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LOBATTO QUADRATURE FORMULAS FOR GENERALIZED GEGENBAUER WEIGHT

Milan A. Kovačević Gradimir V. Milovanović

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ABSTRACT:
The construction of two families of Lobatto type quadrature formulas on $[-1,1]$ with weight function $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$, $\mu,\alpha>-1$, is presented in this paper. This construction is based on the usage of Gauss-Christoffel quadrature formulas and QR algorithm.

LOBATTO-OVE KVADRATURNE FORMULE ZA GENERALISANU GEGENBAUEROVU TEŽINU
U radu se daje konstrukcija dve familije kvadraturnih formula Lobatto-ovog tipa na $[-1,1]$ sa težinskom funkcijom $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$, $\mu,\alpha>-1$. Ova konstrukcija se zasniva na korišćenju Gauss-Christoffelovih kvadratura i QR algoritma.

1. GENERALIZED GEGENBAUER POLYNOMIALS

Let $W_k^{(\alpha,\beta)}$ be the sequence of monic orthogonal polynomials with respect to weight function $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$ $(\mu,\alpha>-1,\ \beta=(\mu-1)/2)$ on $[-1,1]$. These polynomials were introduced by Lascenov in 1953 ([4], see [1, pp.155-6]). It is interesting that J. Radecki "rediscovered" the orthogonal polynomials with this weight in 1980 ([6]). Gegenbauer polynomials C_k^{λ} are a special case of these polynomials $(\mu=0,\ \alpha=\lambda-1/2)$.

The three term recurrence relation for the polynomials $W_k^{(\alpha,\beta)}$ is

$$W_{k+1}^{(\alpha,\beta)}(x)=xW_k^{(\alpha,\beta)}(x)-\Lambda_k W_{k-1}^{(\alpha,\beta)}(x),\quad k=0,1,\ldots, \tag{1.1}$$

where

$$\Lambda_{2m}=\frac{m(m+\alpha)}{(2m+\alpha+\beta)(2m+\alpha+\beta+1)}\qquad \Lambda_{2m-1}=\frac{(m+\beta)(m+\alpha+\beta)}{(2m+\alpha+\beta-1)(2m+\alpha+\beta)}$$

for $m=1,2,\ldots$, except for $\alpha+\beta=-1$ when $\Lambda_1=(\beta+1)/(\alpha+\beta+2)$.

The norm of generalized Gegenbauer polynomials is given by

$$\|W_{2k}^{(\alpha,\beta)}\|^2=\frac{k!}{(k+\alpha+\beta+1)_k}B(k+\alpha+1,k+\beta+1)\ \text{and}\ \|W_{2k+1}^{(\alpha,\beta)}\|^2=\|W_{2k}^{(\alpha,\beta+1)}\|^2.$$

2. CONSTRUCTION OF LOBATTO TYPE FORMULAS

Let us construct the Lobatto type quadrature formula

$$(2.1)\quad \int_{-1}^{1} p(x)f(x)\,dx = Af(-1) + Bf(1) + \sum_{k=1}^{n} C_k f(x_k) + R_{n+2}(f),$$

where $p(x) = |x|^{\mu}(1-x^2)^{\alpha}$, $\mu, \alpha > -1$. In that sense let us observe the following quadrature problem

$$(2.2)\qquad \int_{-1}^{1} p(x)F(x)\,dx = \sum_{k=1}^{n} C_k F(x_k) + R_{n+2}(F),$$

where $F(x) = (1-x^2)g(x)$ and $R(F) \equiv 0$ for each $g \in P_{2n-1}$ (P_{2n-1} is a set of all algebraic polynomials of degree not higher than 2n-1). In other words, the parameters C_k and x_k $(k=1,\ldots,n)$ should be determined in (2.2) such that

$$(2.3)\quad \int_{-1}^{1} (1-x^2)p(x)g(x)\,dx = \sum_{k=1}^{n} C_k(1-x_k^2)g(x_k) \qquad (\forall g \in P_{2n-1}).$$

The formula (2.2) can be interpreted as Gauss-Christoffel quadrature formula with weight $p_1(x) = (1-x^2)p(x) = |x|^{\mu}(1-x^2)^{\alpha+1}$, i.e.,

$$(2.4)\qquad \int_{-1}^{1} p_1(x)g(x)\,dx = \sum_{k=1}^{n} A_k g(x_k) + R_n(g),$$

where x_k $(k=1,\ldots,n)$ are the zeros of the polynomial $W_n^{(\alpha+1,\beta)}$, and A_k $(k=1,\ldots,n)$ are the corresponding Christoffel's numbers. The remainder $R_n(g)$ in the Gauss-Christoffel quadrature formula (2.4) is given by

$$(2.5)\qquad R_n(g) = \frac{\left\| W_n^{(\alpha+1,\beta)} \right\|^2}{(2n)!}\, g^{(2n)}(\xi) \quad (-1<\xi<1),$$

assuming $g \in C^{2n}[-1,1]$.

