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Quadrature Processes in Telecommunications

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Abstract— **In many problems in telecommunications we have to use numerical integration of functions in one or several variables, iterative processes and summation of series. Very often, such problems cannot be solved by standard methods because of their slowly convergence. In this survey, we give an account of some special quadrature methods for such purposes.**

Keywords— **Numerical integration, quadrature processes, acceleration of convergence, strongly oscillatory functions, Fourier analysis, orthogonal polynomials.**

I. Introduction

The methods of numerical integration are very important tools in many problems in telecommunications and related subjects. Very often we meet ill-conditioned problems, the problems with singularities or quasi singularities, the problems with strongly oscillatory functions, etc. Special functions which appear always in the telecommunication problems have integral representations, so that we have to use the numerical integration or some expansions in series. In some cases, the convergence of such processes can be very slowly and we have to use certain techniques for accelerating of such convergence. In many problems, the standard methods require too much computation work and cannot be successfully applied. Therefore, for problems with singularities, for integrals of strongly oscillatory functions and others, there are a large number of special approaches. In this survey we give an account on some special – fast and efficient – quadrature processes. Such methods require a knowledge of orthogonal polynomials.

This paper is organized as follows. In Section 2 we give some basic facts concerning the orthogonality on the real line. The classical orthogonal polynomials and some special functions are included. Section 3 is devoted to the Gaussian quadrature formulas with arbitrary weights and their applications in finding the bit error probability of some communication systems in the presence of noise and interference, as well as in solving certain singular integrals appearing in analysis of antennas. Integration of strongly oscillatory functions including product quadrature formulas is considered in Section 4.

II. Orthogonality on the Real Line

Let $\mathcal{P}_n$ be the set of all algebraic polynomials P ($\not\equiv 0$) of degree at most n and $L^2(a,b)$ be the set of all functions such that $\int_a^b |f(t)|^2 w(t)\,dt < +\infty$, where w is a given *weight* function. The inner product is given by

$$(f,g) = \int_a^b w(t) f(t) g(t)\,dt, \tag{1}$$

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and the norm by $\|f\| = \sqrt{(f,f)} = \left(\int_a^b |f(t)|^2 w(t)\,dt\right)^{1/2}$

A system of polynomials $\{\pi_k\}$, where

$$\pi_k(t) = t^k + \text{lower degree terms}, \quad (\pi_k, \pi_m) = \|\pi_k\|^2 \delta_{km},$$

is called a system of monic orthogonal polynomials with respect to the inner product $(\,.\,,\,.\,)$ given by (1). Here, δ_{km} is Kronecker's delta. Such orthogonal polynomials $\{\pi_k\}$ satisfy a three-term recurrence relation

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k \geq 0, \tag{2}$$

$$\pi_0(t) = 1, \quad \pi_{-1}(t) = 0,$$

with the real coefficients α_k and $\beta_k > 0$. Because of orthogonality, we have that

$$\alpha_k = \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}, \quad \beta_k = \frac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})}. \tag{3}$$

The coefficient β_0, which multiplies $\pi_{-1} = 0$ in three-term recurrence relation may be arbitrary. Sometimes, it is convenient to define it by $\beta_0 = \int_a^b w(t)\,dt$. Then the norm of π_k can be expressed in the form

$$\|\pi_k\| = \sqrt{(\pi_k, \pi_k)} = \sqrt{\beta_0 \beta_1 \cdots \beta_k}\,.$$

Knowing the first n of the coefficients α_k, β_k, $k = 0, 1, \ldots, n-1$, in the recurrence relation (2), we can construct a symmetric tridiagonal matrix in the following way

$$J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{k-1}} \\ 0 & & & \sqrt{\beta_{k-1}} & \alpha_{k-1} \end{bmatrix}. \tag{4}$$

This tridiagonal matrix $J_n = J_n(w)$ is known as the *Jacobi matrix*. The monic polynomial $\pi_n(t)$ can be expressed in the following determinant form (cf. [1])

$$\pi_n(t) = \det(tI_n - J_n),$$

where I_n is the identity matrix of the order n. It is clear that $\pi_n(t) = 0$ if and only if the zeros $\tau_k^{(n)}$ of $\pi_n(t)$ are the same as the eigenvalues of the Jacobi matrix J_n. All zeros of $\pi_n(t)$, $n \geq 1$, are real and distinct and are located in the interior of the interval (a,b).

A very important class of orthogonal polynomials on an interval of orthogonality $(a,b) \in \mathbb{R}$ is constituted by so-called the ***classical orthogonal polynomials***. They are distinguished by several particular properties. Their weight functions $w(t)$ satisfy a differential equation of the first order (see [2]). These polynomials can be classified as the *Jacobi polynomials* $P_k^{(\alpha,\beta)}(t)$ orthogonal on $(-1,1)$ with $w(t)=(1-t)^\alpha(1+t)^\beta$ $(\alpha,\beta>-1)$, the *generalized Laguerre polynomials* $L_k^s(t)$ orthogonal on $(0,+\infty)$ with $w(t)=t^se^{-t}$ $(s>-1)$, and finally as the *Hermite polynomials* $H_k(t)$ which are orthogonal on $(-\infty,+\infty)$ with $w(t)=e^{-t^2}$.

The corresponding coefficients α_k and β_k in three-term recurrence relation (2) for these polynomials (normalized as monic) are given below.

(i) $\hat{P}_k^{(\alpha,\beta)}(t)=2^kk!/((k+\alpha+\beta+1)_k)P_k^{(\alpha,\beta)}(t)$ (monic Jacobi polynomials):

$$\alpha_k=\frac{\beta^2-\alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},$$

$$\beta_k=\frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2((2k+\alpha+\beta)^2-1)}.$$

(ii) $\hat{L}_k^s(t)=(-1)^kL_k^s(t)$ (monic generalized Laguerre polynomials):

$$\alpha_k=2k+s+1,\quad \beta_k=k(k+s).$$

(iii) $\hat{H}_k(t)=2^{-k}H_k(t)$ (monic Hermite polynomials):

$$\alpha_k=0,\quad \beta_k=\frac{k}{2}.$$

The most important case of Jacobi polynomials are the *Chebyshev polynomials of the first kind* $T_k(t)$ $(\alpha=\beta=-1/2)$. The monic polynomials $\hat{T}_0(t)=1$, $\hat{T}_k(t)=2^{1-k}T_k(t)$ $(k\ge 1)$ satisfy (2) with

$$\alpha_k=0,\quad \beta_1=\frac{1}{2},\quad \beta_k=\frac{1}{4}\quad (k\ge 2).$$

In many applications of orthogonal polynomials it is very important to know the recursion coefficients α_k and β_k in an explicit form as in the case of the classical orthogonal polynomials. There are certain non-classical weights when we know also these coefficients. In sequel we mention only three of them:

