DISTRIBUTION OF ZEROS AND INEQUALITIES FOR ZEROS OF ALGEBRAIC POLYNOMIALS

GRADIMIR V. MILOVANOVIĆ
Faculty of Electronic Engineering, Department of Mathematics, P.O. Box 73, 18000 Niš, Yugoslavia
THEMISTOCLES M. RASSIAS
National Technical University of Athens, Department of Mathematics Zografou Campus, 15780 Athens, Greece

Abstract. This paper surveys the zero distribution and inequalities for zeros of algebraic polynomials. Besides the basic facts on the zero distribution we consider the Grace's theorem and many of its applications, the zero distribution for real polynomials, as well as the Eneström-Kakeya theorem for a special class of polynomials. Also, we give some estimates for a number of zeros of a polynomial in a given domain in the complex plane.

1. Introduction

We start with some basic facts on the zero distribution of algebraic polynomials.

Theorem 1.1. *If $P(z)$ is an algebraic polynomial of degree n $(n \geq 1)$, then the equation $P(z) = 0$ has at least one root.*

This is the well-known *fundamental theorem of algebra*. Another variant of this theorem is:

Theorem 1.2. *Every algebraic polynomial of degree n with complex coefficients has exactly n zeros in the complex plane.*

Applying the *principle of the argument* (see [61, pp. 173–175]) to an algebraic polynomial $P(z)$, we obtain

$$\frac{1}{2\pi} \Delta_\Gamma \operatorname{Arg} P(z) = N,$$

where Δ_Γ denotes the variation along the closed contour Γ and N is the number of zeros of the polynomial $P(z)$ interior to Γ, counted with their multiplicities.

Let $P(z)$ be a polynomial of degree n, with m different zeros $z_1, \ldots, z_m$, and their multiplicities $k_1, \ldots, k_m$, respectively. Then we have

$$P(z) = \prod_{\nu=1}^{m} (z - z_\nu)^{k_\nu}, \qquad n = \sum_{\nu=1}^{m} k_\nu. \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 26C10, 30A10, 30C10, 30C15.
Key words and phrases. Zeros of polynomials; Inequalities; Number of zeros in a domain.
The work of the first author was supported in part by the Serbian Scientific Foundation, grant number 04M03.

T.M. Rassias (ed.), Functional Equations and Inequalities, 171–204.
© 2000 *Kluwer Academic Publishers.*

Since

$$P'(z) = P(z)\,\frac{P'(z)}{P(z)} = P(z)F(z),$$

where

$$F(z) = \frac{d}{dz}[\log P(z)] = \sum_{\nu=1}^{m} \frac{k_\nu}{z - z_\nu}, \tag{1.2}$$

the zeros of $P'(z)$, i.e., the *critical points* of $P(z)$, can be separated into two classes. First, there are the points z_ν for which $k_\nu > 1$ as zeros of $P'(z)$, with multiplicities $k_\nu - 1$. Their total multiplicity is

$$\sum_{\nu=1}^{m} (k_\nu - 1) = n - m.$$

Secondly, there are else $m - 1$ zeros of $P'(z)$ which are the zeros of the logarithmic derivative (1.2).

Evidently, if we know the location of the zeros of the polynomial (1.1), then we know *a priori* the location of the first class of zeros of $P'(z)$. However, the location of the second class of zeros of $P'(z)$, i.e., zeros of the function F, given by (1.2), remains as a problem. Some physical, geometric, and function-theoretic interpretation of zeros of F can be found in Marden [56].

In a special case we have the answer to the previous question. Namely, a particular corollary of Rolle's theorem says that any interval I of the real line which contains all the zeros of a polynomial $P(z)$ also contains all the zeros of $P'(z)$. This can be generalized in the sense that I can be replaced by a *line-segment* in the complex plane.

In a general case we have (see Marden [56, p. 22]):

Theorem 1.3. *All the critical points of a non-constant polynomial $P(z)$ lie in the convex hull D of the set of zeros of $P(z)$. If the zeros of $P(z)$ are not collinear, no critical point of $P(z)$ lies on the boundary ∂D of D unless it is a multiple zero of $P(z)$.*

This is a well-known result which was implied in a note of Gauss dated 1836, but it was stated explicitly and proved by Lucas [51]–[54] in 1874. Since Lucas' time, at least thirteen proofs of this Gauss-Lucas theorem have been published. For references see [61].

From the Gauss-Lucas' Theorem 1.3 follows:

Theorem 1.4. *Any circle C which encloses all the zeros of a polynomial $P(z)$ also encloses all the zeros of its derivative $P'(z)$.*

Indeed, if D is the smallest convex polygon enclosing the zeros of $P(z)$, then D lies in C and therefore by Theorem 1.3 all the zeros of $P'(z)$ being in D, also lie in C. It can be proved that Theorems 1.3 and 1.4 are equivalent (cf. Marden [56, p. 23]).

Let $P(z)$ be a real polynomial. Then its non-real zeros can occur only in conjugate imaginary pairs. Constructing the circles (so-called *Jensen circles of* $P(z)$), whose diameters are the line-segments between the pairs of conjugate imaginary zeros of $P(z)$, Jensen [43] stated without proof the following result (cf. Marden [56, p. 26]):

Theorem 1.5. *Let $P(z)$ be a real polynomial. Then each non-real zero of $P'(z)$ lies in or on at least one of the Jensen circles of $P(z)$.*

The proof of this theorem was given by Walsh [83] and later by Echols [22] and Sz.-Nagy [81]. Some other results in this direction can be found in the book of Marden [56].

We mention here an interesting conjecture of Bl. Sendov, better known as Ilieff-Sendov conjecture: *If all zeros of a polynomial $P(z)$ lie in the unit disk $|z| \leq 1$ and if z_0 is any one such zero, then the disk $|z - z_0| \leq 1$ contains at least one zero of $P'(z)$.* For a discussion about this conjecture see [61, pp. 216–243].

In this paper we give an account on some important results in the field on the zero distribution and inequalities for zeros of algebraic polynomials. The paper is organized as follows. Grace's theorem and many applications are considered in Section 2. The zero distribution for real polynomials is analyzed in Section 3. Eneström-Kakeya theorem for a special class of polynomials and its generalizations are studied in Section 4. Finally, in Section 5 we give some estimates for a number of zeros of a polynomial in a given domain in the complex plane.

2. Grace Theorem and Some Applications

Grace [32] introduced the following definition:

Definition 2.1. Two polynomials $A(z)$ and $B(z)$ defined by

$$A(z) = a_0 + \binom{n}{1} a_1 z + \cdots + \binom{n}{k} a_k z^k + \cdots + a_n z^n \tag{2.1}$$

and

$$B(z) = b_0 + \binom{n}{1} b_1 z + \cdots + \binom{n}{k} b_k z^k + \cdots + b_n z^n \tag{2.2}$$

are said to be *apolar* provided that their coefficients satisfy the *apolarity condition*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{2.3}$$

The coefficients of $A(z)$ and $B(z)$ may be real or complex. If $a_r \neq 0$ $(r \geq 0)$ and $a_\nu = 0$ for $\nu = r+1, \ldots, n$, then we regard $z = \infty$ as an $(n-r)$-fold zero of $A(z)$. In case all the coefficients of $A(z)$ are zero, then $A(z)$ is not regarded as a polynomial. Grace [32] discovered the following significant result, which is very useful in the study of the geometry of the zeros of polynomials.

Theorem 2.1. *Let the polynomials $A(z)$ and $B(z)$, defined by (2.1) and (2.2), respectively, be apolar. Let $\alpha_1, \dots, \alpha_n$ be the zeros of $A(z)$ and $\beta_1, \dots, \beta_n$ be the zeros of $B(z)$. If the circular region C contains all of the α_ν, then C must contain at least one of the β_ν.*

Szegő [78] gave a proof of Grace's theorem freed of the invariant-theoretic concepts used by Grace [32], and he also gave several applications. Goodman and Schoenberg [29] obtained a new proof of Grace's theorem by induction on n. Goodman-Schoenberg's approach is the following. By the transform of $A(z)$ under the Möbius transformation

$$z = \frac{aw+b}{cw+d} \qquad \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

they meant the polynomial function defined by

$$\begin{aligned} A^*(w) &\equiv (cw+d)^n A\Big(\frac{aw+b}{cw+d}\Big) \\ &\equiv \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu (aw+b)^\nu (cw+d)^{n-\nu} \\ &\equiv \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu^* w^\nu . \end{aligned}$$

Example. If $A(z) \equiv 1$, then $A^*(w) = (cw+d)^n$ and the n-fold zero of $A(z)$ at $z = \infty$ becomes an n-fold zero of $A^*(z)$ at $w = -d/c$ if $c \neq 0$.

In their inductive proof of Grace's theorem, Goodman and Schoenberg [29] used the following two lemmas:

Lemma 2.2. *Let $A(z)$ and $B(z)$ be apolar polynomials. If the Möbius transformation changes the polynomials (2.1) and (2.2) into*

$$A^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu^* w^\nu \quad \text{and} \quad B^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu^* w^\nu, \tag{2.4}$$

then the polynomials $A^*(w)$ and $B^*(w)$ are also apolar.

Lemma 2.3. *If α is a zero of the polynomial $A(z)$, then its transform β is a zero of the transformed polynomial $A^*(w)$, defined in (2.4).*

Remark 2.1. If neither α nor β takes the value ∞, then we have $\alpha = (a\beta+b)/(c\beta+d)$ and

$$A^*(\beta) = (c\beta+d)^n A\Big(\frac{a\beta+b}{c\beta+d}\Big) = (c\beta+d)^n A(\alpha) = 0.$$

If $\alpha = \infty$ is an r-fold zero of $A(z)$, then $\beta = -d/c$ is an r-fold zero of $A^*(z)$. If $\alpha = a/c$ is an r-fold zero of $A(z)$, then the decomposition used in the proof of Lemma 2.2 proves that $\beta = \infty$ is an r-fold zero of $A^*(z)$.

The complete Goodman-Schoenberg's proof of Grace's theorem can be found in [61, pp. 189–190]. The following applications with their proofs of Grace's theorem can be found in Szegő's paper [78] or the book by Marden [56] (see also Goodman and Schoenberg [29]).

By $A(z)$ and $B(z)$ we consider the polynomials (2.1) and (2.2), and $C(z)$ the related polynomial

$$C(z) \equiv \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k. \tag{2.5}$$

We assume that $A(z)$ and $B(z)$ are apolar.

Theorem 2.4. *Let w be a zero of $C(z)$, and*

$$B^*(z) = z^n B(-w/z),$$

then $A(z)$ and $B^*(z)$ are apolar.

Theorem 2.5. *If all the zeros of $A(z)$ are in $|z| < r$ and all the zeros of $B(z)$ are in $|z| \le \varrho$, then all the zeros of $C(z)$ are in $|z| < r\varrho$. The polynomials $A(z)$, $B(z)$, and $C(z)$ are given by* (2.1), (2.2), *and* (2.5), *respectively.*

Theorem 2.6. *If all the zeros of $A(z)$ are in a closed and bounded convex domain D and all the zeros of $B(z)$ are in $[-1, 0]$, then all the zeros of $C(z)$ lie in D.*

This result, well-known as *Schur-Szegő composite theorem*, has been obtained by Schur [75] and Szegő [78].

The following result is an extension of Rolle's theorem to complex functions:

Theorem 2.7. *Let $P(z)$ be a polynomial of degree n and suppose that $P(1) = 0$ and $P(-1) = 0$. Then the derivative $P'(z)$ has a zero in $|z| \le \cot(\pi/n)$ and this result is best possible. Furthermore $P'(z)$ has a zero in* $\operatorname{Re} z \ge 0$ *and a zero in* $\operatorname{Re} z \le 0$.

