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KRONROD EXTENSIONS OF GAUSSIAN QUADRATURES WITH MULTIPLE NODES

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Abstract — In this paper, general real Kronrod extensions of Gaussian quadrature formulas with multiple nodes are introduced. A proof of their existence and uniqueness is given. In some cases, the explicit expressions of polynomials, whose zeros are the nodes of the considered quadratures, are determined. Very effective error bounds of the Gauss — Turán — Kronrod quadrature formulas, with Gori — Micchelli weight functions, for functions analytic on confocal ellipses, are derived.

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1. Introduction

Let w be an integrable weight function on interval (a, b). It is well-known that the Gauss — Turán quadrature formula with multiple nodes

$$\int_a^b f(t)w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + \bar{R}_{n,s}(f), \quad n \in \mathbb{N}, \quad s \in \mathbb{N}_0, \tag{1.1}$$

is exact for all algebraic polynomials of degree $2(s+1)n-1$ at most. Its nodes τ_ν are the zeros of the corresponding (monic) s-orthogonal polynomial $\pi_{n,s}(t)$ of degree n that minimizes the following integral

$$\varphi(a_0, a_1, \ldots, a_{n-1}) = \int_a^b \pi_n(t)^{2s+2} w(t)\,dt,$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. In order to minimize φ we must have

$$\int_a^b w(t)\pi_n(t)^{2s+1} t^k\,dt = 0, \quad k = 0, 1, \ldots, n-1, \tag{1.2}$$

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which are the corresponding orthogonality relations. For $s = 0$ we have a case of the standard orthogonal polynomials. For more details on Gauss — Turán quadratures see the book [7] and the survey paper [13].

Take now a sequence of nonnegative integers $\sigma = (s_1, s_2, \ldots)$. For any $n \in \mathbb{N}$ we denote the corresponding finite sequence $(s_1, s_2, \ldots, s_n)$ by σ_n and consider the generalization of the Gauss — Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities

$$\int_a^b f(t)w(t)\,dt = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + \bar{R}(f), \tag{1.3}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu = \tau_\nu^{(n,\sigma)}$ $(i = 0, 1, \ldots, 2s_\nu;\ \nu = 1, \ldots, n)$. Such formulas were derived independently by Chakalov and Popoviciu. A deep theoretical progress in this subject was made by Stancu (see [13] and references therein).

In this case, it is important to assume that the nodes $\tau_\nu\ (= \tau_\nu^{(n,\sigma)})$ are ordered, say

$$\tau_1 < \tau_2 < \cdots < \tau_n, \quad \tau_\nu \in [a, b], \tag{1.4}$$

with odd multiplicities $2s_1+1,\ 2s_2+1,\ \ldots,\ 2s_n+1$, respectively, in order to have uniqueness of the Chakalov-Popoviciu quadrature formula (1.3) (cf. Karlin and Pinkus [11]). Then this quadrature formula has the maximum degree of exactness $d_{\max} = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1$ if and only if

$$\int_a^b \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^k\, w(t)\,dt = 0, \quad k = 0, 1, \ldots, n-1. \tag{1.5}$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules was proved by Chakalov, Popoviciu, Morelli and Verna, and the existence and uniqueness subject to (1.4) was proved by Ghizzetti and Ossicini (see [13] for references), and also by Milovanović and Spalević [14].

Conditions (1.4) define the sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathbb{N}_0}$,

$$\pi_{n,\sigma}(t) = \prod_{\nu=1}^{n} (t - \tau_\nu^{(n,\sigma)}), \quad \tau_1^{(n,\sigma)} < \tau_2^{(n,\sigma)} < \cdots < \tau_n^{(n,\sigma)}, \quad \tau_\nu^{(n,\sigma)} \in [a, b],$$

such that

$$\int_a^b \pi_{k,\sigma}(t) \prod_{\nu=1}^{n} (t - \tau_\nu^{(n,\sigma)})^{2s_\nu+1}\, w(t)\,dt = 0, \quad k = 0, 1, \ldots, n-1.$$

These polynomials are called σ-orthogonal polynomials and correspond to the sequence $\sigma = (s_1, s_2, \ldots)$. We will often write simple τ_ν instead of $\tau_\nu^{(n,\sigma)}$. If we have $\sigma = (s, s, \ldots)$, the above polynomials reduce to the s-orthogonal polynomials.

Numerically stable methods for constructing nodes τ_ν and coefficients $A_{i,\nu}$ in Gauss — Turán and Chakalov — Popoviciu quadrature formulas with multiple nodes can be found in [5, 14, 19].

The generalized Chebyshev weight functions $w(t) = w_i(t)$:
(*a*) $w_1(t) = (1 - t^2)^{-1/2}$,
(*b*) $w_2(t) = (1 - t^2)^{1/2+s}$,
(*c*) $w_3(t) = (1 - t)^{-1/2}(1 + t)^{1/2+s}$,
(*d*) $w_4(t) = (1 - t)^{1/2+s}(1 + t)^{1/2}$
will be of interest in the following.

In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial (orthogonal with respect to $w_1(t)$) $T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\,dt, \quad k \geqslant 0.$$

This means that the Chebyshev polynomials T_n are s-orthogonal on $(-1,1)$ for each $s \geqslant 0$. Ossicini and Rosati [23] found three other weight functions $w_i(t)$ $(i=2,3,4)$ for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind U_n, V_n, and W_n, which are defined by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad V_n(t) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, \quad W_n(t) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)},$$

respectively (cf. Gautschi [4]; Gautschi and Notaris [6]), where $t = \cos\theta$. However, these weight functions depend on s (see (b), (c), (d)). It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that in the investigation it is sufficient to study only the first three weights $w_i(t)$, $i = 1,2,3$.

For each $n \in \mathbb{N}$, Gori and Micchelli [9] introduced an interesting class of weight functions defined on $[-1,1]$ for which explicit Gauss — Turán quadrature formulas of all orders can be found. In other words, these classes of weight functions have the peculiarity that the corresponding s-orthogonal polynomials, of the same degree, are independent of s. This class includes certain generalized Jacobi weight functions $w_{n,\mu}(t) = |U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^{\mu}$, where $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$ (Chebyshev polynomial of the second kind) and $\mu > -1$. In this case, the Chebyshev polynomials T_n appear as s-orthogonal polynomials.

