DISCRETE INEQUALITIES OF WIRTINGER'S TYPE

GRADIMIR V. MILOVANOVIĆ and IGOR Ž. MILOVANOVIĆ
Faculty of Electronic Engineering, Department of Mathematics, P.O. Box 73, 18000 Niš, Yugoslavia

Abstract. Various discrete versions of Wirtinger's type inequalities are considered. A short account on the first results in this field given by Fan, Taussky and Todd [10] as well as some generalisations of these discrete inequalities are done. Also, a general method for finding the best possible constants A_n and B_n in inequalities of the form

$$A_n \sum_{k=1}^{n} p_k x_k^2 \le \sum_{k=0}^{n} r_k (x_k - x_{k+1})^2 \le B_n \sum_{k=1}^{n} p_k x_k^2,$$

where $\boldsymbol{p} = (p_k)$ and $\boldsymbol{r} = (r_k)$ are given weight sequences and $\boldsymbol{x} = (x_k)$ is an arbitrary sequence of the real numbers, is presented. Two types of problems are investigated and several corollaries of the basic results are obtained. Further generalisations of discrete inequalities of Wirtinger's type for higher differences are also treated.

1. Introduction and Preliminaries

In the well-known monograph written by Hardy, Littlewood and Pólya [13, pp. 184–187] the following result was mentioned as the Wirtinger's inequality:

Theorem 1.1. *Let f be a periodic function with period (2π) and such that $f' \in L^2(0, 2\pi)$. If $\int_0^{2\pi} f(x)\,dx = 0$ then*

$$\int_0^{2\pi} f(x)^2\,dx \le \int_0^{2\pi} f'(x)^2\,dx, \tag{1.1}$$

with equality in (1.1) *if and only if $f(x) = A\cos x + B\sin x$, where A and B are constants.*

Also, this inequality can be found in the monograph of Beckenbach and Bellman [4, pp. 177–180] and, especially, in one written by Mitrinović in cooperation with Vasić [25, pp. 141–154], including many other inequalities of the same type. The proof of W. Wirtinger was first published in 1916 in the book [5] by Blaschke. However, inequality (1.1) was known before this, though with other conditions on

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the function f. The French and Italian mathematical literature do not mention the name of Wirtinger in connection with this inequality. A historical review on the priority in this subject was given by Mitrinović and Vasić [24] (see also [25–26]). They have mentioned various generalisations and variations of inequality (1.1), as well as possibility of applications of such kind of inequalities in many branches in mathematics as Calculus of Variations, Differential and Integral Equations, Spectral Operator Theory, Numerical Analysis, Approximation Theory, Mathematical Physics, etc. Under some condition of f, there are also many generalisations of (1.1) which give certain estimates of quotients of the form

$$\frac{\int_a^b w(x)f(x)^2\,dx}{\int_a^b f'(x)^2\,dx}, \qquad \frac{\iint_D w(x,y)f(x,y)^2\,dxdy}{\iint_D\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)dxdy},$$

where w is a weight function (in one or two variables) and D is a simply connected plane domain.

There are various discrete versions of Wirtinger type inequalities. In this survey we will deal only with such kind of inequalities.

The paper is organised as follows. In Section 2 we give a summary on the first results in this field given by Fan, Taussky and Todd [10] as well as some generalisations of these discrete inequalities. In Section 3 we present a general method for finding the best possible constants A_n and B_n in inequalities of the form

$$A_n\sum_{k=1}^{n}p_kx_k^2 \le \sum_{k=0}^{n} r_k(x_k-x_{k+1})^2 \le B_n\sum_{k=1}^{n}p_kx_k^2,$$

where $\boldsymbol{p}=(p_k)$ and $\boldsymbol{r}=(r_k)$ are given weight sequences and $\boldsymbol{x}=(x_k)$ is an arbitrary sequence of the real numbers. This method was introduced by authors [19] and later used by other mathematicians (see e.g., [1] and [36]). In the same section we give several corollaries of the basic results. Finally, generalisations of discrete inequalities of Wirtinger's type for higher differences are treated in Section 4.

2. Discrete Fan-Taussky-Todd Inequalities and Some Generalisations

The basic discrete analogues of inequalities of Wirtinger were given by Fan, Taussky and Todd [10]. Their paper has been inspiration for many investigations in this subject. We will mention now three basic results from [10]:

Theorem 2.1. *If $x_1, x_2, \ldots, x_n$ are n real numbers and $x_1 = 0$, then*

$$\sum_{k=1}^{n-1}(x_k-x_{k+1})^2 \ge 4\sin^2\frac{\pi}{2(2n-1)}\sum_{k=2}^{n}x_k^2, \tag{2.1}$$

with equality in (2.1) *if and only if*

$$x_k = A\sin\frac{(k-1)\pi}{2n-1}, \qquad k = 1, 2, \dots, n,$$

where A is an arbitrary constant.

Theorem 2.2. *If $x_0\,(=0)$, $x_1, x_2, \dots, x_n$, $x_{n+1}\,(=0)$ are given real numbers, then*

$$\sum_{k=0}^{n}(x_k - x_{k+1})^2 \geq 4\sin^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}x_k^2, \tag{2.2}$$

with equality in (2.2) *if and only if $x_k = A\sin\dfrac{k\pi}{n+1}$, $k = 1, 2, \dots, n$, where A is an arbitrary constant.*

Theorem 2.3. *If $x_1, x_2, \dots, x_n, x_{n+1}$ are given real numbers such that $x_1 = x_{n+1}$ and*

$$\sum_{k=1}^{n}x_k = 0, \tag{2.3}$$

then

$$\sum_{k=1}^{n}(x_k - x_{k+1})^2 \geq 4\sin^2\frac{\pi}{n}\sum_{k=1}^{n}x_k^2. \tag{2.4}$$

The equality in (2.4) *is attained if and only if*

$$x_k = A\cos\frac{2k\pi}{n} + B\sin\frac{2k\pi}{n}, \qquad k = 1, 2, \dots, n,$$

where A and B are arbitrary constants.

Let A be a real symmetric matrix of the order n, and R be a diagonal matrix of the order n with positive diagonal elements. For the generalised matrix eigenvalue problem

$$A\boldsymbol{x} = \lambda R\boldsymbol{x}, \quad \boldsymbol{x} = [\,x_1 \quad \dots \quad x_n\,]^T, \tag{2.5}$$

the following results are well known (cf. Agarwal [1, Ch. 11]):

1° There exist exactly n real eigenvalues $\lambda = \lambda_\nu$, $\nu = 1, \dots, n$, which need not be distinct.

