Chapter 11

Complex Polynomials Orthogonal on the Semicircle

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11.1. Introduction

Suppose that X is a real linear space of functions, with an inner product $(f, g) : X^2 \to \mathbf{R}$ such that

(a) $(f+g, h) = (f, h) + (g, h)$ (Linearity)

(b) $(\alpha f, g) = \alpha (f, g)$ (Homogeneity)

(c) $(f, g) = (g, f)$ (Symmetry)

(d) $(f, f) > 0,\ (f, f) = 0 \Leftrightarrow f = 0$ (Positivity)

where $f, g, h \in X$ and α is a real scalar.

If X is a complex linear space of functions, then the inner product (f, g) maps X^2 into $\mathbf{C}$ and the requirement (c) is replaced by

(c′) $(f, g) = \overline{(g, f)}$ (Hermitian Symmetry)

where the bar designates the complex conjugate.

A system of polynomials $\{p_k\}$, where

$$p_k(t) = t^k + \text{terms of lower degree} \qquad (k = 0, 1, \ldots)$$

and

$$(p_k, p_m) = 0 \quad (k \neq m), \qquad (p_k, p_m) > 0 \quad (k = m)$$

is called a system of (monic) orthogonal polynomials with respect to the inner product $(\cdot, \cdot)$.

The most common type of orthogonality is with respect to the following inner product

$$(f, g) = \int_{\mathbf{R}} f(t) g(t)\, d\lambda(t),$$

where $d\lambda(t)$ is a nonnegative measure on the real line $\mathbf{R}$ with compact or infinite support, for which all moments

$$\mu_k = \int_{\mathbf{R}} t^k \, d\lambda(t), \quad k = 0, 1, \ldots$$

exist and are finite, and $\mu_0 > 0$. Then the (monic) orthogonal polynomials $\{p_k\}$ satisfy the fundamental three-term recurrence relation

$$(1) \qquad \begin{aligned} &p_{k+1}(t) = (t - a_k)\, p_k(t) - b_k\, p_{k-1}(t), \qquad k = 0, 1, 2, \ldots, \\ &p_{-1}(t) = 0, \quad p_0(t) = 1, \end{aligned}$$

where the coefficients a_k and b_k are given by

$$\begin{aligned} a_k &= \frac{(tp_k, p_k)}{(p_k, p_k)} \qquad (k = 0, 1, \ldots), \\ b_k &= \frac{(p_k, p_k)}{(p_{k-1}, p_{k-1})} \qquad (k = 1, 2, \ldots). \end{aligned}$$

We note that $b_k > 0$ for $k \geq 1$. The coefficient b_0 in (1) is arbitrary, but the definition

$$b_0 = \mu_0 = \int_{\mathbf{R}} d\lambda(t)$$

is sometimes convenient.

Typical examples of such polynomials are the classical orthogonal polynomials of Legendre, Čebyšev, Gegenbauer, Jacobi, Laguerre and Hermite.

An other type of orthogonality is the orthogonality on the unit circle with respect to the inner product

$$(f, g) = \int_0^{2\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})}\, d\mu(\theta), \quad d\mu(\theta) \geqq 0.$$

These polynomials were introduced and studied by Szegö [1]. Monic orthogonal polynomials $\{\Phi_k\}$ on the unit circle satisfy the recurrent relation

$$\Phi_{k+1}(z) = z\, \Phi_k(z) + \Phi_k(0)\, z^k\, \overline{\Phi}_k(1/z), \qquad k = 0, 1, 2, \ldots,$$

which is not of the form (1). For more details consult Nevai [2].

Similarly, we may consider orthogonal polynomials on a rectifiable curve or an arc lying in the complex plane (e.g. Geronimus [3], Szegö [4]). Complex orthogonal polynomials may also be constructed by means of double integrals. Namely, introducing the inner product by

$$(f, g) = \iint_B f(z)\overline{g(z)}\, w(z)\, dx\, dy,$$

for a suitable positive weight function w where B is a bounded region of the complex plane, a system of orthogonal polynomials can be generated (see Carleman [5] and Bochner [6]).