This paper is organized as follows. In Section 2, general real Kronrod extensions of the quadratures with multiple nodes (1.3) (particularly (1.1)) are introduced. A proof of their existence and uniqueness is given. In some cases, the explicit expressions of the polynomials, whose zeros are the nodes of the considered quadratures, are obtained in Section 3. Finally, the very effective error bounds of the Gauss — Turán — Kronrod quadrature formulas, with Gori — Micchelli weight functions, for the functions analytic on confocal ellipses, are considered in Section 4.

2. General real Kronrod extensions of Chakalov — Popoviciu quadratures

Let $\sigma_m^* = (s_1^*, s_2^*, \ldots, s_m^*)$, $s_\mu^* \in \mathbb{N}_0$, $\mu = 1,2,\ldots,m$. Following the well-known idea of Kronrod [3] (see also [21, 22]), we extend formula (1.3) to the interpolatory quadrature formula

$$\int_a^b f(t)w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*} C_{j,\mu}^* f^{(j)}(\tau_\mu^*) + R_{n,m}(f), \tag{2.1}$$

where τ_ν denotes the same nodes as in (1.3), and the new nodes τ_μ^* and new weights $B_{i,\nu}, C_{j,\mu}^*$ are chosen to maximize the degree of exactness of (2.1) which is greater than or equal to

$$\sum_{\nu=1}^{n}(2s_\nu+1) + \sum_{\mu=1}^{m}(2s_\mu^*+1) + m - 1 = 2\biggl(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{m} s_\mu^*\biggr) + n + 2m - 1.$$

We call the quadrature formula (2.1) Chakalov — Popoviciu — Kronrod quadrature formula. A particular case of this formula is the Gauss — Turán — Kronrod quadrature formula, if $s_1 = s_2 = \cdots = s_n = s$.

The well-known Gauss — Kronrod quadrature formula, if $s_1 = s_2 = \cdots = s_n = 0$, $s_1^* = s_2^* = \cdots = s_m^* = 0$, and $m = n+1$, is a particular case of both above-mentioned quadrature formulas. In the theory of Gauss — Kronrod quadrature formulas, the Sieltjes polynomials $E_{n+1}(x)$, whose zeros are the nodes τ_μ^*, namely $E_{n+1}(x) \equiv E_{n+1}(x,w) := \prod_{\mu=1}^{n+1}(t-\tau_\mu^*)$, play an important role. Also, of foremost interest are weight functions for which the Gauss — Kronrod quadrature formula has the property that:

(i) All $n+1$ nodes τ_μ^* are in (a,b) and are simple (i.e., that all zeros of the Stieltjes polynomial $E_{n+1}(x)$ are in (a,b) and are simple).

Also, desirable are weight functions which have in addition to (i) the following properties:

(ii) The *interlacing property*. Namely, the nodes τ_μ^* and τ_ν separate each other (i.e., the $n+1$ zeros of $E_{n+1}(x)$ separate the n zeros of the orthogonal polynomial $\widetilde{P}_n(x) := \prod_{\nu=1}^{n}(t-\tau_\nu)$); and

(iii) all quadrature weights are positive.

On the basis of the above facts, it seems it most natural to consider the Chakalov — Popoviciu — Kronrod quadratures (2.1) in which $m = n+1$, i.e.,

$$\int_a^b f(t)w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*} C_{j,\mu}^* f^{(j)}(\tau_\mu^*) + R_n(f). \tag{2.2}$$

We know that in the general case of quadratures with multiple nodes not all quadrature weights have to be positive. Therefore, for Kronrod extensions of Gaussian quadrature formulas with multiple nodes we cannot consider the property (iii) as desirable.

On the other hand, it is desirable that the nodes τ_μ^*, $\mu = 1,\ldots,n+1$ be all real and ordered, say

$$\tau_1^* < \tau_2^* < \cdots < \tau_{n+1}^*, \quad \tau_\mu^* \in \mathbb{R}, \tag{2.3}$$

as well as satisfy the interlacing property, i.e.,

$$\tau_1^* < \tau_1 < \tau_2^* < \tau_2 < \cdots < \tau_n < \tau_{n+1}^*. \tag{2.4}$$

In the following, we are interested in the Chakalov — Popoviciu — Kronrod quadratures (2.2) in which the nodes τ_μ^*, $\mu = 1,\ldots,n+1$ satisfy property (2.3).

Proposition 2.1. *Let* (1.4), (2.3) *hold. Then the interpoatory quadrature formula* (2.2) *with multiple nodes has the degree of exactness* $2(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^*) + 3n + 1$ *if and only if the orthogonality conditions*

$$\int_a^b \prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1} t^k\, w(t)\,dt = 0, \quad k = 0,1,\ldots,n, \tag{2.5}$$

hold.

Proof. Let $\pi_n(t) \equiv \pi_{n,\sigma}(t) = \prod_{\nu=1}^{n}(t-\tau_\nu)$ be the σ-orthogonal polynomial based on the nodes τ_ν, and $E_{n+1}^{(\sigma^*)}(t) = \prod_{\mu=1}^{n+1}(t-\tau_\mu^*)$ be the corresponding generalized Stieltjes polynomial based on the nodes τ_μ^*. Conditions (2.5) can be reinterpreted in the form

$$\int_a^b E_{n+1}^{(\sigma^*)}(t)\,\pi_{n,\sigma}(t)\,t^k\,\widetilde{w}(t)\,dt = 0, \quad k = 0,1,\ldots,n, \tag{2.6}$$

where $\widetilde{w}(t) = w(t)\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu}\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*}$ is the new implicitly given weight function (see Engels [2, pp. 214–226]). Therefore, because of (2.6), the generalized (monic) Stieltjes polynomial $E_{n+1}^{(\sigma^*)}(t)$ is uniquely determined (cf. [21, p. 145]).