2° Corresponding to each eigenvalue λ_ν there exists an eigenvector $\boldsymbol{x}^\nu$ which can be so chosen that n vectors $\boldsymbol{x}^1, \dots, \boldsymbol{x}^n$ are mutually orthogonal with respect to the matrix $R = \operatorname{diag}(r_{11}, \dots, r_{nn})$, i.e.,

$$(\boldsymbol{x}^i)^T R\boldsymbol{x}^j = \sum_{k=1}^{n} r_{kk}x_k^i x_k^j = 0 \qquad (i \neq j),$$

In particular, these vectors are linearly independent.

3° If A is a tridiagonal real symmetric matrix of the form

$$
H_n(\boldsymbol{a},\boldsymbol{b}) = \begin{bmatrix} a_1 & b_1 & & & O \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ O & & & b_{n-1} & a_n \end{bmatrix}, \tag{2.6}
$$

where $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_{n-1})$ and $b_k^2 > 0$ for $k = 1, \dots, n-1$, then the eigenvalues λ_ν of the matrix A are real and distinct.

4° If $R = I$ and the eigenvalues λ_ν of A are arranged in an increasing order, i.e., $\lambda_1 \le \cdots \le \lambda_n$, then for any vector $\boldsymbol{x} \in \mathbb{R}^n$, we have that

$$
\lambda_1(\boldsymbol{x},\boldsymbol{x}) \le (A\boldsymbol{x},\boldsymbol{x}) \le \lambda_n(\boldsymbol{x},\boldsymbol{x}), \tag{2.7}
$$

where $(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{n} x_k y_k$ is the scalar product of the vectors

$$
\boldsymbol{x} = [\,x_1 \quad \cdots \quad x_n\,]^T \quad \text{and} \quad \boldsymbol{y} = [\,y_1 \quad \cdots \quad y_n\,]^T .
$$

In the case $\lambda_1 < \lambda_2$ the equality $\lambda_1(\boldsymbol{x},\boldsymbol{x}) = (A\boldsymbol{x},\boldsymbol{x})$ holds if and only if $\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}^1$. Similarly, if $\lambda_n > \lambda_{n-1}$ the equality $(A\boldsymbol{x},\boldsymbol{x}) = \lambda_n(\boldsymbol{x},\boldsymbol{x})$ holds if and only if $\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}^n$.

Further, for any vector $\boldsymbol{x}$ orthogonal to $\boldsymbol{x}^1$ ($(\boldsymbol{x},\boldsymbol{x}^1) = 0$), we have

$$
\lambda_2(\boldsymbol{x},\boldsymbol{x}) \le (A\boldsymbol{x},\boldsymbol{x}). \tag{2.8}
$$

If $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4$, then a vector $\boldsymbol{x}$ orthogonal to $\boldsymbol{x}^1$ satisfies the equality $\lambda_2(\boldsymbol{x},\boldsymbol{x}) = (A\boldsymbol{x},\boldsymbol{x})$ if and only if $\boldsymbol{x}$ is a linear combination of $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$.

5° If the real symmetric matrix A is positive definite, i.e., for every nonzero $\boldsymbol{x} \in \mathbb{R}^n$, $(A\boldsymbol{x},\boldsymbol{x}) > 0$, then the eigenvalues λ_ν ($\nu = 1, \dots, n$) are positive. In a particular case when $R = I$ and $A = H_n(\boldsymbol{a},\boldsymbol{b})$ is positive definite, then the eigenvalues λ_ν ($\nu = 1, \dots, n$) can be arranged in a strictly increasing order, $0 < \lambda_1 < \cdots < \lambda_n$.

Note that inequalities (2.1), (2.2) and (2.4) are based on the left inequality in (2.7) (i.e., (2.8)). The right inequality in (2.7) has not been used, so that in [10] we cannot find some opposite inequalities of (2.1), (2.2) and (2.4). As special cases of certain general inequalities, the opposite inequalities of (2.1), (2.2) and (2.4) were first proved in [19] (see also [2]).

Using a method similar to one from [10], Block [6] obtained several inequalities related to (2.1), (2.2) and (2.4), as well as some generalisations of such inequalities. For example, Block has proved the following result:

Theorem 2.4. *For real numbers $x_1, x_2, \dots, x_n (= 0)$, $x_{n+1} = x_1$, the inequality*

$$\sum_{k=1}^{n}(x_k - x_{k+1})^2 \geq 4\sin\frac{\pi}{2n}\sum_{k=1}^{n}x_k^2 \tag{2.9}$$

holds, with equality in (2.9) *if and only if* $x_k = A\sin(k\pi/n)$, $k = 1, 2, \dots, n$, *where* A *is an arbitrary constant.*

A number of generalisations of (2.1), (2.2) and (2.4) were given by Novotna ([27] and [29]). We mention here three of them.

Theorem 2.5. *For real numbers $x_1, x_2, \dots, x_n$ satisfying* (2.3), *the inequality*

$$\sum_{k=1}^{n-1}(x_k - x_{k+1})^2 \geq 4\sin^2\frac{\pi}{2n}\sum_{k=1}^{n}x_k^2 \tag{2.10}$$

holds, with equality in (2.10) *if and only if* $x_k = A\sin((2k-1)\pi/(2n))$, $k = 1, 2, \dots, n$, *where* A *is an arbitrary constant.*

Theorem 2.6. *Let $n = 2m$ and let $x_1, x_2, \dots, x_n, x_{n+1} = x_1$ be real numbers such that* (2.3) *holds. Then*

$$\sum_{k=1}^{n}(x_k - x_{k+1})^2 \geq 4\sin^2\frac{\pi}{n}\sum_{k=1}^{n}x_k^2 + n\sin\frac{\pi}{n}\left(\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\right)(x_m + x_{2m})^2,$$

with equality if and only if

$$x_k = A\cos(2k\pi/n) + B\sin(2k\pi/n), \qquad k = 1, 2, \dots, n,$$

where A and B are arbitrary constants.

Theorem 2.7. *For real numbers $x_1, x_2, \dots, x_n$ satisfying* (2.3), *the inequality*

$$\sum_{k=1}^{n-1}(x_k - x_{k+1})^2 \geq 4\sin^2\frac{\pi}{2n}\sum_{k=1}^{n}x_k^2 + 2n\sin\frac{\pi}{2n}\left(\sin\frac{\pi}{n} - \sin\frac{\pi}{2n}\right)(x_1 + x_n)^2$$

holds, with equality if and only if $x_k = A\sin((2k-1)\pi/(2n))$, $k = 1, 2, \dots, n$, *where* A *is an arbitrary constant.*

Using some appropriate changes, Novotna [27] showed that inequalities (2.1), (2.2) and (2.10) can be obtained from (2.4). She proved the basic Theorem 2.3 using the real trigonometric polynomials. Namely, she used the fact that for every number x_i there exist the Fourier coefficients C_k and C_j^* $(k = 0, 1, \dots, m;\ j = 1, \dots, m-1)$ such that

$$x_i = C_0 + \sum_{k=1}^{m-1}\left(C_k\cos\frac{2\pi ki}{n} + C_k^*\sin\frac{2\pi ki}{n}\right) + (-1)^iC_m, \quad 1 \leq i \leq n.$$

For details on this method see for example [1].