11.2. Orthogonality on the Semicircle

Recently Gautschi and Milovanović [7] (see also [8]) introduced a new type of orthogonality, the so-called orthogonality on the semicircle. The inner product is defined by

$$(1) \qquad (f, g) = \int_\Gamma f(z) g(z) (iz)^{-1} dz,$$

where $\Gamma = \{z \in \mathbf{C} \,|\, z = e^{i\theta},\ 0 \le \theta \le \pi\}$. Alternatively, (1) can be expressed in the form

$$(2) \qquad (f, g) = \int_0^\pi f(e^{i\theta}) g(e^{i\theta})\, d\theta.$$

Notice that this inner product does not satisfy the conditions (c') and (d). Namely, the second factor in (1), i.e. (2), is not conjugated, so that this product does not possess Hermitian symmetry; instead it has property (c).

The corresponding (monic) orthogonal polynomials exist uniquely and satisfy a three-term recurrence relation of the form (1) from 11.1, due to the property $(zf, g) = (f, zg)$.

The general case of orthogonality with the complex weight function w with respect to the inner product

$$(f, g) = \int_\Gamma f(z) g(z) w(z) (iz)^{-1} dz,$$

i.e.

$$(3) \qquad (f, g) = \int_0^\pi f(e^{i\theta}) g(e^{i\theta}) w(e^{i\theta})\, d\theta,$$

was considered by Gautschi, Landau and Milovanović [9].

Let $w : (-1, 1) \to \mathbf{R}^+$ be a weight function, which can be extended to a function $z \mapsto w(z)$ which is regular in the half disc

$$D_+ = \{z \in \mathbf{C} \,|\, |z| < 1,\ \operatorname{Im} z > 0\}.$$

Together with (3) consider the inner product

$$(4)\qquad [f, g] = \int_{-1}^{1} f(x)\, g(x)\, w(x)\, \mathrm{d}x,$$

which is positively definite and therefore generates a unique set of real (monic) orthogonal polynomials $\{p_k\}$:

$$[p_k, p_m] = 0 \quad \text{for} \quad k \neq m \quad \text{and}$$

$$[p_k, p_m] > 0 \quad \text{for} \quad k = m \qquad (k, m = 0, 1, 2, \ldots)$$

On the other hand, the inner product (3) is not Hermitian; the second factor g is not conjugated and the integration is not with respect to the measure $|w(e^{i\theta})|\mathrm{d}\theta$. The existence of corresponding orthogonal polynomials, therefore, is not guaranteed.

We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$[\pi_k, \pi_m] = 0 \quad \text{for} \quad k \neq m \quad \text{and}$$

$$[\pi_k, \pi_m] \neq 0 \quad \text{for} \quad k = m \qquad (k, m = 0, 1, 2, \ldots)$$

where we assume that π_k is monic of degree k.

The existence of the orthogonal polynomials $\{\pi_k\}$ can be established assuming only that

$$(5)\qquad \mathrm{Re}\,(1, 1) = \mathrm{Re} \int_0^{\pi} w(e^{i\theta})\, \mathrm{d}\theta \neq 0.$$

11.3. Existence and Representation of π_n

Assume that the weight function w is positive on $(-1, 1)$, regular in D_+ and such that the integrals (3) and (4) from 11.2 exist for smooth f and g (possibly) as improper integrals. We also assume that the condition (5) of 11.2 is satisfied.

Let C_ε, $\varepsilon > 0$ denote the boundary of D_+ with small circular parts of radius ε and centres at ± 1 spared out and let P be the set of all algebraic polynomials. Then, by Cauchy's theorem, for any $g \in P$ we have

$$(1)\qquad \begin{aligned} 0 &= \int_{C_\varepsilon} g(z)\, w(z)\, \mathrm{d}z \\ &= \Big(\int_{\Gamma_\varepsilon} + \int_{C_{\varepsilon,-1}} + \int_{C_{\varepsilon,+1}} \Big) g(z)\, w(z)\, \mathrm{d}z = \int_{-1+\varepsilon}^{1-\varepsilon} g(x)\, w(x)\, \mathrm{d}x, \end{aligned}$$

where Γ_ε and $C_{\varepsilon,\pm 1}$ are the circular parts of C_ε (with radii 1 and ε respectively). We assume that w is such that for all $g \in P$