Suppose now that the quadrature formula (2.2) has the degree of exactness $2(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^*) + 3n + 1$, and let

$$f(t) = \prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1}\,t^k, \quad k \in \{0,1,\ldots,n\}.$$

Then, we have that

$$\deg(f) \leqslant \sum_{\nu=1}^{n}(2s_\nu+1) + \sum_{\mu=1}^{n+1}(2s_\mu^*+1) + n = 2\Bigg(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^*\Bigg) + 3n + 1.$$

Let us determine, for instance, $f^{(i)}(\tau_1),\ i = 0,1,\ldots,2s_1$. Using the following representation $f(t) = (t-\tau_1)^{2s_1+1}\,u_k(t)$, where

$$u_k(t) = \prod_{\nu=2}^{n}(t-\tau_\nu)^{2s_\nu+1}\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1}\,t^k, \quad k \in \{0,1,\ldots,n\},$$

and the Leibnitz formula, we have

$$f^{(i)}(t) = \sum_{\ell=0}^{i}\binom{i}{\ell}u_k^{(i-\ell)}(t)((t-\tau_1)^{2s_1+1})^{(\ell)}, \quad i = 0,1,\ldots,2s_1.$$

Therefore, $f^{(i)}(\tau_1) = 0$, $i = 0,1,\ldots,2s_1$.

Likewise, we conclude that

$$f^{(i)}(\tau_\nu) = 0; \quad i = 0,1,\ldots,2s_\nu, \quad \nu = 2,\ldots,n,$$

$$f^{(i)}(\tau_\mu^*) = 0; \quad i = 0,1,\ldots,2s_\mu^*, \quad \mu = 1,\ldots,n+1.$$

Using these facts, for the given function $f(t)$ we have that

$$\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*} C_{j,\mu}^* f^{(j)}(\tau_\mu^*) = 0.$$

Because of the latter and since $R_n(f) = 0$, from (2.2) we conclude that (2.5) holds.

Now, let the orthogonality conditions (2.5) hold. Consider an arbitrary polynomial $g(t)$ of degree $\leqslant 2(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^*) + 3n + 1$, which can be represented in the form

$$g(t) = \prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1}\,u_n(t) + v_q(t),$$

where $u_n(t), v_q(t)$ are polynomials of degrees n, q, respectively, and $q = 2(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^* + n)$. As an interpolatory quadrature formula (2.2) is exact for each polynomial of degree $\leqslant 2(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^* + n)$, i.e., $R_n(v_q) = 0$.

Therefore, because of the last facts and (2.5),

$$\int_a^b g(t)\,w(t)\,dt = \int_a^b \prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\prod_{\mu=1}^{n}(t-\tau_\mu^*)^{2s_\mu^*+1}\,u_n(t)\,w(t)\,dt + \int_a^b v_q(t)w(t)\,dt =$$

$$\int_a^b v_q(t)w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} B_{i,\nu}v_q^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*} C_{j,\mu}^* v_q^{(j)}(\tau_\mu^*) + R_n(v_q) =$$

$$\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} B_{i,\nu}v_q^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*} C_{j,\mu}^* v_q^{(j)}(\tau_\mu^*).$$

Because of

$$g^{(i)}(\tau_\nu) = v_q^{(i)}(\tau_\nu); \quad \nu = 1,\dots,n, \quad i = 0,\dots,2s_\nu,$$

$$g^{(j)}(\tau_\mu^*) = v_q^{(j)}(\tau_\mu^*); \quad \mu = 1,\dots,n+1, \quad j = 0,\dots,2s_\mu^*,$$

the previous formula reduces to

$$\int_a^b g(t)\,w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} B_{i,\nu}g^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*} C_{j,\mu}^* g^{(j)}(\tau_\mu^*),$$

which means that the quadrature formula (2.2) has the degree of exactness $2(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{n+1} s_\mu^*) + 3n + 1$. □

Remark 2.1. The above results can also be obtained for a more general case of the quadrature formula (2.2). Namely, for each node $\tau_\mu^* \notin (a,b)$ instead of the multiplicity $2s_\mu^*+1$ the multiplicity n_μ^*+1 ($n_\mu^* \in \mathbb{N}_0$) can be considered, since the factor $(t-\tau_\mu^*)^{n_\mu^*+1}$ holds the same sign on $(-1,1)$. Because of that, the function $\widetilde{\omega}(t)$ remains to be a weight function.

Remark 2.2. For the numerically stable method of calculating the weight coefficients in (2.2) see [19].

3. Explicit expressions of the generalized Stieltjes polynomials

We now study in detail some cases of the quadrature formula (2.2) for a subclass of Gori — Micchelli weight functions, as well as for the generalized Chebyshev weights.

First, consider the subclass of Gori — Micchelli weight functions

$$\omega_{n,\ell}(t) = \left[\frac{U_{n-1}(t)}{n}\right]^{2\ell}(1-t^2)^{\ell-1/2}, \quad \ell \in \{0,1,\dots,s\}. \tag{3.1}$$

In the particular case that $\ell = 0$, (3.1) reduces to the Chebyshev weight function of the first kind $\omega_{n,0}(t) = (1-t^2)^{-1/2}$.

Recall that for the weight functions (3.1) the Chebyshev polynomials of the first kind $T_n(t)$ appear to be s-orthogonal. In this case ($\sigma_n = (s,s,\dots,s)$), the orthogonality conditions (2.5) have the form

$$\int_{-1}^{1} [T_n(t)]^{2s+1}\prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1}\,t^k\,U_{n-1}^{2\ell}(t)(1-t^2)^{\ell-1/2}\,dt = 0, \quad k = 0,1,\dots,n. \tag{3.2}$$

If $\alpha \prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1} = [U_{n-1}(t)]^{2s+1-2\ell}(t^2-1)^{s+1-\ell}$, where α is a normalization constant, then conditions (3.2) take the form

$$\int_{-1}^{1} [U_{n-1}(t)T_n(t)]^{2s+1}\, t^k\, (1-t^2)^{1/2+s}\, dt = 0, \quad k = 0,1,\dots,n,$$

i.e., subject to $2T_n(t)U_{n-1}(t) = U_{2n-1}(t)$ (cf. [21, p. 143, eq. (21)]),

$$\int_{-1}^{1} [U_{2n-1}(t)]^{2s+1}\, t^k\, (1-t^2)^{1/2+s}\, dt = 0, \quad k = 0,1,\dots,n.$$

In fact, the last conditions hold for $k = 0,1,\dots,2n-2$ (see [23]), which means that it has to be $2n-2 \geqslant n$, i.e., $n \geqslant 2$.