New proofs of inequalities (2.1), (2.2) and (2.4) were given by Cheng [8]. His method is based on a connection with discrete boundary problems of the Sturm-Liouville type

$$\begin{aligned} &\Delta\big(p(k-1)\Delta u(k-1)\big) + q(k)u(k) + \lambda r(k)u(k) = 0, \quad k = 1,\dots,n, \\ &u(0) = \lambda u(1), \quad u(n+1) = \beta u(n). \end{aligned} \tag{2.11}$$

For some details of this method see Agarwal [1, Ch. 11]. Another method of proving these inequalities was based on geometric facts in Euclidean space (cf. Shisha [32]).

3. A Spectral Method and Using Orthogonal Polynomials

In this section we consider our method (see [19]) for determining the best constants A_n and B_n in the inequalities

$$A_n \sum_{k=1}^{n} p_k x_k^2 \le \sum_{k=0}^{n} r_k (x_k - x_{k+1})^2 \le B_n \sum_{k=1}^{n} p_k x_k^2, \tag{3.1}$$

under some conditions for a sequence of real numbers $\boldsymbol{x} = (x_k)$, where $\boldsymbol{p} = (p_k)$ and $\boldsymbol{r} = (r_k)$ are given weight sequences. The method is based on the minimal and maximal zeros of certain class of orthogonal polynomials, which satisfy a three-term recurrence relation.

For two N-dimensional real vectors

$$\boldsymbol{z} = [\, z_1 \quad \dots \quad z_N \,]^T \qquad \text{and} \qquad \boldsymbol{w} = [\, w_1 \quad \dots \quad w_N \,]^T$$

we define the usual inner product by $(\boldsymbol{z}, \boldsymbol{w}) = \sum_{k=1}^{N} z_k w_k$ and consider the sums

$$F = \sum_{k=0}^{n} r_k (x_k - x_{k+1})^2 \qquad \text{and} \qquad G = \sum_{k=1}^{n} p_k x_k^2.$$

If we put $\sqrt{p_k}\, x_k = y_k$ $(k = 1,\dots,n)$, then F and G can be transformed in the form

$$F = \sum_{k=0}^{n} \frac{r_k}{p_k p_{k+1}} \big(\sqrt{p_{k+1}}\, y_k - \sqrt{p_k}\, y_{k+1}\big)^2 = (H_N(\boldsymbol{a},\boldsymbol{b})\boldsymbol{y},\boldsymbol{y})$$

and

$$G = \sum_{k=1}^{n} y_k^2 = (\boldsymbol{y},\boldsymbol{y}),$$

where $\boldsymbol{y} \in \mathbb{R}^N$ and $H_N(\boldsymbol{a},\boldsymbol{b})$ is a three-diagonal matrix like (2.6), with $N = n$ or $N = n-1$, depending on the conditions for the sequence $\boldsymbol{x} = (x_k)$. Especially, we will consider the following two cases:

1° $x_0 = x_{n+1} = 0$ and $x_1,\dots,x_n$ are arbitrary real numbers $(N = n)$;

2° $x_1 = 0$ and $x_2,\dots,x_n$ are arbitrary real numbers $(N = n-1)$.

For such three-diagonal matrices we can prove the following auxiliary result ([19]):

Lemma 3.1. *Let $\boldsymbol{p} = (p_k)$ and $\boldsymbol{r} = (r_k)$ be positive sequences and the matrix $H_n(\boldsymbol{a}, \boldsymbol{b})$ be given by (2.6).*

1° *If the sequences $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_{n-1})$ are defined by*

$$\begin{aligned} \boldsymbol{a} &= \Big(\frac{r_0 + r_1}{p_1}, \dots, \frac{r_{n-1} + r_n}{p_n}\Big), \\ \boldsymbol{b} &= \Big(-\frac{r_1}{\sqrt{p_1 p_2}}, \dots, -\frac{r_{n-1}}{\sqrt{p_{n-1} p_n}}\Big), \end{aligned} \tag{3.3}$$

then the matrix $H_n(\boldsymbol{a}, \boldsymbol{b})$ is positive definite.

2° *If the sequences $\boldsymbol{a} = (a_1, \dots, a_{n-1})$ and $\boldsymbol{b} = (b_1, \dots, b_{n-2})$ are defined by*

$$\begin{aligned} \boldsymbol{a} &= \Big(\frac{r_1 + r_2}{p_2}, \dots, \frac{r_{n-2} + r_{n-1}}{p_{n-1}}, \frac{r_{n-1}}{p_n}\Big), \\ \boldsymbol{b} &= \Big(-\frac{r_2}{\sqrt{p_2 p_3}}, \dots, -\frac{r_{n-1}}{\sqrt{p_{n-1} p_n}}\Big), \end{aligned} \tag{3.2}$$

then the matrix $H_{n-1}(\boldsymbol{a}, \boldsymbol{b})$ is positive definite.

We will formulate our results in terms of the monic orthogonal polynomials (π_k) instead of orthonormal polynomials as we made in [19]. Such an approach gives a simpler and nicer formulation than the previous one.

The monic polynomials orthogonal on the real line with respect to the inner product $(f, g) = \int_{\mathbb{R}} f(t) g(t) d\mu(t)$ (with a given measure $d\mu(t)$ on $\mathbb{R}$) satisfy a fundamental three-term recurrence relation of the form

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \tag{3.5}$$

with $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$ (by definition). The coefficients β_k are positive. The coefficient β_0, which multiplies $\pi_{-1}(t) = 0$ in three-term recurrence relation may be arbitrary. Sometimes, it is convenient to define it by $\beta_0 = \int_{\mathbb{R}} d\mu(t)$. Then the norm of π_k can be express in the form

$$\|\pi_k\| = \sqrt{(\pi, \pi_k)} = \sqrt{\beta_0 \beta_1 \cdots \beta_k}\,. \tag{3.6}$$

An interesting and very important property of polynomials $\pi_k(t)$, $k \geq 1$, is the distribution of zeros. Namely, all zeros of $\pi_n(t)$ are real and distinct and are located in the interior of the interval of orthogonality. Let $\tau_\nu^{(n)}$, $\nu = 1, \dots, n$, denote the zeros of $\pi_n(t)$ in an increasing order

$$\tau_1^{(n)} < \tau_2^{(n)} < \cdots < \tau_n^{(n)}. \tag{3.7}$$

It is easy to prove that the zeros $\tau_\nu^{(n)}$ of $\pi_n(t)$ are the same as the eigenvalues of the following tridiagonal matrix

$$J_n = J_n(d\mu) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathrm{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathrm{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

which is known as the *Jacobi matrix*. Also, the monic polynomial $\pi_n(t)$ can be expressed in the following determinant form

$$\pi_n(t) = \det(tI_n - J_n),$$

where I_n is the identity matrix of the order n. For some details on orthogonal polynomials see [17] and [23].

