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A solution to exponential operators

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Abstract The present article provides a solution to the open problem on the exponential type operators, connected with $1+x^2$. We are able to achieve the semi-exponential extension of such operators.

Keywords Semi-exponential operators · Ismail-May operators · · operators.

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1 Operators connected with $1+x^2$

The exponential operators (see [4]) connected with $1+x^2$ are defined by

$$\begin{aligned}(T_\lambda f)(x) &= \int_{-\infty}^{\infty} \kappa_\lambda(x,t) f(t) dt \\ &= \frac{1}{(1+x^2)^{\lambda/2}} \frac{2^{\lambda-2}\lambda}{\pi \Gamma(\lambda)} \int_{-\infty}^{\infty} \left| \Gamma\left(\lambda \frac{1+it}{2}\right)\right|^2 e^{\lambda t \arctan x} f(t)\, dt \quad (1)\end{aligned}$$

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Following A. Tyliba and E. Wachnicki in [7] and M. Horzog [3], very recently Abel et al. [1] captured the extension of exponential type operators, but for the operators (1) authors [1] were not able to find the extension of T_n to its semi-exponential version and they proposed it an open problem. In this note we provide the answer to the open problem, associated with $1+x^2$.

2 Solution to Open Problem

The semi-exponential operators associated with $1+x^2$, with $\beta>0$ the kernel $\kappa_\lambda^\beta(x,t)$ satisfy the following partial differential equation

$$\frac{\partial}{\partial x}\kappa_\lambda^\beta(x,t)=\left[\frac{\lambda(t-x)}{1+x^2}-\beta\right]\kappa_\lambda^\beta(x,t)$$

integrating, we get

$$\log\kappa_\lambda^\beta(x,t)=\lambda t\arctan x-\beta x-\frac{\lambda}{2}\log(1+x^2)+\log A_T^\beta\left(\lambda,t\right)$$

i.e.,

$$\kappa_\lambda^\beta(x,t)=A_T^\beta\left(\lambda,t\right)e^{\lambda t\arctan x-\beta x}\left(1+x^2\right)^{-\lambda/2},$$

where $A_T^\beta\left(\lambda,t\right)$ is constant of integration independent of x. The operator have the form:

$$\left(T_\lambda^\beta f\right)(x)=\frac{e^{-\beta x}}{\left(1+x^2\right)^{\lambda/2}}\int_{-\infty}^{\infty}A_T^\beta\left(\lambda,t\right)e^{\lambda t\arctan x}f(t)\,dt$$

or

$$\left(T_\lambda^\beta f\right)(x)=\frac{e^{-\beta x}}{\left(1+x^2\right)^{\lambda/2}}\int_{-\infty}^{\infty}B_T^\beta\left(\lambda,t\right)e^{t\arctan x}f\left(\frac{t}{\lambda}\right)dt$$

Our target is to find $B_T^\beta\left(\lambda,t\right)$. To have the normalization we need $B_T^\beta\left(\lambda,t\right)$ such that

$$\int_{-\infty}^{\infty}B_T^\beta\left(\lambda,t\right)\frac{e^{t\arctan x}}{\left(1+x^2\right)^{\lambda/2}}dt=e^{\beta x}. \tag{2}$$

The Meixner-Pollaczek polynomials $\{p_n^{(\lambda)}\}_{n=0}^{\infty}$ are defined by the following three-term recurrence relation (cf. [5], [2])

$$(n+1)p_{n+1}^{(\lambda)}(t)=t\,p_n^{(\lambda)}(t)-(n-1+2\lambda)p_{n-1}^{(\lambda)}(t),\quad n=0,1,2,\ldots,$$

with $p_0^{(\lambda)}(t)=1$ and $p_{-1}^{(\lambda)}(t)=0$, where the parameter $\lambda>0$. These polynomials are orthogonal on the real line with respect to the weight function

$$W_\lambda(t)=\frac{1}{2\pi}\left|\Gamma\left(\lambda+i\,\frac{t}{2}\right)\right|^2,$$

and their generating function is

$$G_\lambda(t,x) = \frac{e^{t\arctan x}}{(1+x^2)^\lambda} = \sum_{n=0}^{\infty} p_n^{(\lambda)}(t)x^n. \tag{3}$$

