\sqrt{r_j}\big)+\Phi\big(-1,\mu,1+2\sqrt{r_j}\big)\big]\,,$$

where $\mathfrak K^C_m(2,\boldsymbol r)=\mathfrak K^C_m(s,\boldsymbol r)/2$.

Proof. Setting $s = 2$ in Theorem 8, that is, $q = 1$ in (6.23) the Fox–Wright ${}_2\Psi_1$ one reduces to the Gaussian ${}_2F_1$ function. Moreover, having in mind that

$$ {}_2F_1\left[\begin{array}{c} \frac{\mu}{2},\, \frac{\mu}{2}+\frac12 \\ \frac12 \end{array}\Bigg| \frac{4r_j}{n^2}\right] = \frac{n^\mu}{2}\Big[\frac{1}{(n-2\sqrt{r_j})^\mu} + \frac{1}{(n+2\sqrt{r_j})^\mu}\Big]\,, \tag{6.24}$$

where the formula is valid for $r_j < n^2$ for all $n \in \mathbb{N}$. Accordingly, $r_j \in (0,1)$. Now, summing up the right–hand–side with respect to $n \in \mathbb{N}$ we get:

$$\sum_{n\ge1} \frac{1}{n^\mu}\; {}_2F_1\left[\begin{array}{c} \frac{\mu}{2},\, \frac{\mu}{2}+\frac12 \\ \frac12 \end{array}\Bigg| \frac{4r_j}{n^2}\right] = \frac12 \sum_{n\ge1}\Big[\frac{1}{(n-2\sqrt{r_j})^\mu} + \frac{1}{(n+2\sqrt{r_j})^\mu}\Big]\,,$$

which is a sum of two Hurwitz–Lerch Zetas; these are well defined when the denominator argument $a = -1 + 2\sqrt{r_j}$ is not a negative integer, which means that r_j is different from a perfect square. So the asserted result. The bound for $\big|\widetilde{\mathscr{S}}^{C}_{\mu,\bar{\boldsymbol\nu}}(2,\boldsymbol r)\big|$ also follows. □

Now, we consider the modulus of the hyper–Bessel extension of (4.1), *viz.*

$$\mathscr{S}^J_{\mu,\bar{\boldsymbol\nu}}(\boldsymbol r) = K_m(\boldsymbol r)\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\cdot \mathrm{d}x, \qquad \mu+\boldsymbol\nu^\star > 1\,,$$

and its counterpart version associated with the alternating Mathieu–series integral (4.2):

$$\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol\nu}}(\boldsymbol r) = K_m(\boldsymbol r)\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x+1}\prod_{j=1}^m J^{(\ell_j)}_{\boldsymbol{\nu_j}}(r_j\,x)\cdot \mathrm{d}x; \qquad \mu+\boldsymbol\nu^\star > 0\,,$$

pointing out that $K_m(\boldsymbol r)$ and $\boldsymbol\nu^\star$ denote the same constants as earlier.

In this derivation process the main derivation tool will be either the functional upper bound (6.18), that is its majorized, simplified version (6.19), both presented in Theorem 7.

Theorem 9. *For all $\mu, \boldsymbol\nu^\star > 0$, such that $\mu+\boldsymbol\nu^\star > 1$, $\boldsymbol\ell \in \mathbb{N}^m$ and for all $\boldsymbol r > 0$ we have*

$$\begin{aligned}\big|\mathscr{S}^J_{\mu,\bar{\boldsymbol\nu}}(\boldsymbol r)\big| &\le 2^{-1}K_m(\boldsymbol r)\,\Gamma(\mu+\boldsymbol\nu^\star+1)\prod_{j=1}^m\sum_{k=1}^{\ell_j}\Big(\frac{r_j}{\ell_j+1}\Big)^{\sum_{k=1}^{\ell_j}\nu_{jk}}\frac{1}{\Gamma(\nu_{jk}+1)}\sum_{n\ge1}\frac{1}{n^{\mu+\boldsymbol\nu^\star}}\\ &\times {}_2\Psi_1\left[\begin{array}{c}(\frac{\mu+\boldsymbol\nu^\star}{2},\frac{1}{2(\ell_j+1)}),\,(\frac{\mu+\boldsymbol\nu^\star+1}{2},\frac{1}{2(\ell_j+1)})\\ (\frac12,1)\end{array}\Bigg|\Big(\frac{2\,r_j}{(\ell_j+1)n}\Big)^{\frac{1}{\ell_j+1}}\right].\end{aligned}$$

In the case when $\mu+\boldsymbol\nu^\star > 0$ there holds

$$\begin{aligned}\big|\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol\nu}}(\boldsymbol r)\big| &\le 2^{-1}K_m(\boldsymbol r)\,\Gamma(\mu+\boldsymbol\nu^\star+1)\prod_{j=1}^m\sum_{k=1}^{\ell_j}\Big(\frac{r_j}{\ell_j+1}\Big)^{\sum_{k=1}^{\ell_j}\nu_{jk}}\frac{1}{\Gamma(\nu_{jk}+1)}\sum_{n\ge1}\frac{(-1)^{n-1}}{n^{\mu+\boldsymbol\nu^\star}}\\ &\times {}_2\Psi_1\left[\begin{array}{c}(\frac{\mu+\boldsymbol\nu^\star}{2},\frac{1}{2(\ell_j+1)}),\,(\frac{\mu+\boldsymbol\nu^\star+1}{2},\frac{1}{2(\ell_j+1)})\\ (\frac12,1)\end{array}\Bigg|\Big(\frac{2\,r_j}{(\ell_j+1)n}\Big)^{\frac{1}{\ell_j+1}}\right].\end{aligned}$$

Identical upper bounds hold for $\mathscr{S}^I_{\mu,\bar{\boldsymbol\nu}}(\boldsymbol r)$, and $\widetilde{\mathscr{S}}^I_{\mu,\bar{\boldsymbol\nu}}(\boldsymbol r)$ under the same parameter constraints, respectively.