Comparing (2.2) and (2.4) we find $C_k = A_k/(1-x_k^2)$, $k=1,\ldots,n$, and $R_{n+2}(F) = R_n(g)$.

<u>Lemma 2.1.</u> Suppose

$$H(x) = \begin{cases} (h(x)-h(x_o))/(x-x_o), & x \neq x_o, \\ h'(x_o), & x = x_o, \end{cases}$$

where $h \in C^{m+1}[-1,1]$ and $-1 \le x_o \le 1$. Then $H^{(k)}(x) = h^{(k+1)}(\xi_k)/(k+1)$, $k=0,1,\ldots,m$, where ξ_k is between x and x_o.

<u>Proof.</u> From

$$H(x) = \int_0^1 h'(x_o + t(x-x_o))\,dt,$$

we find

$$H^{(k)}(x) = \int_0^1 t^k h^{(k+1)}(x_o + t(x-x_o))\,dt .$$

Using the mean-value theorem we obtain $H^{(k)}(x) = h^{(k+1)}(\xi_k)/(k+1)$, with ξ_k between x and x_o. ∎

Now, we suppose that function F, defined by $F(x) = (1-x^2)g(x)$, belongs to $C^{2n+2}[-1,1]$. According to the previous lemma, we obtain $g^{(2n)}(\xi) = -F^{(2n+2)}(\eta)/((2n+1)(2n+2))$ $(-1<\eta<1)$, i.e.,

$$\text{(2.6)} \qquad R_{n+2}(F) = R_n(g) = -\frac{\left\|W_n^{(\alpha+1,\beta)}\right\|^2}{(2n+2)!} F^{(2n+2)}(\eta) \quad (-1<\eta<1) .$$

The quadrature formula (2.1) can be obtained now from (2.2). Namely, if we take

$$F(x) = f(x) + \frac{1}{2}(x-1)f(-1) - \frac{1}{2}(x+1)f(1) ,$$

then the formula (2.1) follows from (2.2), where x_k $(k=1,\dots,n)$ are the zeros of the $W_n^{(\alpha+1,\beta)}$, A_k are the Christoffel's numbers, $C_k = A_k/(1-x_k^2)$ $(k=1,\dots,n)$,

$$\text{(2.7)} \qquad A = B = \frac{1}{2}\left(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} - \sum_{k=1}^{n} \frac{A_k}{1+x_k} \right) .$$

The equality $A = B$ holds, because the weight $p_1(x) = |x|^\mu (1-x^2)^{\alpha+1}$ an even function.

Thus, we have

$$\text{(2.8)} \qquad \int_{-1}^{1} |x|^\mu (1-x^2)^\alpha f(x)\,dx = A(f(-1)+f(1)) + \sum_{k=1}^{n} C_k f(x_k) + R_{n+2}(f) ,$$

where x_k, A_k $(k=1,\dots,n)$ are the parameters of Gauss-Christoffel formula (2.4), $C_k = A_k/(1-x_k^2)$, and A is defined by (2.7).

Assuming $f \in C^{2n+2}[-1,1]$, we have

$$\text{(2.9)} \qquad R_{n+2}(f) = R_{n+2}(F) = D_n f^{(2n+2)}(\eta) \quad (-1<\eta<1) ,$$

where $D_n = -\|W_n^{(\alpha+1,\beta)}\|^2/(2n+2)!$. So, we conclude that the Lobatto formula (2.8) has the algebraic degree of precision $2n+1$.