Therefore, in this case ($n \geqslant 2$, $\sigma_{n+1}^* = ((s-\ell)/2, s-\ell, \dots, s-\ell, (s-\ell)/2)$), when the quadrature formula (2.2) has the form ($\tau_1^* = -1, \tau_{n+1}^* = 1$)

$$\int_{-1}^{1} f(t)\, \omega_{n,\ell}(t)\, dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=2}^{n}\sum_{j=0}^{2(s-\ell)} C_{j,\mu}^* f^{(j)}(\tau_\mu^*) + \sum_{j=0}^{s-\ell}(C_{j,1}^* f^{(j)}(-1) + C_{j,n+1}^* f^{(j)}(1)) + R_n(f), \tag{3.4}$$

we have just proved the following statement.

Theorem 3.1. *In the Kronrod extension* (3.4) *of the Gauss — Turán quadrature formula* (1.1) *with the weight function* (3.1), *and for* $n \geqslant 2$, *the corresponding generalized Stieltjes polynomial* $E_{n+1}^{(\sigma^*)}(t)$ ($\sigma_{n+1}^* = ((s-\ell)/2, s-\ell, \dots, s-\ell, (s-\ell)/2)$) *is given by*

$$E_{n+1}^{(\sigma^*)}(t) \equiv (t^2-1)\, U_{n-1}(t),$$

i.e., the nodes τ_μ^*, $\mu = 2,\dots,n$ *are the zeros of* $U_{n-1}(t)$ (*Chebyshev polynomial of the second kind of degree* $n-1$), *and* $\tau_1^* = -1, \tau_{n+1}^* = 1$.

The zeros of $T_n(t)$ and $E_{n+1}^{(\sigma^*)}(t)$ interlace (i.e., satisfy property (2.4)), since

$$2(t^2-1)U_{n-1}(t) = 2(t^2-1)T_n'(t)/n$$

(cf. [24, p. 180, Lemma 1]).

Consider now the generalized Chebyshev weight functions of the second, third and fourth kind.

Let $\sigma_n = (s,\dots,s)$ and $w(t) \equiv w_2(t) = (1-t^2)^{1/2+s}$. In this case, the orthogonality conditions (2.5) reduce to

$$\int_{-1}^{1} [U_n(t)]^{2s+1} \prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1}\, t^k\, (1-t^2)^{1/2+s}\, dt = 0, \quad k = 0,1,\dots,n. \tag{3.5}$$

If $\beta \prod_{\mu=1}^{n+1}(t-\tau_\mu^*)^{2s_\mu^*+1} = T_{n+1}^{2s+1}(t)$, where β is a normalization constant and $\sigma_{n+1}^* = (s,s,\dots,s)$, then conditions (3.5) take the form, since $T_{n+1}(t)U_n(t) = U_{2n+1}(t)/2$,

$$\int_{-1}^{1} [U_{2n+1}(t)]^{2s+1}\, t^k\, (1-t^2)^{1/2+s}\, dt = 0, \quad k = 0,1,\dots,n.$$

In fact, the last conditions hold for $k = 0, 1, \ldots, 2n$.

Therefore, in this case, when the quadrature formula (2.2) has the form

$$\int_{-1}^{1} f(t)\,(1-t^2)^{1/2+s}\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s} C_{j,\mu}^{*} f^{(j)}(\tau_\mu^{*}) + R_n(f), \tag{3.6}$$

we have just proved the following statement.

Theorem 3.2. *In the Kronrod extension* (3.6) *of the Gauss — Turán quadrature formula* (1.1) *with the weight function* $w_2(t) = (1-t^2)^{1/2+s}$ *the corresponding generalized Stieltjes polynomial* $E_{n+1}^{(\sigma^*)}(t)$ $(\sigma_{n+1}^{*} = (s, s, \ldots, s))$ *is given by*

$$E_{n+1}^{(\sigma^*)}(t) \equiv T_{n+1}(t),$$

i.e., the nodes τ_μ^{*}, $\mu = 1, \ldots, n+1$, *are the zeros of* $T_{n+1}(t)$ (*Chebyshev polynomial of the first kind of degree* $n+1$).

It is obvious that in this case the interlacing property (2.4) holds, since it holds for the polynomials $U_n(t), T_{n+1}(t)$.

Further, let $\sigma_n = (s, \ldots, s)$ and $w(t) \equiv w_3(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$. The orthogonality conditions (2.5) reduce in this case to

$$\int_{-1}^{1} [P_n^{(1/2,-1/2)}(t)]^{2s+1} \prod_{\mu=1}^{n+1} (t-\tau_\mu^{*})^{2s_\mu^{*}+1}\, t^k\,(1-t)^{1/2+s}(1+t)^{-1/2}\,dt = 0, \quad k = 0, 1, \ldots, n, \tag{3.7}$$

where $P_n^{(1/2,-1/2)}(t)$ is the Jacobi polynomial orthogonal on $(-1, 1)$ with respect to the weight function $(1-t)^{1/2}(1+t)^{-1/2}$ (see [23]).

If $\gamma \prod_{\mu=1}^{n+1}(t-\tau_\mu^{*})^{2s_\mu^{*}+1} = (1+t)^{s+1}[P_n^{(-1/2,1/2)}(t)]^{2s+1}$, where γ is a normalization constant and $\sigma_{n+1}^{*} = (s/2, s, \ldots, s)$, then conditions (3.7) reduce to the form

$$\int_{-1}^{1} [U_{2n}(t)]^{2s+1}\, t^k\,(1-t^2)^{1/2+s}\,dt = 0, \quad k = 0, 1, \ldots, n,$$

since $P_n^{(1/2,-1/2)}(t)P_n^{(-1/2,1/2)}(t) = \text{const} \cdot U_{2n}(t)$ (cf. [21, p. 147, eq. (33)]).

In fact, these conditions hold for $k = 0, 1, \ldots, 2n-1$.