Proof. Bearing in mind that there are two bounds on disposal for estimating the product of m hyper–Bessel functions in the integrand of $\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$, we start *ad libitum* with the use of the simpler bound (6.19), instead of (6.18). Thus,

$$\begin{aligned}\big|\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})\big| &\le K_m(\boldsymbol{r})\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\prod_{j=1}^m \big|J^{(\ell_j)}_{\boldsymbol{\nu}_j}(r_j\,x)\big|\,\mathrm{d}x\\ &\le K_m(\boldsymbol{r})\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\prod_{j=1}^m \Big(\frac{r_jx}{\ell_j+1}\Big)^{\sum\limits_{k=1}^{\ell_j}\nu_{jk}}\sum_{k=1}^{\ell_j}\frac{\cosh\Big\{2\Big(\dfrac{r_jx}{m+1}\Big)^{\frac{1}{2(m+1)}}\Big\}}{\Gamma(\nu_j+1)}\,\mathrm{d}x\\ &= K_m(\boldsymbol{r})\prod_{j=1}^m\sum_{k=1}^{\ell_j}A_{jk}\sum_{n\ge1}\int_0^\infty \mathrm{e}^{-nx}\,x^{\mu+\boldsymbol{\nu}^\star-1}\,\cosh\Big\{2\Big(\frac{r_jx}{\ell_j+1}\Big)^{\frac{1}{2(\ell_j+1)}}\Big\}\,\mathrm{d}x\,,\end{aligned}$$

where

$$A_{jk} = \Big(\frac{r_j}{\ell_j+1}\Big)^{\sum\limits_{k=1}^{\ell_j}\nu_{jk}}\,\frac{1}{\Gamma(\nu_{jk}+1)}.$$

Employing the Laplace–Mellin transform integral (6.23) for the following specified values of parameters:

$$p=n,\quad \mu+\boldsymbol{\nu}^\star\mapsto\mu,\quad a=2\Big(\frac{r_j}{\ell_j+1}\Big)^{\frac{1}{2(\ell_j+1)}},\quad q=\frac{1}{2(\ell_j+1)}\,,$$

we get

$$\begin{aligned}\big|\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})\big| &\le 2^{-1}K_m(\boldsymbol{r})\,\Gamma(\mu+\boldsymbol{\nu}^\star+1)\prod_{j=1}^m\sum_{k=1}^{\ell_j}A_{jk}\sum_{n\ge1}\frac{1}{n^{\mu+\boldsymbol{\nu}^\star}}\\ &\quad\times{}_2\Psi_1\left[\begin{array}{c}(\frac{\mu+\boldsymbol{\nu}^\star}{2},\frac{1}{2(\ell_j+1)}),\,(\frac{\mu+\boldsymbol{\nu}^\star+1}{2},\frac{1}{2(\ell_j+1)})\\ (\frac12,1)\end{array}\Bigg|\Big(\frac{2\,r_j}{(\ell_j+1)n}\Big)^{\frac{1}{\ell_j+1}}\right].\end{aligned}$$

The Fox–Wright function terms in the upper bound converge for all $\boldsymbol{r}>0$ and all ℓ_j positive integers because

$$\Delta_j = 2-\frac{1}{\ell_j+1}>0;\qquad \ell_j\in\mathbb{N};\, j=1,\cdots,m\,.$$

By similar manipulations we conclude that $\big|\widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})\big|$ has the same upper bound, as well as the modified hyper–Bessel extensions $\mathscr{S}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}^I_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$ of the associated Mathieu- and alternating Mathieu–series integrals, respectively. The absence of moduli in the latter two cases follow from their definitions (4.1) and (4.2). □

7. Log–convexity and Turán type inequalities

In this section we present the log–convexity property and subsequently, Turán type inequality for the multi–parameter Mathieu series and alternating Mathieu series $\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ and $\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$, hyper–Bessel generalized Mathieu series and alternating Mathieu series $\mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r}), \widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$, and Bessel–Clifford generalized Mathieu series and alternating Mathieu series $\mathscr{S}^C_{\mu,\bar{\boldsymbol{\nu}}}(s,\boldsymbol{r})$ and $\widetilde{\mathscr{S}}^C_{\mu,\bar{\boldsymbol{\nu}}}(s,\boldsymbol{r})$.

Both, log-convexity and Turánian property we establish with respect the parameter μ.