1° *Generalized Gegenbauer weight* $w(t)=|t|^\mu(1-t^2)^\alpha$, $\mu,\alpha>-1$, on $[-1,1]$. The (monic) generalized Gegenbauer polynomials $W_k^{(\alpha,\beta)}(t)$, $\beta=(\mu-1)/2$, were introduced by Lascenov [3] (see, also, Chihara [4, pp. 155–156]). These polynomials can be expressed in terms of the Jacobi polynomials,

$$W_{2k}^{(\alpha,\beta)}(t) = \frac{k!}{(k+\alpha+\beta+1)_k}P_k^{\alpha,\beta)}(2t^2-1),$$

$$W_{2k+1}^{(\alpha,\beta)}(t) = \frac{k!}{(k+\alpha+\beta+2)_k}xP_k^{\alpha,\beta+1)}(2t^2-1).$$

Notice that $W_{2k+1}^{(\alpha,\beta)}(t)=tW_{2k}^{(\alpha,\beta+1)}(t)$. Their three-term recurrence relation is

$$W_{k+1}^{(\alpha,\beta)}(t) = tW_k^{(\alpha,\beta)}(t)-\beta_kW_{k-1}^{(\alpha,\beta)}(t),\quad k=0,1,\ldots,$$

$$W_{-1}^{(\alpha,\beta)}(t) = 0,\quad W_0^{(\alpha,\beta)}(t)=1,$$

where

$$\beta_{2k} = \frac{k(k+\alpha)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)},$$

$$\beta_{2k-1} = \frac{(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)},$$

for $k=1,2,\ldots$, except when $\alpha+\beta=-1$; then $\beta_1=(\beta+1)/(\alpha+\beta+2)$. Some applications of these polynomials in numerical quadratures and least square approximation with constraint were given in [5] and [6], respectively.

2° *The hyperbolic weight* $w(t)=1/\cosh t$ on $(-\infty,+\infty)$. The coefficients in three-term recurrence relation are given by

$$\alpha_k=0,\quad \beta_0=\pi,\quad \beta_k=\frac{\pi^2k^2}{4}\quad (k\ge 1).$$

For details and generalizations see Chihara [4, pp. 191–193].

3° *The logistic weight* $w(t)=e^{-t}/(1+e^{-t})^2$ on $(-\infty,+\infty)$. Here we have

$$\alpha_k=0,\quad \beta_0=1,\quad \beta_k=\frac{\pi^2k^4}{4k^2-1}\quad (k\ge 1).$$

A system of orthogonal polynomials for which the recursion coefficients are not known explicitly will be said to be *strong non-classical* orthogonal polynomials. In such cases there are a few known approaches to compute the first n coefficients α_k, β_k, $k=0,1,\ldots,n-1$. These then allow us to compute all orthogonal polynomials of degree $\le n$ by a straightforward application of the three-term recurrence relation (2).

One of approaches for numerical construction of the monic orthogonal polynomials $\{\pi_k\}$ is the *method of moments*, or precisely, *Chebyshev* or *modified Chebyshev algorithm*. The second method makes use of explicit representations (3) in terms of the inner product $(\cdot,\cdot)$. The method is known as the *Stieltjes procedure*. Using a discretization of the inner product by some appropriate quadrature

$$(f,g)\approx(f,g)_N=\sum_{k=1}^{N}w_kf(x_k)g(x_k),\quad w_k>0,$$

the corresponding method is called the *discretized Stieltjes procedure*.

III. Gaussian Quadratures and Applications

Computation of integrals is an important problem. In this section, we consider integrals of the form

$$\int_a^b F(t)\,dt, \tag{5}$$

as well as some multiple integrals, which cannot be solved explicitly. The integration interval can be finite or infinite and the integrand F, generally, can be singular. Sometimes a recursion formula may be found, while in other cases series expansion may be feasible. In the most cases we must use purely numerical methods. If we want to have a good quadrature process with a reasonable convergence, then the integrand should be sufficiently regular. Furthermore, singularities in its first or second derivative can be disturbing. Also, the quasi singularities, i.e., singularities near to the integration interval, cause remarkable decelerate of the convergence. However, this problem could be avoided in some cases by extraction of singularities and their integration in an exact form. For example, if $S(t)$ is a singular part in (5) which can be exact integrated, then the remaining nonsingular part can be computed by some of standard numerical methods. Thus, we have

$$\int_a^b F(t)\,dt = \int_a^b S(t)\,dt - \int_a^b (S(t) - F(t))\,dt.$$

Sometimes, another kind of extractions is also applicable. Namely, in some cases, the integrand can be taken in a multiplicative form $F(t) = w(t)f(t)$, where $w(t)$ is a singular or quasi singular part of $F(t)$ (so-called "heavy" part) and $f(t)$ is an enough smooth part. Under some conditions on $w(t)$ (non-negativity on (a, b), the existence of all moments $\mu_k = \int_a^b t^k w(t)\,dt$, $k = 0, 1, \ldots$, and $\mu_0 > 0$), one can construct the weighted quadrature formulas of the form

$$\int_a^b F(t)\,dt = \int_a^b f(t)w(t)\,dt \approx \sum_{\nu=1}^{n} A_\nu^{(n)} f(\tau_\nu^{(n)}),$$

including the singularity information in the quadrature parameters $A_\nu^{(n)}$ (and $\tau_\nu^{(n)}$, in general). The convergence of such quadratures then depends only on the properties of $f(t)$.

One of the important uses of orthogonal polynomials is in the construction of quadrature formulas of maximum, or nearly maximum, algebraic degree of exactness for integrals involving an arbitrary (nonnegative) weight $w(t)$.

The n-point Gaussian quadrature formula

$$\int_a^b f(t)w(t)\,dt = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + R_n(f), \qquad (6)$$

has maximum algebraic degree of exactness $2n - 1$, in the sense that $R_n(f) = 0$ for all $f \in \mathcal{P}_{2n-1}$. In formula (6), $\tau_\nu = \tau_\nu^{(n)}$ are the *Gauss nodes*, and $\lambda_\nu = \lambda_\nu^{(n)}$ the *Gauss weights* or *Christoffel numbers*. This formula is also known as *Gauss-Christoffel quadrature formula*.

The nodes $\tau_\nu = \tau_\nu^{(n)}$ are the eigenvalues of the symmetric tridiagonal Jacobi matrix $J_n(w)$, given by (4), while the weights $\lambda_\nu = \lambda_\nu^{(n)}$ are given in terms of the first components $v_{\nu,1}$ of the corresponding normalized eigenvectors by

$$\lambda_\nu = \beta_0 v_{\nu,1}^2, \qquad \nu = 1, \ldots, n,$$

where $\beta_0 = \int_a^b w(t)\,dt$. There are well-known and efficient algorithms, such as the QR algorithm with shifts, to compute eigenvalues and eigenvectors of symmetric tridiagonal matrices (cf. [2]).

A simple modification of the previous method can be applied to the construction of Gauss-Radau and Gauss-Lobatto quadrature formulas.