It is easy to prove that (2.8) can be represented in the form

$$(2.10)\quad \int_{-1}^{1} |x|^{\mu}(1-x^2)^{\alpha} f(x)\,dx = A(f(-1)+f(1)) + C_o f(0) + \sum_{k=1}^{m} C_k(f(x_k)+f(-x_k)) + R_{n+2}(f),$$

where $m=\left[n/2\right]$, A is defined by (2.7), $R_{n+2}(f)$ by (2.9), and

$$C_k = \frac{A_k}{1-x_k^2}, \quad x_k > 0, \quad k=1,\dots,m,$$

$$C_o = 0 \quad (n=2m) \quad \text{or} \quad C_o = A_{m+1} \quad (n=2m+1).$$

Since the three term recurrence relation for the polynomials $W_k^{(\alpha+1,\beta)}(x)$, $k=0,1,\dots$, is known, it is possible to determine x_k and A_k $(k=1,\dots,n)$ in Gauss formula (2.4), i.e., in formula (2.10), using QR algorithm (see [3] , [2]). The obtained parameters, for different values of μ and α, are given in Table 2.1. The error constant D_n is given too. Numbers in parentheses denote decimal exponents.

Table 2.1.

n	$\mu=1,\ \alpha=1$		D_n
5	A = 0.00416 6666 6667	C_o = 0.06666 6666 6667	
	x_1 = 0.47596 3149 4780	C_1 = 0.13571 2782 5494	
	x_2 = 0.79410 4487 7608	C_2 = 0.07678 7217 4506	-3.56(-13)
6	A = 0.00250 0000 0000	C_o = 0.	
	x_1 = 0.27017 4062 5470	C_1 = 0.08475 4772 4316	
	x_2 = 0.58907 0255 6048	C_2 = 0.11198 0943 8813	
	x_3 = 0.83964 4097 1558	C_3 = 0.05076 4283 6871	-4.07(-15)
n	$\mu=1,\ \alpha=-0.5$		D_n
5	A = 0.24380 9523 8095	C_o = 0.09523 8095 2381	
	x_1 = 0.55743 0069 1997	C_1 = 0.27682 6047 3616	
	x_2 = 0.88327 8443 5619	C_2 = 0.43174 5381 2099	-1.80(-12)
6	A = 0.20897 9591 8367	C_o = 0.	
	x_1 = 0.31495 1060 8466	C_1 = 0.12948 4966 1689	
	x_2 = 0.67091 8400 9874	C_2 = 0.27970 5391 4893	
	x_3 = 0.91394 1854 3340	C_3 = 0.38183 0050 5051	-2.13(-15)

We note that the formula (2.10) is very useful when $f(-1)=f(1)=0$. For example, this case appears in the least-square approximation with constraint ([5]). The use of quadrature formula in this case requires only n evaluations of function f, and the formula (2.10) integrates the first $2n+1$ powers exactly.

3. QUADRATURES WITH DERIVATIVES

Now, we construct the formula

$$(3.1)\quad \int_{-1}^{1} p(x)f(x)\,dx = Af(-1) + Bf(1) + Df'(-1) + Ef'(1) + \sum_{k=1}^{n} C_k f(x_k) + R_{n+4}(f),$$

where $p(x) = |x|^{\mu}(1-x^2)^{\alpha}$, $\mu,\alpha > -1$. Using the same idea, we consider the following quadrature problem

$$(3.2)\qquad \int_{-1}^{1} p(x)F(x)\,dx = \sum_{k=1}^{n} C_k F(x_k) + R_{n+4}(F)$$

in which $R_{n+4}(F) = 0$ for each $g \in P_{2n-1}$, where $F(x) = (1-x^2)^2 g(x)$. In order to determine x_k and C_k $(k=1,\dots,n)$, we consider (3.2) as Gauss-Christoffel formula with the weight function $p_2(x) = (1-x^2)^2 p(x) = |x|^{\mu}(1-x^2)^{\alpha+2}$, i.e. as

$$(3.3)\qquad \int_{-1}^{1} p_2(x)g(x)\,dx = \sum_{k=1}^{n} A_k g(x_k) + R_n(g),$$

Here x_k $(k=1,\dots,n)$ are the zeros of the $W_n^{(\alpha+2,\beta)}(x)$ and A_k $(k=1,\dots,n)$ are the corresponding Christoffel's numbers. So, we find $C_k = A_k/(1-x_k^2)^2$, $k=1,\dots,n$, and $R_{n+4}(F) = R_n(g)$. Assuming $F \in C^{2n+4}[-1,1]$, we have

$$(3.4)\qquad R_{n+4}(F) = \frac{\|W_n^{(\alpha+2,\beta)}\|^2}{(2n+4)!} F^{(2n+4)}(\xi) \quad (-1<\xi<1).$$