Therefore, in this case, when the quadrature formula (2.2) has the form $(\tau_1^{*} = -1)$

$$\int_{-1}^{1} f(t)\,(1-t)^{1/2+s}(1+t)^{-1/2}\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=2}^{n+1}\sum_{j=0}^{2s} C_{j,\mu}^{*} f^{(j)}(\tau_\mu^{*}) + \sum_{j=0}^{s} C_{j,1}^{*} f^{(j)}(-1) + R_n(f), \tag{3.8}$$

we have just proved the following statement:

Theorem 3.3. *In the Kronrod extension* (3.8) *of the Gauss — Turán quadrature formula* (1.1) *with the weight function* $w_3(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$ *the corresponding generalized Stieltjes polynomial* $E_{n+1}^{(\sigma^*)}(t)$ $(\sigma_{n+1}^{*} = (s/2, s, \ldots, s))$ *is given by*

$$E_{n+1}^{(\sigma^*)}(t) \equiv (t+1)P_n^{(-1/2,1/2)}(t),$$

i.e., the nodes τ_μ^{*}, $\mu = 2, \ldots, n+1$, *are the zeros of* $P_n^{(-1/2,1/2)}(t)$ (*the Chebyshev polynomial of the fourth kind of degree* n), *and* $\tau_1^{*} = -1$.

It is obvious that for this case the interlacing property (2.4) holds, since it holds for the polynomials $P_n^{(1/2,-1/2)}(t)$ and $(t+1)P_n^{(-1/2,1/2)}(t)$.

When the quadrature formula (2.2) has the form ($\tau^*_{n+1}=1$)

$$\int_{-1}^{1} f(t)\,\omega_4(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n}\sum_{j=0}^{2s} C^*_{j,\mu} f^{(j)}(\tau^*_\mu) + \sum_{j=0}^{s} C^*_{j,n+1} f^{(j)}(1) + R_n(f), \tag{3.9}$$

where $\omega_4(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$ is a Chebyshev weight of the fourth kind as in the previous case the following statement can be proved.

Theorem 3.4. *In the Kronrod extension* (3.9) *of the Gauss — Turán quadrature formula* (1.1) *with the weight function* $w_4(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$ *the corresponding generalized Stieltjes polynomial* $E_{n+1}^{(\sigma^*)}(t)$ ($\sigma^*_{n+1} = (s,\ldots,s,s/2)$) *is given by*

$$E_{n+1}^{(\sigma^*)}(t) \equiv (t-1)P_n^{(1/2,-1/2)}(t),$$

i.e., the nodes τ^*_μ, $\mu = 1,\ldots,n$, *are the zeros of* $P_n^{(1/2,-1/2)}(t)$ (*Chebyshev polynomial of the third kind of degree* n), *and* $\tau^*_{n+1} = 1$.

4. Error bounds of Gauss — Turán — Kronrod quadratures with Gori — Micchelli weight functions for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and $\mathcal{D}$ be its interior. Suppose that f is an analytic function in $\mathcal{D}$ and continuous on $\bar{\mathcal{D}}$. Taking any system of m distinct points $\{\xi_1,\ldots,\xi_m\}$ in $\mathcal{D}$ and m nonnegative integers $n_1,\ldots,n_m$, the error in the Hermite interpolating polynomial of f at the point t ($\in \mathcal{D}$) can be expressed in the form (see, e. g., Gončarov [8, Chapter 5])

$$r_m(f;t) = f(t) - \sum_{\nu=1}^{m}\sum_{i=0}^{n_\nu-1} \ell_{i,\nu}(t) f^{(i)}(\xi_\nu) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)\Omega_m(t)}{(z-t)\Omega_m(z)}\,dz, \tag{4.1}$$

where $\ell_{i,\nu}(t)$ denote the fundamental functions of Hermite interpolation and

$$\Omega_m(z) = \prod_{\nu=1}^{m}(z-\xi_\nu)^{n_\nu}. \tag{4.2}$$

Multiplying (4.1) by the weight function $w(t)$ and integrating in t over $(-1,1)$, we get a contour integral representation of the remainder term $R_m(f)$ in the quadrature formula with multiple nodes,

$$R_m(f) = I(f;w) - \sum_{\nu=1}^{m}\sum_{i=0}^{n_\nu-1} A_{i,\nu} f^{(i)}(\xi_\nu) = \frac{1}{2\pi i}\oint_\Gamma K_m(z,w) f(z)\,dz, \tag{4.3}$$

where $A_{i,\nu} = \int_{-1}^{1} \ell_{i,\nu}(t)w(t)\,dt$ and the kernel $K_m(z) = K_m(z,w)$ is given by

$$K_m(z,w) = \frac{\rho_m(z;w)}{\Omega_m(z)},\ \rho_m(z;w) = \int_{-1}^{1} \frac{\Omega_m(t)}{z-t}\,w(t)\,dt, \quad z\in\mathbb{C}\setminus[-1,1]. \tag{4.4}$$

The integral representation (4.3) leads directly to the error estimate

$$|R_m(f)| \leqslant \frac{\bar{\ell}(\Gamma)}{2\pi}\Big(\max_{z\in\Gamma}|K_m(z)|\Big)\Big(\max_{z\in\Gamma}|f(z)|\Big), \tag{4.5}$$

where $\bar{\ell}(\Gamma)$ is the length of the contour Γ.

A general estimate can be obtained by Hölder's inequality. Thus,

$$|R_m(f)| = \frac{1}{2\pi}\Big|\oint_\Gamma K_m(z)f(z)\,dz\Big| \leqslant \frac{1}{2\pi}\Big(\oint_\Gamma |K_m(z)|^p|dz|\Big)^{1/p}\Big(\oint_\Gamma |f(z)|^q|dz|\Big)^{1/q},$$

i.e.,

$$|R_m(f)| \leqslant \frac{1}{2\pi}\,\|K_m\|_p\|\,f\|_q, \tag{4.6}$$

where $1 \leqslant p \leqslant +\infty,\ 1/p + 1/q = 1$, and

$$\|\,f\|_p := \begin{cases} \Big(\oint_\Gamma |f(z)|^p|dz|\Big)^{1/p}, & 1 \leqslant p < +\infty, \\ \max_{z\in\Gamma}|f(z)|, & p = +\infty. \end{cases}$$

The case $p = +\infty$ $(q = 1)$ gives

$$|R_m(f)| \leqslant \frac{1}{2\pi}\Big(\max_{z\in\Gamma}|K_m(z)|\Big)\Big(\oint_\Gamma |f(z)|\,|dz|\Big) \leqslant \frac{\bar{\ell}(\Gamma)}{2\pi}\Big(\max_{z\in\Gamma}|K_m(z)|\Big)\Big(\max_{z\in\Gamma}|f(z)|\Big),$$

i.e., (4.5). On the other hand, for $p = 1$ $(q = +\infty)$ estimate (4.6) reduces to

$$|R_m(f)| \leqslant \frac{1}{2\pi}\Big(\oint_\Gamma |K_m(z)|\,|dz|\Big)\Big(\max_{z\in\Gamma}|f(z)|\Big), \tag{4.7}$$

which is evidently stronger than (4.5) because of the inequality

$$\oint_\Gamma |K_m(z)|\,|dz| \leqslant \bar{\ell}(\Gamma)\Big(\max_{z\in\Gamma}|K_m(z)|\Big).$$

To obtain estimate (4.5) or (4.7), it is necessary to study the magnitude of $|K_m(z)|$ on Γ or the quantity

$$L_m(\Gamma) := \frac{1}{2\pi}\oint_\Gamma |K_m(z)|\,|dz|, \tag{4.8}$$

respectively.