Theorem 10. *Let $\boldsymbol{r} \in \mathbb{R}_+^m$. Then the following assertions are true:*

- *The function $\mu \mapsto \mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ is log–convex on $\mu + \sum\limits_{j=1}^m \nu_j \geq 1$.*
- *The function $\mu \mapsto \widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r})$ is log–convex on $\mu + \sum\limits_{j=1}^m \nu_j \geq 0$.*
- *The function $\mu \mapsto \mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$ is log–convex on $\mu + \sum\limits_{j=1}^m \sum\limits_{k=1}^{\ell_j} \nu_{jk} > 1$.*
- *The function $\mu \mapsto \widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}(\boldsymbol{r})$ is log–convex on $\mu + \sum\limits_{j=1}^m \sum\limits_{k=1}^{\ell_j} \nu_{jk} > 0$.*
- *The function $\mu \mapsto \mathscr{S}^C_{\mu,\bar{\boldsymbol{\nu}}}(s,\boldsymbol{r})$ is log–convex on $s > 0$, $\mu > 1, \bar{\boldsymbol{\nu}} + 1 > 0$.*
- *The function $\mu \mapsto \widetilde{\mathscr{S}}^C_{\mu,\bar{\boldsymbol{\nu}}}(s,\boldsymbol{r})$ is log–convex on $s > 0$, $\mu > 0, \bar{\boldsymbol{\nu}} + 1 > 0$.*

Moreover, for the same parameter range there holds the Turán inequality

$$\mathscr{T}^2_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) \leq \mathscr{T}_{\mu-\alpha,\boldsymbol{\nu}}(\boldsymbol{r})\, \mathscr{T}_{\mu+\alpha,\boldsymbol{\nu}}(\boldsymbol{r})\,, \qquad \mu > \alpha > 0\,, \tag{7.1}$$

where $\mathscr{T}_\mu \in \{\mathscr{S}_{\mu,\boldsymbol{\nu}}, \widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}, \mathscr{S}^J_{\mu,\bar{\boldsymbol{\nu}}}, \widetilde{\mathscr{S}}^J_{\mu,\bar{\boldsymbol{\nu}}}, \mathscr{S}^C_{\mu,\bar{\boldsymbol{\nu}}}, \widetilde{\mathscr{S}}^C_{\mu,\bar{\boldsymbol{\nu}}}\}$.

Proof. Using the integral representation (3.3) by aid of classical Hölder–Rogers inequality for integrals, we have

$$\begin{aligned}
\mathscr{S}_{\lambda\mu_1+(1-\lambda)\mu_2,\boldsymbol{\nu}}(\boldsymbol{r}) &= K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\lambda\mu_1+(1-\lambda)\mu_2-1}}{\mathrm{e}^x-1} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\cdot \mathrm{d}x, \\
&= K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\lambda(\mu_1-1)+(1-\lambda)(\mu_2-1)}}{\mathrm{e}^x-1} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\cdot \mathrm{d}x \\
&= K_m(\boldsymbol{r}) \int_0^\infty \left\{\frac{x^{\mu_1-1}}{\mathrm{e}^x-1} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\right\}^\lambda \left\{\frac{x^{\mu_2-1}}{\mathrm{e}^x-1} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\right\}^{1-\lambda} \cdot \mathrm{d}x \\
&\leq \left\{K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu_1-1}}{\mathrm{e}^x-1} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\cdot \mathrm{d}x\right\}^\lambda \left\{K_m(\boldsymbol{r}) \int_0^\infty \frac{x^{\mu_2-1}}{\mathrm{e}^x-1} \prod_{j=1}^m J_{\nu_j}(r_j\,x)\cdot \mathrm{d}x\right\}^{1-\lambda}.
\end{aligned}$$

This is equivalent to

$$\mathscr{S}_{\lambda\mu_1+(1-\lambda)\mu_2,\boldsymbol{\nu}}(\boldsymbol{r}) \leq \left[\mathscr{S}_{\mu_1,\boldsymbol{\nu}}(\boldsymbol{r})\right]^\lambda \left[\mathscr{S}_{\mu_2,\boldsymbol{\nu}}(\boldsymbol{r})\right]^{1-\lambda}, \tag{7.2}$$

which proves the first assertion for all $\mu_1, \mu_2 > 1$, $\lambda \in [0,1]$ and $\boldsymbol{r} \in \mathbb{R}_+^m$. Next, choosing

$$\mu_1 = \mu - \alpha,\ \mu_2 = \mu + \alpha;\ \mu > \alpha > 0;\ \lambda = \frac{1}{2}$$

in (7.2) we conclude the Turán inequality (7.1) for the extended Mathieu–type series' integral form $\mathscr{S}_{\mu,\boldsymbol{\nu}}$.

In a similar manner, we can prove the next five Turán inequalities. □

8. Applications

The famous American applied mathematician Philip J. Davis (1923–2018) recalled in his book [14, p. 221] the following thought of the not less famous Ukrainian–born mathematician Alexander Ostrowski (1893–1986): "... In the seventeenth and eighteenth centuries, mathematicians tried to express integrals as sums. In the nineteenth century they began to express sums as integrals. So mathematics go in spirals ..." This Ostrowski's quotation can, actually only partially, to describe this our study in presenting several type Mathieu and aligned series by integrals, having in mind in the same time some application ancestry, like: 1. Quantum field theory and the Casimir effect's mathematics; 2. Application to a Fredholm integral equations of the first kind with the non-degenerate kernel in representing higher transcendental special functions of Bessel and hypergeometric type; 3. Modeling of banding and vibration for clamped rectangular plates and membranes of different shape and prisms, following Lamé problem in elasticity and equilibrium (Mathieu, Lauricella, Koialovich, Inglis, Pickett, Meleshko) and in this connection 4. Dynamic response of membranes with arbitrary shape (Caratelli it et al., Nagaya among others).