The quadrature formula (3.1) can be obtained from (3.2) now. Namely, if we put $F(x) = f(x) - H(x)$, where H is the Hermite's interpolation polynomial, i.e.,

$$H(x) = \tfrac{1}{4}(x-1)^2((x+2)f(-1) + (x+1)f'(-1)) + \tfrac{1}{4}(x+1)^2((2-x)f(1) + (x-1)f'(1)),$$

we obtain (3.1), where

$$A = B = \frac{1}{2}\left(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} - \sum_{k=1}^{n} \frac{A_k}{(1-x_k^2)^2} \right),$$

$$D = E = \frac{1}{4}\left(\frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)} - \sum_{k=1}^{n} \frac{A_k}{1-x_k^2} \right),$$

and

$$R_{n+4}(f) = D_n f^{(2n+4)}(\eta) \quad (-1 < \eta < 1),$$

with $D_n = \left\| W_n^{(\alpha+2,\beta)} \right\|^2 / (2n+4)!$. The remainder in this form holds when $f \in C^{2n+4}[-1,1]$.

The formula (3.1) can be expressed in the form

$$(3.5) \quad \int_{-1}^{1} |x|^{\beta}(1-x^2)^{\alpha} f(x)\,dx = A(f(-1) + f(1)) + D(f'(-1) - f'(1)) + \\ + C_o f(0) + \sum_{k=1}^{m} C_k(f(x_k) + f(-x_k)) + R_{n+4}(f),$$

where $m = [n/2]$ and

$$C_k = \frac{A_k}{(1-x_k^2)^2}, \quad x_k > 0, \quad k = 1,\dots,m,$$

$$C_o = 0 \ (n = 2m) \quad \text{or} \quad C_o = A_{m+1} \ (n = 2m+1).$$

The parameters of obtained formula are given in Table 3.1.

Table 3.1.

n	$\mu = 1,\ \alpha = 1$		D_n
4	A = 0.01750 0000 0000	D = 0.00083 3333 3333	
	x_1 = 0.31825 5412 0882	C_1 = 0.11334 5262 4903	
	x_2 = 0.68566 9063 1092	C_2 = 0.11915 4737 5097	5.92(-13)
5	A = 0.01125 0000 0000	D = 0.00041 6666 6667	
	x_0 = 0.	C_0 = 0.05555 5555 5556	
	x_1 = 0.43819 9425 2873	C_1 = 0.12242 2043 6271	
	x_2 = 0.74698 1434 6273	C_2 = 0.08855 0178 5952	8.13(-16)
n	$\mu = 1,\ \alpha = -0.5$		D_n
4	A = 0.42840 8163 2653	D = 0.00870 7482 9932	
	x_1 = 0.36995 8442 6479	C_1 = 0.18035 3176 9663	
	x_2 = 0.76837 7171 6978	C_2 = 0.39123 8659 7684	2.10(-12)
5	A = 0.37616 3265 3061	D = 0.00580 4988 6621	
	x_0 = 0.	C_0 = 0.07407 4074 0741	
	x_1 = 0.49896 8388 1747	C_1 = 0.21981 7276 4594	
	x_2 = 0.82104 0480 5363	C_2 = 0.36698 2421 1974	3.20(-15)

4. EXAMPLE

We consider the integral

$$I = \int_{-1}^{1} |x|\,(1-x^2)\cos\frac{\pi x}{2}\,dx = \frac{48}{\pi^2}\left(\frac{2}{\pi} - \frac{4}{\pi^3} - \frac{1}{6}\right) = 0.31450924354905\ldots,$$

where $p(x) = |x|\,(1-x^2)$, i.e., $\mu = 1$, $\alpha = 1$, and $f(x) = \cos\frac{\pi x}{2}$.

Table 4.1 represents the relative errors in the value of integral I corresponding to the n-point Gauss-Christoffel rule (GC), the Lobatto formula (2.10), and the generalized Lobatto formula (3.5), in that order.

Since $f(-1) = f(1) = 0$ and $f'(-1) = f'(1) = \pi$, all these quadrature rules require n functional evaluations. However, their algebraic degree precision are 2n-1, 2n+1, and 2n+3, respectively.

Table 4.1.

n	G.C	(2.10)	(3.5)
2	2.04(-2)	7.62(-4)	1.80(-5)
3	5.17(-4)	9.16(-6)	1.20(-7)
4	4.60(-6)	4.83(-8)	4.04(-10)
5	3.64(-8)	2.44(-10)	1.38(-12)
6	1.47(-10)	6.88(-13)	6.62(-16)

The results given in Table 4.1.are in good agreement with the previous theoretical results.

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