(2) $$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon,\pm 1}} g(z)\, w(z)\, dz = 0 \qquad (\forall g \in P).$$

Then, if $\varepsilon \to 0$ in (1), we obtain

(3) $$0 = \int_C g(z)\, \omega(z)\, dz = \int_\Gamma g(z)\, \omega(z)\, dz + \int_{-1}^{1} g(x)\, \omega(x)\, dx, \qquad g \in P.$$

The (monic, real) polynomials $\{p_k\}$, orthogonal with respect to the inner product (4) of 11.2, as well as the associated polynomials of the second kind

$$q_k(z) = \int_{-1}^{1} \frac{p_k(z) - p_k(x)}{z - x}\, \omega(x)\, dx \qquad (k = 0, 1, 2, \ldots),$$

are known to satisfy a three-term recurrence relation of the form

(4) $$y_{k+1} = (z - a_k)\, y_k - b_k\, y_{k-1} \qquad (k = 0, 1, 2, \ldots),$$

where

(5) $$y_{-1} = 0,\ y_0 = 1 \text{ for } \{p_k\} \text{ and } y_{-1} = -1,\ y_0 = 0 \text{ for } \{q_k\}.$$

Denote by m_k and μ_k the moments associated with the inner products (4) and (3) of 11.2, respectively

$$m_k = [x^k, 1], \quad \mu_k = (z^k, 1), \quad k \geqq 0,$$

where, in view of (5)

(6) $$b_0 = m_0.$$

THEOREM 1. *Let w be a weight function, positive on $(-1, 1)$, regular in $D_+ = \{z \in \mathbb{C} \,|\, |z| < 1,\ \mathrm{Im}\, z > 0\}$ and such that* (2) *is satisfied and the integrals in* (3) *exist (possibly) as improper integrals. Assume in addition that*

(7) $$\mathrm{Re}(1,1) = \mathrm{Re} \int_0^\pi w\left(e^{i\theta}\right) d\theta \neq 0.$$

Then there exists a unique system of (monic, complex) orthogonal polynomials $\{\pi_k\}$ relative to the inner product (3) *of* 11.2. *Denoting by $\{p_k\}$ the (monic, real) orthogonal polynomials relative to the inner product* (4) *of* 11.2, *we have*

(8) $$\pi_n(z) = p_n(z) - i\,\theta_{n-1}\,p_{n-1}(z) \qquad (n = 0, 1, 2, \ldots),$$

where

(9) $$\theta_{n-1} = \frac{\mu_0\,p_n(0) + iq_n(0)}{i\,\mu_0\,p_{n-1}(0) - q_{n-1}(0)} \qquad (n = 0, 1, 2, \ldots).$$

Alternatively,

(10) $$\theta_n = ia_n + \frac{b_n}{\theta_{n-1}} \qquad (n = 0, 1, 2, \ldots); \quad \theta_{-1} = \mu_0,$$

where a_k, b_k *are the recursion coefficients in* (4) *and* $\mu_0 = (1, 1)$. *In particular, all* θ_n *are real* (*in fact, positive*) *if* $a_n = 0$ *for all* $n \geq 0$. *Finally,*

(11) $$(\pi_n, \pi_n) = \theta_{n-1}[p_{n-1}, p_{n-1}] \neq 0 \qquad (n = 1, 2, \ldots), \qquad (\pi_0, \pi_0) = \mu_0.$$