Taking the contour Γ as a confocal ellipse with foci at the points ∓ 1 and the sum of semi-axes $\varrho > 1$,

$$E_\varrho = \Big\{z \in \mathbb{C}:\ z = \frac{1}{2}(\varrho e^{i\theta} + \varrho^{-1}e^{-i\theta}),\ 0 \leqslant \theta \leqslant 2\pi\Big\}, \tag{4.9}$$

and w as one of the four generalized Chebyshev weight functions $(a)-(d)$, we studied in detail for the Gauss — Turán quadrature formulae ($n_\nu = 2s+1,\ \nu = 1,2,\dots,n;\ s \in \mathbb{N}_0$) estimates (4.5) (see [15], [20]) and (4.7) (see [17] and references therein), respectively.

Since $z = (\xi + \xi^{-1})/2$, $\xi = \varrho e^{i\theta}$, and $|dz| = 2^{-1/2}\sqrt{a_2 - \cos 2\theta}\, d\theta$, where we put

$$a_j = a_j(\varrho) = (\varrho^j + \varrho^{-j})/2, \quad j \in \mathbb{N},\ \varrho > 1, \tag{4.10}$$

(4.8) reduces to

$$L_m(E_\varrho) := \frac{1}{2\pi\sqrt{2}} \int_0^{2\pi} \frac{|\rho_m(z;w)|(a_2 - \cos 2\theta)^{1/2}}{|\Omega_m(z)|}\, d\theta. \tag{4.11}$$

This integral can be evaluated numerically by using a quadrature formula. However, if $w(t)$ is one of the four weight functions $(a)-(d)$, we obtain explicit expressions for $L_m(E_\varrho)$ or for their bounds (see [17]) in the case of Gauss — Turán quadrature formulae.

In the following, for practical reasons, we will take Ω_m to be given in the form (4.2) or to be the same expression multiplied by some number ($\neq 0$), since $K_m(z,w)$ in both cases has the same value (cf. (4.4)).

In the rest of this section, for the Gori — Micchelli weight functions (3.1) in the case of the Gauss — Turán — Kronrod quadratures of the type of (3.4) we succeeded to find explicit expressions for $L_m(E_\varrho) \equiv L_{2n+1}(E_\varrho)$ or for their bounds.

First, we have

$$\rho_{2n+1}(z;w) = \int_{-1}^{1} \left[\frac{U_{n-1}(t)}{n}\right]^{2\ell} (1-t^2)^{\ell-1/2} \frac{[T_n(t)]^{2s+1}[U_{n-1}(t)]^{2s+1-2\ell}(1-t^2)^{s+1-\ell}}{z-t}\, dt =$$

$$\frac{1}{n^{2\ell}} \int_{-1}^{1} (1-t^2)^{s+1/2} \frac{[T_n(t)U_{n-1}(t)]^{2s+1}}{z-t}\, dt = \frac{1}{n^{2\ell}\, 2^{2s+1}} \int_{-1}^{1} (1-t^2)^{s+1/2} \frac{[U_{2n-1}(t)]^{2s+1}}{z-t}\, dt.$$

By substituting $t = \cos\theta$, we obtain (see [15, pp. 1863–1864])

$$\rho_{2n+1}(z;w) = \frac{(-1)^s \pi}{n^{2\ell}\, 2^{4s+1}\, \xi^{2n}} \sum_{\nu=0}^{s} (-1)^\nu \binom{2s+1}{\nu} \frac{1}{\xi^{2(s-\nu)\,2n}}.$$

For this case, we have

$$\Omega_m(z) \equiv \Omega_{2n+1}(z) := (1-z^2)^{s+1-\ell}[T_n(z)]^{2s+1}\,[U_{n-1}(z)]^{2s+1-2\ell} =$$

$$2^{-2s-1}(1-z^2)^{s+1-\ell}[U_{2n-1}(z)]^{2s+1}\,[U_{n-1}(z)]^{-2\ell}.$$

In the following, we will use the known facts

$$1 - z^2 = 1 - \frac{1}{4}(\xi + \xi^{-1})^2 = -\frac{1}{4}(\xi - \xi^{-1})^2, \quad |1-z^2| = \frac{1}{2}(a_2 - \cos 2\theta)^{1/2},$$

$$|T_n(z)| = \frac{1}{\sqrt{2}}(a_{2n} + \cos 2n\theta)^{1/2}, \quad |U_{n-1}(z)| = \left(\frac{a_{2n} - \cos 2n\theta}{a_2 - \cos 2\theta}\right)^{1/2}.$$

On the basis of the last facts, (4.11) reduces to

$$L_{2n+1}(E_\varrho) := \frac{1}{2\pi\sqrt{2}} \int_0^{2\pi} \frac{\dfrac{\pi(a_2-\cos 2\theta)^{1/2}}{\varrho^{2n}\, n^{2\ell}\, 2^{4s+1}} \left| \sum_{\nu=0}^{s} (-1)^\nu \binom{2s+1}{\nu} \frac{1}{\xi^{2(s-\nu)\,2n}} \right| d\theta}{\dfrac{(a_2-\cos 2\theta)^{s+1-\ell}}{2^{3s+2-\ell}} \left(\dfrac{a_{4n}-\cos 4n\theta}{a_2-\cos 2\theta}\right)^{s+1/2} \left(\dfrac{a_{2n}-\cos 2n\theta}{a_2-\cos 2\theta}\right)^{-\ell}}.$$

Putting in order the last expression, we obtain

$$L_{2n+1}(E_\varrho) = \frac{1}{2^{s+\ell+1/2}\, n^{2\ell}\, \varrho^{2n(s+1)}} \int_0^{2\pi} \sqrt{\frac{|\widetilde{W}_s(\varrho^{2n}, 4n\theta)|^2 (a_{2n}-\cos 2n\theta)^{2\ell}}{(a_{4n}-\cos 4n\theta)^{2s+1}}}\, d\theta, \tag{4.12}$$

where $\widetilde{W}_s(\varrho,\theta) := \sum_{\nu=0}^{s} (-1)^\nu \binom{2s+1}{\nu} \varrho^{2\nu-s} e^{i(\nu-s/2)\theta}$ was defined in [17, eq. (5.4)].