8.1. **Quantum field theory and the Casimir effect's mathematics.** The quantum field theory and quantum physics where Mathieu–type series play important roles. So, the $(\boldsymbol{a}, \boldsymbol{\lambda})$–series introduced in [70]

$$S_\mu(\varrho, \mathbf{a}, \boldsymbol{\lambda}) = \sum_{n\geq 0} \frac{a(n)}{(\lambda(n)+\varrho)^\mu}, \qquad (\varrho, \mu > 0) \tag{8.1}$$

where the sequences $a(n), \lambda(n), n \in \mathbb{N}_0$ are non–negative and the monotone $\lambda(n) \uparrow \infty$. This general Mathieu–type series concerns the so–called Zeta–function regularization, that is, the regularized, with the aid of the Riemann Zeta–function, version of a divergent series which appear in derivation of the vacuum expectation value of the energy of a particle field in quantum field theory:

$$\langle 0|T_{00}(s)|0\rangle = \frac{\hbar}{2} \sum_{n=1}^\infty \frac{1}{|\omega_n|^{s-1}},$$

where T_{00} is the zero'th component of the energy-momentum tensor. Similarly, $(\mathbf{a}, \boldsymbol{\lambda})$–series is related with the mathematics of the Casimir effect, in fact the quantum force of attraction between two parallel uncharged conducting plates with vacuum between them. By Casimir's calculations the zeta-regularized version of the energy $\langle E(s)\rangle$ per unit area of a plate is expressed *via* this kind of series:

$$\langle E(s)\rangle = \frac{\hbar A}{(2\pi)^2} \int \mathrm{d}k_x\, \mathrm{d}k_y \sum_{n\geq 0}^\infty \frac{1}{|\omega_n|^{s-1}}, \quad |\omega_n| := c\sqrt{k_x^2 + k_y^2 + (n\pi/a)^2}; \quad |\omega_0| =: c|\vec{k}|\,,$$

being A the area of the metal plates, c the speed of light, $\vec{k}_x, \vec{k}_y$ are the wave vectors in directions parallel to the plates, while a denotes the distance between the plates [9], [10]; consult for further informations about Casimir effect [61] and [73] for its Mathieu series relations.

Now, straightforward transformations and the Kapteyn formula [11, p. 5 Eq. (2.13)]

$$\int_0^\infty \frac{x^\nu\, J_\nu(bx)}{\mathrm{e}^{\pi x}-1}\,\mathrm{d}x = \frac{(2b)^\nu\,\Gamma(\nu+\frac12)}{\sqrt{\pi}} \sum_{n\geq 1} \frac{1}{(n^2\pi^2+b^2)^{\nu+\frac12}}\,,$$

imply the final (seemingly novel) representation result

$$\langle 0|T_{00}(s)|0\rangle = \frac{\hbar}{2(c|\vec{k}|)^{s-1}} + \frac{\hbar\sqrt{a}\,\pi^{\frac{s}{2}}}{c^{s-1}\sqrt{2}^{s}\,\Gamma(\frac{s-1}{2})}\int_0^\infty \frac{x^{\frac{s}{2}-1}\,J_{\frac{s}{2}-1}\big(a\,|\vec{k}|\,x\big)}{\mathrm{e}^{\pi x}-1}\,\mathrm{d}x\,. \tag{8.2}$$

The integral on the right-hand side u (8.2) for $s>2$ can be calculated numerically efficiently with similar quadrature formulas as the one with respect to Einstein's weight function developed by Gautschi and Milovanović [25]. Reviewing this paper in *Mathematical Reviews*, renowned nuclear scientist Gheorghe Adam in his rewiew [MR771039 (86j:65028)], among other things, says, "To the reviewer's knowledge, the present paper provides the first systematic investigation on the derivation of quadrature rules able to provide high-precision accuracy to the above integrals."

In our case the convergence of the corresponding quadrature formulae will be fast if we provide the integration of an entire function $t\mapsto F_s(t)$ with respect to some modified Einstein function $t\mapsto t^\nu/(\mathrm{e}^t-1)$ on $[0,\infty)$, where $0<\nu\leq 1$. Indeed, let $s\in(m,m+1]$, where m is an integer ≥ 2. Then $0<\nu=s-m\leq 1$ and the integral in (8.2) for $s>2$ can be reduced to

$$A\int_0^\infty \frac{t^\nu}{\mathrm{e}^t-1}F_s(t)\,\mathrm{d}x,$$

where $F_s(t)=t^{m-1-\frac{s}{2}}J_{\frac{s}{2}-1}\big(\frac{a}{\pi}\,|\vec{k}|\,t\big)$ is an entire function, and A is a constant.

8.2. **Fredholm integral equation of the first kind and Chaplygin comparison theorem.** The next applications of the Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$–series results concern the Fredholm integral equations of the first kind with the nondegenerate kernel, which ones give the main tool in characterizing the Bessel function of the first kind $J_\nu(x)$ of order ν and the Kummer's confluent hypergeometric function ${}_1F_1(a;c;x)$ in terms of a Bromwich–Wagner contour integral [19, Theorem 1, Eq.(15)], [74, Theorems 19, 20; Corollary 8].

