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EXTREMAL PROBLEMS FOR COEFFICIENTS OF ALGEBRAIC POLYNOMIALS

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Abstract. Extremal problems for coefficients of algebraic polynomials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ are considered. In the other words, under some restrictions of the class of all polynomials of degree n, the upper bounds for $|P^{(k)}(0)|$, which include L^2-norm of P on the real line, are investigated.

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n. We will consider some extremal problems of the form

$$|P^{(k)}(0)| \le C_{n,k}\|P\|.$$

The first result on this subject was given by V. A. Markov [3]. Namely, if

$$\|P\| = \|P\|_\infty = \max_{-1\le x\le 1} |P(x)|$$

and $T_n(x) = \sum_{\nu=0}^{n} t_{n,\nu} x^\nu$ denotes the n-th Chebyshev polynomial of the first kind, Markov proved that

$$|a_k| \le \begin{cases} |t_{n,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is even,} \\ |t_{n-1,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is odd.} \end{cases} \tag{1.1}$$

For $k = n$ (1.1) reduces to the well-known Chebyshev inequality

$$|a_n| \le 2^{n-1}\|P\|_\infty. \tag{1.2}$$

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Under the assumption that $P(1) = 0$ or $P(-1) = 0$, Schur [5] showed that (1.2) can be replaced by

$$|a_n| \le 2^{n-1}\left(\cos\frac{\pi}{4n}\right)^{2n}\|P\|_\infty .$$

This result was extended by Rahman and Schmeisser [4] for polynomials with real coefficients, which have at most $n-1$ distinct zeros in $(-1,1)$.

In L^2-norm

$$\|P\| = \|P\|_2 = \left(\int_{-1}^{1}|P(x)|^2\,dx\right)^{1/2},$$

Labelle [2] proved that

$$|a_k| \le \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!}\left(k+\tfrac{1}{2}\right)^{1/2}\binom{[(n-k)/2]+k+1/2}{[(n-k)/2]}\|P\|_2$$

for all $P \in \mathcal{P}_n$ and $0 \le k \le n$, where the symbol $[x]$ denotes as usual the integral part of x. Equality in this case is attained only for the constant multiples of the polynomial

$$\sum_{\nu=0}^{[(n-k)/2]}(-1)^\nu(4\nu+2k+1)\binom{k+\nu-1/2}{\nu}P_{k+2\nu}(x),$$

where $P_m(x)$ denotes the Legendre polynomial of degree m.

Under restriction $P(1) = 0$, Tariq [6] proved that

$$|a_n| \le \frac{n}{n+1}\cdot\frac{(2n)!}{2^n(n!)^2}\left(\frac{2n+1}{2}\right)^{1/2}\|P\|_2,$$

with equality case

$$P(x) = P_n(x) - \frac{1}{n^2}\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(x).$$

Also, for $k = n-1$, he obtained that

$$|a_{n-1}| \le \frac{(n^2+2)^{1/2}}{n+1}\cdot\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}\left(\frac{2n-1}{2}\right)^{1/2}\|P\|_2, \tag{1.3}$$

with equality case

$$P(x) = \frac{2n+1}{n^2+2}P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2}\sum_{\nu=0}^{n-2}(2\nu+1)P_\nu(x).$$

In the absence of the hypothesis $P(1) = 0$ the factor $(n^2+2)^{1/2}/(n+1)$ appearing on the right-hand side of (1.3) is to be dropped.

In this paper we will consider more general problem including L^2-norm of polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$ and using some restricted polynomial classes.

2. Main Result

Let $d\sigma(x)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k\, d\sigma(x)$, $k = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. In that case, there exists a unique set of orthonormal polynomials $\pi(\cdot) = \pi(\cdot; d\sigma)$, $k = 0, 1, \ldots$, defined by

$$\begin{aligned} \pi_k(x) &= b_k x^k + c_k x^{k-1} + \text{ lower degree terms}, \quad b_k > 0, \\ (\pi_k, \pi_m) &= \delta_{km}, \quad k, m \geq 0, \end{aligned} \tag{2.1}$$

where

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\sigma(x) \qquad (f, g \in L^2(\mathbb{R})).$$

For $P \in \mathcal{P}_n$, we define

$$\|P\| = \sqrt{(P, P)} = \left(\int_{\mathbb{R}} |P(x)|^2\, d\sigma(x) \right)^{1/2}. \tag{2.2}$$

