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A trigonometric orthogonality with respect to a nonnegative Borel measure

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Abstract. In this paper we consider trigonometric polynomials of semi-integer degree orthogonal with respect to a linear functional, defined by a nonnegative Borel measure. By using a suitable vector form we consider the corresponding Fourier sums and reproducing kernels for trigonometric polynomials of semi-integer degree. Also, we consider the Christoffel function, and prove that it satisfies extremal property analogous with the algebraic case.

1. Introduction

A trigonometric polynomial of semi-integer degree $n + 1/2$ is a trigonometric function of the following form

$$\sum_{\nu=0}^{n}\left[c_\nu \cos\left(\nu+\frac{1}{2}\right)x + d_\nu \sin\left(\nu+\frac{1}{2}\right)x\right], \tag{1}$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_n| + |d_n| \neq 0$. The coefficients c_n and d_n are called the leading coefficients.

The orthogonal trigonometric polynomials of semi-integer degree are connected with quadrature rules with an even maximal trigonometric degree of exactness (an odd number of nodes). These quadrature rules have application in the numerical integration of 2π-periodic functions. The first results on orthogonal trigonometric polynomials of semi-integer degree on $[0, 2\pi)$ with respect to a suitable weight function were given in 1959 by Abram Haimovich Turetzkii (see [8]). Such orthogonal systems were studied in detail in [2, 4, 5].

Let us denote by $\mathcal{T}_n$, $n \in \mathbb{N}_0$, the linear space of all trigonometric polynomials of degree less than or equal to n, i.e., the linear span of the following set $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$, by $\mathcal{T}_n^{1/2}$, $n \in \mathbb{N}_0$, the linear space of all trigonometric polynomials of semi-integer degree less than or equal to $n + 1/2$, i.e., the linear span of $\{\cos(k + 1/2)x, \sin(k + 1/2)x : k = 0, 1, \ldots, n\}$, and by $\mathcal{T}$ and $\mathcal{T}^{1/2}$ the set of all trigonometric polynomials and the set of trigonometric polynomials of semi-integer degree, respectively.

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A concept of orthogonality in the space $\mathcal{T}^{1/2}$ was considered more generally in [6], where orthogonal trigonometric polynomials of semi-integer degree with respect to a linear functional, defined on the vector space $\mathcal{T}$, are investigated (the theory of orthogonal algebraic polynomials with respect to a moment functional is well-known (see [1, 3])). This paper is in some sense a continuation of [6], and therefore we briefly repeat some basic facts from [6].

Definition 1.1. Let m_0 be a real number, $\{m_n^C\}$, $\{m_n^S\}$, $n \in \mathbb{N}$, two sequences of real numbers, and let $\mathcal{L}$ be a linear functional defined on the vector space $\mathcal{T}$ by

$$\mathcal{L}[1] = m_0, \quad \mathcal{L}[\cos nx] = m_n^C, \quad \mathcal{L}[\sin nx] = m_n^S, \quad n \in \mathbb{N}.$$

Then $\mathcal{L}$ is called the *moment functional* determined by m_0 and by the sequences $\{m_n^C\}$, $\{m_n^S\}$.

For the brevity, for a 2×2 type matrix $[t_{ij}]$ with trigonometric polynomials as its entries, by $\mathcal{L}[[t_{ij}]]$ we denote the following 2×2 type matrix $[\mathcal{L}[t_{ij}]]$. For each $k \in \mathbb{N}_0$ by $\mathbf{x}^k$ we denote the column vector

$$\mathbf{x}^k = \left[\cos\left(k + \frac{1}{2}\right)x \quad \sin\left(k + \frac{1}{2}\right)x\right]^T.$$

For $k, j \in \mathbb{N}_0$, we define matrices $\mathbf{m}_{k,j}$ by $\mathbf{m}_{k,j} = \mathcal{L}[\mathbf{x}^k(\mathbf{x}^j)^T]$. For each $n \in \mathbb{N}_0$, the matrices $\mathbf{m}_{k,j}$, $k, j = 0, 1, \ldots, n$, are used to define the so-called moment matrix $M_n = [\mathbf{m}_{k,j}]_{k,j=0}^n$, and we denote its determinant by Δ_n, i.e., $\Delta_n = \det M_n$.