The simplest Gaussian formula is the *Gauss-Chebyshev formula*

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\,dt = \frac{\pi}{n} \sum_{\nu=1}^{n} f(\tau_\nu) + R_n(f), \qquad (7)$$

where τ_ν ($\nu = 1, \ldots, n$) are zeros of the Chebyshev polynomial $T_n(t) = \cos(n \arccos t)$, i.e.,

$$\tau_\nu = \cos\frac{(2\nu - 1)\pi}{2n}, \qquad \nu = 1, \ldots, n. \qquad (8)$$

For $f \in C^{2n}[-1, 1]$ the remainder $R_n(f)$ can be represented in the form

$$R_n(f) = \frac{\pi}{2^{2n-1}(2n)!} f^{(2n)}(\xi) \qquad (-1 < \xi < 1).$$

A. Integration of the Error Function

In [7], we considered an integral which appears in telecommunications,

$$P_e = \frac{1}{\pi^m} \int_0^\pi \cdots \int_0^\pi \operatorname{erfc}\Big[c\Big(1 + \sum_{k=1}^{m} c_k \cos\theta_k\Big)\Big]\,d\theta_1 \ldots d\theta_n,$$

where c and c_k are positive constants, and the error function $\operatorname{erfc}(t)$ is defined by

$$w(t) = \operatorname{erfc}(t) = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} e^{-x^2/2}\,dx. \qquad (9)$$

In our calculation, we used the following approximation $(0 \le t < +\infty)$

$$\operatorname{erfc}(t) = (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5) e^{-t^2/2} + \varepsilon, \quad (10)$$

where $x = 1/(1+pt)$, $p = 0.23164189$, and $|\varepsilon| \le 0.75 \times 10^{-7}$. The coefficients a_k are given by:

$$\begin{aligned} a_1 &= 0.127414796, & a_2 &= -0.142248368, \\ a_3 &= 0.7107068705, & a_4 &= -0.7265760135, \\ a_5 &= 0.5307027145. \end{aligned}$$

In order to calculate the error probability P_e we put $x_k = \cos\theta_k$ ($k = 1, \ldots, m$). Then, we get

$$P_e = \frac{1}{\pi^m} \int_{-1}^{1} \frac{dx_1}{\sqrt{1-x_1^2}} \cdots \int_{-1}^{1} \frac{1}{\sqrt{1-x_1^m}} \operatorname{erfc}\Big[c\Big(1 + \sum_{k=1}^{m} c_k x_k\Big)\Big]\,dx_m.$$

Applying (7) successively m times, we obtain

$$P_e = \frac{1}{n^m} \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_m=1}^{n} \Big[c\Big(1 + \sum_{k=1}^{m} c_k \tau_{\nu_k}\Big)\Big] + E_n^{(m)}, \qquad (11)$$

where $E_n^{(m)}$ is the corresponding error.

Taking $f(t) = \text{erfc}(a + bt)$ $(z = a + bt,\ a, b > 0)$, we can find that

$$f^{(2n)}(t) = -\frac{b^{2n}}{\sqrt{2\pi}} \cdot \frac{d^{2n-1}}{dz^{2n-1}}\left(e^{-z^2/2}\right) = \frac{b^{2n}}{2^n\sqrt{\pi}} e^{-s^2} H_{2n-1}(s),$$

where $s = z/\sqrt{2}$ and $H_{2n-1}(s)$ is the Hermite polynomial of degree $2n-1$. Then, for the remainder term in the Gauss-Chebyshev formula, we get

$$r_n = R_n(f) = \frac{\sqrt{\pi}\, b^{2n}}{2^{3n-1}(2n)!} e^{-v^2} H_{2n-1}(v),$$

where $v = (a + b\xi)/\sqrt{2}$ $(-1 < \xi < 1)$. Since (see [8])

$$|H_{2n-1}(v)| \leq |v| e^{v^2/2} \frac{(2n)!}{n!},$$

we conclude that

$$|r_n| \leq \frac{\sqrt{\pi} b^{2n}}{2^{3n-1} n!} |v| e^{-v^2} \leq \pi K_n b^{2n},$$

where we put $1/K_n = 2^{3n-1} n! \sqrt{\pi e}$. Notice that this estimate is not depending on a. By induction, we can prove: *For the remainder $E_n^{(m)}$ in (11) the following estimate*

$$|E_n^{(m)}| \leq \frac{c^{2n}}{2^{3n-1} n! \sqrt{\pi e}} \sum_{k=1}^{m} c_k^{2n} \tag{12}$$

holds. Thus, basing on (11) we have a formula for numerical calculation of the integral P_e in the form

$$P_e \approx P_e^{(n)} = \frac{1}{n^m} \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_m=1}^{n} \left[c\Big(1 + \sum_{k=1}^{m} c_k \tau_{\nu_k}\Big)\right]. \tag{13}$$

If the error in (10) is such that $|\varepsilon| \leq E$, then for the total error in the approximation (13) we have

$$|\varepsilon_T| \leq E + |E_n^{(m)}|.$$

The number of nodes in the Gauss-Chebyshev formula (7) should be taken so that the upper bound of the error $E_n^{(m)}$, given in (12), be the same order as E.

B. Numerical Calculation of the Bit Error Probability

In this subsection we give a concrete example how to calculate the bit error probability of a communication system. In [9] (see also [10]) we considered a phase-coherent communication receiver, supposing the input as a linear combination of the signal with the amplitude A, the interference with the amplitude A_1 and the phase θ, and the Gaussian noise $n(t)$,

$$r(t) = A\cos\omega_0 t + A_1 \cos(\omega_0 t + \theta) + n(t),$$

or, in an equivalent form, as

$$r(t) = AR\cos(\omega_0 t + \psi) + n(t),$$

where

$$R = R(\cos\theta) = \sqrt{1 + 2\eta\cos\theta + \eta^2},$$

$$\psi = \arctan\frac{\eta\sin\theta}{1 + \eta\cos\theta},$$

and $\eta = A_1/A$. Under an assumption of a constant phase in the symbol interval, the conditional error probability for the phase-coherent communication system which uses the phase locked loop (PLL) to provide the synchronisation is given by

$$P_{e/\phi} = \text{erfc}\left[\sqrt{2R_b}\,\cos\phi\right],$$

where $R_b = E/N_0$ (E is the signal energy, N_0 is the single-sided power density spectrum of the Gaussian noise in W/Hz), ϕ is the phase error process, and the function erfc is defined by (9). Using a model from [11] and taking appropriate parameters, after some calculation, we get the steady-state probability density function in the form

$$p(\phi) = \frac{e^{\alpha_0 R\cos\phi}}{2\pi I_0(\alpha_0 R)},$$

where I_0 is the modified Bessel function of the order zero and α_0 is a normalization constant. We put $R_b = R_1 R^2$, where R_1 corresponds to the case when there is no interference. Finally, we obtain the average error probability by averaging over all ϕ and over all θ,

$$P_e = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \text{erfc}\left(\sqrt{2R_1}\,R\cos\phi\right) \frac{e^{\alpha_0 R\cos\phi}}{I_0(\alpha_0 R)}\, d\theta d\phi.$$

By substitutions $x = \cos\phi$ and $y = \cos\theta$ we get

$$P_e = \frac{1}{\pi^2} \int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} \int_{-1}^{1} \frac{e^{\alpha_0 Rx}}{\sqrt{1-x^2}} \frac{\text{erfc}\left(\sqrt{2R_1}\,Rx\right)}{I_0(\alpha_0 R)}\, dx,$$

where $R = R(y) = \sqrt{1 + 2\eta y + \eta^2}$.