In our work we will use the corresponding monic polynomials $P_n^{(\lambda/2)}(t)$ (Note the parameter λ is replaced by $\lambda/2$), which satisfy three-term recurrence relation

$$P_{n+1}^{(\lambda/2)}(t) = t\,P_n^{(\lambda/2)}(t) - n(n-1+\lambda)P_{n-1}^{(\lambda/2)}(t), \quad n = 0,1,2,\ldots, \tag{4}$$

with $P_0^{(\lambda/2)}(t) = 1$ and $P_{-1}^{(\lambda/2)}(t) = 0$. We note that $P_n^{(\lambda/2)}(t) = n!p_n^{(\lambda/2)}(t)$, $n \in N$, so that (3) reduces to

$$G_{\lambda/2}(t,x) = \frac{e^{t\arctan x}}{(1+x^2)^{\lambda/2}} = \sum_{n=0}^{\infty} P_n^{(\lambda/2)}(t)\,\frac{x^n}{n!}. \tag{5}$$

Explicit expressions for these orthogonal polynomials can be given in terms of the Gauss hypergeometric function,

$$P_n^{(\lambda/2)}(t) = (\lambda)_n i^n {}_2F_1\left(\begin{matrix} -n,\ (\lambda+i\,t)/2 \\ \lambda \end{matrix}\ \middle|\ 2\right), \quad n = 0,1,2,\ldots\ . \tag{6}$$

For example,

$$\begin{aligned}
&P_0^{(\lambda/2)}(t) = 1, \quad P_1^{(\lambda/2)}(t) = t, \quad P_2^{(\lambda/2)}(t) = t^2-\lambda, \quad P_3^{(\lambda/2)}(t) = t^3-(3\lambda+2)t,\\
&P_4^{(\lambda/2)}(t) = t^4 - 2(3\lambda+4)t^2 + 3\lambda(\lambda+2),\\
&P_5^{(\lambda/2)}(t) = t^5 - 10(\lambda+2)t^3 + \left(15\lambda^2+50\lambda+24\right)t,\\
&P_6^{(\lambda/2)}(t) = t^6 - 5(3\lambda+8)t^4 + \left(45\lambda^2+210\lambda+184\right)t^2 - 15\lambda(\lambda+2)(\lambda+4),\\
&P_7^{(\lambda/2)}(t) = t^7 - 7(3\lambda+10)t^5 + 7\left(15\lambda^2+90\lambda+112\right)t^3 - 3\left(35\lambda^3+280\lambda^2+588\lambda+240\right)t, \text{ etc.}
\end{aligned}$$

According to (5), Eq. (2) can be written as

$$\int_{-\infty}^{\infty} B_T^{\beta}(\lambda,t)\,G_{\lambda/2}(t,x)dt = e^{\beta x},$$

i.e,

$$\int_{-\infty}^{\infty} B_T^{\beta}(\lambda,t)\left(\sum_{n=0}^{\infty} P_n^{(\lambda/2)}(t)\,\frac{x^n}{n!}\right)dt = \sum_{n=0}^{\infty} \beta^n\frac{x^n}{n!},$$

from where, by comparing the coefficients of x^n, we get

$$\int_{-\infty}^{\infty} B_T^{\beta}(\lambda,t)\,P_n^{(\lambda/2)}(t)\,dt = \beta^n, \quad n = 0,1,2,\ldots\ . \tag{7}$$

In Ismail and May case ($\beta = 0$) the previous relations become

$$\int_{-\infty}^{\infty} B_T^{0}(\lambda,t)\,P_n^{(\lambda/2)}(t)\,dt = \delta_{n,0}, \quad n = 0,1,2,\ldots\ ,$$

where $\delta_{i,j}$ is Kronecker's delta, and evidently $B_T^0(\lambda,t)$ must be equal to $W_{\lambda/2}(t)$, up to some normalization constant.

In a general case we suppose that $B_T^{\beta}(\lambda,t) = \Phi(t;\lambda,\beta)W_{\lambda/2}(t)$, where $t \mapsto \Phi(t;\lambda,\beta)$ is an analytic function for which the following expansion in polynomials $\{P_n^{(\lambda/2)}\}_{n=0}^{\infty}$,

$$\Phi(t;\lambda,\beta) = \sum_{k=0}^{\infty} b_k P_k^{(\lambda/2)}(t), \tag{8}$$

holds, where the coefficients b_k depend on the parameters λ and β. Then (7) becomes

$$\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty} b_k P_k^{(\lambda/2)}(t)\right) P_n^{(\lambda/2)}(t)W_{\lambda/2}(t)\,dt = \beta^n, \quad n = 0,1,2,\ldots\ .$$