When we talk about orthogonal trigonometric polynomials of semi-integer degree, the orthogonality is considered only in terms of trigonometric polynomials of different semi-integer degrees, which means that trigonometric polynomials of the same semi-integer degree have to be orthogonal to all trigonometric polynomials of lower semi-integer degrees, but they may not be orthogonal among themselves. By

$$\mathbf{A}_k(x) = \left[A_{k+1/2}^{(1)}(x) \quad A_{k+1/2}^{(2)}(x)\right]^T, \quad k \in \mathbb{N}_0,$$

we denote the vector whose elements are two linearly independent trigonometric polynomials of semi-integer degree $k+1/2$, and we may also call $\mathbf{A}_k(x)$ a trigonometric polynomial of semi-integer degree $k+1/2$. By

$$\mathcal{S}\{\mathbf{A}_0(x), \ldots, \mathbf{A}_n(x)\} = \left\{A_{1/2}^{(1)}(x), A_{1/2}^{(2)}(x), \ldots, A_{n+1/2}^{(1)}(x), A_{n+1/2}^{(2)}(x)\right\}, \ n \in \mathbb{N}_0,$$

we denote the set consisting of components of the vectors $\mathbf{A}_k(x)$, $k = 0, 1, \ldots, n$. By $\mathbf{0}$ we denote the zero vector $[0 \quad 0]^T$, as well as the 2×2 type zero matrix, which will be clear from the context, and finally, by I we denote the identity matrix of type 2×2 and

$$\hat{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Definition 1.2. Let $\mathcal{L}$ be a moment functional. A sequence of trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n(x)\}_{n=0}^{+\infty}$ is said to be orthogonal with respect to $\mathcal{L}$ if the following conditions are satisfied:

$$\mathcal{L}[\mathbf{x}^k \mathbf{A}_n^T] = 0, \quad k < n; \quad \mathcal{L}[\mathbf{x}^n \mathbf{A}_n^T] = K_n, \tag{2}$$

where K_n, $n \in \mathbb{N}_0$, is an invertible 2×2 type matrix.

A system of orthogonal trigonometric polynomials of semi-integer degree with respect to the moment functional $\mathcal{L}$ exists if and only if $\Delta_n \neq 0$, $n \in \mathbb{N}_0$ ([6, Theorem 2]), and it is uniquely determined by the matrix K_n ([6, Theorem 1]).

A moment functional $\mathcal{L}$ is said to be regular if $\Delta_n \neq 0$ for all $n \in \mathbb{N}_0$, and positive definite if for all $t \in \mathcal{T}^{1/2}$, $t \neq 0$, the following inequality $\mathcal{L}[t^2] > 0$ holds. If a moment functional $\mathcal{L}$ is positive definite, then $\Delta_n > 0$ for all $n \in \mathbb{N}_0$ ([6, Theorem 3]). Therefore, every positive definite moment functional is regular.

For an orthogonal system of trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n\}$ with respect to a regular moment functional $\mathcal{L}$, by $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, we denote the following matrix

$$\boldsymbol{\mu}_n = \mathcal{L}[\mathbf{A}_n\mathbf{A}_n^T]. \tag{3}$$

The matrix $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, given by (3) is symmetric and invertible ([6, Lemma 4]). If $\mathcal{L}$ is a positive definite moment functional, then all of the matrices $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, given by (3), are positive definite ([6, Lemma 5]).

The set $\mathcal{S}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ forms a basis for $\mathcal{T}_n^{1/2}$, $n \in \mathbb{N}_0$ ([6, Lemma 2]).

If $\mathcal{L}$ is a positive definite moment functional, then there exists a system of orthonormal trigonometric polynomials of semi-integer degree $\{\mathbf{A}_n^*(x)\}$ with respect to $\mathcal{L}$, such that the following conditions are satisfied

$$\mathcal{L}[\mathbf{A}_m^*(\mathbf{A}_n^*)^T] = \delta_{m,n}I, \quad m, n \in \mathbb{N}_0,$$

where $\delta_{m,n}$ is Kronecker delta function. The system of orthonormal trigonometric polynomials of semi-integer degree with respect to a positive definite moment functional is uniquely determined up to the multiplication by an orthogonal 2×2 type matrix.