In order to calculate the bit error probability P_e we apply successively the Gauss-Chebyshev formula (7) in n points, so that we get

$$P_e \approx P_e^{(n)} = \frac{1}{n^2} \sum_{\nu=1}^{n} \frac{1}{I_0(\alpha_0 r_\nu)} \sum_{k=1}^{n} \text{erfc}\left(\sqrt{2R_1} r_\nu \tau_k\right) e^{\alpha_0 r_\nu \tau_k},$$

where $r_\nu = R(\tau_\nu) = \sqrt{1 + 2\eta\tau_\nu + \eta^2}$, $\nu = 1, \ldots, n$, and τ_ν are the Chebyshev nodes given by (8). In some cases the convergence of the Gaussian formulas $P_e^{(n)} \to P_e$, when $n \to +\infty$, can be accelerated using some well-known techniques (see [2]).

C. Gaussian Quadratures for Non-Classical Weights

In sequel we mention a few non-classical weights $w(t)$ for which recursion coefficients $\alpha_k = \alpha_k(w)$, $\beta_k = \beta_k(w)$, $k = 0, 1, \ldots, n-1$, have been tabulated in the literature and used in the construction of Gaussian quadratures.

1° *Logarithmic weight* $w(t) = t^\alpha \log(1/t)$, $\lambda > -1$, on $(0, 1)$. Piessens and Branders [12] considered cases when $\alpha = 0, \pm 1/2, \pm 1/3, -1/4, -1/5$ (see also Gautschi [13]).

2° *One-side Hermite weight* $w(t) = \exp(-t^2)$ on $[0, c]$, $0 < c \le +\infty$. The cases $c = 1$, $n = 10$, and $c = +\infty$, $n = 15$, were considered by Steen, Byrne and Gelbard [14] (see also Gautschi [15]).

3° *Error function* on $(0, +\infty)$ given by (9). Gaussian quadratures for $n \le 12$ were considered by Vigneron and Lambin [16].

4° *Reciprocal gamma function* $w(t) = 1/\Gamma(t)$ on $(0, +\infty)$. Gautschi [17] determined the recursion coefficients for $n = 40$ with 20 significant decimal digits (S). This function could be useful as a probability density function in reliability theory (see Fransén [18]).

5° *Einstein's* and *Fermi's weight functions* on $(0, +\infty)$,

$$w_1(t) = \varepsilon(t) = \frac{t}{e^t - 1} \quad \text{and} \quad w_2(t) = \varphi(t) = \frac{1}{e^t + 1}.$$

These functions arise in solid state physics. Gautschi and Milovanović [19] determined the recursion coefficients α_k and β_k, for $n = 40$ with 25 S, and gave an application of the corresponding Gauss-Christoffel quadratures to summation of slowly convergent series.

6° *The hyperbolic weights* on $(0, +\infty)$,

$$w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t}.$$

The recursion coefficients α_k, β_k, for $n = 40$ with 30 S, were obtained by Milovanović [20] and used in summation processes.

7° *The modified exponential weight* on $(-\infty, +\infty)$,

$$w(t) = w^{(\alpha,\beta)}(t) = \frac{e^{-t^2}}{\sqrt{1 + \alpha t + \beta t^2}},$$

where α and β are real parameters such that $\alpha^2 < 4\beta$.

Recently Bandrauk [21] stated a problem[1] of finding a computationally effective approximations for the integral

$$I_{m,n}^{(\alpha,\beta)} = \int_{-\infty}^{+\infty} \hat{H}_m(t)\hat{H}_n(t)w^{(\alpha,\beta)}(t)\,dt, \qquad (14)$$

where $\hat{H}_n(t)$ is the monic Hermite polynomial of degree n. Evidently, for $\alpha = \beta = 0$, this integral expresses the orthogonality of the Hermite polynomials, and $I_{m,n}^{(0,0)} = 0$ for $m \ne n$.

In order to compute the recursion coefficients in three-term recurrence relation (2) for the weight $w^{(\alpha,\beta)}(t)$ on $\mathbb{R}$, we use the discretized Stieltjes procedure, with the discretization based on the Gauss-Hermite quadratures,

$$\int_{-\infty}^{+\infty} P(t)w^{(\alpha,\beta)}(t)\,dt = \int_{-\infty}^{+\infty} \frac{P(t)}{\sqrt{1 + \alpha t + \beta t^2}} e^{-t^2}\,dt \cong \sum_{k=1}^{N} \frac{\lambda_k^H P(\tau_k^H)}{\sqrt{1 + \alpha\tau_k^H + \beta(\tau_k^H)^2}},$$

[1]The original problem was stated with the Hermite polynomials $H_k(t) = 2^k \hat{H}_k(t)$ $(k \ge 0)$.

where P is an arbitrary algebraic polynomial, and τ_k^H and λ_k^H are the parameters of the N-point Gauss-Hermite quadrature formula. We need such a procedure for each of selected pairs (α, β).

The integrand $t \mapsto \hat{H}_m(t)\hat{H}_n(t)w^{(\alpha,\beta)}(t)$ in (14) has $m + n$ zeros in the integration interval and very big oscillations. The case $\alpha = \beta = 1$ and $m = 10$, $n = 15$ is displayed in Figure 1.

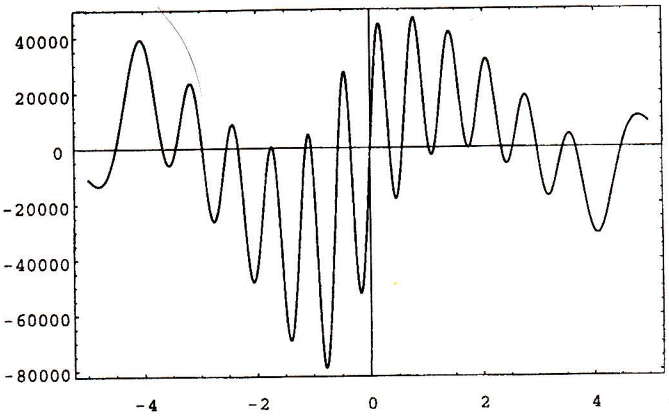

Fig. 1. The case $\alpha = \beta = 1$ and $m = 10$, $n = 15$

Applying the corresponding Gaussian quadratures, with respect to the weight $w^{(\alpha,\beta)}(t)$, to $I_{m,n}^{\alpha,\beta}$ we get approximative formulas

$$I_{m,n}^{\alpha,\beta} \approx Q_{m,n}^{\alpha,\beta} = \sum_{\nu=1}^{N} \lambda_\nu^{(\alpha,\beta)} \hat{H}_m(\tau_\nu^{(\alpha,\beta)})\hat{H}_n(\tau_\nu^{(\alpha,\beta)}). \qquad (15)$$

In the following table we present the obtained results for $\alpha = \beta = 1$ in double precision arithmetic (with machine precision 2.22×10^{-16}) in two cases: $m = 3$, $n = 6$, and $m = 10$, $n = 15$. The number of nodes in quadrature formula (15) was $N = 5, 10, 15, 20$. Numbers in parentheses indicate decimal exponents.