Proof. Assume first that the orthogonal polynomials $\{\pi_k\}$ exist. Putting

$$g(z) = \frac{1}{i}\pi_n(z)z^{k-1}, \qquad 1 \leq k < n$$

in (3) we find

$$0 = \int_{\Gamma} \pi_n(z)\,z^k\,(iz)^{-1}\,w(z)\,dz - i\int_{-1}^{1} \pi_n(x)\,x^{k-1}\,w(x)\,dx$$

$$= (\pi_n, z^k) - i\,[\pi_n, x^{k-1}] = -i\,[\pi_n, x^{k-1}] \qquad (1 \leqq k < n),$$

and hence, upon expanding π_n in the polynomials $\{p_k\}$,

(12) $$\pi_n(z) = p_n(z) - i\,\theta_{n-1}\,p_{n-1}(z) \qquad (n = 0, 1, 2, \ldots).$$

for some constants θ_{n-1}. To determine these constants, put

$$g(z) = [\pi_n(z) - \pi_n(0)]\,(iz)^{-1} = \frac{1}{i}\left\{\frac{p_n(z) - p_n(0)}{z} - i\,\theta_{n-1}\frac{p_{n-1}(z) - p_{n-1}(0)}{z}\right\}$$

in (3) and use the first expression for g to evaluate the first integral, and the second to evaluate the second integral in (3). This gives

(13) $$0 = (\pi_n, 1) - \pi_n(0)\,(1,1) + \frac{1}{i}\,[q_n(0) - i\,\theta_{n-1}\,q_{n-1}(0)] \qquad (n \geqq 1).$$

Since $(\pi_n, 1) = 0$, $(1, 1) = \mu_0$, and using (12) with $z = 0$, we get (9) for $n \geq 1$. Note that the denominator in (9) (and the numerator, for that matter) does not vanish, since Re $\mu_0 \neq 0$

by (7) and $p_k(0)$, $q_k(0)$ cannot vanish simultaneously, $\{p_k\}$ and $\{q_k\}$ being linearly independent solutions of (4). For $n = 0$, (9) yields, by virtue of (5), $\theta_{-1} = \mu_0$.

To show the first relation in (10), replace n by $n + 1$ in (9) and use (4) for $z = 0$, to obtain

$$\begin{aligned}\theta_{n-1} &= \frac{\mu_0 p_{n+1}(0) + iq_{n+1}(0)}{i\mu_0 p_n(0) - q_n(0)} \\ &= \frac{\mu_0[-a_n p_n(0) - b_n p_{n-1}(0)] + i[-a_n q_n(0) - b_n q_{n-1}(0)]}{i\mu_0 p_n(0) - q_n(0)} \\ &= \frac{ia_n[i\mu_0 p_n(0) - q_n(0)] - b_n[\mu_0 p_{n-1}(0) + iq_{n-1}(0)]}{i\mu_0 p_n(0) - q_n(0)} \\ &= ia_n + \frac{b_n}{\theta_{n-1}} \qquad (n \geqq 1).\end{aligned}$$

Using (9) with $n = 1$, (4) with $k = 0$ and (5), yields

$$\theta_0 = \frac{\mu_0(-a_0) + ib_0}{i\mu_0} = ia_0 + \frac{m_0}{\mu_0},$$

since $b_0 = m_0$ (see (6)). Therefore, (10) also holds for $n = 0$.

If all $a_n = 0$, then w is symmetric and we can prove that (see Theorem 1 in 11.6)

$$\mu_0 = (1,1) = \pi w(0) > 0.$$

Hence, using (10) we conclude that θ_n is real.

Conversely, defining π_n by (8) and (9), it follows readily from (3) that for $n \geq 2$,

$$(\pi_n, z^k) = \frac{1}{i}\int_\Gamma \pi_n(z) z^{k-1} w(z)\,dz = i\int_{-1}^{1} \pi_n(x) x^{k-1} w(x)\,dx = 0 \qquad (1 \leqq k < n).$$

and from (13), (9) and (8) for $z = n$ that $(\pi_n, 1) = 0$ for $n \geq 1$. Furthermore,

$$\begin{aligned}(\pi_n, \pi_n) &= \int_\Gamma \pi_n(z) z^n w(z)(iz)^{-1}\,dz \\ &= \frac{1}{i}\int_\Gamma \pi_n(z) z^{n-1} w(z)\,dz = i\int_{-1}^{1} \pi_n(x) x^{n-1} w(x)\,dx \\ &= i\int_{-1}^{1} [p_n(x) - i\theta_{n-1} p_{n-1}(x)]\, x^{n-1} w(x)\,dx \\ &= \theta_{n-1}\int_{-1}^{1} p_{n-1}^2(x) w(x)\,dx,\end{aligned}$$

proving (11).

