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Müntz Orthogonal Polynomials and Their Numerical Evaluation

Gradimir V. Milovanović[1]

Abstract. This paper is devoted to some classes of orthogonal Müntz polynomials on $(0,1)$, their connection with orthogonal rational functions, as well as their numerical computation. Also, we consider some important special cases of such polynomials. For evaluating Müntz polynomials we develop a numerical procedure based on numerical integration in the complex plane.

1. Introduction

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be a complex sequence. We adopt the following definition for x^λ:

$$x^\lambda = e^{\lambda \log x}, \qquad x \in (0,\infty),\ \lambda \in \mathbb{C},$$

and the value at $x=0$ to be the limit of x^λ as $x \to 0$ from $(0,\infty)$ whenever the limit exists, and consider Müntz polynomials as linear combinations of the Müntz system $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$. By $M_n(\Lambda)$ we denote the set of all such polynomials, i.e.,

$$M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\},$$

where the linear span is over the real (or complex) numbers. The union of all $M_n(\Lambda)$ is denoted by $M(\Lambda)$, i.e., $M(\Lambda) = \bigcup_{n=0}^{\infty} M_n(\Lambda)$.

Such generalized polynomials can be orthogonalized and applied to several approximation problems, including quadrature problems. The orthogonal Müntz systems were considered first by the Armenian mathematicians Badalyan [2] and Taslakyan [18]. Recently, they were investigated by McCarthy, Sayre and Shawyer [13] and more completely by Borwein, Erdélyi, and Zhang [4] (see also the recent book [3]).

In this paper we consider orthogonal Müntz polynomials and their numerical evaluation. Section 2 is devoted to some classes of orthogonal Müntz systems on $(0,1)$ and their connection with orthogonal rational functions. Numerical evaluation of Müntz polynomials is considered in §3.

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2. Orthogonal Müntz Polynomials

We investigate two classes of Müntz polynomials which are orthogonal with respect to some inner products. The first of them was introduced by Badalyan [2], and we refer to it as Müntz-Legendre polynomials. The second class was recently defined in [5] and [15].

2.1. Orthogonal Müntz-Legendre polynomials

Let the complex sequence $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be such that $\operatorname{Re}(\lambda_k) > -1/2$ for every $k \in \mathbb{N}_0$ and let $\Lambda_n = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$.

If Γ is a simple contour surrounding all the zeros of the denominator in the rational function

$$W_n(s) = \prod_{k=0}^{n-1} \frac{s + \bar{\lambda}_k + 1}{s - \lambda_k} \cdot \frac{1}{s - \lambda_n} \qquad (n \in \mathbb{N}_0), \tag{2.1}$$

then the Müntz-Legendre polynomials are defined by (see [2], [18], [13], [4], [3])

$$P_n(x) = P_n(x; \Lambda_n) = \frac{1}{2\pi i} \oint_\Gamma W_n(s) x^s \, ds. \tag{2.2}$$

In the case $n = 0$, an empty product in (2.1) should be taken to be equal to 1.

If $\lambda_k \neq \lambda_j$ $(k \neq j)$, by the Cauchy residue theorem, these generalized polynomials can be expressed in power form as

$$P_n(x) = \sum_{k=0}^{n} C_{nk} x^{\lambda_k}, \qquad C_{nk} = \frac{\prod\limits_{\nu=0}^{n-1} (\lambda_k + \bar{\lambda}_\nu + 1)}{\prod\limits_{\substack{\nu=0 \\ \nu \neq k}}^{n} (\lambda_k - \lambda_\nu)}. \tag{2.3}$$

For the Müntz-Legendre polynomials (2.2) the following orthogonality relation holds:

$$(P_n, P_m) = \int_0^1 P_n(x) \overline{P_m(x)} \, dx = \frac{\delta_{nm}}{\lambda_n + \bar{\lambda}_n + 1}.$$

Also, some recurrence relations exist, e.g.,

$$x P_n'(x) - x P_{n-1}'(x) = \lambda_n P_n(x) + (1 + \bar{\lambda}_{n-1}) P_{n-1}(x) \tag{2.4}$$

and

$$P_n(x) = P_{n-1}(x) - (\lambda_n + \bar{\lambda}_{n-1} + 1) x^{\lambda_n} \int_x^1 t^{-\lambda_n - 1} P_{n-1}(t) \, dt \qquad (x \in (0, 1]).$$

It is easy to prove that

$$P_n(1) = 1 \qquad \text{and} \qquad P_n'(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1).$$

In the special case when $\lambda_0 = \lambda_1 = \cdots = \lambda$, (2.2) gives

$$P_n(x; \Lambda_n) = x^\lambda L_n(-(\lambda + \bar{\lambda} + 1) \log x),$$

where $L_n(x)$ is the Laguerre polynomial orthogonal with respect to e^{-x} on $[0,\infty)$ and such that $L_n(0) = 1$.

Taking $x = e^{-t}$, the Müntz-Legendre polynomials can be expressed in terms of a Laplace transform. Namely, we can prove:

Theorem 2.1. *If* $W_n(s)$ *is given by* (2.1) *and*

$$G_n(s) = -W_n(-s) = \prod_{k=0}^{n-1} \frac{s-(\bar{\lambda}_k+1)}{s+\lambda_k} \cdot \frac{1}{s+\lambda_n},$$

then $P_n(e^{-t})$ *is the inverse Laplace transform of* $G_n(s)$, *i.e.,*

$$P_n(e^{-t}) = \mathcal{L}^{-1}[G_n(s)].$$

In the proof of this result we can take, for example, $\alpha > 1/2$, and then prove that

$$P_n(e^{-t}) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} G_n(s) e^{-st}\, ds.$$

An interesting question is connected with the zero distribution of the Müntz-Legendre polynomials for a real sequence Λ. A nice proof of the following result was given in [4].