Walsh [85] obtained the following special case of Grace's theorem and he applied it to derive other related results.

Lemma 2.8. *Let $m_k > 0$, $|\alpha_k| \le 1$ $(k = 1, \dots, n)$, $\sum m_k = 1$, $|z| > 1$. Then the equation in α*

$$\prod_{k=1}^{n} (z - \alpha_k)^{m_k} = z - \alpha \tag{2.6}$$

has a solution α which satisfies $|\alpha| \le 1$. Indeed there exists such a solution α satisfying

$$\min_k \arg[(z - \alpha_k)/z] \le \arg[(z - \alpha)/z] \le \max_k \arg[(z - \alpha_k)/z], \tag{2.7}$$

where these three arguments are values of any $\arg[(z-\beta)/z]$ *chosen continuous for fixed* z *and for all* β *with* $|\beta| \leq 1$.

Lemma 2.8 is true (see Walsh [84, Theorem III]) without the hypothesis $|z| > 1$ if (2.7) is omitted.

As Walsh [85] remarked if the m_k and α_k are fixed in Lemma 2.8 and $|z|$ is large, the point α which depends on z, with $|\alpha| \leq 1$, lies close to the center of gravity of the α_k, as we see by expressing (2.6) as follows

$$\begin{aligned} &\sum m_k \log\left(1-\frac{\alpha_k}{z}\right) \\ &\quad \equiv \sum m_k\left[-\frac{\alpha_k}{z} - \frac{1}{2}\left(\frac{\alpha_k}{z}\right)^2 - \frac{1}{3}\left(\frac{\alpha_k}{z}\right)^3 - \cdots\right] \\ &\quad \equiv \log\left(1-\frac{\alpha}{z}\right) \equiv \left[-\frac{\alpha}{z} - \frac{1}{2}\left(\frac{\alpha}{z}\right)^2 - \frac{1}{3}\left(\frac{\alpha}{z}\right)^3 - \cdots\right]. \end{aligned}$$

Some other related results that Walsh [86] obtained are the following:

Lemma 2.9. *Let* $m_k > 0$, $|\alpha_k| \leq 1$ $(k = 1, \ldots, n)$, $\sum m_k = 1$, $\sum m_k \alpha_k = 0$, $|z| > 1$. *Then there exists an* α *such that* $|\alpha| \leq 1/|z|$, *with*

$$\sum m_k \log\left(1-\frac{\alpha_k}{z}\right) = \log\left(1-\frac{\alpha}{z}\right),$$

where $\arg(1-\alpha/z)$ *may be chosen as in* (2.7).

Lemma 2.10. *Let* $m_k > 0$, $|\alpha_k| \leq 1$ $(k = 1, \ldots, n)$, $\sum m_k = 1$, $|z| > 1$. *Then* α *as defined by the equation*

$$\sum_{k=1}^{n} \frac{m_k}{z-\alpha_k} = \frac{1}{z-\alpha}$$

satisfies $|\alpha| \leq 1$. *Under the same hypotheses and with* $\sum m_k \alpha_k = 0$, *we have* $|\alpha| \leq 1/|z|$.

A relatively immediate application of Lemma 2.9 is the following theorem of Walsh [86].

Theorem 2.11. *Let* $|\alpha_k| \leq 1$ *for* $k = 1, \ldots, n$, *with* $\sum \alpha_k = 0$. *Set*

$$P(z) \equiv \prod (z-\alpha_k) - C,$$

where the constant C *is arbitrary. Then for* $|z| \leq 1$ *all zeros of* $P(z)$ *lie in the* n *circles* $|\, z - C^{1/n} \,| \leq 1$ *and for* $|z| > 1$ *in the* n *lemniscate regions*

$$|\, z(z-C^{1/n}) \,| \leq 1,$$

where $C^{1/n}$ *takes all* n *values.*

Remark. If $|C| < 2^n$ the lemniscate $|\, z(z-C^{1/n}) \,| = 1$ consists of a single Jordan curve, whereas if $|C| > 2^n$ it consists of two mutually exterior ovals contained in the respective closed disks whose centers are zero and $C^{1/n}$, having the common radius $\left[C^{1/n} - (C^{2/n}-4)^{1/2}\right]/2$, a radius less than unity.

Walsh [86] obtained also the following result:

Theorem 2.12. *Suppose that we have*

$$|\alpha_k - a| \le r_1 \quad \text{and} \quad |\beta_k - b| \le r_2 \quad (k = 1, \dots, n),$$

with $\sum \alpha_k = na$ *and* $\sum \beta_k = nb$. *We set*

$$P(z) \equiv \prod (z - \alpha_k) - A \prod (z - \beta_k),$$

where A *is an arbitrary constant. Then if* $A \ne 1$ *all zeros of* $P(z)$ *lie in the* n *loci*

$$\left| z - \frac{a - bA^{1/n}}{1 - A^{1/n}} \right| \le q \mid 1 - A^{1/n} \mid^{-1}, \tag{2.8}$$

where

$$q = r_1 \min\Big(1, \frac{r_1}{|z-a|}\Big) + r_2 |A|^{1/n} \min\Big(1, \frac{r_2}{|z-b|}\Big)$$

and $A^{1/n}$ *is in turn each* n*th root of* A.

If $A = 1$ *and we have*

$$r_1 \min\Big(1, \frac{r_1}{|z-a|}\Big) + r_2 \min\Big(1, \frac{r_2}{|z-b|}\Big) > |a - b|, \tag{2.9}$$

then all zeros of $P(z)$ *lie in the* $n-1$ *loci* (2.8), *where* $A^{1/n}$ *is in turn each* n*th root of unity except unity. If* $A = 1$ *and* (2.9) *is false, we draw no conclusion concerning the location of* z.

Applying the theorem of Grace, the following result was obtained by Szegő [78]:

Theorem 2.13. *Let the polynomial*

$$P(z) = z^n + A_1 z^{n-1} + \cdots + A_n$$

have no zeros in the circular region $|z| \le R$. *Then the "section"*

$$Q(z) = P(z) - z^n = A_1 z^{n-1} + A_2 z^{n-2} + \cdots + A_n$$

has no zeros in the circular region $|z| \le R/2$.

If n is even, the example $P(z) = (z - R)^n$ shows that the circle $|z| \le R/2$ cannot be replaced by a larger concentric circle. However if n is odd, following Szegő, the polynomial $Q(z)$ is different from zero even in the circle $|z| \le (R/2)\sec(\pi/2n)$.

Sz.-Nagy [80] proved the following results:

Theorem 2.14. *Let the polynomial*

$$P(z) = (z - a_1)(z - a_2)\cdots(z - a_n)$$

have no zeros in the circle $|z - \alpha| \leq R$*; and let the polynomial*

$$Q(z) = (z - b_1)(z - b_2)\cdots(z - b_n)$$

have all its zeros in the circle $|z - \alpha| \leq \varrho$, $\varrho < R$. *Then the polynomial* $R(z) = P(z) - \lambda Q(z)$, *for* $|\lambda| \leq t^n$, $0 \leq t < R/\varrho$, *can have no zero in the circle*

$$|z - \alpha| \leq r = \frac{R - \varrho t}{1 + t}\,. \tag{2.10}$$

Proof. We note that for any zero ξ of the polynomial $R(z)$

$$\frac{P(\xi)}{Q(\xi)} = \prod_{k=1}^{n} \frac{\xi - a_k}{\xi - b_k} = \lambda.$$

Therefore $R(z_0) \neq 0$ in every point z_0 where

$$\left|\frac{P(z_0)}{Q(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| \neq |\lambda|.$$

We have that at every point z_0 of the circular region (2.10)

$$|z_0 - a_k| \geq |a_k - \alpha| - |z_0 - \alpha| \geq |a_k - \alpha| - r > R - r = (r + \varrho)t,$$
$$|z_0 - b_k| \leq |z_0 - \alpha| + |b_k - \alpha| \leq r + \varrho,$$

so that

$$\left|\frac{P(z_0)}{Q(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| > \left(\frac{R - r}{r + \varrho}\right)^n = t^n \geq |\lambda|. \quad \square$$

If $b_1 = \cdots = b_n = \alpha$, $|\lambda| = |\varepsilon| = 1$ ($\varrho = 0$, $t = 1$), Sz.-Nagy's theorem 2.14 implies the special case:

Corollary 2.15. *If the polynomial*

$$P(z) = z^n + A_1 z^{n-1} + \cdots + A_n$$

have no zeros in the circular region $|z - \alpha| \leq R$, *then no polynomial* $R(z) = P(z) - \varepsilon(z - \alpha)^n$ *for* $|\varepsilon| \leq 1$ *can have any zeros in the circle* $|z - \alpha| \leq R/2$.

Setting

$$Q(z) = z^n + A_k z^{n-k} = z^{n-k}(z^k + A_k), \quad \alpha = 0, \quad \varrho = |A_k|^{1/k}, \quad \lambda = 1$$

in Theorem 2.14, Sz.-Nagy [80] obtained the following result:

Theorem 2.16. *The polynomial*

$$P(z) = z^n + A_k z^{n-k} + A_{k+1} z^{n-k-1} + \cdots + A_n$$

has at least one zero in the circle $|z| \leq 2r + |A_k|^{1/k}$ *provided the "section"*

$$R(z) = P(z) - z^n - A_k z^{n-k} \equiv A_{k+1} z^{n-k-1} + A_{k+2} z^{n-k-2} + \cdots + A_n$$

has at least one zero on the circle $|z| \leq r$.

Remark. A theorem similar to Theorem 2.14 holds also if the zeros of the polynomials $P(z)$ and $Q(z)$ are in arbitrary circular domains without common points. One of these circular domains is the interior of a circle, the other the exterior or interior of a circle or a half-plane. Corresponding to these cases the following three theorems of Sz.-Nagy [80] can be proved generalizing also some theorems of Szegő [78].

Theorem 2.17. *Let the zeros of the polynomials*

$$P(z) = (z - a_1) \cdots (z - a_n)$$

and

$$Q(z) = (z - b_1) \cdots (z - b_n)$$

be located in the circular regions $|z - \alpha| \geq \varrho_1$ *and* $|z - \beta| \leq \varrho_2$, *respectively. We assume that these regions have no points in common, that is,* $\varrho_1 - \varrho_2 > 0$, $|\beta - \alpha| < \varrho_1 - \varrho_2$. *Then no polynomial*

$$R(z) = P(z) - \varepsilon Q(z) \qquad \text{for} \qquad |\varepsilon| \leq 1,$$

can have a zero in the interior of the ellipse E *with foci at* α *and* β *and with the major axis* $\varrho_1 - \varrho_2$.

Theorem 2.18. *Let the zeros of the polynomials* $P(z)$ *and* $Q(z)$ *be located in the circular regions* $|z - \alpha| \leq \varrho_1$ *and* $|z - \beta| \leq \varrho_2$, *respectively, such that these regions have no points in common, that is,* $|\beta - \alpha| > \varrho_1 + \varrho_2$. *Then no polynomial*

$$R(z) = P(z) - \varepsilon Q(z) \qquad \text{for} \qquad |\varepsilon| = 1,$$

can have a zero in the interior of the hyperbole H *with foci at* α *and* β *and with the real axis* $\varrho_1 + \varrho_2$.