We note from (8) that

$$(14)\qquad (p_n, 1) = i\theta_{n-1}(p_{n-1}, 1) = \cdots = \prod_{\nu=1}^{n} (i\theta_{\nu-1})(1, 1),$$

and, similarly,

$$(p_n, \pi_k) = i\theta_{n-1}(p_{n-1}, \pi_k), \qquad 1 \leqq k < n,$$

which, applied repeatedly, gives

$$(15)\qquad \begin{aligned} (p_n, \pi_k) &= \left(\prod_{\nu=k+1}^{n} i\theta_{\nu-1}\right)(p_k, \pi_k) \\ &= i^{-1}\left(\prod_{\nu=k}^{n} i\theta_{\nu-1}\right)[p_{k-1}, p_{k-1}], \qquad 1 \leqq k \leqq n. \end{aligned}$$

Here, (11) has been used in the last step. From (14) and (15) there follows

$$(16)\qquad p_n(z) = \sum_{k=0}^{n} \left(\prod_{\nu=k+1}^{n} i\theta_{\nu-1}\right)\pi_k(z),$$

the inversion of (8). When $k = n$, the empty product in (16) is to be interpreted as 1.

EXAMPLE 1. $w(z) = 1 + z$.

Here, $\mu_0 = (1, 1) = \pi + 2i$, Re $\mu_0 \neq 0$, so that the orthogonal polynomials $\{\pi_k\}$ exist. Furthermore, $b_0 = m_0 = 2$,

$$a_n = \frac{1}{(2n+1)(2n+3)} \quad (n \geq 0), \qquad b_n = \frac{n(n+1)}{(2n+1)^2} \quad (n \geq 1),$$

so that by (10)

$$\theta_0 = \frac{\pi - 4i}{3(2 - i\pi)}, \qquad \theta_1 = \frac{3\pi + 8i}{5(4 + i\pi)}, \ldots$$

by (4) and (5),

$$p_0(z) = 1, \quad p_1(z) = z - \frac{1}{3}, \quad p_2(z) = z^2 - \frac{2}{3}z - \frac{1}{5}, \quad \ldots,$$

and by (8),

$$\pi_0(z)=1,\quad \pi_1(z)=z-\frac{2}{2-i\pi},\qquad \pi_2=z^2-\frac{i\pi}{4+i\pi}z-\frac{4}{3(4+i\pi)},\ \dots .$$

EXAMPLE 2. $w(z) = z^2$.

Here

$$\mu_0=\int_0^\pi e^{2i\theta}\,d\theta=0$$

so that (7) is violated and thus the polynomials $\{\pi_k\}$ do not exist, even though $w(x) \geq 0$ on $[-1, 1]$ and the polynomials $\{p_k\}$ do exist. It is easily seen that $q_k(0) = 0$ when k is even, so that θ_{n-1} is zero for n even, and undefined for n odd. For an explanation of this example see Theorem 1 in 11.6.

11.4. Recurrence Relation

We assume that

$$\text{(1)}\qquad \operatorname{Re}(1,1)=\operatorname{Re}\int_0^\pi w\left(e^{i\theta}\right)d\theta\neq 0.$$

so that orthogonal polynomials $\{\pi_k\}$ exist. Since $(zf, g) = (f, zg)$, it is known that they must satisfy a three-term recurrence relation

$$\text{(2)}\qquad \begin{aligned}&\pi_{k+1}(z)=(z-ia_k)\,\pi_k(z)-\beta_k\,\pi_{k-1}(z),\qquad k=0,1,2,\dots,\\ &\pi_{-1}(z)=0,\qquad \pi_0(z)=1.\end{aligned}$$