Regarding to the conditions on the sequence $\boldsymbol{x} = (x_k)$, we consider now two important cases:

CASE 1° ($x_0 = x_{n+1} = 0$). If we take $\alpha_{k-1} = -a_k$ and $\sqrt{\beta_k} = -b_k$ (i.e., $\beta_k = b_k^2 > 0$), $k \geq 1$, then we can consider the matrix $H_n(-\boldsymbol{a}, -\boldsymbol{b}) = -H_n(\boldsymbol{a}, \boldsymbol{b})$, defined by (2.6), as a Jacobi matrix for certain class of orthogonal polynomials (π_k). Thus, for every $\boldsymbol{y} \in \mathbb{R}^n$ we have

$$(H_n(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{y}, \boldsymbol{y}) = (-H_n(-\boldsymbol{a}, -\boldsymbol{b})\boldsymbol{y}, \boldsymbol{y}) = (-J_n\boldsymbol{y}, \boldsymbol{y})$$

and

$$-\tau_n^{(n)}(\boldsymbol{y}, \boldsymbol{y}) \leq (-J_n\boldsymbol{y}, \boldsymbol{y}) \leq -\tau_1^{(n)}(\boldsymbol{y}, \boldsymbol{y}),$$

where the zeros $\tau_\nu^{(n)}$, $\nu = 1, \ldots, n$, of $\pi_n(t)$ are given in an increasing order (3.7). On the other hand, putting

$$\boldsymbol{\pi}^*(t) = [\,\pi_0^*(t) \quad \pi_1^*(t) \quad \ldots \quad \pi_{n-1}^*(t)\,]^T \quad \text{and} \quad \boldsymbol{e}_n = [0 \quad 0 \quad \ldots \quad 1]^T,$$

where $\pi_k^*(t) = \pi_k(t)/\|\pi_k\|$, we have (cf. Milovanović [18, p. 178])

$$t\boldsymbol{\pi}^*(t) = J_n\boldsymbol{\pi}^*(t) + \sqrt{\beta_n}\,\pi_n^*(t)\boldsymbol{e}_n.$$

This means that for the eigenvalue $t = \tau_\nu^{(n)}$ of J_n, the corresponding eigenvector is given by $\boldsymbol{\pi}^*(\tau_\nu^{(n)})$. Notice also that the same eigenvector corresponds to the eigenvalue $-\tau_\nu^{(n)}$ of the matrix $-J_n$. Therefore, the following theorem holds.

Theorem 3.2. *Let $\boldsymbol{p} = (p_k)_{k\in\mathbb{N}_0}$ and $\boldsymbol{r} = (r_k)_{k\in\mathbb{N}_0}$ be two positive sequences,*

$$\alpha_{k-1} = -\frac{r_{k-1} + r_k}{p_k}, \qquad \beta_k = \frac{r_k^2}{p_k p_{k+1}} \qquad (k \geq 1),$$

and let (π_k) be a sequence of polynomials satisfying (3.5)*. Then for any sequence of real numbers $x_0\,(=0)$, x_1, ..., x_n, $x_{n+1}\,(=0)$, inequalities*

$$A_n \sum_{k=1}^{n} p_k x_k^2 \leq \sum_{k=0}^{n} r_k (x_k - x_{k+1})^2 \leq B_n \sum_{k=1}^{n} p_k x_k^2, \tag{3.8}$$

hold, with $A_n = -\tau_n^{(n)}$ and $B_n = -\tau_1^{(n)}$, where $\tau_\nu^{(n)}$, $\nu = 1, \ldots, n$, are zeros of $\pi_n(t)$ in an increasing order (3.7).

Equality in the left (*right*) *inequality* (3.8) *holds if and only if*

$$x_k = \frac{C}{\sqrt{p_k}} \cdot \frac{\pi_{k-1}(t)}{\|\pi_{k-1}\|}, \qquad k = 1, \ldots, n,$$

where $t = \tau_n^{(n)}$ ($t = \tau_1^{(n)}$), $\|\pi_k\|$ is given by (3.6) *and C is an arbitrary constant.*

Some corollaries of this theorem are the following results:

Corollary 3.3. *For each sequence of the real numbers x_0 $(= 0)$, $x_1, \dots, x_n$, x_{n+1} $(= 0)$, the following inequalities hold*:

$$(3.9)\qquad 4\sin^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n} x_k^2 \le \sum_{k=0}^{n}(x_k - x_{k+1})^2 \le 4\cos^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n} x_k^2.$$

Equality in the left inequality (3.9) *holds if and only if*

$$x_k = C\sin\frac{k\pi}{n+1}, \qquad k = 1, \dots, n,$$

where C is an arbitrary constant.
Equality in the right inequality (3.9) *holds if and only if*

$$x_k = C(-1)^k\sin\frac{k\pi}{n+1}, \qquad k = 1, \dots, n,$$

where C is an arbitrary constant.

Proof. For $p_k = r_k = 1$ we obtain $\alpha_k = -2$ and $\beta_k = 1$ for each k. Consequently, the recurrence relation (3.5) becomes

$$\pi_{k+1}(t) = (t+2)\pi_k(t) - \pi_{k-1}(t), \quad \pi_0(t) = 1, \ \ \pi_{-1}(t) = 0.$$

Putting $t + 2 = 2x$ and $\pi_k(t) = S_k(x)$, this relation reduces to the three-term recurrence for Chebyshev polynomials of the second kind

$$S_{k+1}(x) = 2xS_k(x) - S_{k-1}(x), \quad S_0(x) = 1, \ \ S_1(x) = 2x.$$

Thus, we have (cf. Milovanović [17, pp. 143–144])

$$(3.10)\qquad \pi_k(t) = S_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}, \qquad \cos\theta = x = \frac{t+2}{2},$$

and therefore the zeros of $\pi_n(t)$ are (in an increasing order)

$$(3.11)\qquad \tau_\nu^{(n)} = -4\sin^2\frac{\theta_\nu}{2}, \quad \theta_\nu = \frac{(n+1-\nu)\pi}{n+1}, \quad \nu = 1, \dots, n.$$

