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ON NUMERICAL EVALUATION OF DOUBLE INTEGRALS OF AN ANALYTIC FUNCTION OF TWO COMPLEX VARIABLES

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Abstract.

A seventh degree rule of the non-product type has been constructed for numerical evaluation of double integrals of an analytic function of two complex variables by choosing a set of 17 points from the set of 25 points needed in the product Birkhoff-Young rule of fifth degree. An asymptotic error estimate for this rule has been determined and the rule has been numerically tested.

1. Introduction.

Birkhoff and Young [3], Lether [7], and Tošić [9] have considered the problem of numerical evaluation of complex definite integrals of an analytic function of one complex variable. In our opinion, the problem of numerical approximation of complex multiple integrals has not received sufficient attention unlike the problem of numerical approximation of real multiple integrals for which exhaustive references have been given in Haber [6], Stroud [8], and Engels [5].

Recently Acharya and Das [1] and Das, Padhy and Acharya [4] have investigated the 25-point fifth degree product rule based on the Birkhoff-Young 5-point fifth degree rule for approximation of the complex double integral $I(f)$

$$I(f) = \int_{L_1}\int_{L_2} f(z^{(1)}, z^{(2)})dz^{(1)}dz^{(2)}, \tag{1.1}$$

where L_j is a directed line segment from the point $z_0^{(j)} - h_j$ to $z_0^{(j)} + h_j$ in the $z^{(j)}$-plane ($j = 1,2$) and f is analytic in the domain $\Omega_1 \times \Omega_2$, where

$$\Omega_j = \{z^{(j)} : |z^{(j)} - z_0^{(j)}| \leqq R_j, \qquad R_j > |h_j|\} \qquad (j = 1,2).$$

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Later for approximation of $I(f)$, Acharya and Das [2] have derived the 17-point fifth degree rule of the non-product type by discarding certain points from the set of 25 points meant for the product lay out based on the 5-point Birkhoff and Young rule [3].

Although problem (1.1) can be reduced to the approximation of two real double integrals over a quadratic region by introduction of two new variables $z^{(j)} = z_0^{(j)} + x_j h_j$, $x_j \in [-1, 1]$ $(j = 1, 2)$, we will solve it directly and simply using complex arithmetic by deriving a quadrature rule of seventh degree using only 17 points.

2. Formulation of the fifth degree rules.

Let k be a non-zero real parameter and $z_m^{(j)} = z_0^{(j)} + k i^{m-1} h_j$, where $i = \sqrt{(-1)}$, $j = 1, 2$ and $m = 1(1)4$. Then $z_3^{(j)}$ and $z_1^{(j)}$ are the end points of the paths of integration L_j $(j = 1, 2)$. Let us introduce the following sets of points:

$$\left.\begin{aligned} P_j &= \{z_0^{(j)}, z_1^{(j)}, z_3^{(j)}\} \\ Q_j &= \{z_0^{(j)}, z_2^{(j)}, z_4^{(j)}\} \\ S_j &= \{z_1^{(j)}, z_4^{(j)}\} \\ T_j &= \{z_2^{(j)}, z_4^{(j)}\} \\ Y_j &= \{z_0^{(j)}\} \end{aligned}\right\} \qquad (j = 1, 2).$$

We consider the following sets each of cardinality 13:

$$A = (P_1 \times P_2) \cup (Y_1 \times T_2) \cup (T_1 \times Y_2),$$
$$B = (Q_1 \times Q_2) \cup (Y_1 \times S_2) \cup (S_1 \times Y_2),$$

where $\times$ denotes Cartesian product. Finally $A \subset \Omega_1 \times \Omega_2$ and $B \subset \Omega_1 \times \Omega_2$ if $k \in (0, 1]$. Let $f(z_p^{(1)}, z_q^{(2)}) \equiv f_{pq}$ $(p, q = 0(1)4)$.