Theorem 2.19. *Let the zeros of the polynomials* $P(z)$ *and* $Q(z)$ *be located in the circular region* $|z - \alpha| \leq \varrho$ *and in the half-plane* S, *respectively, such that these regions have no points in common. Let* K *be a conic section with* α *as focus and the boundary line* L *of the half-plane* S *as the directrix corresponding to* α (*that is, the polar* α). *Then no polynomial* $P(z) - \lambda Q(z)$, *with*

$$|\lambda| \geq t^n = \left(e + \varrho \frac{e+1}{\delta}\right)^n,$$

can have a zero in the interior of the conic section K where e is the numerical eccentricity of K and δ is the distance of α from the line L.

By the interior of a conic section is meant the set of points from which no tangent can be drawn to the given conic section.

Grace's theorem also provides a proof for the following theorem of Schaake and van der Corput [73]:

Theorem 2.20. *Let $f(z_1, \dots, z_n)$ be a linear combination of the elementary symmetric functions of $z_1, \dots, z_n$, i.e.,*

$$f(z_1, \dots, z_n) = a_0 + a_1 \sum z_1 + a_2 \sum z_1 z_2 + \cdots \\ + a_\mu \sum z_1 \cdots z_\mu + a_n z_1 \cdots z_n,$$

and

$$\lambda_n(z_1, \dots, z_n) = \frac{1}{n} \sum_{\mu=0}^{n-1} \binom{n}{\mu}^{-1} \sum z_1 z_2 \cdots z_\mu,$$

then we have the identity

$$f(z_1, \dots, z_n) = \sum_p \lambda_n\Big(\frac{z_1}{p}, \dots, \frac{z_n}{p}\Big) f(p, \dots, p),$$

where p runs through the n-th roots of $z_1 \cdots z_n$. Moreover

$$\sum_p \lambda_n\Big(\frac{z_1}{p}, \dots, \frac{z_n}{p}\Big) = 1,$$

and if $|z_1| = \cdots = |z_n| = 1$ we have $\lambda_n(z_1/p, \dots, z_n/p) \geq 0$.

The proof of this inequality and similar statements can be found in the paper of de Bruijn [8]. A result similar to Theorem 2.20 (so-called *Coincidence Theorem*) was obtained by Walsh [84] (see also Marden [56, p. 62]).

Using a proof similar to Szegő's proof of Grace's theorem, Lee and Yang [50] showed an interesting result, which is an extension to polynomials of degree n of an obvious property of quadratic polynomials:*If $-1 \leq x \leq 1$ then the zeros of $z^2 + 2xz + 1$ lie on the unit circle.* Professor R. Askey pointed out this result in his comment on Szegő's paper [78] in [Gabor Szegő: *Collected Papers*, Vol. I (1915–1927), Birkhäuser, Boston, 1982, p. 534].

Lee and Yang [50] proved:

Theorem 2.21. *The polynomial*

$$\varrho(z) = 1 + P_1 z + P_2 z^2 + \cdots + P_{n-1} z^{n-1} + z^n$$

is reciprocal, i.e., $\varrho(z) = z^n \varrho(z^{-1})$, *and all the roots of* $\varrho(z) = 0$ *lie on the unit circle.*

Now, we mention some results on the location of the zeros of certain composite polynomials.

For the polynomials $P(z)$ and $Q(z)$, defined by

$$P(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu, \quad Q(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu \quad (a_0 b_0 a_n b_m \neq 0),$$

respectively, Aziz [4]–[5] proved:

Theorem 2.22. *If* $m \leq n$ *and the coefficients of the polynomials* $P(z)$ *and* $Q(z)$ *satisfy the apolarity condition*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} b_k a_{n-k} = 0,$$

then the following holds:

1° *If* $Q(z)$ *has all its zeros in* $|z - c| \geq r$, *then* $P(z)$ *has at least one zero in* $|z - c| \geq r$.

2° *If* $P(z)$ *has all its zeros in* $|z - c| \leq r$, *then* $Q(z)$ *has at least one zero in* $|z - c| \leq r$.

As an application of this theorem, Aziz [5] obtained certain generalizations of results of Walsh [84], Szegő [78], de Bruijn [8], and Kakeya [46]. For example, Aziz [5] proved the following result, which is a generalization of Walsh's *Coincidence Theorem* for the case when the circular region C is a circle $|z - c| = r$.

Theorem 2.23. *Let* $(z_1, \dots, z_n) \mapsto G(z_1, \dots, z_n)$ *be a symmetric* n*-linear form of total degree* m, $m \leq n$, *and*

$$C = \{z \in \mathbb{C} \, : \, |z - c| \leq r\}$$

be a circle containing the n *points* $w_1, \dots, w_n$. *Then in* C *there exists a least one point* w *such that* $G(w, \dots, w) = G(w_1, \dots, w_n)$.

Hörmander [35] obtained an extension of Grace's theorem to several variables for homogeneous polynomials defined on a vector space over a field with values in that field. Using symbolic notation he showed how to obtain theorems similar to those de Bruijn obtained in [8].

In his book [56, pp. 68–70] Marden states two theorems which are supposed to be restatements of his results in [55].

Theorem 2.24. *From the given polynomials*

$$P(z) = \sum_{k=0}^{m} a_k z^k\,, \qquad Q(z) = \sum_{k=0}^{n} b_k z^k\,, \tag{2.11}$$

let us form the polynomial

$$R(z) = \sum_{k=0}^{m} a_k g(k) z^k\,. \tag{2.12}$$

If all the zeros of $P(z)$ *lie in the ring*

$$R_0 = \{z \in \mathbb{C} : 0 \le r_1 \le |z| \le r_2 \le +\infty\}, \tag{2.13}$$

and if all the zeros of $Q(z)$ *lie in the annular region* A

$$A = \{z \in \mathbb{C} : 0 \le \rho_1 \le |z|/|z-m| \le \rho_2 \le +\infty\},$$

then all the zeros of $R(z)$ *lie in the ring* R_n

$$R_n = \{z \in \mathbb{C} : r_1 \min(1, \rho_1^n) \le |z| \le r_2 \max(1, \rho_2^n)\}.$$

Theorem 2.24′. *Let* $P(z)$, $Q(z)$, *and* $R(z)$ *be polynomials defined in* (2.11) *and* (2.12), *and let* R_0 *be the ring defined by* (2.13). *If all the zeros of the polynomial* $P(z)$ *lie in the ring* R_0, *then all the zeros of the polynomial* $R(z)$ *lie in the ring*

$$r_1 \min\big[1,\ |Q(0)/Q(m)|\big] \le |z| \le r_2 \max\big[1,\ |Q(0)/Q(m)|\big].$$

Theorem 2.24 is really a part of Marden's corollary in [55, p. 97] whereas Theorem 2.24′ is not included there.

Peretz and Rassias [66] proved that Theorem 2.24′ is false as it stands. They constructed the following counterexample:

Let $P(z) = 1 + 2z + z^2 = (1+z)^2$ and $Q(z) = 1 + 2z - z^2$. Then $n = m = 2$, Q interpolates the coefficients of P at $z = 0, 1, 2$, that is $Q(0) = 1$, $Q(1) = 2$ and $Q(2) = 1$. Thus $R(z) = 1 + 4z + z^2$. The zero (a double zero) of P is -1, so we can take $r_1 = r_2 = 1$. Since $Q(0)/Q(2) = 1$, Theorem 2.24′ asserts that the zeros of R lie in $|z| = 1$. But R vanishes at $-2+\sqrt{3}$, $-2-\sqrt{3}$.

The bounds $r_1\,|Q(0)/Q(m)|$ and $r_2\,|Q(0)/Q(m)|$ appear in the above corollary in Marden [55] but under the restrictions $\rho_2 < 1$, $\rho_1 > 1$ respectively.

The methods of the proofs of these results are different in Marden [55] and Marden [56]. In the first of them he used a direct method whereas in the book [56] Marden used results of Walsh [84] on multilinear symmetric forms (see Theorem 2.23). A corrected form of Theorem 2.24′, without the restrictions $\rho_2 < 1$ or $\rho_1 > 1$, was formulated and proved by Peretz and Rassias [66].

The key to prove Theorem 2.24 is the following transformation on the polynomial $P(z)$ introduced by Marden [56, Lemma (16.2a)]:

$$P_1(z) = \beta_1 P(z) - zP'(z)\,, \qquad \beta_1 \in \mathbb{C}.$$

Actually Marden finds a certain representation for the zeros of $P_1(z)$ with the aid of Walsh's result and then iterates this representation.

Definition 2.2. Let $P(z)$ be a polynomial and $\beta_1, \beta_2, \dots$ a sequence of complex numbers. We define a sequence of polynomials by

$$P_0(z) = P(z)\,, \quad P_k(z) = \beta_k P_{k-1}(z) - zP'_{k-1}(z) \quad \text{for} \quad k = 1, 2, \dots$$

(The sequence $\{\beta_k\}$ will be considered to be fixed.)

Remark. If $P(z) = \sum\limits_{j=0}^{m} a_j z^j$, then

$$P_k(z) = \sum_{j=0}^{m} a_j(\beta_1 - j)\cdots(\beta_k - j)z^j \qquad (k \geq 1)$$

as it can be easily seen.
Marden's representation for the zeros of $P_1(z)$ is included in his Lemma (16.2a) (see Marden [56, p. 69]):
If $\beta_1 \neq m$ and if all the zeros of $P(z)$ lie in a disk C, then every zero Z of $P_1(z)$ may be written as $Z = \xi$ or $Z = [\beta_1/(\beta_1 - m)]\,\xi$, where $\xi \in C$. This follows because $P_1(Z)$ is a symmetric multi-linear form in the zeros of $P(z)$ and so by the above mention Walsh's Coincidence Theorem there is a $\xi \in C$ such that

$$0 = P_1(Z) = \beta_1(Z - \xi)^m - mZ(Z - \xi)^{m-1},$$

which proves the assertion.
In order to state and prove a version of Theorem 2.24′ (see Theorem 2.25 below), Peretz and Rassias [66] introduced the following definition:

Definition 2.3. Let $Q(z) = (\beta_1 - z)\cdots(\beta_n - z)$ and m a given positive integer. We define

$$Q^+(z) = \prod_{\substack{1 \leq j \leq n \\ \mathrm{Re}(\beta_j) \geq m/2}} (\beta_j - z), \qquad Q^-(z) = \prod_{\substack{1 \leq j \leq n \\ \mathrm{Re}(\beta_j) < m/2}} (\beta_j - z),$$

with the understanding that Q^+ or Q^- takes the value 1 if one of the products is empty.

Remark. $Q(z) = Q^+(z)Q^-(z)$, and the zeros of Q^+ are those zeros of Q for which $|\beta/(\beta - m)| \geq 1$.

Theorem 2.25. *Let $P(z)$, $Q(z)$, and $R(z)$ be polynomials defined in* (2.11) *and* (2.12), *and let R_0 be the ring defined by* (2.13). *If all the zeros of the polynomial $P(z)$ lie in the ring R_0, then all the zeros of the polynomial $R(z)$ lie in the ring*

$$r_1 \left| Q^-(0)/Q^-(m) \right| \leq |z| \leq r_2 \left| Q^+(0)/Q^+(m) \right|.$$

Remark. This implies the assertion in Marden's corollary in [55] since if $1 < \rho_1$ then $Q = Q^+$ and if $\rho_2 < 1$ then $Q = Q^-$.