8.3. **The biharmonic differential equation in modeling of elasticity of plates and membranes.** The next application is discovered through considerations of a relevant physical problem, for example, normal modes of vibrating systems of physical and engineering sciences. The two-dimensional problem of stresses in an infinitely long elastic rectangular prism or a thin rectangle subjected to a surface load on its sides is one of the oldest benchmark problems of linear elastostatics, which originates back to Lamé's lectures [42] on the mathematical theory of elasticity. He considered the equilibrium of a three-dimensional elastic parallelepiped under any system of normal forces on its sides. The two–dimensional version of the problem for the elastic rectangle and the general biharmonic problem (which also appears when considering bending of a clamped, thin elastic rectangular plate) also attracted the attention. That two-dimensional problem of an elastic rectangle was developed by Mathieu in his notes, articles, memoir and the monograph on easticity [51, 52, 53, 54]. The main idea of the superposition method for the biharmonic equation in the 2D domain $\mathfrak{D} = [-a, a] \times [-b, b]$ was to consider the sum of two Fourier series on the complete systems of trigonometric functions with respect to coordinates x and y. These series satisfy identically the biharmonic equation on the whole rectangle Ω including all the four edges. The coefficient of a term in one series will depend on all the coefficients of the other series and *vice versa*; the solution reduces to solving the infinite system of linear algebraic equations giving the relations between the coefficients and loading forces. Mathieu [51, 52, 53, 54] studied this problem but without giving any numerical results for stresses for the square plate. Further simplifications, approvals and developments of that theory were found by Koialowitch [38], Lauricella [44, 45, 46, 47], Inglis [29] and Pickett [68] among others. It is worth to mention that only Koialowitch constructed the mathematical solution of the problem and provided some numerical results, see [55, 56]. The whole ancestry of the problem is thoroughly presented in Meleshko's historical survey [57] which contains highly exhaustive references list.

The Neumann problem arise in the analysis of transverse vibrations of an elastic membrane or plate stretched over a planar region $\mathfrak{D} \subset \mathbb{E}^2$, or a prism, parallelopiped in $\mathbb{E}^3$, while the corresponding theory for the plate or prism depends on the biharmonic partial differential expression

$$\Delta^2 = \Delta(\Delta) = \sum_{1\le j,k\le r} \frac{\partial^4}{\partial x_j^2\, \partial x_k^2}\,.$$

For example, when $r = 2$, the clamped plate is studied under the Neumann boundary value problem

$$\Delta^2 U = 0 \text{ in } \mathfrak{D}, \qquad U = 0 \text{ and } \frac{\partial U}{\partial \vec{\boldsymbol{n}}} = 0 \text{ on } \partial\mathfrak{D}\,, \tag{8.3}$$

for U coming from a suitable functions class and where $\partial\mathfrak{D}$ denotes the boundary of $\mathfrak{D}$, [23]. In the case when $\mathfrak{D} = [-a, a] \times [-b, b]$ it was stated that the final representation of the biharmonic function in terms of infinite Fourier series is convergent and satisfies both boundary conditions at all sides of the rectangle, [58]. The resulting infinite system of linear algebraic equations has been treated by a traditional way, named 'the method of reduction' originates back to Fourier, also see [38, 59]. The finiteness of solutions was proved by Mathieu who derived estimates for coefficients in a series form which coincide with $S(r)$ and $S_\mu(r)$. However, these functional series contain trigonometric and hyperbolic sine and cosine function expressions, which alternate their building block's sings, that is, they are in the same time Fourier–Bessel series of Neumann and/or Dini type series (for which consult [3, 91]).

The applied mathematician Schröder [78] has considered the extended 'problem of clamped rectangular plate' for the non–homogeneous Neumann boundary problem case of (8.3):

$$\Delta^2 U = f(x, y) \text{ in } \mathfrak{D}, \qquad U = 0 \text{ and } \frac{\partial U}{\partial \vec{\boldsymbol{n}}} = 0 \text{ on } \partial\mathfrak{D}\,. \tag{8.4}$$

His memoir also contains upper bounds for the Mathieu–type series of the magnitude of $S(r) \le r^{-2}$ derived by the usual series–integral majorization procedure. Schröder method differs from Mathieu's (that is the one used by Lauricella and Koialowitsch), but also includes series of Neumann– and Dini–Bessel type, which terms alternate in signs. Three years later Emersleben [21] for integer

r precise Schröder's bound upon Mathieu series (which result is *de facto* necessary only for convergence purposes in certain by-product Fourier series), in turn his contribution is the first integral expression for $S(r)$, compare (1.4). Section 6 thoroughly extended Emersleben's integral form of the Mathieu series for another generalized multiparameter Mathieu-type series.

It is worth to mention that serious application targeted research is conducted by biharmonic equation's modeling in elasticity theory themata when the domain of the Neumann problems (8.3) and (8.4). So, for $\mathfrak{D}$ being starlike domain, the investigations by Caratelli *et al.* [7, 8] are relevant, while when $\mathfrak{D}$ is of arbitrary shape Nagaya's [63] work can be used; in all three publications the corresponding Fourier–Bessel series belong to the Neumann series family built by the Bessel functions of the first and the second kind.

