AIP Conference Proceedings

Numerical integration of analytic functions

Gradimir V. Milovanović, Dobrilo Đ Tošić, and Miloljub Albijanić

Citation: AIP Conf. Proc. **1479**, 1046 (2012); doi: 10.1063/1.4756325
View online: http://dx.doi.org/10.1063/1.4756325
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=1479&Issue=1
Published by the American Institute of Physics.

Numerical Integration of Analytic Functions

Gradimir V. Milovanović*, Dobrilo Đ. Tošić† and Miloljub Albijanić**

*Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36, P.O. Box 367, 11001 Beograd, Serbia
†Department of Mathematics, Faculty of Electrical Engineering, University of Belgrade, P.O. Box 35-54, 11120 Belgrade, Serbia
**Zavod za udžbenike, Obilićev venac 5, 11001 Beograd, Serbia

Abstract. A weighted generalized N-point Birkhoff–Young quadrature of interpolatory type for numerical integration of analytic functions is considered. Special cases of such quadratures with respect to the generalized Gegenbauer weight function are derived.

Keywords: quadrature formula; weight function; error term; orthogonality; analytic function; nodes; weight coefficients.
PACS: 02.60.Jh, 02.60.Nm

INTRODUCTION

In 1950 Birkhoff and Young [2] proposed a quadrature formula of the form

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z = \frac{h}{15}\Big\{24f(z_0)+4\Big[f(z_0+h)+f(z_0-h)\Big]-\Big[f(z_0+\mathrm{i}h)+f(z_0-\mathrm{i}h)\Big]\Big\}+R_5^{BY}(f)$$

for numerical integration of complex analytic functions in $\Omega=\big\{z : |z-z_0|\le r\big\}$, where $|h|\le r$. This five point quadrature formula is exact for all algebraic polynomials of degree at most five, and its error term can be estimated by

$$|R_5^{BY}(f)|\le \frac{|h|^7}{1890}\max_{z\in S}|f^{(6)}(z)|,$$

where S denotes the square with vertices $z_0+\mathrm{i}^k h$, $k=0,1,2,3$ (see [16] and [3, p. 136]). This error estimate is about four tenths as large as the corresponding error $R_5^{ES}(f)$ for the so-called extended Simpson rule (cf. [13, p. 124])

$$\int_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z = \frac{h}{90}\big\{114f(z_0)+34\big[f(z_0+h)+f(z_0-h)\big]-\big[f(z_0+2h)+f(z_0-2h)\big]\big\}+R_5^{ES}(f),$$

for which we have

$$|R_5^{ES}(f)|\sim\frac{|h|^7}{756}|f^{(6)}(\zeta)|,\qquad 0<\frac{\zeta-(z_0-2h)}{4h}<1.$$

Without loss of generality, the previous Birkhoff–Young formula can be reduced to an integration over $[-1,1]$,

$$I(f)=\int_{-1}^{1} f(z)\,\mathrm{d}z=\frac{8}{5}f(0)+\frac{4}{15}\big[f(1)+f(-1)\big]-\frac{1}{15}\big[f(\mathrm{i})+f(-\mathrm{i})\big]+R_5(f). \tag{1}$$

In 1976 Lether [4] pointed out that the three point Gauss-Legendre quadrature which is also exact for all polynomials of degree at most five, is more precise than (1) and he recommended it for numerical integration. However, Tošić [15] improved the quadrature (1) in a simple way taking its nodes at the points $\pm r$ and $\pm \mathrm{i}r$, with $r\in(0,1)$, instead of ±1 and $\pm\mathrm{i}$, and he derived an one-parametric family of quadrature rules in the form

$$I(f)=2\left(1-\frac{1}{5r^4}\right)f(0)+\left(\frac{1}{6r^2}+\frac{1}{10r^4}\right)\big[f(r)+f(-r)\big]+\left(-\frac{1}{6r^2}+\frac{1}{10r^4}\right)\big[f(\mathrm{i}r)+f(-\mathrm{i}r)\big]+R_5^T(f;r).$$

Numerical Analysis and Applied Mathematics ICNAAM 2012
AIP Conf. Proc. 1479, 1046-1049 (2012); doi: 10.1063/1.4756325

Evidently, for $r = 1$ it reduces to (1) and for $r = \sqrt{3/5}$ to the three point Gauss-Legendre formula. Since the error-term $R_5^T(f;r)$ can be expressed as

$$R_5^T(f;r) = \left(-\frac{2}{3\cdot 6!}r^4 + \frac{2}{7!}\right) f^{(6)}(0) + \left(-\frac{2}{5\cdot 8!}r^4 + \frac{2}{9!}\right) f^{(8)}(0) + \cdots, \tag{2}$$

it is clear that for $r = \sqrt[4]{3/7}$ the first term on the right-hand side in (2) vanishes and the corresponding formula reduces to the modified Birkhoff-Young quadrature rule of the maximum degree of precision seven (named *MF* in [15]),

$$I(f) = \frac{16}{15}f(0) + \frac{1}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right)\left[f(\sqrt[4]{3/7}) + f(-\sqrt[4]{3/7})\right] + \frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\left[f(\mathrm{i}\sqrt[4]{3/7}) + f(-\mathrm{i}\sqrt[4]{3/7})\right] + R_5^{MF}(f),$$

with the error-term $R_5^{MF}(f) = R_5^T(f;\sqrt[4]{3/7}) \approx 1.26\cdot 10^{-6} f^{(8)}(0)$.