Using (8) of 11.3 and (2), for $k \geq 1$ we get

$$p_{k+1}(z)-i\theta_k p_k(z)=(z-ia_k)\,[p_k(z)-i\theta_{k-1}p_{k-1}(z)]-\beta_k\,[p_{k-1}(z)-i\theta_{k-2}p_{k-2}(z)].$$

and substituting here for $zp_k(z)$ and $zp_{k-1}(z)$ the expressions obtained from the basic recurrence relation (4) of 11.3, yields

$$\begin{aligned}&[a_k+i\,(\theta_k-\theta_{k-1}-a_l]\,p_k(z)+[b_k-\beta_k-\theta_{k-1}\,(a_k+ia_{k-1})]\,p_{k-1}(z)\\ &\qquad\qquad +i\,[\beta_k\,\theta_{k-2}-b_{k-1}\,\theta_{k-1}]\,p_{k-2}(z)\equiv 0,\qquad k\geqq 1.\end{aligned}$$

By the linear independence of the polynomials $\{p_k\}$ we conclude that

$$\text{(3)}\qquad \begin{aligned}&a_k+i\,(\theta_k-\theta_{k-1}-a_k)=0,\qquad k\geqq 1,\\ &b_k-\beta_k-\theta_{k-1}\,(a_k+ia_{k-1})=0,\qquad k\geqq 1,\\ &\beta_k\,\theta_{k-2}-b_{k-1}\,\theta_{k-1}=0,\qquad k\geqq 2.\end{aligned}$$

From the last equality in (3) and (10) of 11.3, we get

$$(4)\qquad \beta_k = \frac{\theta_{k-1}}{\theta_{k-2}} b_{k-1} = \theta_{k-1}(\theta_{k-1} - ia_{k-1})$$

for $k > 2$. The first equality in (3) gives

$$(5)\qquad \alpha_k = \theta_k - \theta_{k-1} - ia_k, \qquad k \geqq 1.$$

To verify that (4) also holds for $k = 1$, it suffices to apply the second relation (3) for $k = 1$, in combination with (10) of 11.3 and (5) for $k = 1$. With α_k, β_k thus determined, the second equality in (3) is automatically satisfied, as follows easily from (10) of 11.3. Finally, from

$$\pi_1(z) = z - i\alpha_0 = p_1(z) - i\theta_0 = z - a_0 - i\theta_0,$$

we find

$$(6)\qquad \alpha_0 = \theta_0 - ia_0.$$

Alternatively, by (10) of 11.3 we may write (5) and (6) as

$$(7)\qquad \alpha_k = -\theta_{k-1} + \frac{b_k}{\theta_{k-1}}, \qquad k \geqq 1, \qquad \alpha_0 = \frac{b_0}{\theta_{-1}} = \frac{m_0}{\mu_0}.$$

We have therefore proved:

THEOREM 1. *Under the assumption* (1), *the* (*monic, complex*) *polynomials* $\{\pi_k\}$ *orthogonal with respect to the inner product* (3) *of* 11.2 *satisfy the recurrence relation* (2), *where the coefficients* α_k, β_k *are given by* (5) (*or* (7)) *and* (4), *respectively, with the* θ_n *defined in* (9) (*or* (10)) *of* 11.3.

By comparing the coefficients of z^k on the left and right of (2), we obtain from (5), (6) that

$$\pi_n(z) = z^n - \left(i\theta_{n-1} + \sum_{m=0}^{n-1} a_m\right) z^{n-1} + \cdots \qquad (n \geqq 1).$$

11.5. Jacobi Weight

We consider now the case of the Jacobi weight function

(1) $\quad w(z)=w^{\alpha,\beta}(z)=(1-z)^{\alpha}(1+z)^{\beta}, \qquad \alpha>-1,\ \beta>1,$

where fractional powers are understood in terms of their principal branches. We first obtain the existence of the corresponding orthogonal polynomials $\{\pi_k^{(\alpha,\beta)}\}$.

THEOREM 1. *We have*

(2) $$\mu_0=\mu_0^{(\alpha,\beta)}=\int_0^{\pi} w^{(\alpha,\beta)}(e^{i\theta})\,\mathrm{d}\theta=\pi+i\,\text{v. p.}\int_{-1}^{1}\frac{w^{(\alpha,\beta)}(x)}{x}\,\mathrm{d}x,$$

and hence $\operatorname{Re}\mu_0\neq 0$.