The polynomial $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu \in \mathcal{P}_n$ can be represented in the form

$$P(x) = \sum_{\nu=0}^{n} \alpha_\nu \pi_\nu(x),$$

where

$$\alpha_\nu = (P, \pi_\nu), \qquad \nu = 0, 1, \ldots, n.$$

We note that

$$a_n = \alpha_n b_n, \qquad a_{n-1} = \alpha_n c_n + \alpha_{n-1} b_{n-1}.$$

Since

$$\|P\| = \left(\sum_{\nu=0}^{n} |\alpha_\nu|^2 \right)^{1/2} \geq |\alpha_n|,$$

we have a simple estimate $|a_n| \leq b_n \|P\|$. This inequality can be improved for some restricted classes of polynomials. Because of that, we consider a linear functional $L : \mathcal{P}_n \to \mathbb{C}$, such that

$$M = \sum_{\nu=0}^{n} |L\pi_\nu|^2 > 0, \tag{2.3}$$

and a subset of $\mathcal{P}_n$ defined by

$$W_n = \{P \in \mathcal{P}_n \mid LP = 0,\, \mathrm{dg}\, P = n\}.$$

Using a method given by Giroux and Rahman [1] (see also Tariq [6]), we can prove the following auxiliary result:

Lemma 2.1. *If $P \in W_n$ and $\gamma_0, \gamma_1, \dots, \gamma_n$ are nonnegative numbers such that $\gamma_\mu > \gamma_\nu$ for $\nu = 0, 1, \dots, \mu-1, \mu+1, \dots, n$, then*

$$\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 \le (\gamma_\mu - \gamma) \sum_{\nu=0}^{n} |\alpha_\nu|^2, \tag{2.4}$$

where γ is the unigue root of the equation

$$\sum_{\nu=0}^{n} \frac{|L\pi_\nu|^2}{\gamma_\mu - \gamma_\nu - \gamma} = 0 \tag{2.5}$$

in the interval $(0, \Gamma)$, where

$$\Gamma = \min_{\substack{0 \le \nu \le n \\ \nu \ne \mu}} (\gamma_\mu - \gamma_\nu).$$

Inequality (2.4) is sharp and becomes an equality if and only if $P(x)$ is a constant multiple of the polynomial

$$\sum_{\nu=0}^{n} \frac{\overline{L\pi_\nu}}{\gamma_\mu - \gamma_\nu - \gamma} \pi_\nu(x). \tag{2.6}$$

Proof. Let $P \in W_n$. Then

$$LP = L\Big(\sum_{\nu=0}^{n} \alpha_\nu \pi_\nu\Big) = \sum_{\nu=0}^{n} \alpha_\nu L\pi_\nu = 0. \tag{2.7}$$

Starting from

$$\begin{aligned}\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 &= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} (\gamma_\mu - \gamma_\nu)|\alpha_\nu|^2 \\ &= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2 - \gamma \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu|^2,\end{aligned}$$

and using (2.7), we have

$$\begin{aligned}|\alpha_\mu L\pi_\mu|^2 &= \Big| \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} \alpha_\nu L\pi_\nu \Big|^2 = \Big| \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} \alpha_\nu (L\pi_\nu)(\gamma_\mu - \gamma_\nu - \gamma)^{1/2} (\gamma_\mu - \gamma_\nu - \gamma)^{-1/2} \Big|^2 \\ &\le \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} |\alpha_\nu|^2 (\gamma_\mu - \gamma_\nu - \gamma) \sum_{\substack{\nu=0 \\ \nu\ne\mu}}^{n} \frac{|L\pi_\nu|^2}{\gamma_\mu - \gamma_\nu - \gamma}.\end{aligned}$$

Since γ is the unique root of the equation (2.5), we find that

$$-\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{n}|\alpha_\nu|^2(\gamma_\mu-\gamma_\nu-\gamma)\le\frac{-|\alpha_\mu|^2|L\pi_\mu|^2}{\displaystyle\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{n}\frac{|L\pi_\nu|^2}{\gamma_\mu-\gamma_\nu-\gamma}}=-\gamma|\alpha_\mu|^2,$$

wherefrom

$$\sum_{\nu=0}^{n}\gamma_\nu|\alpha_\nu|^2\le\gamma_\mu\sum_{\nu=0}^{n}|\alpha_\nu|^2-\gamma|\alpha_\mu|^2-\gamma\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{n}|\alpha_\nu|^2,$$

i.e., (2.4).