Let $x = \varrho^{4r}$ ($r > 0$, $\varrho > 1$). Recall that $|\widetilde{W}_s(\varrho^r,\theta)|^2 = \sum_{\ell=0}^{s} (-1)^\ell A_\ell \cos \ell\theta$ (cf. [17, eq. (5.5)]), where

$$A_0 = \frac{1}{x^{s/2}} \sum_{\nu=0}^{s} \binom{2s+1}{\nu}^2 x^\nu \tag{4.13}$$

and

$$A_\ell = \frac{2}{x^{(s-\ell)/2}} \sum_{\nu=0}^{s-\ell} \binom{2s+1}{\nu} \binom{2s+1}{\nu+\ell} x^\nu, \quad \ell = 1, \ldots, s. \tag{4.14}$$

The integrand in (4.12) depends on θ via $\cos 2n\theta$ and $\cos 4n\ell\theta$ ($n \in \mathbb{N}, \ell \in \{1, \ldots, s\}, s \in \mathbb{N}_0$). It is a continuous function of the form $g(2n\theta)$, where

$$g(\theta) \equiv g(\cos\theta, \cos 2\theta, \cos 4\theta, \ldots, \cos 2s\theta).$$

Because of periodicity, it is easy to prove that $\int_0^{2\pi} g(2n\theta)\, d\theta = 2\int_0^{\pi} g(\theta)\, d\theta$. Therefore, $L_{2n+1}(E_\varrho)$ reduces to

$$L_{2n+1}(E_\varrho) = \frac{1}{2^{s+\ell-1/2}\, n^{2\ell}\, \varrho^{2n(s+1)}} \int_0^{\pi} \sqrt{\frac{|\widetilde{W}_s(\varrho^{2n}, 2\theta)|^2 (a_{2n}-\cos\theta)^{2\ell}}{(a_{4n}-\cos 2\theta)^{2s+1}}}\, d\theta. \tag{4.15}$$

Applying Cauchy's inequality to (4.15), we obtain

$$L_{2n+1}(E_\varrho) \leqslant \frac{\sqrt{2}}{2^{s+\ell}\, n^{2\ell}\, \varrho^{2n(s+1)}} \left(\int_0^{\pi} \frac{|\widetilde{W}_s(\varrho^{2n},\theta)|^2}{(a_{4n}-\cos\theta)^{2s+1}}\, d\theta \right)^{1/2} \left(\int_0^{\pi} (a_{2n}-\cos\theta)^{2\ell}\, d\theta \right)^{1/2}. \tag{4.16}$$

Therefore, in our case $r = 2n$, i.e., we take $x = \varrho^{8n}$, since $a_{4n} = (x+1)/(2\sqrt{x})$.

Let (see [17, eq. (5.8)])

$$M_k(\varrho) := \left(\frac{\varrho-\varrho^{-1}}{2}\right)^k P_k\left(\frac{\varrho+\varrho^{-1}}{\varrho-\varrho^{-1}}\right), \tag{4.17}$$

where P_k is the Legendre polynomial of degree k.

We obtain that (cf. [10, eq. 3.661.3])

$$\int_0^{\pi} (a_{2n}-\cos\theta)^{2\ell}\, d\theta = \pi \left(\frac{\varrho^{2n}-\varrho^{-2n}}{2}\right)^{2\ell} P_{2\ell}\left(\frac{\varrho^{2n}+\varrho^{-2n}}{\varrho^{2n}-\varrho^{-2n}}\right) = \pi M_{2\ell}(x^{1/4}). \tag{4.18}$$

Further,

$$\int_0^\pi \frac{|\widetilde{W}_s(\varrho^{2n},\theta)|^2}{(a_{4n}-\cos\theta)^{2s+1}}\,d\theta = \int_0^\pi \frac{\sum_{\ell=0}^{s}(-1)^\ell A_\ell \cos\ell\theta\,d\theta}{(a_{4n}-\cos\theta)^{2s+1}} = \sum_{\ell=0}^{s} A_\ell J_\ell(a_{4n}), \tag{4.19}$$

where

$$J_\ell(a_{4n}) = (-1)^\ell \int_0^\pi \frac{\cos\ell\theta\,d\theta}{(a_{4n}-\cos\theta)^{2s+1}} = \int_0^\pi \frac{\cos\ell\theta\,d\theta}{(a_{4n}+\cos\theta)^{2s+1}}.$$

The integrals $J_\ell(a_{4n})$ (recall that $x=\varrho^{8n}$) we calculate by

$$J_\ell(a_{4n}) = \frac{2^{2s+1}\pi(-1)^\ell x^{s-(\ell-1)/2}}{(x-1)^{4s+1}} \sum_{\nu=0}^{2s} \binom{2s+\nu}{\nu}\binom{2s+\ell}{\ell+\nu}(x-1)^{2s-\nu} \tag{4.20}$$

(see [17, Lemma 4.2], or the book [10, eq. 3.616.7]).

Using (4.20), (4.19) reduces to (see [17, p. 128])

$$\int_0^\pi \frac{|\widetilde{W}_s(\varrho^{2n},\theta)|^2}{(a_{4n}-\cos\theta)^{2s+1}}\,d\theta = \sum_{\ell=0}^{s} A_\ell J_\ell(a_{4n}) = \frac{2^{2s+1}\pi x^{(s+1)/2}}{(x-1)^{4s+1}}\,Q_s(x), \tag{4.21}$$

where $Q_s(x)$ is given by (cf. [17, eq. (4.19)])

$$Q_s(x) := 2\sum_{\ell=0}^{s}{}'(-1)^\ell \Bigg(\sum_{\nu=0}^{s-\ell}\binom{2s+1}{\nu}\binom{2s+1}{\nu+\ell}x^\nu\Bigg)\Bigg(\sum_{\nu=0}^{2s}\binom{2s+\nu}{\nu}\binom{2s+\ell}{\ell+\nu}(x-1)^{2s-\nu}\Bigg).$$

Note that $\deg Q_s(x) = 3s$.