Proof. Let C_{ε}, $\varepsilon>0$, be the contour formed by ∂D_+, with a semicircle of radius r about the origin spared out. Then, by Cauchy's theorem

(3) $$0=\int_{\Gamma}\frac{wz}{iz}\,\mathrm{d}z+\left(\int_{-1}^{-\varepsilon}+\int_{\varepsilon}^{1}\right)\frac{wx}{iz}\,\mathrm{d}x+\int_{C_{\varepsilon}}\frac{wz}{iz}\,\mathrm{d}z,$$

where Γ and c_{ε} are the circular parts of C_{ε} (with radii 1 and ε, respectively). If $\varepsilon\to 0$ in (3) we get

$$0=\mu_0-i\,\text{v. p.}\int_{-1}^{1}\frac{w(x)}{x}\,\mathrm{d}x-\pi w(0),$$

which proves (2).

The following theorem is also proved in [9]:

THEOREM 2. *We have*

$$\pi_n^{(\beta,\alpha)}(z)=(-1)^n\,\overline{\pi}_n^{(\alpha,\beta)}(-z),$$

where $\overline{\pi}_n$ *denotes the polynomial* π_n *with all coefficients conjugated, i.e.* $\overline{\pi}_n(z)=\overline{\pi_n(\overline{z})}$.

11.6. Symmetric Weights and Gegenbauer Weights

THEOREM 1. *If the weight function* w, *in addition to the assumptions stated in* 11.3, *satisfies*

(1) $\quad w(-z)=w(z)$ and $w(0)>0$,

then

(2) $\mu_0 = (1, 1) = \pi w(0),$

and the system of orthogonal polynomials $\{\pi_n\}$ exists uniquely.

Proof. Proceeding as in the proof of Theorem 1 from 11.5, we find

$$0 = \mu_0 - i \text{ v.p.} \int_{-1}^{1} \frac{w(x)}{x} \mathrm{d}x - \pi w(0),$$

where the Cauchy principal value integral on the right vanishes because of symmetry of w. This proves (2).

Under the assumption (1) we have $a_k = 0$ for all $k \geq 0$ in (4) of 11.3. In this case the relations (6), (5) and (4) of 11.4 reduce to

$$\alpha_0 = \theta_0, \qquad \alpha_k = \theta_k - \theta_{k-1}, \qquad \beta_k = \theta_{k-1}^2, \qquad k \geqq 1.$$

Furthermore, by (10) of 11.3

$$\theta_0 = \frac{m_0}{\mu_0}, \qquad \theta_1 = \frac{b_1}{\theta_0},$$

$$\theta_{2m} = \theta_0 \frac{b_2 b_4 \cdots b_{2m}}{b_1 b_3 \cdots b_{2m-1}}, \qquad \theta_{2m+1} = \frac{1}{\theta_0} \frac{b_1 b_3 \cdots b_{2m+1}}{b_2 b_4 \cdots b_{2m}},$$

where $m = 1, 2, \ldots$

In particular, for the Gegenbauer weight

(3) $w(z) = (1 - z^2)^{\lambda - 1/2}, \qquad \lambda > -1/2,$

we have $p_k(z) = \hat{C}_k(z)$ – the monic Gegenbauer polynomials – for which as is well known,

$$b_0 = m_0 = \sqrt{\pi} \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda + 1)},$$

$$b_k = \frac{k(k + 2\lambda - 1)}{4(k + \lambda)(k + \lambda - 1)}, \qquad k \geqq 1.$$

Therefore, by (10) of 11.3,

$$\theta_n = \frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)} \frac{1}{\theta_{n-1}}, \quad n=1, 2, \ldots, \quad \theta_0 = \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\,\Gamma(\lambda+1)},$$

i.e.

$$\theta_n = \frac{1}{\lambda+n} \cdot \frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\lambda+\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\lambda+\frac{n}{2}\right)}, \quad n \geqq 1.$$