In the sequel, by $\{\mathbf{A}_n(x)\}$ we will denote the sequence of the monic orthogonal trigonometric polynomials of semi-integer degree with respect to a regular moment functional $\mathcal{L}$ (existence was proved in [6]), i.e., orthogonal trigonometric polynomials of semi-integer degree of the following form

$$\mathbf{A}_n(x) = \begin{bmatrix} A_{n+1/2}^C(x) \\ A_{n+1/2}^S(x) \end{bmatrix},$$

where $A_{n+1/2}^C(x)$ and $A_{n+1/2}^S(x)$ have the following expanded forms

$$A_{n+1/2}^C(x) = \cos\left(n + \frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[c_\nu^{(n)} \cos\left(\nu + \frac{1}{2}\right)x + d_\nu^{(n)} \sin\left(\nu + \frac{1}{2}\right)x\right], \tag{4}$$

$$A_{n+1/2}^S(x) = \sin\left(n + \frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[f_\nu^{(n)} \cos\left(\nu + \frac{1}{2}\right)x + g_\nu^{(n)} \sin\left(\nu + \frac{1}{2}\right)x\right], \tag{5}$$

for some real coefficients $c_\nu^{(n)}$, $d_\nu^{(n)}$, $f_\nu^{(n)}$ and $g_\nu^{(n)}$, $\nu = 0, 1, \ldots, n-1$.

In [6, §3] it was proved that $\{\mathbf{A}_n(x)\}$ satisfies the following three-term recurrence relation

$$2\cos x\mathbf{A}_n = \mathbf{A}_{n+1} + \boldsymbol{\alpha}_n^C\mathbf{A}_n + \boldsymbol{\beta}_n^C\mathbf{A}_{n-1}, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}, \tag{6}$$

with $\boldsymbol{\alpha}_n^C = \mathcal{L}[2\cos x\mathbf{A}_n\mathbf{A}_n^T]\boldsymbol{\mu}_n^{-1}$, $n \in \mathbb{N}_0$ and $\boldsymbol{\beta}_n^C = \boldsymbol{\mu}_n\boldsymbol{\mu}_{n-1}^{-1}$, $n \in \mathbb{N}$, $\boldsymbol{\beta}_0^C = \boldsymbol{\mu}_0$, as well as the three-term recurrence relation

$$2\sin x\mathbf{A}_n = -\hat{I}\mathbf{A}_{n+1} + \boldsymbol{\alpha}_n^S\mathbf{A}_n + \boldsymbol{\beta}_n^S\mathbf{A}_{n-1}, \quad n = 0, 1, \ldots; \quad \mathbf{A}_{-1} = \mathbf{0}, \tag{7}$$

where $\boldsymbol{\alpha}_n^S = \mathcal{L}[2\sin x\mathbf{A}_n\mathbf{A}_n^T]\boldsymbol{\mu}_n^{-1}$, $n \in \mathbb{N}_0$, $\boldsymbol{\beta}_n^S = \boldsymbol{\mu}_n\hat{I}\boldsymbol{\mu}_{n-1}^{-1}$, $n \in \mathbb{N}$.

If $\mathcal{L}$ is a positive definite moment functional, then by $\{\mathbf{A}_n^*(x)\}$ we will denote the sequence of the orthonormal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$, given by $\mathbf{A}_n^*(x) = \boldsymbol{\nu}_n^{-1}\mathbf{A}_n(x)$, where the matrix $\boldsymbol{\nu}_n$ is the positive square root of the matrix $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$ (see [6, 9]). As it was said, $\{\mathbf{A}_n(x)\}$ is a sequence of the monic orthogonal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$.

The paper is organized as follows. In the Section 2 an orthogonality with respect to a moment functional, defined by using a nonnegative Borel measure on $[-\pi, \pi)$, is considered. Special attention is devoted to the cases of the Borel measures determined by a symmetric weight function as well as by a π-periodic weight function. Fourier orthogonal series and reproducing kernels are investigated in Section 3. Also, it is proved that the corresponding Christoffel function satisfies extremal property analogous with the algebraic case.