Using (4.18), (4.21), (4.16) reduces to

$$L_{2n+1}(E_\varrho) \leqslant \frac{\pi}{2^{\ell-1}\,n^{2\ell}}\sqrt{M_{2\ell}(\varrho^{2n})}\,\Phi_s(x), \tag{4.22}$$

where $\Phi_s(x) = \sqrt{Q_s(x)/(x-1)^{4s+1}}$.

Thus, we have just proved the following statement.

Theorem 4.1. *Let $x=\varrho^{8n}$, $n\geqslant 2$, and a_j, A_0 and A_k be defined by* (4.10), (4.13) *and* (4.14), *respectively. Then, for the Gori — Micchelli weight functions* (3.1), *we have that*

$$L_{2n+1}(E_\varrho) = \frac{1}{2^{s+\ell-1/2}\,n^{2\ell}\,\varrho^{2n(s+1)}} \int_0^\pi \sqrt{\frac{\sum_{k=0}^{s}(-1)^k A_k\cos 2k\theta\,\left(a_{2n}-\cos\theta\right)^{2\ell}}{(a_{4n}-\cos 2\theta)^{2s+1}}}\,d\theta. \tag{4.23}$$

Moreover, an estimate of the form (4.22) *holds.*

Example 4.1. Consider the weight function

$$\omega_{n,1}(t) = \frac{U_{n-1}^2(t)}{n^2}\,(1-t^2)^{1/2}$$

which belongs to class (3.1) with $\ell = 1$ and the corresponding Gauss — Turán — Kronrod quadrature formula (3.4). Figure 4.1 presents the value of $\log_{10}(L_{2n+1}(E_\varrho))$, for $s = 2$, as a function of ϱ (solid line), when $n = 5$ (left) and $n = 10$ (right). The value of $L_{2n+1}(E_\varrho)$ was calculated by (4.23). We also present the corresponding graphs $\varrho \mapsto \log_{10}(\pi\sqrt{M_{2\ell}(\varrho^{2n})}\Phi_s(x)/(2^{\ell-1}n^{2\ell}))$ (see (4.22)), for $\ell = 1, s = 2$, by dashed lines. As we can see, the error bound (4.22) is very precise especially for larger values of n and ϱ.

Remark 4.1. Some of the introduced Gauss — Turán — Kronrod quadrature formulas in Section 3, as well as L^1-error bounds of the type of (4.7) for integrands analytic on confocal ellipses (4.9), were considered a few years ago in [16] (see also [25]). Namely, the Gauss — Turán quadratures of Lobatto type (3.4) (with (3.5)) from [16] for $k = m = 0$

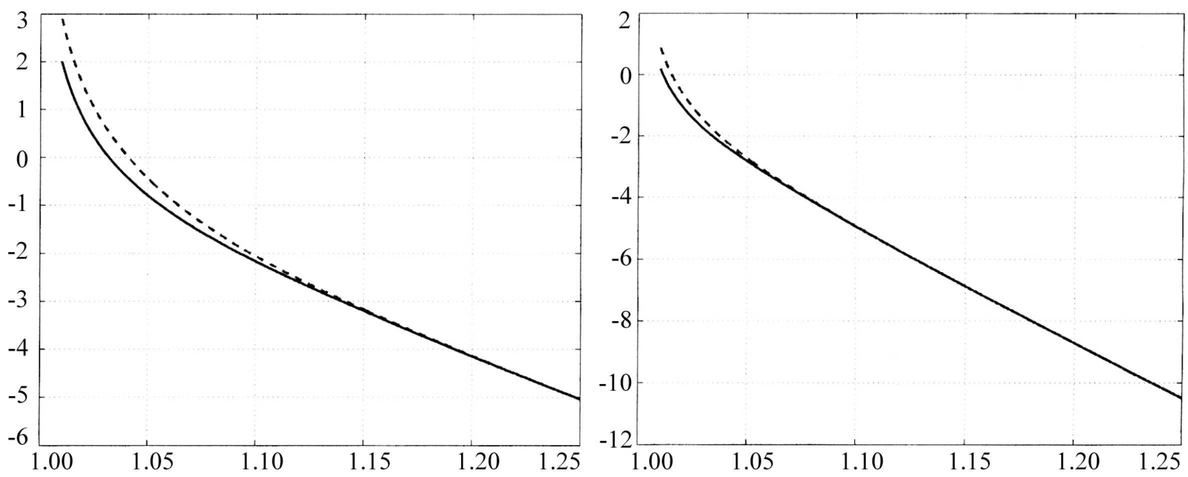

Fig. 4.1. Log_{10} of the values $L_{2n+1}(E_\varrho)$ (solid line) and their bounds given by (4.22) (dashed line) for $n = 5$ (left) and $n = 10$ (right), for $\varrho \in [1.01, 1.25]$

reduce to our Gauss — Turán — Kronrod quadratures (3.4) with $\ell = 0$ ($n \geqslant 2$). Also, these quadrature formulas appear as a particular case ($m = r = 2s+2$) of the quadrature formulas considered by Shi (see [25, Eqs. (1.1)–(1.6)]). The Gauss — Turán quadratures (3.4) (with (3.5)) from [16] for $k = m = s + 1$ (see also [17]) reduce to our Gauss — Turán — Kronrod quadratures (3.6). The Gauss — Turán quadratures of the Radau type (3.4) (with (3.5)) from [16] for $k = s + 1, m = 0$ ($k = 0, m = s + 1$) reduce to the Gauss — Turán — Kronrod quadratures (3.8) ((3.9)) in this paper. Finally, the quadrature formula (3.4) for $\ell = s$ ($n \geqslant 2$) is a new formula in the class of Kronrod extensions of Gauss — Turán quadrature formulas introduced by Li [12] (see also [18]), for which explicit expressions of the generalized Stieltjes polynomials are known.

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