Proof of Theorem 2.25. We may assume that

$$Q(z) = (\beta_1 - z) \cdots (\beta_n - z).$$

By Lemma (16.2a) in Marden [56, p. 69] any zero Z of

$$P_1(z) = \sum_{j=0}^{m} a_j(\beta_1 - j)z^j$$

can be written as $Z = \xi$ or $Z = \left[\beta_1/(\beta_1 - m)\right]\xi$, where $|\xi| \leq r_2$. Thus

$$|Z| \leq r_2 \max\{1, |\beta_1/(\beta_1 - m)|\}.$$

Applying the same lemma once more with $P_1(z)$ in place of $P(z)$ and $P_2(z)$ in place of $P_1(z)$ we conclude that any zero Z of

$$P_2(z) = \sum_{j=0}^{m} a_j(\beta_1 - j)(\beta_2 - j)z^j$$

satisfies

$$|Z| \leq r_2 \max\{1, |\beta_1/(\beta_1 - m)|\} \max\{1, |\beta_2/(\beta_2 - m)|\}.$$

We iterate the above n times to conclude that any zero Z of $P_n(z) = R(z)$ satisfies

$$\begin{aligned} |Z| \leq r_2 \prod_{j=1}^{n} \max\{1, |\beta_j/(\beta_j - m)|\} &= r_2 \prod_{\substack{1 \leq j \leq n \\ |\beta_j/(\beta_j - m)| \geq 1}} |\beta_j/(\beta_j - m)| \\ &= r_2 \prod_{\substack{1 \leq j \leq n \\ \mathrm{Re}(\beta_j) \geq m/2}} |\beta_j/(\beta_j - m)| \\ &= r_2 \left| Q^+(0)/Q^+(m) \right| . \end{aligned}$$

Similarly we obtain by the fact that the zeros of P lie in $|z| \geq r_1$ that any zero Z of $R(z)$ satisfies

$$\begin{aligned} |Z| \geq r_1 \prod_{j=1}^{n} \min\{1, |\beta_j/(\beta_j - m)|\} &= r_1 \prod_{\substack{1 \leq j \leq n \\ |\beta_j/(\beta_j - m)| < 1}} |\beta_j/(\beta_j - m)| \\ &= r_1 \left| Q^-(0)/Q^-(m) \right| . \quad \square \end{aligned}$$

The method of Peretz and Rassias [66] of proving Theorem 2.25 can be used to prove other similar results. We will give such an example:

Definition 2.4. Let $P(z)$ be a polynomial and $\beta_1, \beta_2, \dots$ a sequence of complex numbers. We define a sequence of polynomials by

$$\bar{P}_0(z) = P(z)\,, \quad \bar{P}_k(z) = \beta_k \bar{P}_{k-1}(z) - z^2 \bar{P}''_{k-1}(z)\,, \quad k = 1, 2, \dots$$

(The sequence $\{\beta_k\}$ will be considered to be fixed.)

Remark. If $P(z) = \sum\limits_{j=0}^{m} a_j z^j$ then

$$\bar{P}_k(z) = \sum_{j=0}^{m} a_j \left[\beta_1 - j(j-1)\right] \cdots \left[\beta_k - j(j-1)\right] z^j \quad \text{for} \quad k \geq 1.$$

Lemma 2.26. *If $\sqrt{\beta_1} \neq \pm\sqrt{m(m-1)}$ and if all the zeros of $P(z)$ lie in a disk C, then every zero Z of $\bar{P}_1(z)$ may be written as*

$$Z = \xi \qquad \text{or} \qquad Z = \pm\left[\sqrt{\beta_1}/\left(\sqrt{\beta_1} \mp \sqrt{m(m-1)}\right)\right]\xi,$$

where $\xi \in C$.

Proof. $\bar{P}_1(Z)$ is a symmetric multi-linear form in the zeros of $P(z)$ and so by the Walsh's Coincidence Theorem there is a $\xi \in C$ such that

$$0 = \bar{P}_1(Z) = \beta_1 (Z - \xi)^m - m(m-1) Z^2 (Z - \xi)^{m-2},$$

which proves the assertion. $\square$

Definition 2.5. Let $Q(z) = (\beta_1 - z) \cdots (\beta_n - z)$ and m a given positive integer. We define

$$Q^{+1/2}(z) = \prod_{\substack{1 \leq j \leq n \\ \mathrm{Re}\{\sqrt{\beta_j}\} \geq m(m-1)/2}} \left(\sqrt{\beta_j} - z\right)$$

and

$$g^{-1/2}(z) = \prod_{\substack{1 \leq j \leq n \\ 0 \leq \mathrm{Re}\{\sqrt{\beta_j}\} < m(m-1)/2}} \left(\sqrt{\beta_j} - z\right)$$

such that if a product is empty, we take it to be equal to 1, and wherever we take the square roots $\sqrt{\beta_j}$ to be such that $\mathrm{Re}\{\sqrt{\beta_j}\} \geq 0$ for $1 \leq j \leq n$.

Using the previous results and definitions, Peretz and Rassias [66] proved:

Theorem 2.27. *If all the zeros of the polynomial $P(z)$, defined by* (2.11), *lie in the ring*

$$R_0 = \left\{z \in \mathbb{C} \,:\, 0 \leq r_1 \leq |z| \leq r_2 \leq +\infty\right\}$$

and if $Q(z)$ *is a polynomial of degree* n, *then all the zeros of*

$$R(z) = \sum_{k=0}^{m} a_k Q\big(k(k-1)\big) z^k$$

lie in the ring

$$r_1 \,\big|\, [Q^{-1/2}(0)/Q^{-1/2}(\sqrt{m(m-1)})] \,\big| \le |z| \le r_2 \,\big|\, [Q^{+1/2}(0)/Q^{+1/2}(\sqrt{m(m-1)})] \,\big| \,.$$

Remark. It is obvious how to obtain similar results by using the transformations $\beta_1 P(z) - z^s f^{(s)}(z)$ for $1 \le s \le \operatorname{dg} P(z)$.

3. Distribution of Zeros of Real Polynomials

In 1916 Pólya [67] considered two polynomials

$$P(x) = \sum_{\nu=0}^{m} a_\nu x^\nu \quad \text{and} \quad h(x) = \sum_{\nu=0}^{n} b_\nu x^\nu$$

of degree m and n, respectively, with only real zeros, and proved the following results:

Theorem 3.1. *Let* $n \ge m$ *and let the zeros of the polynomial* $h(x)$ *be all negative. Then the real algebraic curve*

$$F(x,y) \equiv b_0 P(y) + b_1 x P'(y) + \cdots + b_m x^m P^{(m)}(y) = 0$$

has m *intersection points with each line* $sx - ty + u = 0$, *where* $s \ge 0$, $t \ge 0$, $s + t > 0$ *and* u *is real.*

As Pólya noted this theorem gives a unified proof of three important special cases regarding composite polynomials:

1° For $x = 1$ it gives a special case of the Hermite-Paulain theorem (see Obreschkoff [64, Satz 3.1]);

2° For $y = 0$ it gives a theorem of Schur (see Obreschkoff [64, Satz 7.4]);

3° For $x = y$ it gives a result of Pólya and Schur [68, p. 107].

Craven and Csordas [11] investigated some of the properties of the curve

$$F(x,y) \equiv \sum_{\nu=0}^{n} b_\nu x^\nu P^{(\nu)}(y) = 0, \tag{3.1}$$

without restriction on $P(y)$, and when the polynomial

$$h(x) = \sum_{\nu=0}^{n} b_\nu x^\nu \qquad (b_n = 1,\, n \ge 1) \tag{3.2}$$

has only real zeros. Their main result shows that no branch of (3.1) can pass through two distinct zeros of $P(y)$ on the y-axis.

Theorem 3.2. *Let $h(x)$ be a real polynomial with only real zeros, given by* (3.2), *and let $P(y)$ be an arbitrary polynomial.*

If $b_0 \neq 0$ each branch of the real curve (3.1) *which intersects the y-axis will intersect the y-axis in exactly one point and will intersect each vertical line $x = c$, where c is an arbitrary constant.*

If $b_0 = 0$, the conclusion still holds for all branches which do not coincide with the y-axis. Furthermore, if two branches which cross the y-axis intersect at a singular point (x_0, y_0) not on the y-axis, then these branches are in fact components of the form $y - y_0 = 0$, and thus coincide as horizontal line.

As a direct extension of Theorem 3.1 to arbitrary polynomials $P(y)$, Craven and Csordas [11] proved the following result:

Theorem 3.3. *Let $h(x)$ be a real polynomial with only real negative zeros, given by* (3.2), *and let $P(y)$ be an arbitrary polynomial with r real zeros and degree at most n. Then the real algebraic curve* (3.1) *has at least r intersection points with each line $sx - ty + u = 0$, where $s \geq 0$, $t \geq 0$, $s + t > 0$ and u is real.*

Removing the restriction on degree of $P(y)$ they also proved:

Theorem 3.4. *Let $h(x)$ be a real polynomial, given by* (3.2), *with only real non-positive zeros, and let $P(y)$ be an arbitrary polynomial with r real zeros. Then the real algebraic curve* (3.1) *has at least r intersection points with every line of positive slope.*

We mention now four interesting corollaries of the previous results:

Corollary 3.5. *Let $P(x)$ be a real polynomial and $h(x)$ a real polynomial with only real zeros, given by* (3.2). *Then the polynomial $Q(x) = \sum\limits_{\nu=0}^{n} b_\nu P^{(\nu)}(x)$ has at least as many real zeros as $P(x)$. If $P(x)$ has only real zeros, then every multiple zero of $Q(x)$ is also a multiple zero of $P(x)$.*

This is the Hermite-Paulain theorem. Notice that it is a corollary of Theorem 3.2, where $Q(x) \equiv F(1, x)$.

Corollary 3.6. *Let $h(x)$ be a real polynomial with only real zeros, all of the same sign or zero, defined by* (3.2), *and let $P(x) = \sum\limits_{\nu=0}^{m} a_\nu x^\nu$, where $m \leq n$. Then the polynomial $Q(x) = \sum\limits_{\nu=0}^{n} \nu! a_\nu b_\nu x^\nu$ has at least as many real zeros as $P(x)$.*

By a slight change in the hypotheses of this corollary, Craven and Csordas [11] obtained the result separately for the positive and negative zeros.

Corollary 3.7. *Let $h(x)$ be a real polynomial with only real negative zeros, defined by* (3.2), *and let $P(x) = \sum\limits_{\nu=0}^{m} a_\nu x^\nu$, where $m \leq n$. Then the polynomial*

$$Q(x) = \sum_{\nu=0}^{n} \nu! a_\nu b_\nu x^\nu$$

has at least as many positive (negative) zeros as $P(x)$ has positive (negative) zeros. The multiplicity of zero as a zero is the same.

Setting $y = x$, from Theorem 3.4 one has:

Corollary 3.8. *Let $h(x)$ be a real polynomial, given by (3.2), with only real non-positive zeros, and let $P(y)$ be an arbitrary real polynomial. Then the polynomial*

$$g(x) = \sum_{\nu=0}^{n} b_\nu x^\nu P^{(\nu)}(x)$$

has at least as many real zeros as $P(x)$.

In the sequel we deal with the inequality

$$Z_C(T[P(x)]) \le Z_C(P(x)), \tag{3.3}$$

where $Z_C(P(x))$ denotes the number of non-real zeros of a real polynomial $P(x)$, counting their multiplicities, and where T is a linear transformation. If T is the differentiation operator, i.e., $T = D = d/dx$, then (3.3) is a consequence of the Rolle's theorem. If $h(x)$ is a real polynomial with only real zeros and $T = h(D)$, then (3.3) follows from Corollary 3.5. There are many other linear transformations T which possess the property (3.3).