2. Orthogonality with respect to a nonnegative Borel measure

Let $\mathrm{d}\mu$ be a nonnegative Borel measure on $\mathbb{R}$ with an infinite set as its support. Then by

$$\mathcal{L}[t] = \int t(x)\,\mathrm{d}\mu(x), \quad t \in \mathcal{T},$$

is given the positive definite linear functional. Let us suppose also that the measure $\mathrm{d}\mu$ is such that it has finite moments of all orders, i.e.,

$$0 < m_0 = \mathcal{L}[1] = \int \mathrm{d}\mu(x) < +\infty,$$

$$m_n^C = \mathcal{L}[\cos nx] < +\infty, \quad m_n^S = \mathcal{L}[\sin nx] < +\infty, \quad n \in \mathbb{N}.$$

In such cases we will refer to the measure $\mathrm{d}\mu$ instead of to the positive definite linear functional $\mathcal{L}$.

Of a special interest are examples of $\mathcal{L}$ expressible as integrals with respect to a nonnegative weight function w on some interval of length 2π, vanishing there only on a set of measure zero. Here, we choose the interval $[-\pi, \pi)$. The orthogonality on any other interval $[L, L+2\pi)$ can be reduced to $[-\pi, \pi)$ by a linear transformation (see Corollary 2.1 and Theorem 3.3 in [4]). Thus, we consider

$$\mathcal{L}[t] = \int_{-\pi}^{\pi} t(x) w(x)\,\mathrm{d}x, \quad t \in \mathcal{T},$$

and refer it to the weight function w.

According to the previously introduced notation, by

$$\mathbf{A}_n(x) = \left[A_{n+1/2}^C(x) \quad A_{n+1/2}^S(x)\right]^T$$

we denote the monic orthogonal trigonometric polynomial of semi-integer degree $n + 1/2$ with respect to a weight function w on $[-\pi, \pi)$.

2.1. Symmetric weight function

The first important case is an orthogonality with respect to a symmetric weight function w, such that $w(-x) = w(x)$, $x \in (-\pi, \pi)$. It was proved in [4] that for such a weight function, orthogonal trigonometric polynomials of semi-integer degree have a quite simple structure, i.e., $A_{n+1/2}^C(x)$ depends only on cosine functions, and $A_{n+1/2}^S(x)$ depends only on sine functions, i.e.,

$$A_{n+1/2}^C(x) = \sum_{\nu=0}^{n} c_\nu^{(n)} \cos\left(\nu + \frac{1}{2}\right)x, \quad c_n^{(n)} = 1$$

and

$$A_{n+1/2}^S(x) = \sum_{\nu=0}^{n} g_\nu^{(n)} \sin\left(\nu + \frac{1}{2}\right)x, \quad g_n^{(n)} = 1.$$

Since $A_{n+1/2}^C(x)$, $n \in \mathbb{N}_0$, is an even function and $A_{n+1/2}^S(x)$, $n \in \mathbb{N}_0$, is an odd function, $\mathcal{L}[A_{n+1/2}^C(x) A_{n+1/2}^S(x)]$ must be equal to zero. Therefore, all of the matrices $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, are diagonal.

The recurrence relations (6) and (7) can be written in the following forms

$$\mathbf{A}_{n+1} = 2\cos x \mathbf{A}_n - \boldsymbol{\alpha}_n^C \mathbf{A}_n - \boldsymbol{\beta}_n^C \mathbf{A}_{n-1}, \quad n = 0, 1, \ldots, \tag{8}$$

and

$$\mathbf{A}_{n+1} = -2\sin x \hat{I} \mathbf{A}_n + \hat{I}\boldsymbol{\alpha}_n^S \mathbf{A}_n + \hat{I}\boldsymbol{\beta}_n^S \mathbf{A}_{n-1}, \quad n = 0, 1, \ldots, \tag{9}$$

where $\mathbf{A}_{-1} = \mathbf{0}$. It can be easily seen that for a symmetric weight function w on $[-\pi, \pi)$, all of the following matrices $\boldsymbol{\alpha}_n^C$, $\boldsymbol{\beta}_n^C$, $\hat{I}\boldsymbol{\alpha}_n^S$ and $\hat{I}\boldsymbol{\beta}_n^S$ are diagonal.