TABLE I

GAUSSIAN APPROXIMATION OF THE INTEGRAL $I_{m,n}^{\alpha,\beta}$

N	$Q_{3,6}^{1,1}$	$Q_{10,15}^{1,1}$
5	2.63168167926273(-1)	-4.01134148759825(4)
10	2.63168167926273(-1)	3.20721013272847(4)
15	2.63168167926273(-1)	-2.06784419769247(4)
20	2.63168167926273(-1)	-2.06784419769247(4)

Since the N-point Gaussian quadrature formula (15) has maximum algebraic degree of exactness $2N - 1$, we see that obtained results are exact for every N such that $2N - 1 \ge m + n$.

D. Singular Integrals in Analysis of a Monopole Antenna

A numerical procedure for a class of singular integrals which appear in the analysis of a monopole antenna, coaxially located along the axis of a infinite conical reflector was considered by Milovanović, Surutka, and Janković in [22]. Namely, they considered the integral

$$I(a,\nu)=\int_0^a \frac{j_\nu(x)}{x}\sin(a-x)\,dx, \tag{16}$$

where $j_\nu(x)$ is the spherical Bessel function of the index ν, defined by

$$j_\nu(x)=\frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^\nu\sum_{k=0}^{+\infty}\frac{(-1)^k(x/2)^{2k}}{k!\Gamma(\nu+k+3/2)},$$

and the index ν is a solution of the equation

$$P_\nu(\cos\theta_1)=0, \tag{17}$$

where $P_\nu(\cos\theta)$ is the Legendre function of the the first kind defined by

$$P_\nu(\cos\theta)=\frac{\sqrt{2}}{\pi}\int_0^\theta\frac{\cos(\nu+1/2)\varphi}{\sqrt{\cos\varphi-\cos\theta}}\,d\vartheta, \tag{18}$$

and θ_1 is the flare angle of the cone. Equation (17) has an infinite number of solutions ν_k $(k\in\mathbb{N})$.

Since

$$\lim_{x\to0+}\frac{j_\nu(x)}{x}=\begin{cases}0, & \nu>1,\\ 1/3, & \nu=1,\\ +\infty, & \nu<1,\end{cases}$$

we see that the integrand in (16) is singular when $\nu<1$. This case occurs when $\theta_1>90°$. Namely, then the first solution of (17) is less than 1 ($\nu_1<1$). An analysis of this equation was done in [23] (see also [24]).

The integration problem (16) was solved in [22] by extraction of singularity in the form

$$I(a,\nu)=C_\nu(a)\frac{a^\nu}{\nu}+\int_0^a\frac{j_\nu(x)\sin(a-x)-C_\nu(a)x^\nu}{x}\,dx,$$

where $C_\nu(a)=2^{-\nu-1}\sqrt{\pi}\sin a/\Gamma(\nu+3/2)$. For calculation of the spherical Bessel function the authors used a procedure given in [24].

We give here an alternative procedure for (16) using only Gaussian quadratures. In our approach we take an integral representation of the Bessel functions.

Since

$$j_\nu(z)=\sqrt{\frac{\pi}{2z}}\,J_{\nu+1/2}(z),$$

using the following representation for the cylindric Bessel functions (see [25, p. 360, Eq. 9.1.20])

$$J_\nu(z)=\frac{2(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^1(1-t^2)^{\nu-1/2}\cos(zt)\,dt,$$

which holds for $\mathrm{Re}\,\nu>-1/2$, we find

$$j_\nu(x)=\frac{(x/2)^\nu}{2\Gamma(\nu+1)}\int_{-1}^1(1-t^2)^\nu\cos(xt)\,dt$$

and then

$$I(a,\nu)=A\int_0^a\left(\frac{x}{2}\right)^{\nu-1}\sin(a-x)dx\int_{-1}^1(1-t^2)^\nu\cos(xt)\,dt,$$

i.e.,

$$I(a,\nu)=A\int_{-1}^1(1-t^2)^\nu G_\nu(t)\,dt,$$

where $A=1/(4\Gamma(\nu+1))$ and

$$G_\nu(t)=\int_0^a\left(\frac{x}{2}\right)^{\nu-1}\sin(a-x)\cos(xt)\,dx\quad(\nu>0).$$

After integration by parts, this formula reduces to

$$G_\nu(t)=\frac{2}{\nu}\int_0^a\left(\frac{x}{2}\right)^\nu\big[\cos(a-x)\cos xt+t\sin(a-x)\cos xt\big]dx.$$

Changing variables $x=a(1-\xi^2)$ $(\xi\ge0)$, we get

$$G_\nu(t)=\frac{8}{\nu}\left(\frac{a}{2}\right)^{\nu+1}\int_0^1\xi(1-\xi^2)^\nu g(\xi,t)\,d\xi,$$

where

$$g(\xi,t)=\cos[a\xi^2]\cos[at(1-\xi^2)]+t\sin[a\xi^2]\sin[at(1-\xi^2)].$$

Notice that $g(\pm\xi,\pm t)=g(\xi,t)$. Because of that, we have

$$I(a,\nu)=\frac{(a/2)^{\nu+1}}{\nu\Gamma(\nu+1)}\int_{-1}^1\int_{-1}^1 w^{(1,\nu)}(\xi)w^{(0,\nu)}(t)g(\xi,t)\,d\xi dt,$$

where $w^{(\mu,\nu)}(t)=|t|^\mu(1-t^2)^\nu$ is the generalized Gegenbauer weight (see 1° in Sect. II). The construction of the corresponding Gaussian quadratures is very simple in this case with regard to the knowledge of recursion coefficients in an explicit form. Here also, there is a convenience in a number of the integrand evaluations. Since the integrand is even, we can get the Gaussian quadrature of degree of exactness $4N-1$, taking only N (positive) points $\tau_1^{(\mu,\nu)},\dots,\tau_N^{(\mu,\nu)}$, as zeros of the polynomial $W_{2N}^{(\alpha,\beta)}(t)$, where $\alpha=\nu$, $\beta=(\mu-1)/2$ (see 1° in Sect. II). Thus,

$$\int_1^1 w^{(\mu,\nu)}(t)\phi(t)\,dt\approx Q_N^{(\mu,\nu)}(\phi)=2\sum_{i=1}^N A_k^{(\mu,\nu)}\phi(\tau_k^{(\mu,\nu)}),$$

and we finally get

$$I(a,\nu)\approx I_N(a,\nu)=\frac{4(a/2)^{\nu+1}}{\nu\Gamma(\nu+1)}\sum_{i=1}^N\sum_{j=1}^N A_iB_jg(x_i,y_j),$$

where, because of simplicity, we put

$$A_k=A_k^{(1,\nu)},\quad x_k=\tau_k^{(1,\nu)},\quad B_k=A_k^{(0,\nu)},\quad y_k=\tau_k^{(0,\nu)},$$

for $k=1,\dots,n$. This quadrature formula is based on N^2 nodes and gives good approximation of the integral $I(\pi/2,\nu)$. The obtained results rounded to 12 decimal places, for $a=\pi/2$ and $\nu=0.1(0.1)1.0$, are displayed in Table II. We used our quadrature formula for $N=7$. All digits in approximation $I_7(\pi/2,\nu)$ are correct.