Thus, the best constants in (3.9) are

$$A_n = -\tau_n^{(n)} = 4\sin^2\frac{\pi}{2(n+1)}$$

and

$$B_n = -\tau_1^{(n)} = 4\sin^2\frac{n\pi}{2(n+1)} = 4\cos^2\frac{\pi}{2(n+1)}.$$

Since $\|S_k\| = \sqrt{\pi/2}$ for each k, using (3.10) and (3.11) we find the extremal sequences for the left and the right inequality in (3.9). For example, for the right inequality we have

$$\frac{\pi_{k-1}(\tau_1^{(n)})}{\|\pi_{k-1}\|} = \frac{\sin k\theta_1}{\sin\theta_1} = \frac{1}{\sin\theta_1}\sin\Big(k\frac{n\pi}{n+1}\Big) = -\frac{\cos k\pi}{\sin\theta_1}\sin\frac{k\pi}{n+1},$$

from which follows

$$x_k = C(-1)^k \sin\frac{k\pi}{n+1} \qquad (k = 1, \dots, n),$$

where C is an arbitrary constant. □

Remark 3.1. Theorem 2.2 is contained in Corollary 3.3.

In a more general case we can take

$$p_k = (a+bk)^2 \quad \text{and} \quad r_k = (a+bk)(a+b(k+1)),$$

with $a, b \geq 0$. When $b = 0$ we obtain Corollary 3.3. However, if $b \neq 0$, because of homogeneity in (3.8), it is enough to put $b = 1$. In that case, we obtain the same polynomials as in Corollary 3.3.

Corollary 3.4. *For each sequence of the real numbers x_0 $(= 0)$, $x_1, \dots, x_n, x_{n+1}$ $(= 0)$, the following inequalities*

$$\begin{aligned}(3.12)\quad 4\sin^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}(k+a)^2x_k^2 &\leq \sum_{k=0}^{n}(k+a)(k+a+1)(x_k - x_{k+1})^2\\ &\leq 4\cos^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}(k+a)^2x_k^2\end{aligned}$$

hold, where $a \geq 0$.

Equality in the left inequality (3.12) *holds if and only if*

$$x_k = \frac{C}{k+a}\sin\frac{k\pi}{n+1}, \qquad k = 1, \dots, n,$$

where C is an arbitrary constant.

Equality in the right inequality (3.12) *holds if and only if*

$$x_k = \frac{C(-1)^k}{k+a}\sin\frac{k\pi}{n+1}, \qquad k = 1, \dots, n,$$

where C is an arbitrary constant.

Remark 3.2. The corresponding inequalities for $a = 0$ were considered in [19].

Corollary 3.5. *For each sequence of the real numbers $x_0\,(=0)$, $x_1, \dots, x_n$, x_{n+1} $(=0)$, we have*

$$A_n \sum_{k=1}^{n} x_k^2 \le \sum_{k=0}^{n} k(x_k - x_{k+1})^2 \le B_n \sum_{k=1}^{n} x_k^2, \tag{3.12}$$

where A_n and B_n are minimal and maximal zeros of the monic Laguerre polynomial $L_n(x)$, respectively.

Equality in the left (right) inequality (3.12) *holds if and only if*

$$x_k = CL_{k-1}(x)/(k-1)! \qquad (k = 1, \dots, n),$$

where $x = A_n$ $(x = B_n)$ and C is an arbitrary constant.

In this case we have $\alpha_k = -(2k+1)$ and $\beta_k = k^2$, so that the relation (3.5) becomes

$$\pi_{k+1}(t) = (t + 2k + 1)\pi_k(t) - k^2\pi_{k-1}(t).$$

Putting $t = -x$ and $\pi_k(-x) = (-1)^k L_k(x)$, this relation reduces to one, which corresponds to the monic Laguerre polynomials orthogonal on $(0, +\infty)$ with respect to the measure $d\mu(x) = e^{-x}\,dx$. The norm of $L_k(x)$ is given by $\|L_k\| = k!$.

In a more general case we can take

$$r_0 = 0, \quad r_k = \frac{1}{B(s+1,k)}, \quad p_k = \frac{1}{(k+s)B(s+1,k)} \quad (k \ge 1), \tag{3.13}$$

where $s > -1$ and $B(p,q)$ is the beta function ($B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, Γ is the gamma function). Then we have $\alpha_k = -(2k+s+1)$ and $\beta_k = k(k+s)$, and the corresponding recurrence relation, after changing variable $t = -x$ and $\pi_k(-x) = (-1)^k L_k^s(x)$, becomes

$$L_{k+1}^s(x) = (x - (2k+s+1))L_k^s(x) - k(k+s)L_{k-1}^s(x), \tag{3.14}$$

where $L_k^s(x)$, $k = 0, 1, \dots$, are the generalised monic Laguerre polynomials orthogonal on $(0, +\infty)$ with respect to the measure $d\mu(x) = x^s e^{-x}\,dx$. Thus, we have the following result:

Corollary 3.6. *Let $s > -1$ and let $\boldsymbol{r} = (r_k)_{k\in\mathbb{N}_0}$ and $\boldsymbol{p} = (p_k)_{k\in\mathbb{N}}$ be given by* (3.13). *For each sequence of real numbers $x_0\,(=0)$, $x_1, \dots, x_n$, $x_{n+1}\,(=0)$, we have*

$$A_n \sum_{k=1}^{n} p_k x_k^2 \le \sum_{k=0}^{n} r_k(x_k - x_{k+1})^2 \le B_n \sum_{k=1}^{n} p_k x_k^2, \tag{3.15}$$

where A_n and B_n are minimal and maximal zeros of the monic generalised Laguerre polynomial $L_n^s(x)$, respectively.

Equality in the left (right) inequality (3.15) *holds if and only if*

$$x_k = \frac{CL_{k-1}^s(x)}{\sqrt{(k-1)!\Gamma(k+s)}} \qquad (k = 1, \dots, n),$$

where $x = A_n$ $(x = B_n)$ and C is an arbitrary constant.