This formula was extended by Milovanović and Đorđević [14] to the following quadrature rule of interpolatory type

$$I(f) = Af(0) + B\left[f(r_1)+f(-r_1)\right] + C\left[f(\mathrm{i}r_1)+f(-\mathrm{i}r_1)\right] + D\left[f(r_2)+f(-r_2)\right] + E\left[f(\mathrm{i}r_2)+f(-\mathrm{i}r_2)\right] + R_9(f;r_1,r_2),$$

where $0 < r_1 < r_2 < 1$. For $r_1 = r_1^* = \sqrt[4]{(63-4\sqrt{114})/143}$ and $r_2 = r_2^* = \sqrt[4]{(63+4\sqrt{114})/143}$, this formula has the algebraic precision $d = 13$, with the error-term $R_9(f;r_1^*,r_2^*) \approx 3.56\cdot 10^{-14} f^{(14)}(0)$.

Quadrature formulae of Birkhoff–Young type for analytic functions have been investigated in several papers [1], [9], [10], [11], [12]. These formulas can also be used to integrate real harmonic functions (see [2]). In addition, we mention also that Lyness and Delves [6] and Lyness and Moler [7], and later Lyness [5], developed formulae for numerical integration and numerical differentiation of complex functions.

In this paper we consider a generalized quadrature formula of Birkhoff–Young type for integrating analytic functions with respect to a given weight function.

GENERALIZED BIRKHOFF–YOUNG QUADRTAURE FORMULA

In this section we consider the following N-point generalized quadrature formula of interpolatory type

$$I(f) := \int_{-1}^{1} f(z)w(z)\,\mathrm{d}z = Q_N(f) + R_N(f), \tag{3}$$

for numerical integration of analytic functions, which are analytic in the unit disk $\Omega = \{z : |z| \le 1\}$. The weight function $w : (-1,1) \to \mathbb{R}^+$ is an arbitrary even positive function, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z)\mathrm{d}z$, $k = 0,1,\ldots,$ exist. Notice that $\mu_{2k+1} = 0$ and $\mu_{2k} > 0$ for each $k \in \mathbb{N}_0$. The quadrature formula (3) has the nodes at the zeros of a monic polynomial of degree N,

$$\omega_N(z) = z^\nu p_{n,\nu}(z^4) = z^\nu \prod_{k=1}^{n}(z^4 - r_k), \quad 0 < r_1 < \cdots < r_n < 1, \tag{4}$$

i.e.,

$$Q_N(f) = \sum_{j=0}^{\nu-1} C_j f^{(j)}(0) + \sum_{k=1}^{n}\left\{A_k\left[f(x_k)+f(-x_k)\right] + B_k\left[f(\mathrm{i}x_k)+f(-\mathrm{i}x_k)\right]\right\}, \quad x_k = \sqrt[4]{r_k}, \ k = 1,\ldots,n,$$

where $n = [N/4]$ and $\nu = N - 4[N/4]\,(\in \{0,1,2,3\})$, i.e., $N = 4n+\nu$. The corresponding remainder term is denoted by $R_N(f)$. Practically, we consider four subclasses of these quadratures, $Q_{4n+\nu}(f)$, $\nu = 0,1,2,3$. Notice that in $Q_{4n}(f)$ the first sum is empty. Also, in order to have $Q_N(f) = I(f) = 0$ for $f(z) = z$, it must be $C_1 = 0$, so that $Q_{4n+1}(f) \equiv Q_{4n+2}(f)$.

Recently Milovanović [10] has proved the existence and uniqueness of these quadratures $Q_N(f)$, with a maximal degree of precision $d = 6n + s$, where $n = [N/4]$, $\nu = N - 4[N/4] \in \{0,1,2,3\}$, and

$$s = \begin{cases} \nu - 1, & \nu = 0,2, \\ \nu, & \nu = 1,3, \end{cases} \tag{5}$$

as well as a characterization of such generalized quadratures in terms of multiple orthogonal polynomials, using the the orthogonality conditions [10, Theorem 4.3]

$$\int_0^1 t^k p_{n,\nu}(t^2) t^{s/2} w(\sqrt{t})\,\mathrm{d}t = 0, \quad k = 0,1\ldots,n-1. \tag{6}$$

In this paper we give a method for numerical construction of quadratures $Q_{4n+\nu}(f)$, $\nu = 0,1,2,3$ using the moments of the weight function.