Following Craven and Csordas [10] we take a sequence of real numbers $\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$ and, for $P(x) = \sum\limits_{\nu=0}^{n} a_\nu x^\nu$, we define $\Gamma[P(x)]$ by

$$\Gamma[P(x)] = \sum_{\nu=0}^{n} a_\nu \gamma_\nu x^\nu. \tag{3.4}$$

Let $Q(x)$ be a real polynomial with only real negative zeros and $\Gamma = \{Q(\nu)\}_{\nu=0}^{+\infty}$. Then Laguerre's theorem (see Craven and Csordas [10]) asserts that

$$Z_C(\Gamma[P(x)]) = Z_C\Big(\sum_{\nu=0}^{n} a_\nu Q(\nu) x^\nu\Big) \le Z_C(P(x)),$$

where $P(x) = \sum\limits_{\nu=0}^{n} a_\nu x^\nu$ is an arbitrary real polynomial.

The real sequences $\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$ for which $\Gamma[P(x)]$ has only real zeros whenever $P(x)$ is a real polynomial with only real zeros have been characterized by Pólya and Schur [68]. They called such a sequence Γ as a *multiplier sequence of the first kind*. Also they introduced a *multiplier sequence of the second kind* if Γ takes every real polynomial $P(x)$, all of whose zeros are real and of the same sign, into a polynomial all of whose zeros are real. Notice that the above mentioned sequence $\Gamma = \{Q(\nu)\}_{\nu=0}^{+\infty}$ is a multiplier sequence of the first kind.

The following characterization was given Pólya and Schur [68] (see also Hille [34] and Iserles, Nørsett, and Saff [40]):

Theorem 3.9. *Let $\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$, $\gamma_0 \neq 0$, be a sequence of real numbers and*

$$\Phi(z) = \sum_{\nu=0}^{+\infty} \frac{1}{\nu!} \gamma_\nu z^\nu. \tag{3.5}$$

1° *Then in order that Γ be a multiplier sequence of the first kind it is necessary and sufficient that the series* (3.5) *converge in the whole plane, and that the entire function $\Phi(z)$ or $\Phi(-z)$ can be represented in the form*

$$Ce^{\sigma z} \prod_{n=1}^{+\infty} \Big(1 + \frac{z}{z_n}\Big),$$

where $C \in \mathbb{R}$, $\sigma \geq 0$, $z_n > 0$, and $\sum_{n=1}^{+\infty} z_n^{-1} < +\infty$.

2° *In order that the sequence Γ be a multiplier sequence of the second kind it is necessary and sufficient that the series* (3.5) *converge in the whole plane, and that the entire function $\Phi(z)$ can be represented in the form*

$$Ce^{-\alpha z^2 + \beta z} \prod_{n=1}^{+\infty} \Big(1 - \frac{z}{z_n}\Big) e^{z/z_n},$$

where $\alpha \geq 0$, $\beta, C, z_n \in \mathbb{R}$, and $\sum_{n=1}^{+\infty} z_n^{-2} < +\infty$.

An equivalent characterization, again due to Pólya and Schur [68], is in terms of the *Jensen polynomials*

$$g_n(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} \gamma_\nu z^\nu \qquad (n = 0, 1, \dots).$$

Theorem 3.10. *$\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$ is a multiplier sequence of the first kind if and only if all Jensen polynomials have only real zeros, all of the same sign.*

A family of multiplier sequences of the first kind which depend continuously on a parameter was introduced by Craven and Csordas [10]. Using that, they obtained the following fundamental inequality.

Theorem 3.11. *Let $\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$ be a multiplier sequence of the first kind and let $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ be an arbitrary real polynomial of degree n. Then*

$$Z_C(\Gamma[P(x)]) \leq Z_C(P(x)), \tag{3.6}$$

where $\Gamma[P(x)]$ is defined by (3.4).

Craven and Csordas [10] completely characterized all real sequences $\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$ which satisfy inequality (3.6) for all real polynomials $P(x)$. Namely, they proved that *Γ is a multiplier sequence of the first kind if and only if* (3.6) *holds for any real polynomial $P(x)$.*

Also, we mention here the following consequence of the previous theorem, as an extension of the *Schur-Szegő composite theorem* (Theorem 2.6).

Theorem 3.12. *Let $h(x) = \sum_{\nu=0}^{n} b_\nu x^\nu$ be a real polynomial of degree n with only real negative zeros and let $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ be an arbitrary real polynomial of degree n. Then*

$$Z_C\Big(\sum_{\nu=0}^{n} a_\nu b_\nu x^\nu\Big) \le Z_C(P(x)).$$

We mention here also an useful result (cf. Obreschkoff [65, p. 107]):

Theorem 3.13. *Let $a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with only real zeros and let $z \mapsto \Phi(z)$ be an entire function of the second kind without positive zeros. Then the polynomial*

$$a_0\Phi(0) + a_1\Phi(1)z + \cdots + a_n\Phi(n)z^n$$

has only real zeros.

Craven and Csordas [12] gave also a characterization of the sequences $\{\gamma_\nu\}_{\nu=0}^{+\infty}$ with the property that, for any complex polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ and convex region D containing the origin and the zeros of $P(z)$, the zeros of $\sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu$ again lie in D. Many applications and related results can be also found in this paper as well as in [13]–[15]. The case $D = \{z \in \mathbb{C} : |z| \le 1\}$ was considered in [47].

There are many so-called *zero-mapping transformations* which map polynomials with zeros in a certain interval into polynomials with zeros in another interval. A general technique for the construction of such transformations was developed by Iserles and Nørsett [39]. It is based on the theory of bi-orthogonal polynomials that has been also developed by Iserles and Nørsett [38].

Let D and E be two real intervals that need not be distinct, $\Gamma = \{\gamma_\nu\}_{\nu=0}^{+\infty}$ be a given real multiplier sequence of the first kind, and $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$. Consider the transformation T defined by $T[P(x)] = \Gamma[P(x)]$, i.e.,

$$T\Big\{\sum_{\nu=0}^{n} a_\nu x^\nu\Big\} = \sum_{\nu=0}^{n} \gamma_\nu a_\nu x^\nu.$$

Then, given that all the zeros of $P(x)$ are real, all the zeros of $T[P(x)]$ will also be real.

Iserles and Nørsett [39] introduced sixteen zero-mapping transformations (see also [61, pp. 213–215]). An alternative technique for generating transformations with predictable behaviour of zeros can be developed from the work of Al-Salam and Ismail [1]. Namely, let ψ be a Laplace transform of a non-negative function and assume that it is analytic and with non-zero derivatives at the origin. Then the transformation

$$T\left\{\sum_{\nu=0}^{n} q_\nu x^\nu\right\} = \sum_{\nu=0}^{n} \frac{q_\nu}{\psi^{(\nu)}(0)}(-x)_\nu$$

maps polynomials with real zeros into polynomials with real zeros. Here, $(a)_\nu$ denotes the standard Pochhammer's symbol:

$$(a)_0 = 1, \quad (a)_\nu = (a)_{\nu-1}(a+\nu-1), \quad \nu \geq 1.$$

More details of the proof and examples are given in [40] and [41].

At the end of this section we mention a result on the zero distribution of a class of polynomials associated with the generalized Hermite polynomials.

The sequence of polynomials $\{h_{n,m}^{\lambda}(x)\}_{n=0}^{+\infty}$, where λ is a real parameter and m is an arbitrary positive integer, was studied in [20]. For $m = 2$, the polynomial $h_{n,m}^{\lambda}(x)$ reduces to $H_n(x,\lambda)/n!$, where $H_n(x,\lambda)$ is the Hermite polynomial with a parameter. For $\lambda = 1$, $h_{n,2}^{1}(x) = H_n(x)/n!$, where $H_n(x)$ is the classical Hermite polynomial. Taking $\lambda = 1$ and $n = mN + q$, where $N = [n/m]$ and $0 \leq q \leq m-1$, Đorđević [20] introduced the polynomials $P_N^{(m,q)}(t)$ by $h_{n,m}^{1}(x) = (2x)^q P_N^{(m,q)}((2x)^m)$, and proved that they satisfy an $(m+1)$-term linear recurrence relation of the form

$$\sum_{i=0}^{m} A_N(i,q) P_{N+1-i}^{(m,q)}(t) = B_N(q) t P_N^{(m,q)}(t), \tag{3.7}$$

where $B_N(q)$ and $A_N(i,q)$ $(i = 0,1,\ldots,m)$ are constants depending only on N, m and q. Recently, Milovanović [58] determined the explicit expressions for the coefficients in (3.7) using some combinatorial identities.

An explicit representation of the polynomial $P_N^{(m,q)}(t)$ can be given in the form (see [20], [58]),

$$P_N^{(m,q)}(t) = \sum_{k=0}^{N} (-1)^{N-k} \frac{t^k}{(N-k)!(q+mk)!}, \tag{3.8}$$

where $m \in \mathbb{N}$ and $q \in \{0,1,\ldots,m-1\}$.

Using Theorem 3.13 with the function $\Phi(z) = \Gamma(z+1)/\Gamma(mz+q+1)$, where $\Gamma(z)$ is the gamma function, Milovanović and Stojanović [62] proved the following result:

Theorem 3.14. *The polynomial $P_N^{(m,q)}(t)$ defined by (3.8), where $m \in \mathbb{N}$ and $q \in \{0,1,\ldots,m-1\}$, has only real and positive zeros for every $N \in \mathbb{N}$.*

For some other classes of polynomials and the corresponding zero distribution see [21], [59], [60].

4. Eneström-Kakeya Theorem and Its Generalizations

For polynomials with positive coefficients Eneström [24] and Kakeya [45] proved (see also Henrici [33, p. 462] and Marden [56, p. 136]):

Theorem 4.1. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be any polynomial whose coefficients satisfy*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then $P(z)$ *has no zeros for* $|z| > 1$.

This theorem has been extended and sharpened in various ways (cf. Hurwitz [36], Egerváry [23], Tomić [82], Krishnaiah [49], Cargo and Shisha [9], Joyal, Labelle, and Rahman [44], Govil and Rahman [31], Jain [42], Govil and Jain [30], Anderson, Saff, and Varga [2]–[3], Dilcher [18], Kovačević and Milovanović [48], etc.). In this subsection we will mention some of them.

Let I, E, ∇ be standard difference operators defined by (cf. Milovanović [57])

$$\mathrm{I}\, a_k = a_k \quad \mathrm{E}\, a_k = a_{k+1}, \quad \nabla a_k = a_k - a_{k-1}$$

and let

$$\nabla^\alpha = \left(\mathrm{I} - \mathrm{E}^{-1}\right)^\alpha = \sum_{\nu=0}^{+\infty} (-1)^\nu \binom{\alpha}{\nu} \mathrm{E}^{-\nu}.$$

Cargo and Shisha [9] proved:

Theorem 4.2. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *such that* $a_0 \geq 0$ *and* $a_\nu \geq 0$, $\nabla^\alpha a_\nu \leq 0$ $(\nu = 1, \ldots, n)$ *for a given* α $(0 < \alpha \leq 1)$, *then* $P(z)$ *has no zeros in* $|z| < 1$.