The corresponding orthogonal polynomials can be represented in the form

$$\pi_k(z) = \hat{C}_k^\lambda(z) - i\theta_{k-1}\hat{C}_{k-1}^\lambda(z)$$

and their norm is given by

$$\|\pi\| = \sqrt{\pi}\,(\theta_0\theta_1 \ldots \theta_{k-1}) = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\lambda+\frac{k}{2}\right)}{\Gamma(\lambda+k)}.$$

In particular, we have:

1° The Legendre case $\lambda = 1/2$:

$$\theta_k = \left(\frac{\Gamma((k+2)/2)}{\Gamma((k+1)/2)}\right)^2, \quad k \geqq 0;$$

2° The Čebyšev case $\lambda = 0$: $\theta_0 = 1$, $\theta_k = 1/2$, $k \geq 1$;

3° The Čebyšev case of the second kind $\lambda = 1$: $\theta_k = 1/2$, $k \geq 0$.

The Legendre case is considered in detail in [7], and Gegenbauer case, including various applications to numerical analysis, in [10].

11.7. The zeros of $\pi_n(z)$

It follows from (2) of 11.4 that the zeros of $\pi_n(z)$ are the eigenvalues of the (complex, tridiagonal) matrix

$$J_n = \left\| \begin{matrix} i\alpha_0 & 1 & & & O \\ \beta_1 & i\alpha_1 & 1 & & \\ & \beta_2 & i\alpha_2 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & \beta_{n-1} & i\alpha_{n-1} \end{matrix} \right\|.$$

The elements of J_n are easily computed using (6), (5), (4) of 11.4 and (10) of 11.3, since the recursion coefficients a_k, b_k for the orthogonal polynomials $\{p_k\}$ are known.

If the weight w is symmetric, then $\beta_k = \theta_{k-1}^2$ is positive, and J_n can be transformed into a real matrix. Indeed, a similarity transformation with the diagonal matrix

$$D_n = \operatorname{diag}(1,\, i\theta_0,\, i^2\theta_0\theta_1,\, i^3\theta_0\theta_1\theta_2, \ldots) \in \mathbf{C}^{n\times n},$$

transforms the complex matrix J_n into the real nonsymmetric tridiagonal matrix

$$A_n = -iD_n^{-1} J_n D_n = \left\|\begin{matrix} a_0 & \theta_0 & & & & O \\ -\theta_0 & a_0 & \theta_1 & & & \\ & -\theta_0 & a_2 & \ddots & & \\ & & \ddots & \ddots & \theta_{n-2} & \\ O & & & & -\theta_{n-2} & a_{n-1} \end{matrix}\right\|,$$

with eigenvalues $\eta_\nu = -i\zeta_\nu$. Using the EISPACK subroutine HQR (see [11]) we can evaluate all the eigenvalues η_ν $(\nu = 1, \ldots, n)$ of A_n, and then all the zeros $\zeta_\nu = i\eta_\nu$ $(\nu = 1, \ldots, n)$ of $\pi_n(z)$.

A theorem on the distribution of zeros of $\pi_n(z)$ in the case when the weight function w is symmetric, i.e.

$$w(-z) = w(z) \quad \text{and} \quad w(0) > 0$$

is proved in [9]. We shall quote here a particular, but important, result regarding the distribution of zeros of the polynomials $\pi_n^\lambda(z)$, orthogonal with respect to the complex Gegenbauer weight

$$w(z) = (1-z^2)^{-1/2}, \quad \lambda > -1/2.$$

THEOREM 1. *If $\lambda > -1/2$, all the zeros of $\pi_n^\lambda(z)$ are simple and for $n \geq 2$ they belong to the upper unit half disc $D_+ = \{z \in \mathbf{C} \mid |z| < 1,\ \operatorname{Im} z > 0\}$.*

REMARK 1. The problem of distribution of zeros for general weight functions is not solved. The case of Jacobi weights (see (1) of 11.5) is particularly interesting. Numerical experiments indicate that all the zeros belong to the half disc D_+.

REMARK 2. For the Gegenbauer case a second order linear differential equation is found in [9] whose particular solution is the polynomial $\pi_n^\lambda(z)$. Applications of those polynomials to numerical integration and numerical differentiation of analytic functions are given in [10].

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