Table III shows the relative errors in approximations $I_N(\pi/2,\nu)$ for $N=2(1)6$ and again $\nu=0.1(0.1)1.0$. As we can see, the convergence of approximations is fast and we can take relatively small N in order to get a satisfactory result.

TABLE II

APPROXIMATION OF $I(\pi/2,\nu)$ FOR $\nu = 0.1(0.1)1.0$

ν	Approximation $I_7(\pi/2,\nu)$
0.1	9.092660539259
0.2	4.113983342491
0.3	2.470467111313
0.4	1.661658513482
0.5	1.187153595723
0.6	0.879930124888
0.7	0.668250458550
0.8	0.516135176348
0.9	0.403518784385
1.0	0.318309886184

TABLE III

RELATIVE ERRORS IN APPROXIMATIONS $I_N(\pi/2,\nu)$ FOR $\nu = 0.1(0.1)1.0$ AND $N = 2(1)6$

ν	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
0.1	9.2(−3)	1.5(−4)	1.3(−6)	7.6(−9)	3.0(−11)
0.2	8.2(−3)	1.3(−4)	1.1(−6)	6.3(−9)	2.5(−11)
0.3	7.2(−3)	1.1(−4)	9.4(−7)	5.3(−9)	2.1(−11)
0.4	6.5(−3)	9.5(−5)	8.0(−7)	4.4(−9)	1.7(−11)
0.5	5.8(−3)	8.3(−5)	6.9(−7)	3.7(−9)	1.4(−11)
0.6	5.2(−3)	7.3(−5)	5.9(−7)	3.1(−9)	1.2(−11)
0.7	4.6(−3)	6.4(−5)	5.1(−7)	2.6(−9)	9.8(−12)
0.8	4.2(−3)	5.6(−5)	4.4(−7)	2.2(−9)	8.2(−12)
0.9	3.8(−3)	4.9(−5)	3.8(−7)	1.9(−9)	6.9(−12)
1.0	3.4(−3)	4.4(−5)	3.3(−7)	1.6(−9)	5.8(−12)

E. *Application to Legendre Function of the First Order*

Numerical calculation of the Legendre function of the first order is also possible using Gaussian quadratures. We start with Dirichlet-Mehler integral representation (18). The functions $P_\nu(x)$ satisfy the three-term recurrence relation

$$(\nu+2)P_{\nu+2}(t) = (2\nu+3)tP_{\nu+1}(t) - (\nu+1)P_\nu(t). \quad (19)$$

When ν is an nonnegative integer, the functions $P_\nu(t)$ reduce to the Legendre polynomials orthogonal on $(-1,1)$.

The integrand in (18) is quasi-singular at $\theta = 0$, i.e., when $t = 1$. Therefore, we use an extraction in the form

$$\begin{aligned} P_\nu(\cos\theta) &= \cos[(\nu+1/2)\theta]P_{-1/2}(\cos\theta) \\ &\quad + \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(\nu+1/2)\varphi - \cos(\nu+1/2)\theta}{\sqrt{\cos\varphi - \cos\theta}}\, d\theta, \end{aligned}$$

and then we change variables $\varphi = \theta(1-x^2)$ in order to get an integral on $(0,1)$. Thus, we find

$$P_\nu(\cos\theta) = \frac{2}{\pi}\cos[(\nu+1/2)\theta]K\Big(\sin\frac{\theta}{2}\Big) + \frac{4}{\pi}\int_0^1 S(\theta,x)\,dx,$$

where

$$S(\theta,x) = \frac{(\theta x)\sin[(\nu+1/2)(\theta-\xi)]\sin\xi}{\sin^{1/2}(\theta-\xi)\sin^{1/2}\xi}, \quad \xi = \frac{\theta x^2}{2},$$

and K is the complete elliptic integral of the first kind.

TABLE IV

MAXIMAL ABSOLUTE ERRORS IN CALCULATION OF $P_\nu(\cos\theta)$ FOR $0 \le \theta \le \vartheta$ AND $0 \le \nu < 2$, WHEN $N = 5$ AND $N = 10$

N	$\vartheta = 60°$	$\vartheta = 90°$	$\vartheta = 120°$	$\vartheta = 150°$
5	8.9(−7)	3.1(−6)	1.7(−4)	1.5(−3)
10	4.7(−13)	5.9(−13)	7.1(−11)	1.5(−9)

For numerical calculation of the integral $\int_0^1 S(\theta,x)\,dx$ we use the standard N-point Gauss-Legendre quadrature formula transformed before to $(0,1)$, while for the complete elliptic integral

$$K(\sin\alpha) = \int_0^{\pi/2} (1-\sin^2\alpha\sin^2\theta)^{-1/2}\, d\theta$$

we use the well-known process of the arithmetic-geometric mean (cf. [25, pp. 598–599]). An analysis of this quadrature process shows that we must take $N = 20$ in the Gauss-Legendre rule in order to get the values of $P_\nu(\cos\theta)$ for $0 \le \nu < 2$ and $0 \le \theta < \pi$ with an absolute error less than 10^{-10}. Some computational problems can occur when $\theta \to \pi$. By certain restrictions on θ, for example $0 \le \theta \le \vartheta < \pi$, our approximation for $P_\nu(\cos\theta)$ gives better results. The corresponding maximal absolute errors in calculation of $P_\nu(\cos\theta)$ are given in Table IV.

When the index $\nu \ge 2$ it is convenient to use three-term recurrence relation (19), starting by two values $P_\mu(\cos\theta)$ and $P_{\mu+1}(\cos\theta)$, where $0 \le \mu < 1$. One similar procedure was considered in [23].