CASE 2° $(x_1 = 0)$. Here, in fact, we consider the inequalities

$$A_n \sum_{k=1}^{n} p_k x_k^2 \le \sum_{k=1}^{n-1} r_k (x_k - x_{k+1})^2 \le B_n \sum_{k=1}^{n} p_k x_k^2, \tag{3.16}$$

for any sequence of the real numbers x_1 $(= 0)$, x_2, $\dots$, x_n.
Using Lemma 3.1 (Part 2°) we put $N = n - 1$,

$$\alpha_{k-1} = -\frac{r_k + r_{k+1}}{p_{k+1}}, \qquad \beta_k = \frac{r_{k+1}^2}{p_{k+1}p_{k+2}} \qquad (k \ge 1), \tag{3.17}$$

and also $\alpha_{k-1} = -a_k$, $\sqrt{\beta_k} = -b_k$ $(k \ge 1)$. Taking

$$\boldsymbol{\pi}^*(t) = [\,\pi_0^*(t) \quad \pi_1^*(t) \quad \dots \quad \pi_{n-2}^*(t)\,]^T \quad \text{and} \quad \boldsymbol{e}_{n-1} = [0 \quad 0 \quad \dots \quad 1]^T,$$

where $\pi_k^*(t) = \pi_k(t)/\|\pi_k\|$, we have, as in the previous case,

$$t\boldsymbol{\pi}^*(t) = J_{n-1}\boldsymbol{\pi}^*(t) + \sqrt{\beta_{n-1}}\,\pi_{n-1}^*(t)\boldsymbol{e}_{n-1},$$

but now

$$H_{n-1}(\boldsymbol{a}, \boldsymbol{b}) = -H_{n-1}(-\boldsymbol{a}, -\boldsymbol{b}) = -J_{n-1} - \frac{r_n}{p_n} D_{n-1},$$

where $D_{n-1} = \operatorname{diag}(0, \dots, 0, 1)$. So, we obtain that

$$H_{n-1}(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{\pi}^*(t) + t\boldsymbol{\pi}^*(t) = \Big(\sqrt{\beta_{n-1}}\pi_{n-1}^*(t) - \frac{r_n}{p_n}\,\pi_{n-2}^*(t)\Big)\boldsymbol{e}_{n-1},$$

from which we conclude that the eigenvalues of $H_{n-1}(\boldsymbol{a}, \boldsymbol{b})$, in notation $\lambda_\nu = -\tau_\nu$, $\nu = 1, \dots, n-1$, are the zeros of the polynomial

$$\sqrt{\beta_{n-1}}\pi_{n-1}^*(t) - \frac{r_n}{p_n}\,\pi_{n-2}^*(t). \tag{3.18}$$

The corresponding eigenvectors are $\boldsymbol{\pi}^*(\tau_\nu)$.
Since $\|\pi_{n-1}\| = \|\pi_{n-2}\|\sqrt{\beta_{n-1}}$, the polynomial (3.18) can be reduced to one represented in terms of the monic polynomials,

$$R_{n-1}(t) = \pi_{n-1}(t) - \frac{r_n}{p_n}\,\pi_{n-2}(t). \tag{3.19}$$

Theorem 3.7. *Let $p = (p_k)_{k\in\mathbb{N}}$ and $r = (r_k)_{k\in\mathbb{N}}$ be two positive sequences, α_{k-1} and β_k ($k \geq 1$) be given by* (3.17), *and let (π_k) be a sequence of polynomials satisfying* (3.5). *Then for any sequence of real numbers $x_1\,(=0)$, x_2, …, x_n, inequalities* (3.16) *hold, with $A_n = \min\limits_{\nu}\{-\tau_\nu\}$ $B_n = \max\limits_{\nu}\{-\tau_\nu\}$, where τ_ν, $\nu = 1, \dots, n-1$, are zeros of the polynomial $R_{n-1}(t)$ given by* (3.19).

Equality in the left (right) inequality (3.16) *holds if and only if*

$$x_1 = 0, \quad x_k = \frac{C}{\sqrt{p_k}} \cdot \frac{\pi_{k-2}(t)}{\|\pi_{k-2}\|}, \qquad k = 2, \dots, n,$$

where $t = -A_n$ ($t = -B_n$), $\|\pi_k\|$ is given by (3.6) *and C is an arbitrary constant.*

Some corollaries of this theorem are the following results:

Corollary 3.8. *For each sequence of real numbers $x_1\,(=0)$, x_2, …, x_n, the following inequalities hold:*

$$4\sin^2\frac{\pi}{2(2n-1)}\sum_{k=2}^{n} x_k^2 \leq \sum_{k=1}^{n-1}(x_k - x_{k+1})^2 \leq 4\cos^2\frac{\pi}{2n-1}\sum_{k=2}^{n} x_k^2. \tag{3.20}$$

Equality in the left inequality (3.20) *holds if and only if*

$$x_k = C\sin\frac{(k-1)\pi}{2n-1}, \qquad k = 1, \dots, n,$$

where C is an arbitrary constant.

Equality in the right inequality (3.20) *holds if and only if*

$$x_k = C(-1)^k\sin\frac{2(k-1)\pi}{2n-1}, \qquad k = 1, \dots, n,$$

where C is an arbitrary constant.

Here we have (as in Corollary 3.3) that

$$\pi_k(t) = S_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}, \qquad t + 2 = 2x,$$

and

$$R_{n-1}(t) = S_{n-1}(x) - S_{n-2}(x) = \frac{\cos((2n-1)\theta/2)}{\cos(\theta/2)},$$

and therefore

$$\tau_\nu = -4\sin^2\frac{\nu\pi}{2n-1}, \qquad \nu = 1, \dots, n-1.$$

Corollary 3.9. *Let $s > -1$ and let $r = (r_k)_{k\in\mathrm{N}}$ and $p = (p_k)_{k\in\mathrm{N}}$ be given by*

$$(3.21)\qquad r_1 = 0,\quad r_{k+1} = \frac{1}{B(s+1,k)},\quad p_{k+1} = \frac{1}{(k+s)B(s+1,k)}\qquad (k \geq 1).$$

For each sequence of real numbers $x_1\ (=0)$, $x_2, \ldots, x_n$, we have

$$(3.22)\qquad \sum_{k=1}^{n-1} r_k(x_k - x_{k+1})^2 \leq B_n \sum_{k=2}^{n} p_k x_k^2,$$

where B_n is a maximal zero of the monic generalised Laguerre polynomial $L_{n-2}^{s+1}(x)$. Equality in (3.22) holds if and only if

$$(3.23)\qquad x_1 = 0,\quad x_k = C(-1)^k \frac{L_{k-2}^s(B_n)}{\Gamma(k+s-1)},\quad k = 2, \ldots, n,$$

where C is an arbitrary constant.

Proof. Taking $\pi_k(-x) = (-1)^k L_k^s(x)$, with (3.21) we obtain the recurrence relation (3.14), so that the polynomial (3.19) becomes

$$\begin{aligned} R_{n-1}(t) &= \pi_{n-1}(t) - (n+s-1)\pi_{n-2}(t) \\ &= (-1)^{n-1}\big(L_{n-1}^s(-t) + (n+s-1)L_{n-2}^s(-t)\big) \\ &= (-1)^n t L_{n-2}^{s+1}(-t). \end{aligned}$$

Thus, B_n is a maximal zero of the monic generalised Laguerre polynomial $L_{n-2}^{s+1}(x)$. Evidently, $A_n = 0$.