Equality case is attained when

$$\alpha_\nu(\gamma_\mu-\gamma_\nu-\gamma)^{1/2}=A\frac{\overline{L\pi_\nu}}{(\gamma_\mu-\gamma_\nu-\gamma)^{1/2}}\qquad(\nu=0,1,\ldots,\mu-1,\mu+1,\ldots,n),$$

where $A=$ const, i.e. when

$$P(x)=\sum_{\nu=0}^{n}\alpha_\nu\pi_\nu(x)=A\sum_{\nu=0}^{n}\frac{\overline{L\pi_\nu}}{\gamma_\mu-\gamma_\nu-\gamma}\pi_\nu(x).\quad\square$$

Theorem 2.2. *If* $P\in W_n$ *then*

$$|a_n|\le b_n\sqrt{1-\frac{1}{M}\,|L\pi_n|^2}\;\|P\|,\tag{2.8}$$

where M *is given by* (2.3)*. Inequality* (2.8) *is sharp and becomes an equality if and only if* $P(x)$ *is a constant multiple of the polynomial*

$$\pi_n(x)-\frac{L\pi_n}{M-|L\pi_n|^2}\sum_{\nu=0}^{n-1}\overline{(L\pi_\nu)}\pi_\nu(x).\tag{2.9}$$

Proof. Putting $\gamma_n=1$ and $\gamma_\nu=0$, $\nu=0,1,\ldots,n-1$, from (2.5) follows that $\gamma=|L\pi_n|^2/M$. Then (2.4) reduces to

$$|\alpha_n|^2\le\Big(1-\frac{1}{M}|L\pi_n|^2\Big)\|P\|^2,\tag{2.10}$$

i.e. (2.8), because $a_n=\alpha_n b_n$. Also, then (2.6) becomes (2.9). $\square$

Similarly, if we put $\gamma_k=1$ and $\gamma_\nu=0$, $\nu\neq k$, from (2.4) follows

$$|\alpha_k|^2\le\Big(1-\frac{1}{M}|L\pi_k|^2\Big)\|P\|^2.\tag{2.11}$$

Since $a_{n-1} = \alpha_n c_n + \alpha_{n-1} b_{n-1}$, we have

$$|a_{n-1}| \le |\alpha_n||c_n| + |\alpha_{n-1}||b_{n-1}|.$$

Using (2.10) and (2.11), for $k = n-1$, we can obtain an inequality for $|a_{n-1}|$, but it is not sharp in general case. However, if $c_n = 0$, we have

$$|a_{n-1}| \le b_{n-1}\Big(1 - \frac{1}{M}|L\pi_{n-1}|^2\Big)^{1/2}\|P\|. \tag{2.12}$$

Precisely, the following result is valid:

Theorem 2.3. *Let* $\mu_{2k-1} = \int_{\mathbb{R}} x^{2k-1} d\sigma(x) = 0$, $k = 1, 2, \ldots,$ *and* M *is given by* (2.3). *If* $P \in W_n$, *then inequality* (2.12) *holds, with equality case for the constant multiples of the polynomial*

$$\pi_{n-1}(x) - \frac{L\pi_{n-1}}{M - |L\pi_{n-1}|^2} \sum_{\substack{\nu=0 \\ \nu \ne n-1}}^{n} \overline{(L\pi_\nu)}\pi_\nu(x).$$

Under conditions of this theorem, for the orthogonal polynomials π_k the following property

$$\pi_k(-x) = (-1)^k \pi_k(x)$$

holds.

Further, we will consider the simple functional L, defined by $LP = P'(\alpha)$, where α is a given number. Similarly, it could be considered the functional L, defined by $LP = P'(\alpha)$, etc.