According to (4), the polynomial $p_{n,\nu}(t^2)$ can be expressed in the form

$$p_{n,\nu}(t^2) = \sum_{j=0}^{n} (-1)^j \sigma_j t^{2(n-j)}, \tag{7}$$

where σ_j are the so-called *elementary symmetric functions*, defined by $\sigma_j = \sum_{(k_1,\ldots,k_j)} r_{k_1} \cdots r_{k_j}$, $j = 1,\ldots,n$, and the summation is performed over all combinations $(k_1,\ldots,k_j)$ of the basic set $\{1,\ldots,n\}$. Thus, $\sigma_1 = r_1 + r_2 + \cdots + r_n$, $\sigma_2 = r_1 r_2 + \cdots + r_{n-1} r_n$, …, $\sigma_n = r_1 r_2 \cdots r_n$, and for the convenience we put $\sigma_0 = 1$. Starting from the orthogonality conditions (6), we obtain the following system of linear equations

$$\sum_{j=0}^{n} (-1)^j \sigma_j \int_0^1 t^{k+2(n-j)+s/2} w(\sqrt{t})\,\mathrm{d}t = 0, \quad k = 0,1\ldots,n-1,$$

i.e.,

$$\sum_{j=1}^{n} (-1)^{j-1} m_{k,j} \sigma_j = m_{k,0}, \quad k = 0,1\ldots,n-1, \tag{8}$$

where $m_{k,j} = \int_0^1 t^{k+2(n-j)+s/2} w(\sqrt{t})\,\mathrm{d}t = 2\int_0^1 z^{2k+4(n-j)+s+1} w(z)\,\mathrm{d}z$. Since $s+1$ is always an even number (see (5)), the coefficients in the previous system of equations can be expressed only in terms of the moments of the weight function w. Namely, $m_{k,j} = \mu_{2k+4(n-j)+s+1}$, $k = 0,1,\ldots,n-1$; $j = 0,1,\ldots,n$.

TABLE 1. The values of σ_k, $k = 1,\ldots,n$, for $n = 1,2,3,4$ for the Legendre weight $w(z) = 1$ and $w(z) = |z|$

weight		Legendre ($\gamma = 0,\ \alpha = 0$)				$w(z) = \lvert z\rvert$ ($\gamma = 1,\ \alpha = 0$)			
n	ν	σ_1	σ_2	σ_3	σ_4	σ_1	σ_2	σ_3	σ_4
1	0	1/5				1/3			
	1,2	3/7				1/2			
	3	5/9				3/5			
2	0	70/99	1/33			4/5	1/15		
	1,2	126/143	15/143			20/21	1/7		
	3	66/65	7/39			15/14	3/14		
3	0	99/85	63/221	1/221		5/4	5/14	1/84	
	1,2	429/323	693/1615	7/323		7/5	1/2	1/30	
	3	195/133	1287/2261	15/323		84/55	7/11	2/33	
4	0	260/161	38610/52003	660/7429	5/7429	56/33	28/33	4/33	1/495
	1,2	204/115	14586/15295	1716/10925	9/2185	24/13	756/715	28/143	1/143
	3	1292/675	8398/7245	572/2415	11/1035	180/91	180/143	40/143	15/1001

In the case of the generalized Gegenbauer weight function $w(z) = |z|^{\gamma}(1-z^2)^{\alpha}$, $\gamma,\alpha > -1$ (see [8, pp. 147–148]), we obtain

$$m_{k,j} = \int_0^1 t^{k+2(n-j)+\beta} (1-t)^{\alpha}\,\mathrm{d}t = \frac{\Gamma(\alpha+1)\Gamma(k+2(n-j)+\beta+1)}{\Gamma(k+2(n-j)+\alpha+\beta+2)},$$

where $\beta = (s+\gamma)/2$. The coefficients σ_k, $k = 1,\ldots,n$, of the node polynomial for some specific parameters γ and α and $n \le 4$ are presented in Tables 1 and 2. The determination of the weight coefficients A_k, B_k, C_j in the quadrature sum $Q_N(f)$ is a linear problem and it can be solved by interpolation (cf. [8, §5.1]).