Theorem 2.2. *For real numbers* $\lambda_\nu > -1/2$ $(\nu = 0, 1, \dots)$ *the Müntz-Legendre polynomial* $P_n(x;\Lambda_n)$ *has exactly* n *distinct zeros in* $(0,1)$, *and it changes sign at each of these zeros. Furthermore, the zeros of the polynomials*

$$P_{n-1}(x;\Lambda_{n-1}) \qquad \text{and} \qquad P_n(x;\Lambda_n)$$

in $(0,1)$ *strictly interlace.*

Now we consider the important special case where

$$\lambda_{2k} = \lambda_{2k+1} = k \qquad (k = 0, 1, \dots).$$

Namely, we take $\lambda_{2k} = k$ and $\lambda_{2k+1} = k + \varepsilon$ $(k = 0, 1, \dots)$, where ε decreases to zero. The corresponding limit process leads to orthogonal Müntz polynomials with logarithmic terms. Then, (2.1) becomes

$$W_n(s) = \begin{cases} \displaystyle\prod_{\nu=0}^{m-1} \left(\frac{s+\nu+1}{s-\nu}\right)^2 \frac{1}{s-m}, & \text{when } n = 2m, \\ \displaystyle\prod_{\nu=0}^{m} \left(\frac{s+\nu+1}{s-\nu}\right)^2 \frac{1}{s+m+1}, & \text{when } n = 2m+1. \end{cases}$$

Applying the Cauchy residue theorem to the integral in (2.2), with this rational function, we obtain the following representation for the corresponding Müntz polynomials:

$$P_n(x) = R_n(x) + S_n(x) \log x \qquad (n = 0, 1, \dots), \tag{2.5}$$

where $R_n(x)$ and $S_n(x)$ are algebraic polynomials of degree $[n/2]$ and $[(n-1)/2]$, respectively, i.e.,

$$R_n(x) = \sum_{\nu=0}^{[n/2]} a_\nu^{(n)} x^\nu, \qquad S_n(x) = \sum_{\nu=0}^{[(n-1)/2]} b_\nu^{(n)} x^\nu. \tag{2.6}$$

Notice that $P_n(1) = R_n(1) = 1$. The first few Müntz polynomials (2.5) are:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= 1 + \log x,\\
P_2(x) &= -3 + 4x - \log x,\\
P_3(x) &= 9 - 8x + 2(1+6x)\log x,\\
P_4(x) &= -11 - 24x + 36x^2 - 2(1+18x)\log x,\\
P_5(x) &= 19 + 276x - 294x^2 + 3(1+48x+60x^2)\log x,\\
P_6(x) &= -21 - 768x + 390x^2 + 400x^3 - 3(1+96x+300x^2)\log x.
\end{aligned}$$

The following theorem gives explicit expressions for the coefficients of the polynomials (2.6) for arbitrary n.

Theorem 2.3. *If n is an even number, $n = 2m$, we have*

$$a_\nu^{(2m)} = -\binom{m+\nu}{m}^2 \binom{m}{\nu}^2 \left[\frac{2m+1}{2\nu+1} + 2(m-\nu) \sum_{\substack{j=0\\ j\neq\nu}}^{m-1} \frac{2j+1}{(j-\nu)(j+\nu+1)}\right]$$

and

$$b_\nu^{(2m)} = -(m-\nu)\binom{m+\nu}{m}^2 \binom{m}{\nu}^2,$$

for each $0 \le \nu \le m-1$. *For* $\nu = m$ *we have*

$$a_m^{(2m)} = \binom{2m}{m}^2 \qquad \textit{and} \qquad b_m^{(2m)} = 0.$$

If n is an odd number, $n = 2m+1$, we have

$$a_\nu^{(2m+1)} = \binom{m+\nu}{m}^2 \binom{m}{\nu}^2 \left[\frac{2m+1}{2\nu+1} + 2(m+\nu+1) \sum_{\substack{j=0\\ j\neq\nu}}^{m} \frac{2j+1}{(j-\nu)(j+\nu+1)}\right]$$

and

$$b_\nu^{(2m+1)} = (m+\nu+1)\binom{m+\nu}{m}^2 \binom{m}{\nu}^2,$$

for each $0 \le \nu \le m$.

A simple proof of Theorem 2.3 can be obtained from the residue theorem. An explicit expression for $S_n(x)$, $n = 2m+1$, is given in [3, Theorem A.2.1].

These orthogonal Müntz polynomials can be used in the proof of the irrationality of $\zeta(3)$ and of other familiar numbers (see [3, pp. 372–381] and [19]).

Putting $\lambda_k + \beta/2$ instead of λ_k, $k = 0, 1, \ldots$, in the sequence Λ, we can define a kind of Müntz-Jacobi polynomials $P_n^{(\beta)}(x)$ by

$$P_n^{(\beta)}(x) = \frac{x^{-\beta/2}}{2\pi i} \oint_\Gamma W_n^{(\beta)}(s) x^s \, ds, \tag{2.7}$$

where

$$W_n^{(\beta)}(s) = \prod_{k=0}^{n-1} \frac{s + \bar{\lambda}_k + \beta/2 + 1}{s - \lambda_k - \beta/2} \cdot \frac{1}{s - \lambda_n - \beta/2}.$$

Then, the following result holds:

Theorem 2.4. *Let $\beta \in \mathbb{R}$ and $\operatorname{Re} \lambda_k > -(\beta+1)/2$ for each $k \in \mathbb{N}_0$. Then*

$$(P_n^{(\beta)}, P_m^{(\beta)}) = \int_0^1 P_n^{(\beta)}(x) \overline{P_m^{(\beta)}(x)} \, x^\beta \, dx = \frac{\delta_{nm}}{\lambda_n + \bar{\lambda}_n + \beta + 1}.$$

The proof of this result and other properties of $P_n^{(\beta)}(x)$ will be given elsewhere.