For constructing the 13-point rules of degree five, we take A and then B as the sets of interpolating points. With A as the set of interpolating points the following 13-point rule for approximation of $I(f)$ is proposed:

$$\begin{aligned} I(f) \cong Q_1(f) = h_1 h_2[&a_0 f_{00} + a_1(f_{10} + f_{01} + f_{03} + f_{30}) \\ &+ a_2(f_{20} + f_{02} + f_{04} + f_{40}) + a_3(f_{11} + f_{13} + f_{31} + f_{33})]. \end{aligned} \tag{2.1}$$

Similarly with B as the set of interpolating points the following 13-point rule for the numerical approximation of $I(f)$ is proposed:

$$\begin{aligned} I(f) \cong Q_2(f) = h_1 h_2[&b_0 f_{00} + b_1(f_{10} + f_{01} + f_{03} + f_{30}) \\ &+ b_2(f_{20} + f_{02} + f_{04} + f_{40}) + b_3(f_{22} + f_{24} + f_{42} + f_{44})]. \end{aligned} \tag{2.2}$$

Due to the symmetry in both rules it is noted that Q_1 and Q_2 are exact for monomials

$$f(z^{(1)}, z^{(2)}) = (z^{(1)} - z_0^{(1)})^\alpha (z^{(2)} - z_0^{(2)})^\beta \tag{2.3}$$

when at least one of α or β is an odd positive integer. The coefficients a and b are determined by setting $(\alpha, \beta) = (0,0)$, $(2,0)$, $(2,2)$ and $(4,0)$ in the formulas (2.2) and (2.3) and making both Q_1 and Q_2 exact for the corresponding monomials. This leads to the following systems of equations:

$$\left\{\begin{aligned} a_0 + 4a_1 + 4a_2 + 4a_3 &= 4 \\ a_1 - \ a_2 + 2a_3 &= 2/3k^2 \\ a_3 &= 1/9k^4 \\ a_1 + \ a_2 + 2a_3 &= 2/5k^4, \end{aligned}\right. \tag{2.4}$$

and

$$\left\{\begin{aligned} b_0 + 4b_1 + 4b_2 + 4b_3 &= 4 \\ b_1 - \ b_2 - 2b_3 &= 2/3k^2 \\ b_3 &= 1/9k^4 \\ b_1 + \ b_2 + 2b_3 &= 2/5k^4. \end{aligned}\right. \tag{2.5}$$

It may also be noted that the cases $(\alpha, \beta) = (0,2)$ and $(0,4)$ yield the same equations as the cases $(\alpha, \beta) = (2,0)$ and $(4,0)$ respectively. Solving the equations in (2.4) and (2.5) we get

$$a_0 = 4 - \frac{52}{45k^4}, \qquad a_1 = \frac{1}{3k^2} - \frac{1}{45k^4}, \qquad a_2 = \frac{1}{5k^4} - \frac{1}{3k^2}, \qquad a_3 = \frac{1}{9k^4}; \tag{2.6}$$

$$b_0 = 4 - \frac{52}{45k^4}, \quad b_1 = \frac{1}{3k^2} + \frac{1}{5k^4}, \qquad b_2 = -\frac{1}{45k^4} - \frac{1}{3k^2}, \qquad b_3 = \frac{1}{9k^4}. \tag{2.7}$$

Thus we have:

THEOREM 2.1. *Each of the* 13-*points rules* Q_1 *and* Q_2 *for the complex double integral* $I(f)$ *of the analytic function* f *in* $\Omega_1 \times \Omega_2$ *has degree five for any value* $k \in (0,1]$.