2.2. π-periodic weight function

The second interesting case is of a π-periodic weight function, i.e., $w(x) = w(x+\pi)$, $x \in [-\pi, 0)$. The following theorem was proved in [2].

Theorem 2.1. *If the weight function is periodic with the period π, then for the orthogonal trigonometric polynomials of semi-integer degree the following equalities*

$$A^S_{k+1/2}(x) = (-1)^{k+1} A^C_{k+1/2}(x+\pi), \quad A^C_{k+1/2}(x) = (-1)^k A^S_{k+1/2}(x+\pi), \quad k \in \mathbb{N}_0,$$

hold.

Using Theorem 2.1, the following result can be easily proved.

Corollary 2.2. *In a case of a π-periodic weight function the following equalities*

$$\mathcal{L}[(A^C_{n+1/2}(x))^2] = \mathcal{L}[(A^S_{n+1/2}(x))^2], \quad \mathcal{L}[A^C_{n+1/2}(x)A^S_{n+1/2}(x)] = 0,$$

$$\mathcal{L}[2\cos x (A^C_{n+1/2}(x))^2] = -\mathcal{L}[2\cos x (A^S_{n+1/2}(x))^2],$$

$$\mathcal{L}[2\sin x (A^C_{n+1/2}(x))^2] = -\mathcal{L}[2\sin x (A^S_{n+1/2}(x))^2]$$

hold.

Lemma 2.3. *In a case of a π-periodic weight function we have the following:*

- *all of the matrices $\boldsymbol{\beta}^C_n$ in* (8) *and $\hat{\mathrm{I}}\boldsymbol{\beta}^S_n$ in* (9) *are diagonal with the equal entries on the main diagonal;*
- *all of the matrices $\boldsymbol{\alpha}^C_n$ in* (8) *and $\hat{\mathrm{I}}\boldsymbol{\alpha}^S_n$ in* (9) *are symmetric with the opposite numbers as entries on the main diagonal.*

Proof. According to Corollary 2.2, in a case of a π-periodic weight function all of the matrices $\boldsymbol{\mu}_n$, $n \in \mathbb{N}_0$, are diagonal with the equal entries on the main diagonal. By using that fact, Corollary 2.2 and obtained formulas for the recursion coefficients for the monic orthogonal trigonometric polynomials of semi-integer degree, it is easy to get what is stated by direct calculation. □

3. Fourier orthogonal series and reproducing kernels

Let $L^2(\mathrm{d}\mu)$ denotes a Hilbert space of measurable functions f for which

$$\int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}\mu(x) < +\infty,$$

and let define the linear functional $\mathcal{L}$ on $L^2(\mathrm{d}\mu)$ by

$$\mathcal{L}[f] = \int f(x)\,\mathrm{d}\mu(x), \quad f \in L^2(\mathrm{d}\mu). \tag{10}$$

The inner product $(\cdot,\cdot)$ is given by

$$(f,g) = \mathcal{L}[fg] = \int f(x)g(x)\,\mathrm{d}\mu(x), \quad f,g \in L^2(\mathrm{d}\mu).$$

Let $\{\mathbf{A}^*_n\}_{n\in\mathbb{N}_0}$, $\mathbf{A}^*_n = [A^{*(1)}_{n+1/2} \quad A^{*(2)}_{n+1/2}]^T$, be a sequence of orthonormal trigonometric polynomials with respect to $\mathcal{L}$. Such a sequence exists, because $\mathcal{L}$ is a positive definite functional. For any function $f \in L^2(\mathrm{d}\mu)$, we can consider its Fourier orthogonal series with respect to $\{\mathbf{A}^*_n\}$:

$$f \sim \sum_{n=0}^{\infty} \left(a^{(1)}_n(f) A^{*(1)}_{n+1/2} + a^{(2)}_n(f) A^{*(2)}_{n+1/2} \right), \quad a^{(i)}_n(f) = \mathcal{L}[f A^{*(i)}_{n+1/2}],\ i = 1,2,$$

or, using the vector notation:

$$f \sim \sum_{n=0}^{\infty} \mathbf{a}_n^T(f)\mathbf{A}_n^*, \quad \mathbf{a}_n(f) = \int f(x)\mathbf{A}_n^*(x)\,\mathrm{d}\mu(x) = \mathcal{L}[f\mathbf{A}_n^*]. \tag{11}$$