In the special case $\lambda_k = k$ ($k = 0, 1, \ldots$), the generalized polynomials (2.7) reduce to the classical Jacobi polynomials $\tilde{P}_n^{(0,\beta)}$ ($\beta > -1$) shifted to $[0, 1]$. Then

$$\begin{aligned} P_n^{(\beta)}(x) = \tilde{P}_n^{(0,\beta)}(2x-1) &= (-1)^n \binom{n+\beta}{n} {}_2F_1(-n, n+\beta+1; \beta+1; x) \\ &= (-1)^n \binom{n+\beta}{n} \sum_{k=0}^{n} \frac{(-n)_k (n+\beta+1)_k}{(\beta+1)_k} \cdot \frac{x^k}{k!} \\ &= \sum_{k=0}^{n} C_{nk}^{(\beta)} x^k, \end{aligned}$$

where

$$C_{nk}^{(\beta)} = \frac{(-1)^{n-k}}{k!(n-k)!} \prod_{\nu=0}^{n-1} (\beta + 1 + k + \nu).$$

Here, ${}_2F_1$ is the hypergeometric function, $(p)_k$ is defined by $(p)_k = \Gamma(p+k)/\Gamma(p)$, and Γ is the gamma function. For $\beta = 0$ these polynomials reduce to the Legendre polynomials shifted to $[0, 1]$.

Remark 2.5. *It would be interesting to construct the Müntz-Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ orthogonal with respect to the inner product*

$$(f, g) = \int_0^1 f(x) \overline{g(x)} \, (1-x)^\alpha x^\beta \, dx.$$

2.2. Another class of orthogonal Müntz polynomials

Recently, we defined an external operation for the Müntz polynomials from $M(\Lambda)$ (see [5] and [15]). Namely, for $\alpha, \beta \in \mathbb{C}$ we define

$$x^\alpha \odot x^\beta = x^{\alpha\beta} \qquad (x \in (0,\infty)),$$

and then for polynomials $P \in M_n(\Lambda)$ and $Q \in M_m(\Lambda)$, i.e.,

$$P(x) = \sum_{i=0}^{n} p_i x^{\lambda_i} \qquad \text{and} \qquad Q(x) = \sum_{j=0}^{m} q_j x^{\lambda_j}, \tag{2.8}$$

we define

$$(P \odot Q)(x) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_i q_j x^{\lambda_i \lambda_j}. \tag{2.9}$$

Under the restrictions that for each i and j we have

$$|\lambda_i| > 1, \qquad \operatorname{Re}(\lambda_i \bar{\lambda}_j - 1) > 0, \tag{2.10}$$

we can introduce a new inner product for Müntz polynomials (2.8) (see [15]),

$$[P, Q] = \int_0^1 (P \odot \overline{Q})(x)\, \frac{dx}{x^2}\,, \tag{2.11}$$

where $(P \odot Q)(x)$ is defined by (2.9).

Under the conditions (2.10), we defined (see [15]) the Müntz polynomials $Q_n(x) \equiv Q_n(x; \Lambda_n)$, $n = 0, 1, \dots$, orthogonal with respect to the inner product (2.11). These polynomials are associated with the rational functions

$$W_n(s) = \frac{\prod_{\nu=0}^{n-1}(s - 1/\bar{\lambda}_\nu)}{\prod_{\nu=0}^{n}(s - \lambda_\nu)} \qquad (n = 0, 1, \dots) \tag{2.12}$$

in the sense that

$$Q_n(x) = \frac{1}{2\pi i} \oint_\Gamma W_n(s) x^s \, ds, \tag{2.13}$$

where the simple contour Γ surrounds all the points λ_ν $(\nu = 0, 1, \dots, n)$. We note that the functions (2.12) form a system known as Malmquist system of rational functions (see Walsh [20, p. 305], Djrbashian [6]–[8]), which are orthogonal on the unit circle $|s| = 1$ with respect to the inner product

$$(u, v) = \frac{1}{2\pi i} \oint_{|s|=1} u(s)\overline{v(s)}\, \frac{ds}{s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta})\overline{v(e^{i\theta})}\, d\theta. \tag{2.14}$$

This generalizes the Szegő class of polynomials orthogonal on the unit circle (see Szegő [17, pp. 287–295]).

The following theorem gives the orthogonality relation for the polynomials $Q_n(x)$.

Theorem 2.6. *Under the conditions* (2.10) *on the sequence* Λ, *the Müntz polynomials* $Q_n(x)$, $n = 0, 1, \ldots$, *defined by* (2.13), *are orthogonal with respect to the inner product* (2.11), *i.e.*,

$$[Q_n, Q_m] = \frac{1}{(|\lambda_n|^2 - 1)|\lambda_0\lambda_1 \cdots \lambda_{n-1}|^2}\,\delta_{n,m}.$$

The proof of this theorem is based on the orthogonality of the Malmquist system of rational functions (2.12). Namely, we can prove that

$$[Q_n, Q_m] = (W_n, W_m),$$

where the inner products $[\,\cdot\,,\,\cdot\,]$ and $(\,\cdot\,,\,\cdot\,)$ are given by (2.11) and (2.14), respectively.

Assuming that $\lambda_i \neq \lambda_j$ $(i \neq j)$, we get a representation of (2.13) in the form

$$Q_n(x) = \sum_{k=0}^{n} A_{n,k} x^{\lambda_k}, \qquad A_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_\nu)}{\prod\limits_{\substack{\nu=0\\ \nu\neq k}}^{n} (\lambda_k - \lambda_\nu)} \quad (k = 0, 1, \ldots, n). \tag{2.15}$$

Now we mention some recurrence relations for the polynomials $Q_n(x)$.