Because of orthogonality

$$\langle P_k^{(\lambda/2)}, P_n^{(\lambda/2)}\rangle = \int_{-\infty}^{\infty} P_k^{(\lambda/2)}(t)P_n^{(\lambda/2)}(t)W_{\lambda/2}(t)\,dt = \|P_n^{(\lambda/2)}\|^2\delta_{k,n},$$

we conclude that

$$b_n\|P_n^{(\lambda/2)}\|^2 = \beta^n, \quad n = 0,1,2,\ldots\ .$$

In order to find $\|P_n^{(\lambda/2)}\|^2$ we use the recurrence coefficients $\beta_k = k(k-1+\lambda)$, $k = 1,2,\ldots$ from (4), so that (cf. [6, p. 97])

$$\|P_n^{(\lambda/2)}\|^2 = \beta_0\beta_1\cdots\beta_n = \frac{n!\Gamma(\lambda+n)}{2^{\lambda-1}},$$

where

$$\beta_0 := \int_{-\infty}^{\infty} W_{\lambda/2}(t)\,dt = \frac{\Gamma(\lambda)}{2^{\lambda-1}}.$$

In this way, we obtain the coefficients b_k in the expansion (8) as

$$b_k = \frac{2^{\lambda-1}\beta^k}{k!\Gamma(\lambda+k)}, \quad k = 0,1,2,\ldots\ ,$$

as well as

$$B_T^{\beta}(\lambda,t) = W_{\lambda/2}(t)\Phi(t;\lambda,\beta) = \frac{2^{\lambda-2}}{\pi}\left|\Gamma\left(\frac{\lambda+i\,t}{2}\right)\right|^2 \sum_{k=0}^{\infty}\frac{\beta^k}{k!\Gamma(\lambda+k)}P_k^{(\lambda/2)}(t).$$

Then the semi-exponential operators connected with $1+x^2$ have the form:

$$\left(T_\lambda^{\beta}f\right)(x) = \frac{2^{\lambda-2}e^{-\beta x}}{\pi}\sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{\lambda+i\,t}{2}\right)\right|^2 P_n^{(\lambda/2)}(t)G_{\lambda/2}(t,x)f\left(\frac{t}{\lambda}\right)dt.$$

Note that for $\beta = 0$ we have

$$B_T^0(\lambda,t) = \frac{2^{\lambda-2}}{\pi\Gamma(\lambda)}\left|\Gamma\left(\frac{\lambda+i\,t}{2}\right)\right|^2.$$

3 Moments

Using the generating function (5) we have

$$\begin{aligned}\left(T_\lambda^\beta f\right)(x) = \frac{2^{\lambda-2}e^{-\beta x}}{\pi} \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \\ \times \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{\lambda+i\,t}{2}\right)\right|^2 P_n^{(\lambda/2)}(t) P_m^{(\lambda/2)}(t)\, f\left(\frac{t}{\lambda}\right) dt.\end{aligned}$$

For $f(t) = e_s(t) = t^s, s = 0,1,2$ we obtain the moments for the semi-exponential operators as follows:

$$\begin{aligned}\left(T_\lambda^\beta e_0\right)(x) &= 2^{\lambda-1}e^{-\beta x} \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \left\langle P_n^{(\lambda/2)}, P_m^{(\lambda/2)} \right\rangle \\ &= 2^{\lambda-1}e^{-\beta x} \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \cdot \frac{x^n}{n!} \cdot \frac{n!\Gamma(\lambda+n)}{2^{\lambda-1}} \\ &= 1\end{aligned}$$

and

$$\left(T_\lambda^\beta e_1\right)(x) = \frac{2^{\lambda-1}e^{-\beta x}}{\lambda} \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \left\langle tP_n^{(\lambda/2)}, P_m^{(\lambda/2)} \right\rangle.$$