Taking only monotonicity of the coefficients of a polynomial, Joyal, Labelle, and Rahman [44] proved:

Theorem 4.3. *If* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of (exact) degree* n $(n \geq 1)$ *such that* $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$, *then* $P(z)$ *has all its zeros in the disk*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{4.1}$$

Proof. Consider the polynomial $z^n Q(1/z)$, where

$$Q(z) = a_n z^{n+1} + (1-z)P(z) = a_0 + \sum_{k=1}^{n} (\nabla a_k) z^k$$

and $\nabla a_k = a_k - a_{k-1}$. For $|z| \leq 1$ we have

$$|z^n Q(1/z)| \leq |a_0| + \left|\sum_{k=1}^{n} (\nabla a_k) z^k\right| \leq |a_0| + \sum_{k=1}^{n} \nabla a_k = |a_0| + a_n - a_0,$$

i.e., $|Q(1/z)| \le (|a_0| + a_n - a_0)/|z|^n$. Replacing z by $1/z$ we obtain that

$$|Q(z)| \le (|a_0| + a_n - a_0)|z|^n$$

for $|z| \ge 1$. Also, for $|z| \ge 1$ we have

$$\begin{aligned} |(z-1)P(z)| &= |a_n z^{n+1} - Q(z)| \\ &\ge |a_n|\,|z|^{n+1} - |Q(z)| \\ &\ge |z|^n\big(|a_n|\,|z| - (|a_0| + a_n - a_0)\big). \end{aligned}$$

Since $a_n - a_0 \ge |a_n| - |a_0|$ we note that $R = (|a_0| + a_n - a_0)/|a_n| \ge 1$. Supposing $|z| > R$ we conclude that $|(z-1)P(z)| > 0$, i.e., the polynomial $P(z)$ has no zeros for $|z| > R$. □

For $a_0 > 0$ the disk (4.1) becomes $|z| \le 1$, i.e., Theorem 4.3 reduces to Theorem 3.4.

If $\lambda > 0$, taking $\lambda^{n-k}a_k$ instead of a_k $(k = 0, 1, \dots, n)$ in Theorem 4.3, we can formulate the following statement:

Theorem 4.4. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of (exact) degree n $(n \ge 1)$ such that $a_k - \lambda a_{k-1} \ge 0$ $(k = 1, \dots, n)$ for some $\lambda > 0$, then $P(z)$ has all its zeros in the disk*

$$|z| \le \frac{a_n - a_0\lambda^n + |a_0|\lambda^n}{\lambda |a_n|}. \tag{4.2}$$

A direct proof of this result was given by Kovačević and Milovanović [48]. Also, they gave some comments regarding this result. For example, if $a_0 < 0$ and $a_n > 0$, the bound given by (4.2) has the minimal value $n/((n-1)\lambda^*)$, if

$$\lambda = \lambda^* = \left(\frac{a_n}{2a_0(1-n)}\right)^{1/n}.$$

Thus, the best estimate for zeros of a polynomial, according to Theorem 4.4, can be obtained when the polynomial coefficients satisfy the conditions $a_k - \lambda^* a_{k-1} \ge 0$ $(k = 1, \dots, a_n)$. Notice that the conditions of Theorem 4.3 and $\lambda^* < 1$ imply that $a_k - \lambda^* a_{k-1} \ge 0$ for every k.

Dewan and Govil [17] showed that the disk given by (4.1) can be replaced by an annulus with a smaller outer radius. More precisely, they proved the following result:

Theorem 4.5. *Under conditions of Theorem 4.3, the polynomial $P(z)$ has all its zeros in the annulus (perhaps degenerate)*

$$R_2 \le |z| \le R_1,$$

where

$$R_1 = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left[\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right]^{1/2},$$

$$R_2 = \frac{R_1^2}{2M_2^2}\left[-b(M_2 - |a_0|) + \left(b^2(M_2 - |a_0|)^2 + 4|a_0|R_1^{-2}M_2^3\right)^{1/2}\right]$$

and

$$M_1 = a_n - a_0 + |a_0|, \qquad M_2 = R_1^n(|a_n|R_1 + a_n - a_0),$$
$$c = a_n - a_{n-1}, \qquad b = a_1 - a_0 .$$

Moreover

$$0 \le R_2 \le 1 \le R_1 \le \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

An equivalent statement of Theorem 4.1, due in fact to Eneström [24], is the following (cf. Anderson, Saff, and Varga [2]):

Theorem 4.6. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $n \ge 1$, *be any polynomial with* $a_k > 0$ *for all* $0 \le k \le n$. *Setting*

$$\alpha = \alpha[P] = \min_{0 \le k < n} \frac{a_k}{a_{k+1}}, \quad \beta = \beta[P] = \max_{0 \le k < n} \frac{a_k}{a_{k+1}}, \tag{4.3}$$

then all zeros of $P(z)$ *are contained in the annulus*

$$\alpha \le |z| \le \beta. \tag{4.4}$$

It is interesting to ask whether both inequalities in (4.4) are sharp, in the sense that polynomials with positive coefficients can be found having zeros either on $|z| = \alpha$ or on $|z| = \beta$. An affirmative answer was given by Hurwitz [37]. He showed that such extremal polynomials have a very special characterization. Using the Perron-Frobenius theory of non-negative matrices, Anderson, Saff, and Varga [2] gave a new proof of Hurwitz result, including some corrections and showed that the zeros of a particular set of polynomials fill out the Eneström-Kakeya annulus (4.4) in a precise manner. They used the following notation: $\mathcal{P}_n$ denotes the set of all complex polynomials of degree exactly n, and

$$\mathcal{P}_n^+ = \left\{P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \;\middle|\; a_\nu > 0 \text{ for all } 0 \le \nu \le n\right\}$$

It is clear, if

$$\varrho(P_n) = \max_{\nu}\{|z_\nu|\} \quad (P_n(z_\nu) = 0)$$

denotes the *spectral radius* of any polynomial $P_n(z)$ of degree at least unity, then it follows from (4.4) that

$$(\forall P_n(z) \in \mathcal{P}_n^+) \qquad \beta[P_n] \ge \varrho(P_n) \qquad (n \ge 1).$$

For $P_n(z) \in \mathcal{P}_n^+$, Anderson, Saff, and Varga [2] set

$$\overline{S} = \overline{S}[P_n] = \{\nu \in S_n \,:\, \beta a_{n+1-\nu} - a_{n-\nu} > 0,\ a_{-1} = 0\},$$

$$\underline{S} = \underline{S}[P_n] = \{\nu \in S_n \,:\, a_{\nu-1} - \alpha a_\nu > 0,\ a_{n+1} = 0\},$$

where $S_n = \{1, 2, \dots, n+1\}$ and α and β for $P_n(z)$ are defined in (4.3). Note that these sets are non-empty, since $n+1$ is an element of both sets. Also associated with $P_n(z) \in \mathcal{P}_n^+$, they introduced the positive integers

$$\overline{k} = \overline{k}[P_n] = \gcd\{\nu \in \overline{S}\}, \quad \underline{k} = \underline{k}[P_n] = \gcd\{\nu \in \underline{S}\}.$$

With this notation, Anderson, Saff, and Varga [2] proved the following result:

Theorem 4.7. *For any $P_n(z) \in \mathcal{P}_n^+$ with $n \geq 1$, all the zeros of $P_n(z)$ lie in the annulus (4.4). Moreover, $P_n(z)$ can vanish on $|z| = \beta$ if and only if $\overline{k} > 1$. If $\overline{k} > 1$, the zeros of $P_n(z)$ on $|z| = \beta$ are all simple, and are precisely given by*

$$\beta \exp\Bigl(\frac{2\pi i \nu}{\overline{k}}\Bigr) \qquad (\nu = 1, 2, \dots, \overline{k} - 1),$$

and $P_n(z)$ has the form

$$P_n(\beta z) = \bigl(1 + z + z^2 + \cdots + z^{\overline{k}-1}\bigr) Q_m\bigl(z^{\overline{k}}\bigr),$$

where $Q_m(w) \in \mathcal{P}_m^+$. If $m \geq 1$, then all the zeros of $Q_m(w)$ lie in $|w| < 1$, and $\beta[Q_m] \leq 1$.
Similarly, $P_n(z)$ can vanish on $|z| = \alpha$ if and only if $\underline{k} > 1$. If $\underline{k} > 1$, the zeros of $P_n(z)$ on $|z| = \alpha$ are simple and given precisely by

$$\alpha \exp\Bigl(\frac{2\pi i \nu}{\underline{k}}\Bigr) \qquad (\nu = 1, 2, \dots, \underline{k} - 1),$$

and $P_n(z)$ has the form

$$z^n P_n(\alpha / z) = \bigl(1 + z + z^2 + \cdots + z^{\underline{k}-1}\bigr) R_m\bigl(z^{\underline{k}}\bigr),$$

where $R_m(w) \in \mathcal{P}_m^+$. If $m \geq 1$, then all the zeros of $R_m(w)$ lie in $|w| < 1$ and $\beta[R_m] \leq 1$.

Anderson, Saff, and Varga [3] was extended the classical Eneström-Kakeya theorem to the case of any complex polynomial having no zeros on the ray $[0, +\infty)$. They showed that this extension is sharp in the sense that, given such a complex polynomial $P_n(z)$ of degree $n \geq 1$, a sequence of polynomials $\{Q_{m_\nu}(z)\}_{\nu=0}^{+\infty}$ can be found for which the classical Eneström-Kakeya theorem, applied to the products $Q_{m_\nu}(z) P_n(z)$, yields the maximum of the moduli of the zeros of $P_n(z)$, when $\nu \to +\infty$. Also, Anderson, Saff, and Varga [3] described a computational algorithm, based on linear programming, for improving the Eneström-Kakeya upper bound.

5. Number of Zeros in a Given Domain

We consider here a few domains in the complex plane, starting with a simple case when that domain is the real line.

Let r denote the number of real zeros, taking multiplicity into account, of a polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad (a_0 a_n \neq 0). \tag{5.1}$$

Under conditions that

$$|a_0| \geq b, \quad |a_n| \geq b, \quad |a_k| \leq a \quad (k = 1, \ldots, n-1),$$

Bloch and Pólya proved the following inequality

$$r < A_1(a,b) \frac{n \log \log n}{\log n},$$

where the constant A_1 depends only on a and b. A few years later Schmidt [74] proved the sharper inequality

$$r^2 < A_2(a,b) n \log \frac{an}{b},$$

where A_2 depends on a and b, and the still sharper one

$$r^2 \leq A_3 n \log R, \tag{5.2}$$

where A_3 is a positive constant and

$$R = \frac{|a_0| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}. \tag{5.3}$$

Schmidt's detailed proof has never been published because Schur [76] found an elementary short proof for it. Also Schur proved that $A_3 = 4$ is the best constant in (5.2) and that

$$r^2 - 2r \leq 2n \log \frac{Q}{2},$$

where

$$Q = \frac{|a_0|^2 + \cdots + |a_n|^2}{|a_0 a_n|}.$$

Szegő [79] proved that

$$r(r+1) < 4(n+1) \log R$$

and

$$r(r+1) + (p-q)^2 < 4(n+1) \log M$$

where p and q denote the numbers of positive and negative zeros, respectively, of $P(z)$, and

$$M = \max_{|z|=1} \frac{|P(z)|}{\sqrt{|a_0 a_n|}}.$$

Erdős and Turán [25] gave a short proof of an inequality for the number of positive zeros:

Theorem 5.1. *If $P(z)$, given by* (5.1), *has p positive zeros, then*

$$p^2 \leq 2n \log R,$$

where R is defined by (5.3).