Theorem 2.7. *Suppose that* Λ *is a complex sequence satisfying* (2.10). *Then the polynomials* $Q_n(x)$, *defined by* (2.13), *satisfy the following relations:*

$$xQ_n'(x) = xQ_{n-1}'(x) + \lambda_n Q_n(x) - (1/\bar{\lambda}_{n-1})Q_{n-1}(x),$$

$$xQ_n'(x) = \lambda_n Q_n(x) + \sum_{k=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_k) Q_k(x),$$

$$xQ_n''(x) = (\lambda_n - 1)Q_n'(x) + \sum_{k=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_k) Q_k'(x),$$

$$Q_n(1) = 1, \quad Q_n'(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_k),$$

$$Q_n(x) = Q_{n-1}(x) - (\lambda_n - 1/\bar{\lambda}_{n-1}) x^{\lambda_n} \int_x^1 t^{-\lambda_n - 1} Q_{n-1}(t)\, dt \quad (x \in (0, 1]).$$

One particular result for (2.13) when $\lambda_\nu \to \lambda$ for each ν, may be interesting:

Corollary 2.8. *Let* $Q_n(x)$ *be defined by* (2.13) *and let* $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$. *Then*

$$Q_n(x) = x^\lambda L_n\big(-(\lambda - 1/\bar{\lambda}) \log x\big),$$

where $L_n(x)$ *is the Laguerre polynomial.*

Also, for a real sequence Λ such that

$$1 < \lambda_0 < \lambda_1 < \cdots \tag{2.16}$$

we have:

Theorem 2.9. *Let* Λ *be a real sequence satisfying* (2.16). *Then the polynomial* $Q_n(x)$, *defined by* (2.13), *has exactly* n *simple zeros in* $(0,1)$ *and no other zeros in* $[1,\infty)$.

3. Numerical Evaluation of Müntz Polynomials

A direct evaluation of Müntz polynomials $P_n(x)$ (or $Q_n(x)$) in the power form (2.3) (or (2.15)) can be problematic in finite arithmetic, especially when n is a large number and x is close to 1. The polynomial coefficients C_{nk} (or A_{nk}) become very large numbers when n increases, but their sums are always equal to 1. (Recall that $P_n(1) = 1$ and $Q_n(1) = 1$.) In order to illustrate this fact we consider a special class of Müntz polynomials determined by (2.5). Their coefficients are given in Theorem 2.3.

3.1. A special case of Müntz polynomials

Let $\mathbf{a}_n$ and $\mathbf{b}_n$ be the vectors of coefficients of the polynomials $R_n(x)$ and $S_n(x)$, defined by (2.6), i.e.,

$$\mathbf{a}_n = \begin{bmatrix} a_0^{(n)} & a_1^{(n)} & \cdots & a_{[n/2]}^{(n)} \end{bmatrix}^T, \qquad \mathbf{b}_n = \begin{bmatrix} b_0^{(n)} & b_1^{(n)} & \cdots & b_{[(n-1)/2]}^{(n)} \end{bmatrix}^T.$$

Using Theorem 2.3, we can calculate these vectors. For $n = 10$ and $n = 20$ we have

$$\mathbf{a}_{10} = \frac{1}{3}\begin{bmatrix} -134 & -52020 & -999810 & -1133440 & 1994895 & 190512 \end{bmatrix}^T,$$
$$\mathbf{b}_{10} = -5\begin{bmatrix} 1 & 720 & 26460 & 125440 & 79380 \end{bmatrix}^T$$

and

$$\mathbf{a}_{20} = \frac{1}{63}\begin{bmatrix} -7318 \\ -52049250 \\ -24527715300 \\ -2114001489600 \\ -48491344751850 \\ -337299299349012 \\ -625811341034880 \\ 163660745064960 \\ 674793629510715 \\ 173135700710830 \\ 2150491110768 \end{bmatrix}, \qquad \mathbf{b}_{20} = -10\begin{bmatrix} 1 \\ 10890 \\ 7056720 \\ 824503680 \\ 26512946460 \\ 286339821768 \\ 1131219048960 \\ 1633930721280 \\ 775478838420 \\ 85336948840 \end{bmatrix},$$

respectively. The absolute values of some coefficients increase very fast. For example, the vectors $\mathbf{a}_{30}$ and $\mathbf{b}_{30}$ are

$$\frac{1}{6006}\begin{bmatrix} -1189751 \\ -44524636800 \\ -114037231854000 \\ -57424250116396800 \\ -8712106595849556900 \\ -497343278165994868608 \\ -12111056103238468936560 \\ -134308661572934174177280 \\ -686397947699727236368395 \\ -1495971634625290991808640 \\ -818037563553451371290160 \\ 1213101132351099611692800 \\ 1461592959488813080647900 \\ 439304716456359806505600 \\ 33189655142727694141200 \\ 144513038735768102400 \end{bmatrix}, \quad -15\begin{bmatrix} 1 \\ 53760 \\ 176729280 \\ 110279070720 \\ 20527415148240 \\ 1445130026436096 \\ 44257107059605440 \\ 642345550526085120 \\ 4645714010933775780 \\ 17129875530027420160 \\ 32118516618801412800 \\ 29659336430994816000 \\ 12512532556825938000 \\ 2060815557834432000 \\ 90230418791064000 \end{bmatrix},$$

respectively.

Using Horner's scheme for evaluating the values of $P_n(x)$, written in the form (where $b_{n/2}^{(n)} = 0$ for n even)

$$P_n(x) = \sum_{\nu=0}^{[n/2]} c_\nu^{(n)} x^\nu, \qquad c_\nu^{(n)} = a_\nu^{(n)} + b_\nu^{(n)} \log x,$$

we obtain numerical results heavily affected by errors. Relative errors in the values of $P_n(x)$, for $n = 10(10)40$ and some selected values of x, obtained by using D-arithmetic (with machine precision $\approx 2.22 \times 10^{-16}$), are presented in Table 3.1. Numbers in parentheses indicate decimal exponents.