Since

$$\begin{aligned} \frac{1}{\sqrt{p_k}} \cdot \frac{\pi_{k-2}(-B_n)}{\|\pi_{k-2}\|} &= \sqrt{\frac{(k+s-1)\Gamma(s+1)(k-2)!}{\Gamma(k+s)}} \cdot \frac{(-1)^{k-2} L_{k-2}^s(B_n)}{\sqrt{(k-2)!\Gamma(k+s-1)}} \\ &= (-1)^k \frac{\sqrt{\Gamma(s+1)}}{\Gamma(k+s-1)} L_{k-2}^s(B_n), \end{aligned}$$

we obtain the extremal sequence (3.23) for which the equality is attained in (3.22). $\square$

Remark 3.3. A few members of the monic generalised Laguerre polynomials $L_k^{s+1}(x)$ are

$$\begin{aligned} &L_0^{s+1}(x) = 1, \\ &L_1^{s+1}(x) = x - (s+2), \\ &L_2^{s+1}(x) = x^2 - 2(s+3)x + (s+2)(s+3), \\ &L_3^{s+1}(x) = x^3 - 3(s+4)x^2 - (s+3)(s+12)x - (s+2)(s+3)(s+4). \end{aligned}$$

It is not difficult to show that $B_3 = s+2$, $B_4 = s+3+\sqrt{s+3}$.

Remark 3.4. For $s = 0$ the inequality (3.22) reduces to (see [19])

$$\sum_{k=1}^{n-1}(k-1)(x_k - x_{k+1})^2 \le B_n \sum_{k=2}^{n} x_k^2,$$

where B_n is a maximal zero of the monic generalised Laguerre polynomial $L_{n-2}^1(x)$.

Remark 3.5. If for every k we take $x_k = (-1)^k a_k$ the inequalities (3.1) become

$$A_n \sum_{k=1}^{n} p_k|a_k|^2 \le \sum_{k=0}^{n} r_k|a_k + a_{k+1}|^2 \le B_n \sum_{k=1}^{n} p_k|a_k|^2.$$

Moreover, these inequalities are valid for complex numbers too.

At the end of this section we mention some results of Losonczi [15]. He considered inequalities of the form

$$\alpha_i^{\pm} \sum_{k=0}^{n} |x_k|^2 \le \sum\nolimits^{i} |x_k \pm x_{k+m}|^2 \le \beta_i^{\pm} \sum_{k=0}^{n} |x_k|^2, \tag{3.24}$$

where $x_0, x_1, \dots, x_n$ are real or complex numbers, $1 \le m \le n$, summation symbols defined by:

$$\sum\nolimits^{1} = \sum_{k=0}^{n-m},$$

$$\sum\nolimits^{2} = \sum_{k=0}^{n} \quad \text{with } x_{n+1} = \cdots = x_{n+m} = 0,$$

$$\sum\nolimits^{3} = \sum_{k=-m}^{n-m} \quad \text{with } x_{-m} = \cdots = x_{-1} = 0,$$

$$\sum\nolimits^{4} = \sum_{k=-m}^{n} \quad \text{with } x_{-m} = \cdots = x_{-1} = 0 = x_{n+1} = \cdots = x_{n+m},$$

$\alpha_i^{\pm}$, $\beta_i^{\pm}$ $(i = 1, 2, 3, 4)$ are constants and either the $+$ or the $-$ sign is taken. It is easy to see that the cases $i = 2$ and $i = 3$ are the same apart from the notation of the variables x_k. Hence there are 6 different cases in (3.24) corresponding to $i = 1, 2$ or $i = 3, 4$ and the $+$ or $-$ sign. Losonczi found the best constants $\alpha_i^{\pm}$ and $\beta_i^{\pm}$ in all cases and it was based on the determination of eigenvalues of some suitable Hermitian matrices.

Theorem 3.10. *Let n and m be fixed natural numbers $(1 \le m \le n)$ and $r = [n/m]$. The inequalities (3.24) hold for every real or complex numbers $x_0, x_1, \dots, x_n$, with the best constants*:

$$\alpha_1^+ = \alpha_1^- = 0, \quad \beta_1^+ = \beta_1^- = 4\cos^2 \frac{\pi}{2(r+1)};$$

$$\alpha_2^+ = \alpha_2^- = \alpha_3^+ = \alpha_3^- = 4\sin^2 \frac{\pi}{2(2r+3)},$$

$$\beta_2^+ = \beta_2^- = \beta_3^+ = \beta_3^- = 4\cos^2 \frac{\pi}{2r+3};$$

$$\alpha_4^+ = \alpha_4^- = 4\sin^2 \frac{\pi}{2(r+2)}, \quad \beta_4^+ = \beta_4^- = 4\cos^2 \frac{\pi}{2(r+2)}.$$

Remark 3.6. In connection with extremal properties of nonnegative trigonometric polynomials Szegő [33] and Egerváry and Szász [9] proved that for every complex numbers $x_0, x_1, \ldots, x_n$ the inequalities

$$-\gamma \sum_{k=0}^{n} |x_k|^2 \leq \sum_{k=0}^{n-m} (x_k \bar{x}_{k+m} + \bar{x}_k x_{k+m}) \leq \gamma \sum_{k=0}^{n} |x_k|^2 \tag{3.25}$$

holds, with the best constant $\gamma = 2\cos(\pi/(r+2))$, where $r = [n/m]$. The case $m = 1$ was previously proved by Fejér [11]. It is clear that the inequalities (3.25) are related to (3.24).

4. Inequalities for Higher Differences

In this section we give a short account on generalisations of Wirtinger's type inequalities to higher differences. The first results for the second difference were proved by Fan, Taussky and Todd [10]:

Theorem 4.1. *If $x_0\,(=0)$, $x_1, x_2, \ldots, x_n$, $x_{n+1}\,(=0)$ are given real numbers, then*

$$\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \geq 16 \sin^4 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} x_k^2, \tag{4.1}$$

with equality in (4.1) *if and only if* $x_k = A \sin \dfrac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$, *where A is an arbitrary constant.*

Theorem 4.2. *If $x_0, x_1, \ldots, x_n, x_{n+1}$ are given real numbers such that $x_0 = x_1$, $x_{n+1} = x_n$ and* (2.3) *holds, then*

$$\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \geq 16 \sin^4 \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2. \tag{4.2}$$

The equality in (4.2) *is attained if and only if*

$$x_k = A \cos \frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n,$$

where A is an arbitrary constant.