In order to prove this result we suppose that $z_k = r_k e^{i\theta_k}$ $(k = 1, 2, \ldots, n)$ be zeros of $P(z)$, i.e.,

$$P(z) = a_0 \prod_{k=1}^{n} (z - r_k e^{i\theta_k})$$

and put

$$q(z) = \prod_{k=1}^{n} (z - e^{i\theta_k}).$$

It is easy to see that

$$r|z - e^{i\theta}|^2 \leq |z - re^{i\theta}|^2 \qquad (r \geq 0),$$

when $|z| = 1$. Then, using this inequality we have

$$|q(z)|^2 = \prod_{k=1}^{n} |\, z - e^{i\theta_k}|^2 \leq \frac{|a_0|^2}{|a_0|^2 r_1 r_2 \cdots r_n} \prod_{k=1}^{n} |\, z - r_k e^{i\theta_k}|^2,$$

i.e.,

$$|q(z)|^2 \leq \frac{|P(z)|^2}{|a_0 a_n|},$$

whenever $|z| = 1$.

Put $C = \{z \in \mathbb{C} : |z| = 1\}$,

$$\|f\|_C = \max_{|z|=1} |f(z)| \qquad \text{and} \qquad \|f\|_{[a,b]} = \max_{x \in [a,b]} |f(x)|.$$

Since $P(z)$ has p positive real zeros, we see that the polynomial $q(z)$ has p zeros at 1. Using the change of variables $x = z + z^{-1}$ applied to $z^n q(z^{-1}) q(z)$ we can prove that (see [7, pp. 17–18])

$$\begin{aligned}
\|q\|_C^2 &\geq \min_{b_k} \|(z-1)^p (z^{n-p} + b_{n-p-1} z^{n-p-1} + \cdots + b_1 z + b_0\|_C^2 \\
&\geq \min_{c_k} \|x^p (x^{n-p} + c_{n-p-1} x^{n-p-1} + \cdots + c_1 z + c_0\|_{[0,4]} \\
&= 4^n \min_{d_k} \|x^p (x^{n-p} + d_{n-p-1} x^{n-p-1} + \cdots + d_1 z + d_0\|_{[0,1]} \\
&\geq \frac{4^n}{\sqrt{2n+1}\binom{2n}{n+p}}.
\end{aligned}$$

Finally, using the inequality

$$\frac{p^2}{n} \leq \log\left(\frac{4^n}{\sqrt{2n+1}\binom{2n}{n+p}}\right),$$

we obtain $p^2 \leq 2n \log R$.

A refinement of this theorem was recently given in [7]:

Theorem 5.2. *Every algebraic polynomial $P(z)$ of the form* (5.1), *with* $|a_n| = 1$ *and* $|a_k| \le 1$ $(k = 0, 1, \ldots, n-1)$, *has at most* $\lfloor \frac{16}{7}\sqrt{n} \rfloor + 4$ *zeros at* 1.

Theorem 5.3. *If the zeros of the polynomial* $P(z)$, *given by* (5.1), *are denoted by*

$$z_\nu = r_\nu \exp(i\varphi_\nu) \qquad (\nu = 1, \ldots, n),$$

then for every $0 \le \alpha < \beta \le 2\pi$ *we have*

$$\Big| \sum_{\nu \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \Big| < 16\sqrt{n \log R},$$

where R is given by (5.3) *and the index set* $I(\alpha, \beta)$ *is defined by*

$$I(\alpha, \beta) = \{\nu \in \{1, \ldots, n\} \,:\, \alpha \le \varphi_\nu \le \beta\}.$$

The content of this theorem can be expressed by saying that the zeros of a polynomial are uniformly distributed in the different angles with vertex at the origin if the coefficients "in the middle" are not too large compared with the extreme ones. In the case $a_0 = a_1 = \cdots = a_n$ the uniform distribution is of course much more perfect than is expressed in the previous theorem and represents the ideal case, but this theorem shows that if all coefficients satisfy the condition

$$n^{-\lambda} \le |a_\nu| \le n^\lambda \qquad (\nu = 0, 1, \ldots, n), \tag{5.4}$$

then $R \le (n+1)n^{2\lambda} < (n+1)^{2\lambda+1}$, i.e.,

$$\Big| \sum_{\nu \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \Big| < 16\sqrt{2\lambda + 1}\sqrt{n \log(n+1)}\,.$$

Hence a rather radical change of the coefficients restricted only by (5.4) cannot "spoil" the uniformly dense distribution of the zeros in angles very much (cf. Erdős and Turán [25]).
Erdős and Turán [25] deduced from Theorem 5.3 the inequality (5.2), even in a slightly sharpened form.
For number of zeros in a circle, Singh [77] has proved the following results:

Theorem 5.4. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial with real or complex coefficients such that for* $n \ge 2$

$$|a_0| + |a_1| + \cdots + |a_{n-1}| \le n|a_n|$$

and

$$2^{(n+1)/2} r^n (n+1) \Big| \frac{a_n}{a_0} \Big| < 1, \qquad r \ge \frac{1}{2}.$$

Then at least one zero of $P(z)$ lies outside the circle $|z| = r$.

Theorem 5.5. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *satisfy*

$$\min_{0\le\nu\le n} |a_\nu| \ge 1 \qquad \textit{and} \qquad \max_{0\le\nu<n} |a_\nu| \ge |a_n|.$$

Then

$$n(R/K) \le \frac{2\log\{(n+1)|a_n|R^n\}}{\log K} \qquad (K>1),$$

where $n(x)$ *is the number of zeros of* $P(z)$ *for* $|z| \le x$ *and*

$$R = \max\left\{ \left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \left|\frac{a_{n-3}}{a_n}\right|^{1/3}, \dots \right\}.$$

Rahman [69] improved Singh's results in the following theorem.

Theorem 5.6. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial with real or complex coefficients and let the conditions of Theorem* 5.4 *be satisfied, then the least number of zeros of* $P(z)$ *lying outside the circle* $|z| = r$ *is* $(n+1)/2$ *or* $(n/2)+1$ *according as* n *is an odd or an even positive integer. Under the same conditions as in Theorem* 5.5 *one also has*

$$n(R/K) \le \frac{\log\{(n+1)|a_n|R^n\}}{\log K} \tag{5.5}$$

for every $K > 1$.

The following result is due to Mohammad [63]:

Theorem 5.7. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *such that* $a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 > 0$, *then the number of zeros of* $P(z)$ *in* $|z| \le 1/2$ *does not exceed*

$$1 + \frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

As a generalization of Theorem 5.7, Dewan [16] proved:

Theorem 5.8. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *with complex coefficients such that*

$$|\arg a_\nu - \beta| \le \alpha \le \pi/2 \qquad (\nu = 0, 1, \dots, n)$$

for some real β, *and*

$$|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_1| \ge |a_0|,$$

then the number of zeros of $P(z)$ *in* $|z| \le 1/2$ *does not exceed*

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{\nu=0}^{n-1}|a_\nu|}{|a_0|}.$$

Theorem 5.9. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *with complex coefficients. If* $\operatorname{Re} a_\nu = \alpha_\nu$, $\operatorname{Im} a_\nu = \beta_\nu$, *for* $\nu = 0, 1, \ldots, n$ *and* $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0$, *then the number of zeros of* $P(z)$ *in* $|z| \leq 1/2$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{\nu=0}^{n} |\beta_\nu|}{|a_0|}.$$

The previous three theorems were generalized by Bidkham and Dewan [6] for different classes of polynomials which in turn also refine upon them.
The zero distribution of the trinomial

$$T(x) = mx^n - nx^m + n - m, \qquad n > m > 0 \tag{5.6}$$

recently has been investigated by Dilcher, Nulton, and Stolarsky [19].
Let $C = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\gcd(m, n)$ be the greatest common divisor of the integers m and n.

Theorem 5.10. *Let* $a > b > 0$ *be real numbers and* $n > m > 0$ *be integers. Then the number of zeros of*

$$P(z) = bz^n - az^m + a - b$$

strictly inside C *is* $m - \gcd(m, n)$ *if* $a/b \geq n/m$, *and* m *if* $a/b < n/m$.

As a consequence of this result, Dilcher, Nulton, and Stolarsky [19] proved:

Corollary 5.11. *If* $n > m > 0$ *are two relatively prime integers, then the trinomial* $T(z)$ *in* (5.6) *has* $m-1$ *zeros strictly inside* C, $n-m-1$ *zeros strictly outside* C, *and a double zero at* $z = 1$.

For $n \geq 3$ the zeros of $T(z)/(z-1)^2$ lie in the following annuli:

1° For $m = 1$

$$1 + (n-2)^{-1} \leq |z| \leq [2(n-1)]^{1/(n-1)};$$

2° For $2 \leq m \leq n-2$

$$\begin{aligned} \max\left\{(2m)^{-1/m}, 1 - \frac{1}{n-m}\sqrt{2n/m}\right\} &\leq |z| \\ &\leq \min\left\{[2(n-m)]^{1/(n-m)}, 1 + \frac{1}{m}\sqrt{2n/(n-m)}\right\}; \end{aligned}$$

3° For $m = n-1$

$$[2(n-1)]^{-1/(n-1)} \leq |z| \leq 1 - (n-1)^{-1}.$$

Recently, Gleyse and Moflih [28] gave a new algebraic proof and method for the exact computation of the number of zeros of a real polynomial inside the unit disk.

In their investigation, they used the Brown transform and Schur-Cohn transforms of a real polynomial (see also [26] and [27]).

Different results about the number of zeros in a half-plane, in a sector, or in a given circle, the reader can find in the book of Marden [56].

References

[1] W. A. Al-Salam and M. E. H. Ismail, *Polynomials orthogonal with respect to discrete convolution,* J. Math. Anal. Appl. **55** (1976), 125–139.

[2] N. Anderson, E. B. Saff, and R. S. Varga, *On the Eneström-Kakeya theorem and its sharpness,* Linear Algebra Appl. **28** (1979), 5–16.

[3] N. Anderson, E. B. Saff, and R. S. Varga, *An extension of the Eneström-Kakeya theorem and its sharpness,* SIAM J. Math. Anal. **12** (1981), 10–22.

[4] A. Aziz, *On the zeros of composite polynomials,* Pacific J. Math. **103** (1982), 1–7.

[5] A. Aziz, *On the location of the zeros of certain composite polynomials,* Pacific J. Math. **118** (1985), 17–26.

[6] M. Bidkham and K. K. Dewan, *On the zeros of a polynomial,* In: Numerical Methods and Approximation Theory III (Niš, 1987) (G. V. Milovanović, ed.), Univ. Niš, Niš, 1988, pp. 121–128.

[7] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities,* Springer-Verlag, New York, 1995.

[8] N. G. de Bruijn, *On the zeros of a polynomial and of its derivative,* Nederl. Akad. Wetensch. Proc. **49** (1946), 1037–1044 [= Indag. Math. **8** (1946), 635–643].

[9] G. T. Cargo and O. Shisha, *Zeros of polynomials and fractional order differences of their coefficients,* J. Math. Anal. Appl. **7** (1963), 176–182.

[10] T. Craven and G. Csordas, *An inequality for the distribution of zeros of polynomials and entire functions,* Pacific J. Math. **95** (1981), 263-280.

[11] T. Craven and G. Csordas, *On the number of real roots of polynomials,* Pacific J. Math. **102** (1982), 15–28.

[12] T. Craven and G. Csordas, *The Gauss-Lucas theorem and Jensen polynomials,* Trans. Amer. Math. Soc. **278** (1983), 415–427.

[13] T. Craven and G. Csordas, *Location of zeros, Part I: Real polynomials and entire functions,* Illinois J. Math. **27** (1983), 244–278.