TABLE 3.1. Relative errors in the values $P_n(x)$ in D- (and Q-) arithmetic

x	$n = 10$	$n = 20$	$n = 30$	$n = 40$	Q-arith. ($n = 40$)
10^{-3}	3.08(−15)	9.92(−14)	2.31(−12)	1.03(−10)	
10^{-2}	2.84(−14)	2.44(−11)	5.72(−9)	1.12(−6)	
0.1	1.52(−12)	8.29(−7)	8.06(−3)	7.62(+1)	
0.2	8.38(−12)	3.21(−6)	4.10(−1)	2.43(+5)	9.05(−14)
0.5	3.42(−10)	3.83(−4)	1.93(+2)	5.49(+11)	5.08(−7)
0.9	5.34(−10)	1.11(−2)	7.28(+4)	9.24(+12)	4.95(−5)
1.0	2.13(−10)	5.13(−3)	4.89(+4)	4.81(+11)	8.41(−5)

As we can see, the values obtained for $n \geq 30$ are quite wrong, excluding cases when x is very close to zero. When $n = 10$ and $n = 20$, at $x = 1$ we lost approximatively 6 and 13 decimal digits, respectively. Also, when we used

Q-arithmetic (machine precision $\approx 1.93 \times 10^{-34}$ on the MICROVAX 3400), for $n = 40$ and $x = 1$ we lost about 30 digits (see the last column in Table 3.1). Notice that the shapes of the curves $y = R_n(x)$ and $y = S_n(x) \log x$ (Fig. 1) are very similar at first sight, but we know that the sum $R_n(x) + S_n(x) \log x$ represents the Müntz polynomial $P_n(x)$ which changes its sign n times on $[0, 1]$. Its zeros are more densely distributed around 0 than in other parts of the interval $[0, 1]$. In Figs. 2 and 3 we display $P_{20}(x)$ on the intervals $[0.05, 1]$ (14 zeros), $[10^{-3}, 0.05]$ (4 zeros), and $[0, 10^{-3}]$ (two zeros).

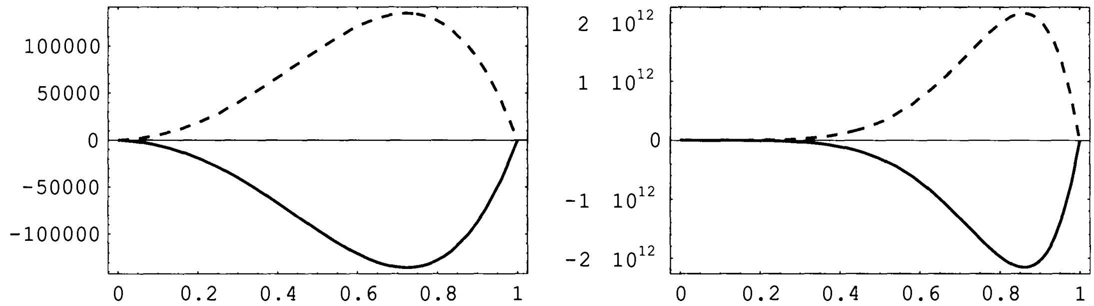

FIGURE 1. Graphics $x \mapsto R_n(x)$ (solid line) and $x \mapsto S_n(x) \log x$ (broken line) for $n = 10$ and $n = 20$

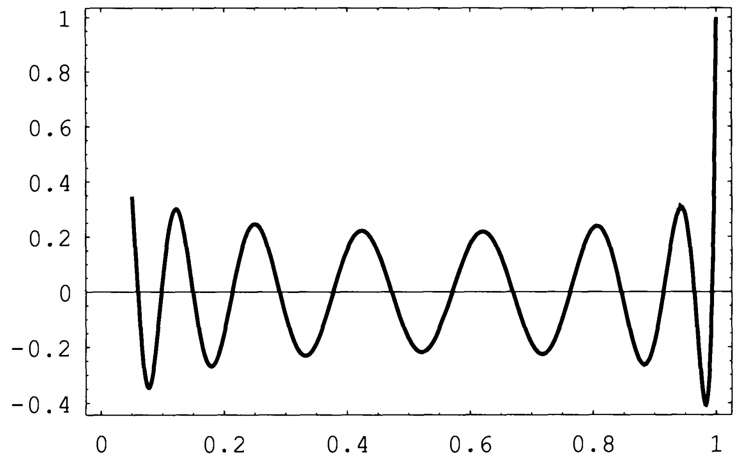

FIGURE 2. The Müntz polynomial $P_{20}(x) = R_{20}(x) + S_{20}(x) \log x$ on $[0.05, 1]$

Before concluding this subsection we mention that the Müntz polynomials (2.5) have a logarithmic behaviour in the neighbourhood of zero, i.e.,

$$P_{2m}(x) \sim -m \log x, \qquad P_{2m+1}(x) \sim (m+1) \log x \qquad (x \to 0+).$$

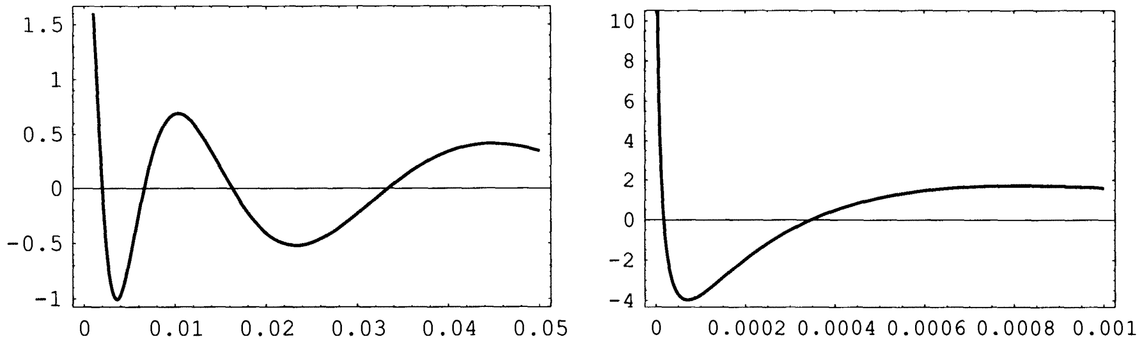

FIGURE 3. The Müntz polynomial $P_{20}(x)$ on $[0.001, 0.05]$ and $[0, 0.001]$

3.2. A numerical method for evaluating Müntz polynomials

In this subsection we give a stable numerical method for evaluating the values of the Müntz-Legendre polynomials defined by (2.1) and (2.2), i.e.,

$$P_n(x) = \frac{1}{2\pi i} \oint_\Gamma W_n(s) x^s \, ds, \qquad W_n(s) = \prod_{\nu=0}^{n-1} \frac{s + \bar{\lambda}_\nu + 1}{s - \lambda_\nu} \cdot \frac{1}{s - \lambda_n}. \tag{3.1}$$

For evaluating Müntz polynomials $Q_n(x)$, defined in §2.2, we can use the same procedure with the rational function (2.12).