Using three-term recurrence relation (4) it reduces to

$$\begin{aligned}\left(T_\lambda^\beta e_1\right)(x) &= \frac{2^{\lambda-1}e^{-\beta x}}{\lambda} \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \langle P_{n+1}^{(\lambda/2)} + \beta_n P_{n-1}^{(\lambda/2)}, P_m^{(\lambda/2)} \rangle \\ &= \frac{2^{\lambda-1}e^{-\beta x}}{\lambda} \left\{ \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \langle P_{n+1}^{(\lambda/2)}, P_m^{(\lambda/2)} \rangle \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \beta_n \langle P_{n-1}^{(\lambda/2)}, P_m^{(\lambda/2)} \rangle \right\} \\ &= \frac{2^{\lambda-1}e^{-\beta x}}{\lambda} \left\{ \sum_{n=0}^{\infty} \frac{\beta^n}{n!\Gamma(\lambda+n)} \cdot \frac{x^{n+1}}{(n+1)!} \cdot \frac{(n+1)!\Gamma(\lambda+n+1)}{2^{\lambda-1}} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{\beta^{n+1}}{(n+1)!\Gamma(\lambda+n+1)} \cdot \frac{x^n}{n!} \cdot (n+1)(n+\lambda) \cdot \frac{n!\Gamma(\lambda+n)}{2^{\lambda-1}} \right\} \\ &= \frac{e^{-\beta x}}{\lambda} \left\{ x(\lambda+\beta x)e^{\beta x} + \beta e^{\beta x} \right\} \\ &= x + \frac{\beta}{\lambda}(1+x^2).\end{aligned}$$

Similarly, for $f(t) = e_2(t) = t^2$ we have

$$\left(T_\lambda^\beta e_2\right)(x) = \frac{2^{\lambda-1}e^{-\beta x}}{\lambda^2}\sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\sum_{m=0}^{\infty}\frac{x^m}{m!}\left\langle tP_n^{(\lambda/2)}, tP_m^{(\lambda/2)}\right\rangle,$$

i.e.,

$$\begin{aligned}
\left(T_\lambda^\beta e_2\right)(x) &= \frac{2^{\lambda-1}e^{-\beta x}}{\lambda^2}\sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\sum_{m=0}^{\infty}\frac{x^m}{m!}\left\langle P_{n+1}^{(\lambda/2)} + \beta_n P_{n-1}^{(\lambda/2)}, P_{m+1}^{(\lambda/2)} + \beta_m P_{m-1}^{(\lambda/2)}\right\rangle \\
&= \frac{2^{\lambda-1}e^{-\beta x}}{\lambda^2}\left\{\sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\sum_{m=0}^{\infty}\frac{x^m}{m!}\left\langle P_{n+1}^{(\lambda/2)}, P_{m+1}^{(\lambda/2)}\right\rangle\right. \\
&\quad + \sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\sum_{m=1}^{\infty}\frac{x^m}{m!}\beta_m\left\langle P_{n+1}^{(\lambda/2)}, P_{m-1}^{(\lambda/2)}\right\rangle \\
&\quad + \sum_{n=1}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\sum_{m=0}^{\infty}\frac{x^m}{m!}\beta_n\left\langle P_{n-1}^{(\lambda/2)}, P_{m+1}^{(\lambda/2)}\right\rangle \\
&\quad \left. + \sum_{n=1}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\sum_{m=1}^{\infty}\frac{x^m}{m!}\beta_n\beta_m\left\langle P_{n-1}^{(\lambda/2)}, P_{m-1}^{(\lambda/2)}\right\rangle\right\} \\
&= \frac{2^{\lambda-1}e^{-\beta x}}{\lambda^2}\left\{\sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\cdot\frac{x^n}{n!}\cdot\frac{(n+1)!\Gamma(\lambda+n+1)}{2^{\lambda-1}}\right. \\
&\quad + \sum_{n=0}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\cdot\frac{x^{n+2}}{(n+2)!}(n+2)(n+1+\lambda)\frac{(n+1)!\Gamma(\lambda+n+1)}{2^{\lambda-1}} \\
&\quad + \sum_{n=2}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\cdot\frac{x^{n-2}}{(n-2)!}n(n-1+\lambda)\frac{(n-1)!\Gamma(\lambda+n-1)}{2^{\lambda-1}} \\
&\quad \left. + \sum_{n=1}^{\infty}\frac{\beta^n}{n!\Gamma(\lambda+n)}\cdot\frac{x^n}{n!}n^2(n-1+\lambda)^2\frac{(n-1)!\Gamma(\lambda+n-1)}{2^{\lambda-1}}\right\} \\
&= x^2 + \frac{1}{\lambda^2}(1+x^2)\left[\beta^2(1+x^2) + 2\beta(1+\lambda)x + \lambda\right].
\end{aligned}$$

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