9. Discussion. Remarks. Open problems

A. The notation $\Omega(z), z \in \mathbb{C}$ stands for the so–called Butzer–Flocke–Hauss (complete) Omega function introduced in [4, Definition 7.1], [5] in the form

$$\Omega(z) := 2\int_{0+}^{\frac{1}{2}} \sinh(zu)\cot(\pi u)\,\mathrm{d}u, \qquad z \in \mathbb{C}\,.$$

It is the Hilbert transform $\mathscr{H}_1[\mathrm{e}^{-zx}](0)$ at zero of the 1–periodic function $\left(\mathrm{e}^{-zx}\right)_1$, defined by the periodic extension of the exponential function e^{-zx}, $|x| < \frac{1}{2}$, $z \in \mathbb{C}$, thus

$$\Omega(z) = \mathscr{H}_1\left[\mathrm{e}^{-zx}\right](0) = \mathrm{PV}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{zu}\,\cot(\pi u)\,\mathrm{d}u\,.$$

Another expressions for complete BHF Omega function $\Omega(x)$ are given by Butzer *et al.* [6]:

$$\Omega(x) = \frac{2}{\pi}\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{1}{\mathrm{e}^t+1}\cos\left(\frac{xt}{2\pi}\right)\mathrm{d}t\,, \qquad x \in \mathbb{R}\,, \tag{9.1}$$

by Tomovski and Pogány for the real argument complete BHF Ω function which reads [90, p. 10, Theorem 3.3]

$$\Omega(x) = 2\sqrt{\frac{2}{\pi}}\sinh\left(\frac{x}{2}\right)\,\mathrm{PV}\int_0^\infty \sinh\left(\frac{xt}{\pi}\right)\,\tan t\,\mathrm{d}t\,. \tag{9.2}$$

By extensions in the integrand of the Butzer–Flocke–Hauss Omega function which is intimately connected to the generalized Mathieu series (consult the extensive study by Butzer and Pogány [5]) we are faced with a new territory of ideas and series/integral conclusion upon the structure of these kind generalizations.

Inspired by (9.1), we can write

$$\Omega(x) = \frac{\sqrt{x}}{\pi}\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{\sqrt{t}}{\mathrm{e}^t+1}\,J_{-\frac{1}{2}}\left(\frac{xt}{2\pi}\right)\mathrm{d}t$$

having in mind that $\cos(z) = \sqrt{\pi z/2}\,J_{-\frac{1}{2}}(z)$, then we can proceed with generalization considering hyper–Bessel function, Bessel–Clifford function and/or their products instead of $J_{-\frac{1}{2}}$ in the kernel.

On the other hand the well–known equality $\sinh(z) = \sqrt{\pi z/2}\,I_{\frac{1}{2}}(z)$ can be used for extending the integrand of the representation formula (9.2).

Finally, a linear ODE was obtained [6, Theorem 1], whose particular solution is the Butzer–Flocke–Hauss complete real–parameter Omega function $\Omega(z), z \in \mathbb{C}$ [4, Definition 2.1], which is associated with the complex–index Bernoulli function $B_\alpha(z)$ and with the complex–index Euler function $E_\alpha(z)$. This is accomplished with the aid of an integral representation of the alternating Mathieu–type series $\widetilde{T}(r) = \sum\limits_{n\geq 0} (-1)^{n-1}(n^2+r^2)^{-2}$, $r \in \mathbb{R}$.

Moreover, it is worth to mention that the use of the Chaplygin comparison theorem, actually a highly powerful differential inequality's use was exploited in discussing the magnitudes of bounding inequalities for the BFH Ω function by constant use of the alternating Mathieu series and generalized Mathieu eries, see [75].

B. The multi–index Mittag–Leffler function's power series definition is [35, p. 1888, Eqs. (7–8)]

$$E^{(n)}_{(\rho_j),(\mu_j)}(z) = {}_1\Psi_n\left[\begin{array}{c} (1,1) \\ (\mu_1,\rho_1), \cdots (\mu_n,\rho_n) \end{array} \middle| z \right] = \sum_{k\geq 0} \frac{z^k}{\prod\limits_{j=1}^n \Gamma(\mu_j+\rho_j k)};$$

the parameters range follow the conditions declared for the Fox–Wright generalized hypergemoetric function. Obviously, close connections are there between Bessel-Clifford function and the multi–parameter Mittag–Leffler function [16, 33, 36, 66, 88]. So, the integral representation of multi–parameter Mathieu and alternating Mathieu series become

$$\mathscr{S}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = \frac{\pi^{\frac m2}\prod\limits_{j=1}^m r_j^{\nu_j-\frac12}}{2^{\frac m2+\sum\limits_{j=1}^m \nu_j}} \int_0^\infty \frac{x^{\mu+\sum\limits_{j=1}^m \nu_j-1}}{\mathrm{e}^x-1} \prod_{j=1}^m E^{(2)}_{(1,1),(\nu_j+1,1)}\left(-\frac{r_j^2x^2}{4}\right)\mathrm{d}x,$$

and

$$\widetilde{\mathscr{S}}_{\mu,\boldsymbol{\nu}}(\boldsymbol{r}) = \frac{\pi^{\frac m2}\prod\limits_{j=1}^m r_j^{\nu_j-\frac12}}{2^{\frac m2+\sum\limits_{j=1}^m \nu_j}} \int_0^\infty \frac{x^{\mu+\sum\limits_{j=1}^m \nu_j-1}}{\mathrm{e}^x+1} \prod_{j=1}^m E^{(2)}_{(1,1),(\nu_j+1,1)}\left(-\frac{r_j^2x^2}{4}\right)\mathrm{d}x,$$

where the so–called Dzrbashjan's (binomial/two–parameter) Mittag-Leffler type function notation is used. For $m = 1$, the integral representations (3.1) and (3.2) reduce to the familiar Mathieu series and alternating Mathieu series (1.4) and (1.6) by virtue of (1.9).