Our method is based on a direct evaluation of the contour integral in (3.1). First we take the contour $\Gamma = \Gamma_R = C_R \cup L_R$ (see Fig. 4). Thus, C_R is a semicircle with center at $\sigma < -1/2$ and radius R such that all poles of $W_n(s)$ are inside the contour Γ_R, and L_R is the straight line $s = \sigma + it$, $-R \leq t \leq R$. (Notice that the function $W_n(s)$ has only real poles marked by crosses in Fig. 4 for $n = 5$.)

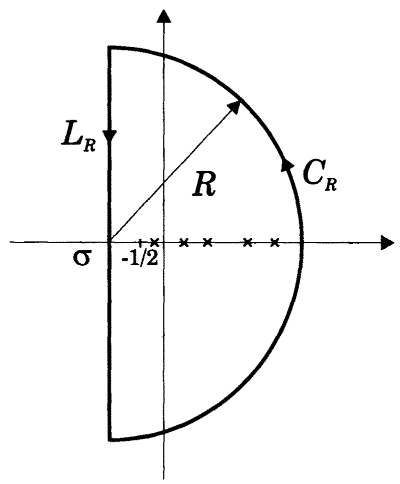

FIGURE 4. The contour of integration for the integral in (3.1)

Lemma 3.1. *We have* $\int_{C_R} W_n(s) x^s \, ds \to 0$ *when* $R \to \infty$.

Proof. Let $s \in C_R$, i.e., $s = \sigma + Re^{i\theta}$, $-\pi \le \theta \le \pi$. For a sufficiently large R, there exists a positive constant $M > 1$ such that $|W_n(s)| \le M/R$. Indeed, this follows from

$$
\begin{aligned}
|W_n(s)| &= \left| \prod_{\nu=0}^{n-1} \frac{\sigma + \bar{\lambda}_\nu + Re^{i\theta} + 1}{\sigma - \lambda_\nu + Re^{i\theta}} \right| \cdot \left| \frac{1}{\sigma - \lambda_n + Re^{i\theta}} \right| \\
&= \frac{1}{R} \cdot \frac{1}{\left| 1 + \dfrac{\sigma - \lambda_n}{R} e^{-i\theta} \right|} \prod_{\nu=0}^{n-1} \left| \frac{1 + \dfrac{\sigma + \bar{\lambda}_\nu + 1}{R} e^{-i\theta}}{1 + \dfrac{\sigma - \lambda_\nu}{R} e^{-i\theta}} \right| .
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\left| \int_{C_R} W_n(s) x^s \, ds \right| &\le \int_{-\pi/2}^{\pi/2} |W_n(s)| \cdot \left| e^{(\sigma + Re^{i\theta}) \log x} \right| R \, d\theta \\
&\le M e^{\sigma \log x} \int_{-\pi/2}^{\pi/2} e^{-R \cos\theta \log(1/x)} \, d\theta .
\end{aligned}
$$

Using the Jordan inequality $\cos\theta > 1 - 2\theta/\pi \quad (0 < \theta < \pi/2)$ and putting $\omega = \log(1/x) > 0$, we get

$$
\left| \int_{C_R} W_n(s) x^s \, ds \right| \le 2M e^{-\sigma\omega} \int_0^{\pi/2} e^{-R\omega(1 - 2\theta/\pi)} \, d\theta = \frac{M\pi e^{-\sigma\omega}}{R\omega} \left(1 - e^{-R\omega} \right) \to 0
$$

when $R \to \infty$. □

Thus, when $R \to \infty$, integration along the contour Γ_R reduces to integration over the line L_R, so that

$$
P_n(x) = \frac{1}{2\pi i} \int_{\sigma + i\infty}^{\sigma - i\infty} W_n(s) x^s \, ds = -\frac{x^\sigma}{2\pi} \int_{-\infty}^{\infty} W_n(\sigma + it) e^{-i\omega t} \, dt,
$$

i.e.,

$$
P_n(x) = \frac{x^\sigma}{2\pi} \int_{-\infty}^{\infty} \left[-\frac{1}{\omega} W_n\Big(\sigma - i \frac{t}{\omega} \Big) \right] e^{it} \, dt .
$$

Since

$$
-\frac{1}{\omega} W_n\Big(\sigma - i\frac{t}{\omega} \Big) = -\frac{1}{\omega} \prod_{\nu=0}^{n-1} \frac{\sigma - it/\omega + \bar{\lambda}_\nu + 1}{\sigma - it/\omega - \lambda_\nu} \cdot \frac{1}{\sigma - it/\omega - \lambda_n} = \frac{1}{i} f_n(t;\omega),
$$

where

$$
f_n(t;\omega) = \prod_{\nu=0}^{n-1} \frac{t + i(\sigma + \bar{\lambda}_\nu + 1)\omega}{t + i(\sigma - \lambda_\nu)\omega} \cdot \frac{1}{t + i(\sigma - \lambda_n)\omega}, \tag{3.2}
$$

we obtain

$$
P_n(x) = \frac{x^\sigma}{2\pi i} \int_{-\infty}^{\infty} f_n(t;\omega) e^{it} \, dt .
$$

This gives the following result:

Theorem 3.2. *Let $\sigma < -1/2$, $f_n(t;\omega)$ be defined by* (3.2), *and*

$$\varphi_n(t;\omega) = \frac{1}{2i}\Big(f_n(t;\omega)e^{it} + f_n(-t;\omega)e^{-it}\Big). \tag{3.3}$$

Then the Müntz-Legendre polynomials can be represented in the integral form

$$P_n(x) = \frac{x^\sigma}{\pi}\int_0^\infty \varphi_n(t;\omega)\,dt. \tag{3.4}$$

In the sequel we consider the case when the sequence Λ is real. An important corollary of Theorem 3.2 is the following result:

Theorem 3.3. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be a real sequence such that $\lambda_k > -1/2$ for every $k \in \mathbb{N}_0$, $f_n(t;\omega)$ be defined by* (3.2), *and $\sigma < -1/2$. Then the Müntz-Legendre polynomials have the following integral representation:*

$$P_n(x) = \frac{x^\sigma}{\pi}\mathrm{Im}\left\{\int_0^\infty f_n(t;\omega)e^{it}\,dt\right\}. \tag{3.5}$$

In this case we have $f_n(-t;\omega) = \overline{f_n(t;\omega)}$, and then (3.3) becomes $\varphi_n(t;\omega) = \mathrm{Im}\left\{f_n(t;\omega)e^{it}\right\}$, and (3.4) gives (3.5). The poles of $f_n(t;\omega)$ are then purely imaginary, $i(\lambda_\nu - \sigma)\omega$, $\nu = 0, 1, \dots, n$, and located on the positive part of the imaginary axis, because of $\lambda_\nu > \sigma$ and $\omega = \log(1/x) > 0$.

In order to calculate the integral in (3.5) (or in (3.4)), we use the following idea on complex integration (see [14, Theorem 2.2]). We select a positive number $a = m\pi$ ($m \in \mathbb{N}$) and put

$$\begin{aligned}\int_0^\infty f_n(t;\omega)e^{it}\,dt &= \int_0^a f_n(t;\omega)e^{it}\,dt + \int_a^{+\infty} f_n(t;\omega)e^{it}\,dt\\ &= L_1(f_n(\,\cdot\,;\omega)) + L_2(f_n(\,\cdot\,;\omega)).\end{aligned}$$

Here,

$$L_1(f_n(\,\cdot\,;\omega)) = \int_0^{a=m\pi} f_n(t;\omega)e^{it}\,dt = \sum_{k=1}^m \int_{(k-1)\pi}^{k\pi} f_n(t;\omega)e^{it}\,dt,$$

i.e.,

$$L_1(f_n(\,\cdot\,;\omega)) = \pi\int_0^1\left[\sum_{k=1}^m f_n\big(\pi(\xi+k-1);\omega\big)e^{i\pi(\xi+k-1)}\right]d\xi, \tag{3.6}$$

and

$$L_2(f_n(\,\cdot\,;\omega)) = \int_a^{+\infty} f_n(t;\omega)e^{it}\,dt.$$

Since $f_n(z;\omega)$ is a holomorphic function in the region $D = \big\{z \in \mathbb{C} \mid \mathrm{Re}\,z \geq a > 0,\ \mathrm{Im}\,z \geq 0\big\}$ and $|f_n(z;\omega)| \leq A/|z|$ when $|z| \to +\infty$, for some positive constant A, we can prove

$$L_2(f_n(\,\cdot\,;\omega)) = ie^{ia}\int_0^{+\infty} f_n(a+iy;\omega)e^{-y}\,dy. \tag{3.7}$$

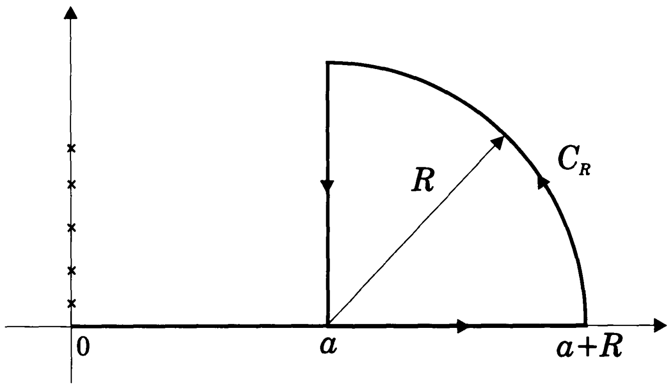

FIGURE 5. The contour of integration for the integral $L_2(f_n(\cdot\,;\omega))$

Indeed, if we take a closed contour of integration as in Fig. 5, consisting of the real segment $[a, a+R]$, the circular arc C_R, and the line segment joining the points $a+iR$ and a, we get, by Cauchy's residue theorem,

$$\int_a^{a+R} f_n(t;\omega)e^{it}\,dt + \int_0^{\pi/2} \left[f_n(z;\omega)e^{iz}\right]_{z=a+Re^{i\theta}} Rie^{i\theta}\,d\theta + i\int_R^0 f_n(a+iy;\omega)e^{i(a+iy)}\,dy = 0.$$

Using Jordan's lemma, we obtain the following estimate for the integral over the circular arc C_R,

$$\left|\int_0^{\pi/2} \left[f_n(z;\omega)e^{iz}\right]_{z=a+Re^{i\theta}} Rie^{i\theta}\,d\theta\right| \le \frac{\pi}{2}\cdot\frac{A}{\sqrt{a^2+R^2}}\left(1-e^{-R}\right) \to 0$$

when $R \to +\infty$. Thus we conclude that (3.7) holds.