IV. INTEGRATION OF OSCILLATING FUNCTIONS AND PRODUCT INTEGRATION RULES

In this section we consider integrals of the form

$$I(f,K) = I(f(\cdot),K(\cdot;x)) = \int_a^b w(t)f(t)K(t;x)\,dt, \quad (20)$$

where (a,b) is an interval on the real line, which may be finite or infinite, $w(t)$ is a given weight function as before, and the kernel $K(t;x)$ is a function depending on a parameter x and such that it is highly oscillatory or has singularities on the interval (a,b) or in its nearness. Usually, an application of standard quadrature formulas to $I(f;K)$ requires a large number of nodes and too much computation work in order to achieve a modest degree of accuracy. A few typical examples of such kernels are:

1° Oscillatory kernel $K(t;x) = e^{ixt}$, where $x = \omega$ is a large positive parameter. In this class we have Fourier integrals over $(0,+\infty)$ (Fourier transforms)

$$F(f;\omega) = \int_0^{+\infty} t^\mu f(t)e^{i\omega t}\,dt \qquad (\mu > -1)$$

or Fourier coefficients

$$c_k(f) = a_k(f) + ib_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{ikt}\,dt, \quad (21)$$

where $\omega = k \in \mathbb{N}$. There are also some other oscillatory integral transforms like the Bessel transforms

$$H_m(x) = \int_0^{+\infty} t^\mu f(t) H_\nu^{(m)}(\omega t)\, dt \quad (m = 1, 2), \tag{22}$$

where ω is a real parameter and $H_\nu^{(m)}(t)$, $m = 1, 2$, are the Hankel functions (see Wong [26]). Also, we mention here a type of integrals involving Bessel functions

$$I_\nu(f;\omega) = \int_0^{+\infty} e^{-t^2} J_\nu(\omega t) f(t^2) t^{\nu+1}\, dt, \quad \nu > -1, \tag{23}$$

where ω is a large positive parameter.

2° Logarithmic singular kernel $K(t;x) = \log|t - x|$, where $a \le x \le b$.

3° Algebraic singular kernel $K(t;x) = |t - x|^\alpha$, where $\alpha > -1$ and $a < x < b$.

Also, we mention here an important case when $K(t;x) = 1/(t - x)$, where $a < x < b$ and the integral (20) is taken to be a Cauchy principal value integral.

In this section we consider only integration of oscillatory functions.

A. A Summary on Standard Methods

The earliest formulas for numerical integration of rapidly oscillatory function are based on the piecewise approximation by the low degree polynomials of $f(x)$ on the integration interval. The resulting integrals over subintervals are then integrated exactly. A such method was obtained by Filon [27]. The error estimate was given by Håvie [28] and Ehrenmark [29]. There are many improvements of this method. For example, Flinn [30] used fifth-degree polynomials in order to approximate $f(x)$ taking values of function and values of its derivative at the points x_{2k-2}, x_{2k-1}, and x_{2k}, and Stetter [31] used the idea of approximating the transformed function by polynomials in $1/t$.

The construction of Gaussian formulae for oscillatory weights has also been considered (cf. Gautschi [32], Piessens [33], [34], [35]). Defining nonnegative functions on $[-1, 1]$,

$$u_k(t) = \frac{1}{2}(1 + \cos k\pi t), \quad v_k(t) = \frac{1}{2}(1 + \sin k\pi t),$$

the Fourier coefficients (21) can be expressed in the form

$$a_k(f) = 2\int_{-1}^{1} f(\pi t) u_k(t)\, dt - \int_{-1}^{1} f(\pi t)\, dt$$

and

$$b_k(f) = 2\int_{-1}^{1} f(\pi t) v_k(t)\, dt - \int_{-1}^{1} f(\pi t)\, dt.$$

Now, the Gaussian formulae can be obtained for the first integrals on the right-hand side in these equalities. For $k = 1(1)12$ Gautschi [32] obtained n-point Gaussian formulas with 12 decimal digits when $n = 1(1)8$, $n = 16$, and $n = 32$. We mention, also, that for the interval $[0, +\infty)$ and the weight functions $w_1(t) = (1 + \cos t)(1 + t)^{-(2n-1+s)}$ and $w_2(t) = (1 + \sin t)(1 + t)^{-(2n-1+s)}$, $n = 1(1)10$, $s = 1.05(0.05)4$, the n-point formulas were constructed by Krilov and Kruglikova [36].

Quadrature formulas for the Fourier and the Bessel transforms (22) were derived by Wong [26].

Other formulas are based on the integration between the zeros of $\cos mx$ or $\sin mx$ (cf. [37]–[41]). In general, if the zeros of the oscillatory part of the integrand are located in the points x_k, $k = 1, 2, \ldots, m$, on the integration interval $[a, b]$, where $a \le x_1 < x_2 < \cdots < x_m \le b$, then we can calculate the integral on each subinterval $[x_k, x_{k+1}]$ by an appropriate rule. A Lobatto rule is good for this purpose (see Davis and Rabinowitz [37, p. 121]) because of use the end points of the integration subintervals, where the integrand is zero, so that more accuracy can be obtained without additional computation.

There are also methods based on the Euler and other transformations to sum the integrals over the trigonometric period (cf. Longman [42], Hurwitz and Zweifel [43]).

B. Product Integration Rules

Consider the integral (20) with a "well-behaved" function f on (a, b). The main idea in the method of product integration is to determine the adverse behaviour of the kernel K in an analytic form.

Let $\pi_k(\cdot)$, $k = 0, 1, \ldots$, be orthogonal polynomials with respect to the weight $w(t)$ on (a, b), and let λ_ν and τ_ν ($\nu = 1, \ldots, n$) be Christoffel numbers and nodes, respectively, of the n-point Gaussian quadrature formula (6). Further, let $L_n(f;\cdot)$ be the Lagrange interpolation polynomial for the function f, based on the zeros of $\pi_n(t)$, i.e.,

$$L_n(f;t) = \sum_{\nu=1}^{n} f(\tau_\nu)\ell_\nu(t),$$

where $\ell_\nu(t) = \pi_n(t)/((t - \tau_\nu)\pi_n'(\tau_\nu))$, $\nu = 1, \ldots, n$. Expanding it in terms of orthogonal polynomials $\{\pi_\nu\}$, we have

$$L_n(f;t) = \sum_{\nu=0}^{n-1} a_\nu \pi_\nu(t),$$

where the coefficients a_ν, $\nu = 0, 1, \ldots, n-1$, are given by

$$a_\nu = \frac{1}{\|\pi_\nu\|^2}(L_n(f;\cdot), \pi_\nu) = \frac{1}{\|\pi_\nu\|^2}\int_a^b w(t) L_n(f;t)\pi_\nu(t)\, dt.$$

Since the degree of $L_n(f;\cdot)\pi_\nu(\cdot) \le 2n - 2$, we can apply Gaussian formula (6), and then

$$a_\nu = \frac{1}{\|\pi_\nu\|^2}\sum_{k=1}^{n} \lambda_k f(\tau_k)\pi_\nu(\tau_k), \tag{24}$$

because of $L_n(f;\tau_k) = f(\tau_k)$ for each $k = 1, \ldots, n$.