C. The constraints of the inequality (6.20) actually can be weaken. Namely, rewriting $a_1\cdots a_n \leq a_1+\cdots+a_n$ into

$$a_n(a_1\cdots a_{n-1}-1) \leq a_1+\cdots+a_{n-1},$$

we see that (6.20) holds true when

$$a_1\cdots a_{n-1} < 1,\, a_n > 0; \quad \text{or} \quad a_1\cdots a_{n-1} > 1,\, a_n < \frac{a_1+\cdots+a_{n-1}}{a_1\cdots a_{n-1}-1}\,.$$

We conclude that when **no more then one** $a_j > 1$, then the inequality (6.20) is valid as well. This comment allows further possibilities to enlarge the parameter space in Theorem 7.

D. The set of bounding inequalities for Bessel–function family members is extensive, excluding the Bessel–Clifford functions and the hyper–Bessel and modified hyper–Bessel functions for the interval $x > j_{\nu_m,1}^{m+1}$ which, according to our best knowledge, do not exist; here $j_{\nu_m,1}$ stands for the first positive zero of the hyper–Bessel function $J_{\boldsymbol{\nu}}^{(m)}(x)$, consult the introductory part of section 6.

In this respect immediately arise two important questions: **(a)** to derive suitable functional bounding inequalities for the Bessel–Clifford functions and the hyper–Bessel and modified hyper–Bessel functions, and **(b)** the sharpness issues for the derived upper bounds concerning the extended Mathieu–type series integrals. In turn, the case **(b)** implies a new ask: the upper bounds magnitude with respect to large parameters $\mu, \bar{\boldsymbol{\nu}}$ and $\boldsymbol{r}$.

As to the problem **(a)** we give an answer by Theorem 7; also consult in this respect C. Having in mind the hybrid upper bound (6.8) introduced by Srivastava and Pogány in [86] for Bessel functions of the first kind, we can combine another existing bounds like the ones by Krasikov, Landau, Lommel, Minkashisundaram–Szász, Olenko, and Sitnik for the annulus and another suitable ones for the remaining part of the domain.

In our exposition we decide for Paris' bound (6.13) which holds for the unit interval. This implies that our approach suggests several possible bounds; the 'best bound' contest could be resolved by the smallest magnitude study, for instance.

Hybrid bounds for the Bessel–Clifford, and hyper–Bessel functions we can build following the traces of aforementioned method by Srivastava and Pogány. Considering the bound (6.10) by Aktaş *et al.* from one side for the values of the argument in annulus and the offered bounds from Theorem 7, we have on disposal the hybrid functional upper bound for the hyper–Bessel function $J_{\boldsymbol{\nu}}^{(m)}(x)$ as

$$\begin{aligned}\big|J_{\boldsymbol{\nu}}^{(m)}(x)\big| \le \mathfrak{H}_m^J(x) = \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m \nu_j} \Bigg\{ &\frac{1}{A(\boldsymbol{\nu})}\, \mathrm{e}^{-\frac{x^{m+1}}{(m+1)^{m+1}\,A(\boldsymbol{\nu})}} \cdot \chi_{\left(0, j_{\nu_m,1}^{m+1}\right]}(x) \\ &+ \sum_{j=1}^m \frac{1}{\Gamma(\nu_j+1)} \cosh\left[2\left(\frac{x}{m+1}\right)^{\frac{1}{2(m+1)}}\right] \cdot \chi_{\left(j_{\nu_m,1}^{m+1}, \infty\right)}(x) \Bigg\}; \quad A(\boldsymbol{\nu}) = \prod_{j=1}^m \Gamma(\nu_j+2)\,,\end{aligned}$$

where $\chi_S(x)$ denotes the characteristic function of the set S.

On the other hand the Redheffer–type inequality (6.11) implies the hybrid bound

$$\begin{aligned}\big|J_{\boldsymbol{\nu}}^{(m)}(x)\big| \le \mathfrak{K}_m^J(x) = \Big(\frac{x}{m+1}\Big)^{\sum\limits_{j=1}^m \nu_j} \Bigg\{ &\frac{j_{\nu_m,1}^{m+1} - x^{m+1}}{j_{\nu_m,1}^{m+1} \prod\limits_{j=1}^m \Gamma(\nu_j+1)} \cdot \chi_{\left(0, j_{\nu_m,1}^{m+1}\right)}(x) \\ &+ \sum_{j=1}^m \frac{1}{\Gamma(\nu_j+1)} \cosh\left[2\left(\frac{x}{m+1}\right)^{\frac{1}{2(m+1)}}\right] \cdot \chi_{\left[j_{\nu_m,1}^{m+1}, \infty\right)}(x) \Bigg\}.\end{aligned}$$

The derivation of hybrid bound for the Bessel–Clifford function $C_{\boldsymbol{\nu}}^{(m)}(x)$ and its applications we leave to the interested reader.