A converse inequality of (4.1) was proved by Lunter [16], Yin [36] and Chen [7] (see also Agarwal [1]).

Theorem 4.3. *If $x_0\,(=0)$, $x_1, x_2, \ldots, x_n$, $x_{n+1}\,(=0)$ are given real numbers, then*

$$\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \leq 16 \cos^4 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} x_k^2, \tag{4.3}$$

with equality in (4.3) *if and only if* $x_k = A(-1)^k \sin \dfrac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$, *where A is an arbitrary constant.*

Chen [7] also proved the following result:

Theorem 4.4. *If $x_0, x_1, \dots, x_n, x_{n+1}$ are given real numbers such that $x_0 = x_1$ and $x_{n+1} = x_n$, then*

$$\sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 \le 16\cos^4\frac{\pi}{2n}\sum_{k=1}^{n} x_k^2,$$

with equality holding if and only if

$$x_k = A(-1)^k \sin\frac{(2k-1)\pi}{n}, \qquad k = 1, 2, \dots, n,$$

where A is an arbitrary constant.

Proof. In this case, the $n \times n$ symmetric matrix corresponding to the quadratic form

$$F_2 = \sum_{k=0}^{n-1}(x_k - 2x_{k+1} + x_{k+2})^2 = (H_{n,2}\boldsymbol{x}, \boldsymbol{x})$$

is

$$H_{n,2} = \begin{bmatrix} 2 & -3 & 1 & & & & \\ -3 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -3 \\ & & & & 1 & -3 & 2 \end{bmatrix}.$$

This matrix is the square of the $n \times n$ matrix

$$H_n = H_{n,1} = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}. \tag{4.4}$$

The eigenvalues of H_n are

$$\lambda_\nu = \lambda_\nu(H_n) = 4\cos^2\frac{(n-\nu+1)\pi}{2n}, \qquad \nu = 1, \dots, n,$$

and therefore, the largest eigenvalue of H_n is

$$\lambda_n(H_n) = 4\cos^2\frac{\pi}{2n} > \lambda_{n-1}(H_n).$$

The corresponding eigenvector is $\boldsymbol{x}^n = [\,x_{1n} \quad x_{2n} \quad \dots \quad x_{nn}\,]^T$, where

$$x_{\nu n} = (-1)^\nu \sin\frac{(2\nu-1)\pi}{2n}, \qquad \nu = 1, 2, \dots, n.$$

Thus, the largest eigenvalue of $H_{n,2}$ is

$$\lambda_n(H_{n,2}) = 16\cos^4\frac{\pi}{2n} > \lambda_{n-1}(H_{n,2}),$$

and the associated eigenvector is $\boldsymbol{x}^n$. $\square$

Remark 4.1. Notice that the minimal eigenvalue of the matrix H_n (and also $H_{n,2}$) is $\lambda_1 = 0$. Therefore, the condition (2.3) must be included in Theorem 4.2 and the best constant is the square of the relevant eigenvalue

$$\lambda_2 = 4\cos^2\frac{(n-1)\pi}{2n} = 4\sin^2\frac{\pi}{2n}.$$

For any n-dimensional vector $\boldsymbol{x} = [x_1\ x_2\ \dots\ x_n]^T$, Pfeffer [30] introduced a periodically extended n-vector by setting $x_{i+rn} = x_i$ for $i = 1, 2, \dots, n$ and $r \in \mathbb{N}$, and used the mth difference of $\boldsymbol{x}$ given by $\boldsymbol{x}^{(m)} = [\Delta^m x_1\ \Delta^m x_2\ \dots\ \Delta^m x_n]^T$, where

$$\Delta^m x_i = \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}x_{i-[m/2]+r}, \qquad 1 \le i \le n,$$

in order to prove the following result:

Theorem 4.5. *If $\boldsymbol{x}$ is a periodically extended n-vector and (2.3) holds, then*

$$(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(m)}) \ge \left(4\sin^2\frac{\pi}{n}\right)^m (\boldsymbol{x}, \boldsymbol{x}),$$

with equality case if and only if $\boldsymbol{x}$ is the periodic extension of a vector of the form $C_1\boldsymbol{u} + C_2\boldsymbol{v}$, where

$$\boldsymbol{u} = [u_1\quad u_2\quad \dots\quad u_n]^T \qquad \text{and} \qquad \boldsymbol{v} = [v_1\quad v_2\quad \dots\quad v_n]^T$$

have the following components

$$u_k = \cos\frac{2k\pi}{n}, \qquad v_k = \sin\frac{2k\pi}{n}, \qquad k = 1, \dots, n,$$

and C_1 and C_2 are arbitrary real constants.

Recently we have studied inequalities of the form (see [21])

$$A_{n,m}\sum_{k=1}^{n}x_k^2 \le \sum_{k=l_m}^{u_m}(\Delta^m x_k)^2 \le B_{n,m}\sum_{k=1}^{n}x_k^2, \tag{4.5}$$

where $l_m = 1 - [m/2]$, $u_m = n - [m/2]$ and

$$\Delta^m x_k = \sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+m-i}.$$

The quadratic form $F_m = \sum\limits_{k=l_m}^{u_m} (\Delta^m x_k)^2$ for $m = 1$ reduces to

$$F_1 = x_1^2 + \sum_{k=2}^{n-1} 2x_k^2 + x_n^2 - 2\sum_{k=1}^{n-1} x_k x_{k+1},$$

with corresponding tridiagonal symmetric matrix $H_n = H_{n,1}$ given by (4.4). Under conditions

$$x_s = x_{1-s}, \qquad x_{n+1-s} = x_{n+s} \qquad (l_m \leq s \leq 0)$$

we proved that the corresponding matrix of the quadratic form F_m is exactly the mth power of the matrix $H_n = H_{n,1}$ so that the best constant in the right inequality (4.5) is given by

$$B_{n,m} = 4^m \cos^{2m} \frac{\pi}{2n}.$$

Evidently, $A_{n,m} = 0$. However, by restriction (2.3), the best constant in the left inequality (4.5) is given by

$$A_{n,m} = 4^m \sin^{2m} \frac{\pi}{2n}.$$

For other generalisations of discrete Wirtinger's inequalities for higher differences see [6], [16], [31] and [34]. There are also generalisations for multidimensional sequences and partial differences (see [6] and [28]). Finally, we mention that there exist some types of non-quadratic Wirtinger's inequalities (cf. [6], [10] and [12]) as well as discrete inequalities of Opial's type (cf. [3], [14], [20], [22], [35]).

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