Putting $L_n(f;t)$ in (20) instead of $f(t)$ we obtain

$$I(f, K) = Q_n(f;x) + R_n^{PR}(f;x),$$

where

$$Q_n(f;x) = \int_a^b w(t)L_n(f;t)K(t;x)\,dt,$$

i.e.,

$$Q_n(f;x) = \sum_{\nu=0}^{n-1} a_\nu \int_a^b w(t)\pi_\nu(t)K(t;x)\,dt \qquad (25)$$

and $R_n^{PR}(f;x)$ is the corresponding remainder. By $b_\nu(x)$ we denote the integrals in (25),

$$b_\nu(x) = \int_a^b w(t)\pi_\nu(t)K(t;x)\,dt, \quad \nu = 0,1,\dots,n-1. \quad (26)$$

Finally, we obtain so-called the *product integration rule*

$$Q_n(f;x) = \sum_{\nu=0}^{n-1} a_\nu b_\nu(x), \qquad (27)$$

where the coefficients a_ν and $b_\nu(x)$ are given by (24) and (26), respectively. Another form of (27) is

$$Q_n(f;x) = \sum_{k=1}^{n} \Lambda_k(x) f(\tau_k), \qquad (28)$$

where

$$\Lambda_k(x) = \lambda_k \sum_{\nu=0}^{n-1} \frac{1}{\|\pi_\nu\|^2}\pi_\nu(\tau_k)b_\nu(x), \quad k = 1,\dots,n.$$

As we mentioned on the beginning of this subsection, it is very important in this method to have $b_\nu(x)$ in an analytic form. It is very convenient if we have a Fourier expansion of the kernel $K(\cdot;x)$ in terms of orthogonal polynomials π_ν,

$$K(t;x) = \sum_{\nu=0}^{+\infty} B_\nu(x)\pi_\nu(t).$$

Because of (26), we see that $B_\nu(x) = b_\nu(x)/\|\pi_\nu\|^2$.

Let $K_n(\cdot;x)$ be the best L^2-approximation of $K(\cdot;x)$ in $\mathcal{P}_{n-1}$, i.e.,

$$K_n(t;x) = \sum_{\nu=0}^{n-1} \frac{b_\nu(x)}{\|\pi_\nu\|^2}\pi_\nu(t). \qquad (29)$$

We can see that the product integration rule (27), i.e., (28), is equivalent to the Gaussian rule applied to the function $f(\cdot)K_n(\cdot;x)$. Indeed, since $\Lambda_k(x) = \lambda_k K_n(\tau_k;x)$, we have

$$Q_n^G(f(\cdot)K_n(\cdot;x)) = \sum_{k=1}^{n} \lambda_k f(\tau_k)K_n(\tau_k;x) = Q_n(f;x).$$

In some applications $K_n(\tau_k;x)$ can be computed conveniently by Clenshaw's algorithm based on the recurrence relation (2) for the orthogonal polynomials π_ν.

In some cases we know analytically the coefficients in an expansion of (29). Now, we give some of such examples.

In [44, p. 560] we used

$$\int_{-1}^{1} C_k^\lambda(t)e^{i\omega t}(1-t^2)^{\lambda-1/2}\,dt = i^k \frac{2\pi\Gamma(2\lambda+k)}{k!\Gamma(\lambda)(2\omega)^\lambda} J_{k+\lambda}(\omega),$$

where $C_k^\lambda(t)$ ($\lambda > -1/2$) is the Gegenbauer polynomial of degree k. Taking this exact value of the integral we find the following expansion of $e^{i\omega t}$ in terms of Gegenbauer polynomials,

$$K(t;\omega) = e^{i\omega t} \sim \left(\frac{2}{\omega}\right)^\lambda \Gamma(\lambda) \sum_{k=0}^{+\infty} i^k(k+\lambda)J_{k+\lambda}(\omega)C_k^\lambda(t),$$

where $x \in [-1,1]$. In this case, (28) reduces to the product rule with respect to the Gegenbauer weight.

In some special cases we get: (1) For $\lambda = 1/2$ – the method of Bakhvalov-Vasil'eva [45]; (2) For $\lambda = 0$ and $\lambda = 1$ – the method of Patterson [46]. An approximation by Chebyshev polynomials was considered by Piessens and Poleunis [47].

Taking the expansion

$$e^{i\omega t} \sim e^{-(\omega/2)^2} \sum_{k=0}^{+\infty} i^k \frac{(\omega/2)^k}{k!} H_k(t), \quad |t| < +\infty,$$

where H_k is the Hermite polynomial of degree n, we can calculate integrals of the form $\int_{-\infty}^{+\infty} e^{-t^2}e^{i\omega t}f(t)\,dt$. In a similar way we can use the expansion

$$e^{i\omega t^2} \sim \sum_{k=0}^{+\infty} \frac{(i\omega)^k}{k!2^{2k}(1-i\omega)^{k+1/2}} H_{2k}(x), \quad |t| < +\infty.$$

Consider now the integral $I_\nu(f;\omega)$ given by (23), which can be reduced to the following form

$$\begin{aligned} I_\nu(f;\omega) &= \frac{1}{2}\int_0^{+\infty} e^{-t}J_\nu(\omega\sqrt{t})f(t)t^{\nu/2}\,dt \\ &= \frac{1}{2}\int_0^{+\infty} t^\nu e^{-t}\big[t^{-\nu/2}J_\nu(\omega\sqrt{t})\big]f(t)t^{\nu/2}\,dt, \end{aligned}$$

where we put the oscillatory kernel in the brackets. Using the monic generalized Laguerre polynomials $\hat{L}_n^\nu(t)$, which are orthogonal on $(0,+\infty)$ with respect to the weight $t^\nu e^{-t}$, we get the expansion

$$t^{-\nu/2}J_\nu(\omega\sqrt{t}) \sim \left(\frac{\omega}{2}\right)^\nu e^{-(\omega/2)^2} \sum_{k=0}^{+\infty} \frac{(-1)^k(\omega/2)^{2k}}{k!\Gamma(k+\nu+1)} \hat{L}_n^\nu(t).$$

Thus, in this case the the coefficients (26) become

$$b_k(\omega) = (-1)^k\left(\frac{\omega}{2}\right)^{\nu+2k} e^{-(\omega/2)^2}.$$

In 1979 Gabutti [48] investigated in details the case $\nu = 0$. Using a special procedure in D-arithmetic on an IBM 360/75 computer he illustrated the method taking an example with $f(t) = \sin t$ and $\omega = 20$.

At the end we mention that it is possible to find exactly $I_\nu(f;\omega)$ when $f(t) = e^{i\alpha t}$. Namely,

$$I_\nu(e^{i\alpha t};\omega) = \frac{1}{2}\left(\frac{\omega}{2}\right)^\nu \frac{1}{(1-i\alpha)^{\nu+1}} \exp\left[-\frac{(\omega/2)^2}{1-i\alpha}\right].$$

The imaginary part of this gives the previous example. An asymptotic behaviour of this integral was investigated by Frenzen and Wong [49] (see also Gabutti [50] and Gabutti and Lepora [51]).

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