Let us denote the product $\mathbf{a}_n^T(f)\mathbf{A}_n^*$ by $A_n(f)$. Then

$$A_n(f;x) = \int f(y)A_n(x,y)\,\mathrm{d}\mu(y), \quad A_n(x,y) = (\mathbf{A}_n^*(x))^T\mathbf{A}_n^*(y).$$

The n-th partial sum of the Fourier expansion (11) can be written as follows

$$s_n(f;x) = \sum_{k=0}^{n} \mathbf{a}_n^T(f)\mathbf{A}_n^* = \int K_n(x,y)f(y)\,\mathrm{d}\mu(y), \tag{12}$$

where the function $(x,y) \mapsto K_n(x,y)$ is defined by

$$K_n(x,y) = \sum_{k=0}^{n} A_k(x,y). \tag{13}$$

It is easy to see that $A_n(x,y) = A_n(y,x)$ and then also $K_n(x,y) = K_n(y,x)$. The function K_n is called the *reproducing kernel*. The reason for that name lies in the following simple property.

Theorem 3.1. *For all $t \in \mathcal{T}_n^{1/2}$ we have the following representation*

$$t(x) = \mathcal{L}[K_n(x,\cdot)t(\cdot)] = \int K_n(x,y)t(y)\,\mathrm{d}\mu(y).$$

Proof. Let $\{\mathbf{A}_n^*\}$ be a sequence of orthonormal trigonometric polynomials of semi-integer degree. Every $t \in \mathcal{T}_n^{1/2}$ can be expanded in terms of the basis $\mathcal{S}\{\mathbf{A}_0^*, \mathbf{A}_1^*, \ldots, \mathbf{A}_n^*\}$ in the following way

$$t(x) = \sum_{k=0}^{n} \mathbf{c}_k^T(t)\mathbf{A}_k^*,$$

where, because of orthogonality, we have $\mathbf{c}_k(t) = \mathcal{L}[t\mathbf{A}_k^*]$. Thus, we get

$$t(x) = \sum_{k=0}^{n} (\mathcal{L}[t\mathbf{A}_k^*])^T\mathbf{A}_k^* = \mathcal{L}[K_n(x,\cdot)t(\cdot)],$$

what was stated. □

Notice that for $x \in \mathbb{R}$ and $n \geq 0$ one has

$$K_n(x,x) = \sum_{k=0}^{n} A_k(x,x) = \sum_{k=0}^{n} (\mathbf{A}_k^*(x))^T\mathbf{A}_k^*(x) = \sum_{k=0}^{n} \|\mathbf{A}_k^*(x)\|^2 > 0,$$

where $\|\cdot\|$ denotes Euclidean norm on $\mathbb{R}^2$. The reciprocal of this function is the *Christoffel function*

$$\lambda_n(x) = (K_n(x,x))^{-1}, \tag{14}$$

and it satisfies the extremal property given in the following statement.

Theorem 3.2. *Let $\mathcal{L}$ be a positive definite linear functional. Then, for an arbitrary point $x \in \mathbb{R}$ the following equality*

$$\lambda_n(x) = \min\left\{\mathcal{L}[t^2] : t(x) = 1, t \in \mathfrak{T}_n^{1/2}\right\}$$

holds.

Proof. Let $\{\mathbf{A}_n^*\}$ be a sequence of orthonormal trigonometric polynomials of semi-integer degree with respect to $\mathcal{L}$. If $t \in \mathfrak{T}_n^{1/2}$ it can be written as follows

$$t(x) = \sum_{k=0}^{n} \mathbf{a}_k^T(t)\mathbf{A}_k^*(x), \quad \mathbf{a}_k(t) = \mathcal{L}[t\mathbf{A}_k^*].$$

By the orthogonality we have

$$\mathcal{L}[t^2] = \sum_{k=0}^{n} \mathbf{a}_k^T(t)\mathbf{a}_k(t) = \sum_{k=0}^{n} \|\mathbf{a}_k(t)\|^2.$$