Finally, for $a = m\pi$, (3.7) becomes

$$L_2(f_n(\cdot\,;\omega)) = (-1)^m \int_0^\infty \psi_n(y;\omega)e^{-y}\,dy, \tag{3.8}$$

where

$$\psi_n(y;\omega) = if_n(a+iy;\omega) = \prod_{\nu=0}^{n-1} \frac{y+(\sigma+\lambda_\nu+1)\omega-ia}{y+(\sigma-\lambda_\nu)\omega-ia}\cdot\frac{1}{y+(\sigma-\lambda_n)\omega-ia}.$$

Theorem 3.4. *Under the conditions of Theorem 3.3, the Müntz-Legendre polynomials have a computable representation*

$$P_n(x) = \frac{x^\sigma}{\pi}\mathrm{Im}\,\left\{L_1(f_n(\cdot\,;\omega)) + L_2(f_n(\cdot\,;\omega))\right\}, \tag{3.6}$$

where $L_1(f_n(\cdot\,;\omega))$ and $L_2(f_n(\cdot\,;\omega))$ are given by (3.6) and (3.8), respectively.

In the numerical implementation we use the Gauss-Legendre rule on $(0,1)$ and the Gauss-Laguerre rule for calculating $L_1(f_n(\,\cdot\,;\omega))$ and $L_2(f_n(\,\cdot\,;\omega))$, respectively. Numerical experiments show that a convenient choice for the parameter σ is $\lambda_{\min} - \pi/\omega$, where $\lambda_{\min} = \min\{\lambda_0, \lambda_1, \dots\}$.

In order to calculate the relative errors in Table 3.1, we used the previous numerical procedure for evaluating $P_n(x)$ with machine precision (in D-arithmetic).

Remark 3.5. *At the Oberwolfach Meeting on "Applications and Computation of Orthogonal Polynomials" (March, 1998), we also presented a stable numerical method for constructing the generalized Gaussian quadratures using orthogonal Müntz systems, but it is not included in this paper because of limited space. Some references in that direction are* [1], [9], [10], [11], [12], *and* [16].

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References

[1] I.V. Andronov, *Application of orthogonal polynomials in the boundary-contact value problems,* VIII Simposium sobre polinomios ortogonales y sus aplicaciones (Sevilla, Sept. 22–26, 1997), Book of Abstracts, Facultad de Matemáticas, Universidad de Sevilla, 1997, p. 32.

[2] G.V. Badalyan, *Generalization of Legendre polynomials and some of their applications,* Akad. Nauk. Armyan. SSR Izv. Fiz.-Mat. Estest. Tekhn. Nauk, **8** (5) (1955), 1–28 and **9** (1) (1956), 3–22 (Russian, Armenian summary).

[3] P. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities,* Graduate Texts in Mathematics **161**, Springer-Verlag, New York, 1995.

[4] P. Borwein, T. Erdélyi, and J. Zhang, *Müntz systems and orthogonal Müntz-Legendre polynomials,* Trans. Amer. Math. Soc., **342** (1994), 523–542.

[5] B. Danković, G.V. Milovanović, and S. Lj. Rančić, *Malmquist and Müntz orthogonal systems and applications,* in: Th.M. Rassias, Ed., *Inner product spaces and applications*, Pitman Res. Notes Math. Ser. **376**, Longman, Harlow, 1997, 22–41.

[6] M.M. Djrbashian, *Orthogonal systems of rational functions on the circle with given set of poles,* Dokl. Akad. Nauk SSSR, **147** (1962), 1278–1281.

[7] M.M. Djrbashian, *Orthogonal systems of rational functions on the circle,* Akad. Nauk. Armyan. SSR Izv. Mat., **1**(1) (1966), 3–24 and **1**(2) (1966), 106–125 (Russian).

[8] M.M. Djrbashian, *A survey on the theory of orthogonal systems and some open problems,* in: P. Nevai, Ed., *Orthogonal polynomials — theory and practice*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **294**, Kluwer, Dordrecht, 1990, 135–146.

[9] W. Gautschi, *A survey of Gauss-Christoffel quadrature formulae,* in: P.L. Butzer and F. Fehér, Eds., *E.B. Christoffel*, Birkhäuser, Basel, 1981, 72–147.

[10] C.G. Harris and W.A.B. Evans, *Extension of numerical quadrature formulae to cater for end point singular behaviours over finite intervals,* Internat. J. Comput. Math., **6** (1977), 219–227.

[11] S. Karlin and W.J. Studden, *Tchebycheff systems with applications in analysis and statistics,* Pure and Applied Mathematics **15**, John Wiley Interscience, New York, 1966.

[12] J. Ma, V. Rokhlin, and S. Wandzura, *Generalized Gaussian quadrature rules for systems of arbitrary functions,* SIAM J. Numer. Anal., **33** (1996), 971–996.

[13] P.C. McCarthy, J.E. Sayre, and B.L.R. Shawyer, *Generalized Legendre polynomials,* J. Math. Anal. Appl., **177** (1993), 530–537.

[14] G.V. Milovanović, *Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures,* Comput. Math. Appl., **36** (1998), 19–39.

[15] G.V. Milovanović, B. Danković, and S. Lj. Rančić, *Some Müntz orthogonal systems,* J. Comput. Appl. Math., **99** (1998), 299–310.

[16] T.J. Stieltjes, *Sur une généralisation de la théorie des quadratures mécaniques,* C.R. Acad. Sci. Paris, **99** (1884), 850–851.

[17] G. Szegő, *Orthogonal polynomials,* Amer. Math. Soc. Colloq. Publ. **23**, 4th ed., Amer. Math. Soc., Providence, RI, 1975.

[18] A.K. Taslakyan, *Some properties of Legendre quasipolynomials with respect to a Müntz system,* Mathematics, Èrevan Univ., Èrevan, **2** (1984), 179–189 (Russian, Armenian summary).

[19] W. Van Assche, *Approximation theory and analytic number theory,* Special Functions and Differential Equations (Madras), 1997, to appear.

[20] J.L. Walsh, *Interpolation and approximation by rational functions in the complex plane,* 5th ed., Amer. Math. Soc. Colloq. Publ. **20**, Amer. Math. Soc., Providence, RI, 1969.

Gradimir V. Milovanović
Faculty of Electronic Engineering
Department of Mathematics
University of Niš
P.O. Box 73
18000 Niš
Serbia, Yugoslavia
E-mail address: `grade@elfak.ni.ac.yu`