At the end, as an example, we mention only a quadrature rule with respect to the Chebyshev weight of the second kind of the precision $d = 15$,

$$\int_{-1}^{1} f(z)\sqrt{1-z^2}\,\mathrm{d}z \approx \frac{131\pi}{462} f(0) + \frac{5\pi}{616} f''(0) + + \sum_{k=1}^{2} \Big\{ A_k \big[f(\sqrt[4]{r_k}) + f(-\sqrt[4]{r_k}) \big] + B_k \big[f(\mathrm{i}\sqrt[4]{r_k}) + f(-\mathrm{i}\sqrt[4]{r_k}) \big] \Big\},$$

TABLE 2. The values of σ_k, $k=1,\ldots,n$, for $n=1,2,3,4$ for the Chebyshev weight of the first and second kind

weight		Chebyshev I $(\gamma=0,\ \alpha=-1/2)$				Chebyshev II $(\gamma=0,\ \alpha=1/2)$			
n	ν	σ_1	σ_2	σ_3	σ_4	σ_1	σ_2	σ_3	σ_4
1	0	3/8				1/8			
	1,2	5/8				5/16			
	3	35/48				7/16			
2	0	7/8	7/128			7/12	7/384		
	1,2	21/20	21/128			3/4	9/128		
	3	33/28	33/128			99/112	33/256		
3	0	297/224	99/256	33/4096		33/32	55/256	11/4096	
	1,2	143/96	143/256	143/4096		143/120	429/1280	143/10240	
	3	13/8	1287/1792	143/2048		117/88	117/256	65/2048	
4	0	39/22	117/128	65/512	39/32768	65/44	39/64	65/1024	13/32768
	1,2	85/44	221/192	221/1024	221/32768	85/52	51/64	119/1024	85/32768
	3	323/156	969/704	323/1024	1615/98304	323/182	4845/4928	323/1792	1615/229376

where $r_{1,2}=\left(99\pm\sqrt{3333}\right)/224$, and the corresponding numerical values of the coefficients (A_1,A_2,B_1,B_2) are

$$\left(0.2670187613792136,\ 0.06912330953826695,\ 0.004040430200888034,\ -0.0001832961844082505\right).$$

Applying this rule to the integral $I=\int_{-1}^{1}\mathrm{e}^z\sqrt{1-z^2}\,\mathrm{d}z=\pi I_1(1)$, where $I_1(z)$ is the modified Bessel function of the first kind, we obtain the approximative value 1.7754996892121809179, with the relative error $1.63\cdot10^{-17}$.

ACKNOWLEDGMENTS

The authors were supported in part by the Serbian Ministry of Education and Science (Project: Approximation of integral and differential operators and applications, grant number #174015).

REFERENCES

1. M. Acharya, B.P. Acharya, S. Pati, Numerical evaluation of integrals of analytic functions, *Internat. J. Comput. Math.* **87**, No. 12, 2747–2751 (2010).
2. G. Birkhoff, D. M. Young, "Numerical quadrature of analytic and harmonic functions", *J. Math. Phys.* **29**, 217–221 (1950).
3. P. J. Davis, P. Rabinowitz, *Methods of Numerical Integration*, New York: Academic Press, 1975.
4. F. Lether, On Birkhoff-Young quadrature of analytic functions, *J. Comput. Appl. Math.* **2**, 81–84 (1976).
5. J.N. Lyness, Quadrature methods based on complex function values, *Math. Comp.* **23**, 601–619 (1969).
6. J. N. Lyness, L. M. Delves, On numerical contour integration round a closed contour, *Math. Comp.* **21**, 561–577 (1967).
7. J.N. Lyness, C.B. Moler, Numerical differentiation of analytic functions, *SIAM J. Numer. Anal.* **4**, 202–210 (1967).
8. G. Mastroianni, G.V. Milovanović, *Interpolation Processes – Basic Theory and Applications*, Springer Monographs in Mathematics, Berlin – Heidelberg: Springer – Verlag, 2008.
9. M.T. McGregor, On a modified Birkhoff-Young quadrature formula for analytic functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **3**, 13–16 (1992).
10. G.V. Milovanović, Numerical quadratures and orthogonal polynomials, *Stud. Univ. Babeş-Bolyai Math.* **56**, 449–64 (2011).
11. G.V. Milovanović, Generalized quadrature formulae for analytic functions, *Appl. Math. Comput.* **218**, 8537–8551 (2012).
12. G.V. Milovanović, A.S. Cvetković, M. Stanić, A generalized Birkhoff-Young-Chebyshev quadrature formula for analytic functions, *Appl. Math. Comput.* **218**, 944–948 (2011).
13. J. B. Scarborough, *Numerical Mathematical Analysis*, Baltimore: The Johns Hopkins Press, 1930.
14. G. V. Milovanović, R. Ž. Đorđević, On a generalization of modified Birkhoff-Young quadrature formula, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 735 – No. 762, 130–134 (1982).
15. D. Đ. Tošić, A modification of the Birkhoff-Young quadrature formula for analytic functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 602 – No. 633, 73–77 (1978).
16. D. M. Young, An error bound for the numerical quadrature of analytic functions, *J. Math. Phys.* **31**, 42–44 (1952).