The problem **(b)** has an interesting consequence and another benefits. Namely, the computable power series representation formulae in terms of higher transcendental functions which consist the matter of Theorems 1–4 and all their corollaries are in fact summation formulae when we reverse their point of view. On the other hand Theorems 5-9 give functional upper bounds for the integrals represented by the series/summations which are *a fortiori* upper bounds for the mentioned highly complicated sums.

E. Propositions 1 and 2 present novel results in Laplace–Mellin transform family of the products of m hyper–Bessel functions of not necessarily equal size parameters, complementing the appropriate sections of Laplace transform collections like Erdélyi *et al.* [22] and Prudnikov *et al.* [76].

Next, new integral Laplace–Mellin transform formula (6.23) was obtained for the cosine hyperbolic function with general power argument, reads as follows:

$$\int_0^\infty \mathrm{e}^{-px} x^{\mu-1} \cosh\big(ax^q\big)\,\mathrm{d}x = \frac{\Gamma(\mu+1)}{2\,p^\mu}\,{}_2\Psi_1\left[\begin{array}{c} (\frac{\mu}{2},q),\,(\frac{\mu}{2}+\frac12,q) \\ (\frac12,1)\end{array}\middle| \frac{4^{q-1}\,a^2}{p^{2q}}\right],$$

where $\mu+1>0$, $\Re(p)>0$, $q\in(0,1)$, which follows from the convergence condition for the Fox–Wright ${}_2\Psi_1$ function as $\Delta = 2-2q>0$. Else, for $q=1$ the series converges for $|a|<\Re(p)$; this can be seen from the constraint $\rho=1$ around (2.2) or by the asymptotic behavior of the integrand for large x. This transform is an important extension of the related formula listed in the monograph by Prudnikov *et al.* [76, p. 60, Eq. 17], where only the *positive rational* exponent $q\in\mathbb{Q}_+$ was considered, and the formula is in the Meijer G function setting given, which use we systematically avoid in the whole exposition; consult also [77] in this respect.

It is worth to mention that the case $q=1$ of the previous formula reduces to a sum of two algebraic functions, *viz.* (6.24).

F. The integral representations of hyper–Bessel and modified hyper–Bessel functions are reported for instance in [16, p. 32, Theorem 3]

$$J_{\boldsymbol\nu}^{(m)}(z) = c_m\left(\frac{z}{m+1}\right)^{\sum\limits_{j=1}^m \nu_j}\int_0^1 G^{m,0}_{m,m}\left[t\,\middle|\,\begin{array}{c}\boldsymbol\nu\\ \frac{\boldsymbol j}{m+1}-1\end{array}\right]\cos_{m+1}\big[(m+1)z\,t^{1/(m+1)}\big]\,\mathrm{d}t\,,$$

$$I_{\boldsymbol\nu}^{(m)}(z) = c_m\left(\frac{z}{m+1}\right)^{\sum\limits_{j=1}^m \nu_j}\int_0^1 G^{m,0}_{m,m}\left[t\,\middle|\,\begin{array}{c}\boldsymbol\nu\\ \frac{\boldsymbol j}{m+1}-1\end{array}\right]h_{m+1}\big[(m+1)z\,t^{1/(m+1)}\big]\,\mathrm{d}t\,,$$

where $z,\nu_j\in\mathbb{C}$ and $\Re(\nu_j)>-1$, $j=\overline{1,m}$, the constant

$$c_m = (2\pi)^{-\frac{m}{2}}\sqrt{m+1}\,,$$

while the generalized cosine function of the order $r\in\mathbb{N}$ reads

$$\cos_r(w) = \sum_{k\ge0}\frac{(-1)^k w^{rk}}{(rk)!} = {}_0F_{r-1}\left[\begin{array}{c} - \\ \frac1n,\frac2n,\dots,\frac{n-1}{n}\end{array}\middle| -\left(\frac{z}{r}\right)^r\right], \quad \cos_1(w)\equiv\cos(w).$$

In the integrand the Meijer G function terminology is used. In turn, the modified hyper–Bessel functions' integral representation contains the generalized hyperbolic $h_{m+1}(w) = \cos_{m+1}(-w)$ function in the kernel, [16, p. 31, Definition 3].

We also notice that the multi–index Mittag–Leffler function is related with the Delerue hyper–Bessel function by the formula [34, p. 1132, Eq. (16)].

Finally, we re-call the integral representation formula for the Bessel–Clifford function [28, p. 17, §5] (also see [16]) which reads:

$$C_{\boldsymbol{\nu}}^{(m)}(z) = c_m \int_0^1 G_{m,m}^{m,0}\left[t \,\middle|\, \begin{matrix} \boldsymbol{\nu} \\ \frac{\boldsymbol{j}}{m+1} - 1 \end{matrix} \right] \cos_{m+1}\left[(m+1)z\, t^{1/(m+1)}\right] \mathrm{d}t\,.$$

These integral expressions could be also used in another fashion approach for the tasks realized in our present exposition.

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