If x is a fixed number and $t \in \mathfrak{T}_n^{1/2}$ such that $t(x) = 1$, then, by Cauchy's inequality (see [7]), we get

$$\begin{aligned} 1 &= t^2(x) = \left(\sum_{k=0}^{n} \mathbf{a}_k^T(t)\mathbf{A}_k^*(x)\right)^2 \leq \left(\sum_{k=0}^{n} \|\mathbf{a}_k(t)\|\|\mathbf{A}_k^*(x)\|\right)^2 \\ &\leq \sum_{k=0}^{n} \|\mathbf{a}_k(t)\|^2 \sum_{k=0}^{n} \|\mathbf{A}_k^*(x)\|^2 = \mathcal{L}[t^2]K_n(x,x), \end{aligned}$$

from which follows

$$\mathcal{L}[t^2] \geq \frac{1}{K_n(x,x)} = \lambda_n(x).$$

Equality is attained only if the sequences $\{\mathbf{a}_k(t)\}_{k=0}^n$ and $\{\mathbf{A}_k^*(x)\}_{k=0}^n$ are proportional, i.e., if $\mathbf{a}_k(t) = \gamma\mathbf{A}_k^*(x)$, $k = 0, 1, \ldots, n$, for some real constant γ. Then we have

$$1 = t(x) = \sum_{k=0}^{n} \mathbf{a}_k^T(t)\mathbf{A}_k^*(x) = \sum_{k=0}^{n} \gamma(\mathbf{A}_k^*(x))^T\mathbf{A}_k^*(x) = \gamma K_n(x,x).$$

Thus, equality holds if and only if $\mathbf{a}_k(t) = (K_n(x,x))^{-1}\mathbf{A}_k^*(x)$. □

Theorem 3.3. *Let $\mathcal{L}$ be a positive definite moment functional given by* (10) *and $f \in L^2(\mathrm{d}\mu)$. Then, among all trigonometric polynomials of semi-integer degree t in $\mathfrak{T}_n^{1/2}$, the value $\mathcal{L}[|f - t|^2]$ becomes minimal if and only if $t(x) = s_n(f, x)$.*

Proof. Let $\mathcal{S}\{\mathbf{A}_1^*, \ldots, \mathbf{A}_n^*\}$ be an orthonormal basis of $\mathfrak{T}_n^{1/2}$. For any $t \in \mathfrak{T}_n^{1/2}$ there exist $\mathbf{b}_k$, $k = 0, 1, \ldots, n$, such that

$$t(x) = \sum_{k=0}^{n} \mathbf{b}_k^T\mathbf{A}_k^*(x).$$

According to the orhonormal property of $\{\mathbf{A}_k^*\}$ and (11) we have

$$\begin{aligned} 0 &\leq \mathcal{L}[|f - t|^2] = \mathcal{L}[f^2] - 2\sum_{k=0}^{n} \mathbf{b}_k^T\mathcal{L}[f\mathbf{A}_k^*] + \sum_{k=0}^{n} \mathbf{b}_k^T\mathbf{b}_k \\ &= \mathcal{L}[f^2] - 2\sum_{k=0}^{n} \mathbf{b}_k^T\mathbf{a}_k(f) + \sum_{k=0}^{n} \mathbf{b}_k^T\mathbf{b}_k \\ &= \mathcal{L}[f^2] - \sum_{k=0}^{n} \mathbf{a}_k^T(f)\mathbf{a}_k(f) + \sum_{k=0}^{n} \left(\mathbf{a}_k^T(f)\mathbf{a}_k(f) + \mathbf{b}_k^T\mathbf{b}_k - 2\mathbf{b}_k^T\mathbf{a}_k(f)\right). \end{aligned}$$

The third term on the right hand side in previous inequality is nonnegative according to Cauchy's inequality, and it vanishes for $\mathbf{b}_k = \mathbf{a}_k(f)$, $k = 0, 1, \ldots, n$. Therefore, the value $\mathcal{L}[|f - t|^2]$ is minimal if and only if $t(x) = s_n(f, x)$. $\square$

Remark 3.4. From the previous theorem in case when minimum is attained we get *Bessel's inequality*:

$$\sum_{k=0}^{+\infty} (\mathbf{a}_k(f))^T \mathbf{a}_k(f) \leq \mathcal{L}[f^2].$$

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