[14] T. Craven and G. Csordas, *Location of zeros, Part II: Ordered fields,* Illinois J. Math. **27** (1983), 279–299.

[15] T. Craven and G. Csordas, *Problems and theorems in the theory of multiplier sequences,* Serdica Math. J. **22** (1996), 515–524.

[16] K. K. Dewan, *Extremal properties and coefficient estimates for polynomials with restricted zeros and on location of zeros of polynomials,* Ph. D. Thesis, I. I. T., Delhi, New Delhi, 1980.

[17] K. K. Dewan and N. K. Govil, *On the Eneström-Kakeya theorem,* J. Approx. Theory **42** (1984), 239–244.

[18] K. Dilcher, *A generalization of the Eneström-Kakeya theorem,* J. Math. Anal. Appl. **116** (1986), 473–488.

[19] K. Dilcher, J. D. Nulton, and K. B. Stolarsky, *The zeros of a certain family of trinomials,* Glasgow Math. J. **34** (1992), 55–74.

[20] G. B. Đorđević, *Generalized Hermite polynomials,* Publ. Inst. Math. (Beograd) (N.S.) **53 (67)** (1993), 69–72.

[21] G. B. Đorđević and G. V. Milovanović, *Polynomials related to the generalized Hermite polynomials,* Facta Univ. Ser. Math. Inform. **8** (1993), 35–43.

[22] W. H. Echols, *Note on the roots of the derivative of a polynomial,* Amer. Math. Monthly **27** (1920), 299–300.

[23] J. Egerváry, *On a generalization of a theorem of Kakeya,* Acta Sci. Math. (Szeged) **5** (1931), 78–82.

[24] G. Eneström, *Härledning af en allmän formel för antalet pensionärer ...,* Öfv. af. Kungl. Vetenskaps-Akademiens Förhandlingen, No. 6, Stockholm, 1893.

[25] P. Erdős and P. Turán, *On the distribution of roots of polynomials,* Ann. of Math. **51** (1950), 105–119.

[26] B. Gleyse, *A Schur's modified sequence and the number of zeros of a real polynomial in the unit disk: testing its numerical quality,* Appl. Math. Lett. **12**, No. 6 (1999), 57–60.

[27] B. Gleyse and M. Moflih, *Exact computation of the number of zeros of a real polynomial in the open unit disk by a determinant representation,* Comput. Math. Appl. **38**, No. 11/12 (1999), 257–263.

[28] B. Gleyse and M. Moflih, *On the exact computation of the number of zeros of a real polynomial in the open unit disk: an algebraic approach,* Comput. Math. Appl. **39**, No. 1/2 (2000), 209–214.

[29] A.W. Goodman and I.J. Schoenberg, *A proof of Grace's theorem by induction,* Honam Math. J. **9** (1987), 1–6.

[30] N.K. Govil and V.K. Jain, *On the Eneström-Kakeya theorem II,* J. Approx. Theory **22** (1978), 1–10.

[31] N.K. Govil and Q.I. Rahman, *On the Eneström-Kakeya theorem,* Tôhoku Math. J. **20** (1968), 126–136.

[32] J.H. Grace, *The zeros of a polynomial,* Proc. Cambridge Philos. Soc. **11** (1901), 352–357.

[33] P. Henrici, *Applied and Computational Complex Analysis,* Vol. 1, Wiley, New York, 1974.

[34] E. Hille, *Analytic Function Theory,* Vol. II, Chelsea, New York, 1962.

[35] L. Hörmander, *On a theorem of Grace,* Math. Scand. **2** (1954), 55–64.

[36] A. Hurwitz, *Ueber die Nullstellen der Bessel'schen Function,* Math. Ann. **33** (1889), 246–266.

[37] A. Hurwitz, *Über einem Satz des Herrn Kakeya,* Tôhoku Math. J. **4** (1913/14), 89–93.

[38] A. Iserles and S.P. Nørsett, *On the theory of bi-orthogonal polynomials,* Trans. Amer. Math. Soc. **306** (1988), 455–474.

[39] A. Iserles and S.P. Nørsett, *Zeros of transformed polynomials,* SIAM J. Math. Anal. **21** (1990), 483–509.

[40] A. Iserles, S.P. Nørsett, and E.B. Saff, *On transformations and zeros of polynomials,* Rocky Mountain J. Math. **21** (1991), 331–357.

[41] A. Iserles and E.B. Saff, *Zeros of expansions in orthogonal polynomials,* Math. Proc. Cambridge Philos. Soc. **105** (1989), 559–573.

[42] V.K. Jain, *On the location of zeros of polynomials,* Ann. Univ. Mariae Curie-Skłodowska Sect. A **30** (1976), No. 5, 43–48.

[43] J.L.W.V. Jensen, *Recherches sur la théorie des équations,* Aequationes Math. **36** (1913), 181–185.

[44] A. Joyal, G. Labelle, and Q.I. Rahman, *On the location of zeros of polynomials,* Canad. Math. Bull. **10** (1967), 53–63.

[45] S. Kakeya, *On the limits of the roots of an algebraic equation with positive coefficients,* Tôhoku Math. J. **2** (1912), 140–142.

[46] S. Kakeya, *On algebraic equations having the roots of limited magnitude,* Proc. Phys.-Math. Soc. Japan (3) **3** (1921), 94–100.

[47] I.M. Kasandrova and M.D. Kostova, *On a class of multiplier sequences,* Serdica **19** (1993), 228–233.

[48] M.A. Kovačević and I.Ž. Milovanović *On a generalization of the Eneström-Kakeya theorem,* PU.M.A. Ser. A **3** (1992), 43–47.

[49] P.V. Krishnaiah, *On Kakeya's theorem,* J. London. Math. Soc. **30** (1955), 314–319.

[50] T.D. Lee and C.N. Yang, *Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model,* Phys. Rev. **87** (1952), 410–419.

[51] F. Lucas, *Propriétés géométriques des fractions rationnelles*, C.R. Acad. Sci. Paris **77** (1874), 431–433; **78** (1874), 140–144; **78** (1874), 180–183; **78** (1874), 271–274.

[52] F. Lucas, *Sur une application de la méchanique rationalle a à la théorie des équations*, C.R. Acad. Sci. Paris **89** (1879), 224–226.

[53] F. Lucas, *Géométrie des polynômes*, J. École Polytech. (1) **46** (1879), 1–33.

[54] F. Lucas, *Statique des polynômes*, Bull. Soc. Math. France **17** (1888), 2–69.

[55] M. Marden, *The zeros of certain composite polynomials*, Bull. Amer. Math. Soc. **49** (1943), 93–100.

[56] M. Marden, *Geometry of Polynomials*, Math. Surveys **3**, Amer. Math. Soc. Providence, R. I., 1966.

[57] G. V. Milovanović, *Numerical Analysis, I*, 3rd ed., Naučna Knjiga, Belgrade, 1991 (Serbian).

[58] G. V. Milovanović, *Recurrence relation for a class of polynomials associated with the generalized Hermite polynomials*, Publ. Inst. Math. (Beograd) (N.S.) **54 (68)** (1993), 35–37.

[59] G. V. Milovanović and G. B. Đorđević, *On some properties of Humbert's polynomials*, Fibonacci Quart. **25** (1987), 356–360.

[60] G. V. Milovanović and G. B. Đorđević, *On some properties of Humbert's polynomials*, Facta Univ. Ser. Math. Inform. **6** (1991), 23–30.

[61] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topic in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore – New Jersey – London – Hong Kong, 1994.

[62] G. V. Milovanović and N. M. Stojanović, *Zero distribution of a class of polynomials associated with the generalized Hermite polynomials*, Facta Univ. Ser. Math. Inform. **9** (1994), 51–54

[63] Q. G. Mohammad, *On the zeros of the polynomials*, Amer. Math. Monthly **72** (1965), 631–633.

[64] N. Obreschkoff, *Verteilung und Berechnung der Nullstellen Reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[65] N. Obreschkoff, *Zeros of Polynomials*, Bulgar. Acad. Sci., Sofia, 1963 (Bulgarian).

[66] R. Peretz and Th. M. Rassias, *Some remarks on theorems of M. Marden concerning the zeros of certain composite polynomials*, Complex Variables **18** (1992), 85–89.

[67] G. Pólya, *Über algebraische Gleichungen mit nur reelen Wurzeln*, Vierteljschr. Naturforsch. Ges. Zürich **61** (1916), 546–548.

[68] G. Pólya and J. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math. **144** (1914), 89–113.

[69] Q. I. Rahman, *On the zeros of a class of polynomials*, Proc. Nat. Inst. Sci. India **22** (1956), 137–139.

[70] Th. M. Rassias, *On certain properties of polynomials and their derivative*, In: Topics in Mathematical Analysis (Th. M. Rassias, ed.), World Scientific Publishing Company, Singapore, 1989, pp. 758–802.

[71] Th. M. Rassias, *Zeros of polynomials and their derivative*, Rev. Roumaine Math. Pures Appl. **36** (1991), 441–478.

[72] Th. M. Rassias and H. M. Srivastava, *Some recent advances in the theory of the zeros and critical points of a polynomial*, In: Topics in Polynomials of One and Several Variables and Their Applications: A Mathematical Legacy of P. L. Chebyshev (1821–1894) (Th. M. Rassias, H. Srivastava, and A. Yanushauskas, eds.), World Scientific Publishing Company, Singapore, 1993, pp. 463–481.

[73] G. Schaake and J. G. van der Corput, *Ungleichungen für Polynome und trigonometrische Polynome*, Compositio Math. **2** (1935), 321–361.

[74] E. Schmidt, *Über algebraische Gleichungen von Pólya-Bloch-Typus*, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1932, pp. 321–328.

[75] J. Schur, *Zwei Sätze úber algebraische Gleichungen mit lauter rellen Wurzeln*, J. Reine Angew. Math. **144** (1914), 75–88.

[76] J. Schur, *Untersuchungen über algebraische Gleichungen. I. Bemerkungen zu einem Satz von E. Schmidt,* Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1933, pp. 403–428.

[77] S. K. Singh, *On the zeros of a class of polynomials,* Proc. Nat. Inst. Sci. India **19** (1953), 601–603.

[78] G. Szegő, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen,* Math. Z. **13** (1922), 28–55.

[79] G. Szegő, *Bemerkungen zu einem Satz von E. Schmidt über algebraische Gleichungen,* Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1934, pp. 86–98.

[80] G. (J. v.) Sz.-Nagy, *Ueber geometrische Relationen zwischen den Wurzeln einer Algebraischen Gleichung und ihrer Derivierten,* Jber. Deutsch. Math.-Verein. **27** (1918), 44–48.

[81] G. (J. v.) Sz.-Nagy, *Zur Theorie der algebraischen Gleichungen,* Jber. Deutsch. Math.-Verein. **31** (1922), 238–251.

[82] M. Tomić, *Généralisation et démonstration géométrique de certains théorèmes de Fejér et Kakeya,* Acad. Serbe Sci. Publ. Inst. Math. **2** (1948), 146–156.

[83] J. L. Walsh, *On the location of the roots of the derivative of a rational function,* Ann. of Math. **22** (1920), 128–144.

[84] J. L. Walsh, *On the location of the roots of certain types of polynomials,* Trans. Amer. Math. Soc. **24** (1922), 163–180.

[85] J. L. Walsh, *On Pellet's theorem concerning the roots of a polynomial,* Ann. of Math. (2) **26** (1924), 59–64.

[86] J. L. Walsh, *A theorem of Grace on the zeros of polynomials, revisited,* Proc. Amer. Math. Soc. **15** (1964), 354–360.