We next examine the error estimates $E_1(f)$ and $E_2(f)$ given as

$$I(f) = Q_1(f) + E_1(f) \quad \text{and} \tag{2.8}$$

$$I(f) = Q_2(f) + E_2(f) \tag{2.9}$$

associated with the rules Q_1 and Q_2 respectively, assuming f to be analytic in the domain $\Omega_1 \times \Omega_2$. Expanding f in Taylor series about $(z_0^{(1)}, z_0^{(2)}) \in \Omega_1 \times \Omega_2$ we obtain

$$f(z^{(1)}, z^{(2)}) = \sum_{\mu, \nu \geq 0} \frac{1}{\mu! \nu!} f^{\mu+\nu} (z^{(1)} - z_0^{(1)})^\mu (z^{(2)} - z_0^{(2)})^\nu, \quad \text{where} \tag{2.10}$$

$$f^{\mu+\nu} \doteq \frac{\partial^{\mu+\nu} f(z_0^{(1)}, z_0^{(2)})}{\partial^{\mu} z^{(1)} \partial^{\nu} z^{(2)}}. \tag{2.11}$$

Inserting (2.10) into (2.8) and (2.9) we get, in view of (2.6) and (2.7):

$$E_1(f) = h_1 h_2 \Bigg\{ \left(\frac{3-7k^4}{3780} (h_1^6 f^{6+0} + h_2^6 f^{0+6}) + \frac{3-5k^2}{540} (h_1^4 h_2^2 f^{4+2} + h_1^2 h_2^4 f^{2+4}) \right)$$

$$+ \left(\frac{5-9k^4}{453600} (h_1^8 f^{8+0} + h_2^8 f^{0+8}) + \frac{9k^4-25}{32400k^4} h_1^4 h_2^4 f^{4+4} \right.$$

$$\left. + \frac{3k^4-7}{12960k^4} (h_1^6 h_2^2 f^{6+2} + h_1^2 h_2^6 f^{2+6}) \right) + \ldots \Bigg\} \quad \text{and}$$

$$E_2(f) = h_1 h_2 \Bigg\{ \left(\frac{3-7k^2}{3780} (h_1^6 f^{6+0} + h_2^6 f^{0+6}) + \frac{3+5k^2}{540} (h_1^4 h_2^2 f^{4+2} + h_1^2 h_2^4 f^{2+4}) \right)$$

$$+ \left(\frac{5-9k^4}{453600} (h_1^8 f^{8+0} + h_2^8 f^{0+8}) + \frac{9k^4-25}{32400k^4} h_1^4 h_2^4 f^{4+4} \right.$$

$$\left. + \frac{3k^4-7}{12960k^4} (h_1^6 h_2^6 f^{6+2} + h_1^2 h_2^6 f^{2+6}) \right) + \ldots \Bigg\}.$$

It may be observed that by setting $k = \sqrt{0.6}$ in the rule Q_1 given by (2.1) it boils down to the 9-point product rule which can be obtained by multiplying together the 3-point Gauss-Legendre rules investigated by Lether [7] for the one dimensional contour integral of an analytic function of a single complex variable.

3. Generation of the seventh degree formula.

We now generate from Q_1 and Q_2 a seventh degree formula Q for approximation of the double integral $I(f)$.

If Q is a seventh degree formula involving a certain set of points C as nodes, where $C \subset (P_1 \cup Q_1) \times (P_2 \cup Q_2)$ for approximating $I(f)$, i.e.,

$$I(f) = Q(f) + E(f),$$

then the error E must not involve partial derivatives $f^{\mu+\nu}$ of order $\mu+\nu < 8$. To meet this requirement we first set $k^4 = 3/7$ in both Q_1 and Q_2 and then multiply (2.8) by $(3+5k^2)$ and (2.9) by $(3-5k^2)$ and subtract. As a result of this all sixth order partial derivatives are eliminated in $E(f)$ and we are led to

$$Q(f) = \left(\frac{3+5k^2}{10k^2} Q_1 - \frac{3-5k^2}{10k^2} Q_2\right) f \text{ and } E(f) = \left(\frac{3+5k^2}{10k^2} E_1 - \frac{3-5k^2}{10k^2} E_2\right) f,$$

where $k^4 = 3/7$. It is pertinent to note that the set of nodes C in formula Q is given by $C = A \cup B$ and $|C| = 17$. The rule Q and its error E are given by

$$\begin{aligned} Q(f) = \frac{h_1 h_2}{135} \Big\{ & 176 f_{00} + (28 + 8\sqrt{21})(f_{10} + f_{01} + f_{30} + f_{03}) \\ & + (28 - 8\sqrt{21})(f_{20} + f_{02} + f_{40} + f_{04}) \\ & + (\tfrac{35}{2} + \tfrac{7}{2}\sqrt{21})(f_{11} + f_{13} + f_{31} + f_{33}) + (\tfrac{35}{2} - \tfrac{7}{2}\sqrt{21})(f_{22} + f_{24} + f_{42} + f_{44}) \Big\} \end{aligned}$$

and

$$\begin{aligned} E(f) = h_1 h_2 \Big\{ & \frac{1}{396900} (h_1^8 f^{8+0} + h_2^8 f^{0+8}) \\ & - \frac{148}{675} h_1^4 h_2^4 f^{4+4} - \frac{1}{972} (h_1^6 h_2^2 f^{6+2} + h_1^2 h_2^6 f^{2+6}) + \dots \Big\}, \text{ where} \end{aligned} \tag{3.1}$$

$$f_{pq} = f(z_p^{(1)}, z_q^{(2)}) \qquad (p, q = 0(1)4),$$

$$z_m^{(j)} = z_0^{(j)} + (3/7)^{1/4} i^{m-1} h_j \qquad (m = 1(1)4, \qquad j = 1, 2)$$

and $f^{\mu+\nu}$ has been defined by (2.11). From the expansion (3.1) the following results are evident:

THEOREM 3.1. *The 17-point rule Q for the numerical approximation of the complex double integral $I(f)$ of an analytic function of two variables has degree seven.*

THEOREM 3.2. *If $h_2 = \lambda h_1$ and f is analytic in $\Omega_1 \times \Omega_2$, then $|E(f| \sim A|h_1|^{10}$, where*

$$A = |\lambda| \left| \frac{1}{396900} (f^{8+0} + \lambda^8 f^{0+8}) - \frac{\lambda^2}{27} \left\{ \frac{1}{36} (f^{6+2} + \lambda^4 f^{2+6}) + \frac{148}{25} \lambda^2 f^{4+4} \right\} \right|.$$

4. Numerical results.

For numerical verification of the 17-point seventh degree rule Q, defined by (3.1), we consider the following simple example

$$I = \int_{z^{(2)} = -i\zeta}^{i\zeta} \int_{z^{(1)} = -\zeta}^{\zeta} \exp(z^{(1)} + z^{(2)}) dz^{(1)} dz^{(2)} = 4i \sin\zeta \sinh\zeta$$

for $\zeta = 0.6(0.1)1.0$, i.e. for different magnitudes of h_1 and h_2. The results are compared by the rules Q' (the 25-point product Birkhoff and Young rule [1]) and Q'' (the 17-point fifth degree rule [2]). The table shows the results (numbers in parentheses denote decimal exponents). We note that the rule Q is most accurate.

ζ	Rule	Approximate value of I/i	Relative error
0.6	Q	1.43792690	1.99(−7)
	Q'	1.43792459	−1.41(−6)
	Q''	1.43793131	3.27(−6)
0.7	Q	1.95477350	6.90(−7)
	Q'	1.95476268	−4.85(−6)
	Q''	1.95479408	1.12(−5)
0.8	Q	2.54835808	2.02(−6)
	Q'	2.54831693	−1.41(−5)
	Q''	2.54843630	3.27(−5)
0.9	Q	3.21640939	5.19(−6)
	Q'	3.21627571	−3.64(−5)
	Q''	3.21666343	8.42(−5)
1.0	Q	3.95563869	1.21(−5)
	Q'	3.95525509	−8.49(−5)
	Q''	3.95636752	1.96(−4)

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