For the functional $LP = P(\alpha)$, we have

$$M = \sum_{\nu=0}^{n} |\pi_\nu(\alpha)|^2.$$

Taking a real α and using the Christoffel-Darboux identity, we find

$$M = \sum_{\nu=0}^{n} \pi_\nu(\alpha)^2 = \lambda_{n+1}\big(\pi'_{n+1}(\alpha)\pi_n(\alpha) - \pi'_n(\alpha)\pi_{n+1}(\alpha)\big),$$

where λ_{n+1} is the coefficient in the recurrence relation for orthonormal polynomials

$$\lambda_{n+1}\pi_{n+1}(x) = (x - \tau_n)\pi_n(x) - \lambda_n \pi_{n-1}(x), \quad n = 0, 1, \ldots, n, \tag{2.13}$$
$$\pi_0(x) = 1/\sqrt{\mu_0}, \quad \pi_{-1}(x) = 0.$$

If we put

$$\pi_n(x) = b_n x^n + c_n x^{n-1} + \cdots, \quad b_n > 0, \tag{2.14}$$

then $b_n = \lambda_{n+1} b_{n+1}$, i.e. $b_n = b_0/(\lambda_1\lambda_2\cdots\lambda_n)$.

In next sections we will consider the measures of the classical orthogonal polynomials (Gegenbauer, generalized Laguerre and Hermite polynomials).

3. Gegenbauer Case

Here, we have $d\sigma(x) = (1-x^2)^{\lambda-1/2}dx,\ \lambda > -1/2$.

Let $\{C_\nu^\lambda\}$ be the sequence of Gegenbauer polynomials orthogonal with respect to the measure $d\sigma(x)$ on $(-1,1)$. Their generating functions is

$$(1-2tx+t^2)^{-\lambda} = \sum_{\nu=0}^{+\infty} C_\nu^\lambda(x)t^\nu, \qquad \lambda \neq 0,$$

and three-term recurrence relation

$$(\nu+1)C_{\nu+1}^\lambda(x) = 2(\nu+\lambda)xC_\nu^\lambda(x) - (\nu+2\lambda-1)C_{\nu-1}^\lambda(x),$$

with $C_0^\lambda(x) = 1$ and $C_1^\lambda(x) = 2\lambda x$.

In the sequel we will also need the formulae

$$C_\nu^\lambda(1) = \binom{\nu+2\lambda-1}{\nu} = \frac{(2\lambda)_\nu}{\nu!} \quad \text{and} \quad C_\nu^\lambda(-x) = (-1)^\nu C_\nu^\lambda(x).$$

Of interest is the limit behavior of $C_\nu^\lambda(x)$ when $\lambda \to 0$, given by

$$\lim_{\lambda\to 0} \frac{C_\nu^\lambda(x)}{\lambda} = \frac{2}{\nu}T_\nu(x), \qquad \nu = 1,2,\dots, \tag{3.1}$$

where $T_\nu(x)$ denotes the Chebyshev polynomial of the first kind.

Using the norm of C_ν^λ,

$$h_\nu = \|C_\nu^\lambda\|^2 = \frac{\lambda}{(\nu+\lambda)\Lambda(\lambda)}C_\nu^\lambda(1), \qquad \Lambda(\lambda) = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\Gamma(\lambda+1/2)},$$

where Γ is the gamma function, we can obtain the coefficients λ_n in the corresponding recurrence relation for the orthonormal Gegenbauer polynomials as well as the coefficients b_n in (2.14). Thus,

$$\lambda_n = \frac{1}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}}, \quad b_n = 2^n\sqrt{\Lambda(\lambda)\frac{(\lambda)_n(\lambda+1)_n}{n!(2\lambda)_n}},$$

where

$$(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.$$

It is interesting to consider two cases: $\alpha = 1$ and $\alpha = 0$ (The case $\alpha = -1$ is the same as the case $\alpha = 1$).

From Theorem 2.2 follows:

Corollary 3.1. *If* $P(x)=\sum_{\nu=0}^{n}a_\nu x^\nu$ *is a polynomial of degree* n *such that* $P(1)=0$*, then*

$$|a_n| \le 2^{n+\lambda}\Gamma(n+\lambda)\sqrt{\frac{(n+\lambda)(2n+2\lambda-1)}{2\pi(n-1)!(2n+2\lambda+1)\Gamma(n+2\lambda+1)}}\,\|P\|\,, \tag{3.2}$$

where

$$\|P\| = \left(\int_{-1}^{1}|P(x)|^2\,d\sigma(x)\right)^{1/2}.$$

The inequality (3.2) *reduces to an equality if and only if*

$$P(x)=A\Big(C_n^\lambda(x)-\frac{2(2\lambda+1)}{n(2n+2\lambda-1)}\sum_{\nu=0}^{n-1}(\nu+\lambda)C_\nu^\lambda(x)\Big),$$

where C_ν^λ *is the Gegenbauer polynomial of degree* ν *and* $A=\text{const}$*.*

Similarly, from Theorem 2.3 follows:

Corollary 3.2. *Under conditions of the above corollary, we have*

$$|a_{n-1}| \le 2^{n+\lambda-1}\Gamma(n+\lambda-1)\sqrt{\frac{q_n(\lambda)(n+\lambda-1)(2n+2\lambda-1)}{2\pi(n-1)!(2n+2\lambda+1)\Gamma(n+2\lambda+1)}}\,\|P\|\,, \tag{3.3}$$

where $q_n(\lambda)=(2\lambda+1)(n+2\lambda)+n(n-2)$*. Inequality* (3.3) *becomes an equality if and only if*

$$P(x)=A\Big(C_{n-1}^\lambda(x)-\frac{2n(2\lambda+1)}{(2n+2\lambda-1)q_n(\lambda)}\sum_{\substack{\nu=0\\ \nu\ne n-1}}^{n}(\nu+\lambda)C_\nu^\lambda(x)\Big),$$

where $A=\text{const}$*.*

In particular, for $\lambda=1/2$, these corollaries reduce to the Legendre case, which was investigated by Tariq [6].

In the Chebyshev case of the first kind ($\lambda=0$), (3.2) and (3.3) reduce to the inequalities

$$|a_n| \le \frac{2^n}{\sqrt{2\pi}}\sqrt{\frac{2n-1}{2n+1}}\,\|P\| \tag{3.4}$$

and

$$|a_{n-1}| \le \frac{2^{n-1}}{\sqrt{2\pi}}\sqrt{\frac{2n-1}{2n+1}}\,\|P\|\,, \tag{3.5}$$

respectively, where

$$\|P\| = \left(\int_{-1}^{1} |P(x)|^2 \frac{dx}{\sqrt{1-x^2}} \right)^{1/2}.$$

In the absence of the hypothesis $P(1) = 0$ the factor $\sqrt{(2n-1)/(2n+1)}$ appearing on the right-hand side of (3.4) and (3.5) is to be dropped.

Using the limit process (3.1), we obtain the equality case in (3.4) when

$$P(x) = A\left(T_n(x) - \frac{2}{2n-1}\left(\frac{1}{2} + \sum_{\nu=1}^{n-1} T_\nu(x) \right) \right), \qquad A = \text{const}.$$

Similarly, the equality in (3.5) is attained for

$$P(x) = A\left(\frac{2n+1}{2n-1} T_{n-1}(x) - \frac{2}{2n-1}\left(\frac{1}{2} + \sum_{\nu=1}^{n} T_\nu(x) \right) \right),$$

where $A =$ const.

The case $\alpha = 0$, i.e. the case with restriction $P(0) = 0$, can be reduced to the case of non-restricted class of polynomials of degree $n-1$, with the measure $d\sigma(x) = x^2(1-x^2)^{\lambda-1/2}dx$.

4. Generalized Laguerre Case

In this case we have $d\sigma(x) = x^s e^{-x}$, $s > -1$ on $(0, +\infty)$.

Using the generalized Laguerre polynomials $L_n^s(x)$, given by

$$L_n^s(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (s+\nu+1)_{n-\nu} x^\nu,$$

we can obtain the corresponding orthonormal polynomials $\pi_n(x)$, which satisfy the three-term recurrence relation (2.13), with

$$\lambda_n = \sqrt{n(n+s)} \qquad \text{and} \qquad \tau_n = 2n + s + 1.$$

The corresponding coefficient b_n in (2.14) is given by $b_n = 1 \Big/ \sqrt{n!\Gamma(n+s+1)}$.

Taking $\alpha = 0$, from Theorem 2.2 follows:

Corollary 4.1. *If* $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ *is a polynomial of degree* n *such that* $P(0) = 0$, *then*

$$|a_n| \le \frac{\|P\|}{\sqrt{(n-1)!\Gamma(n+s+2)}}, \tag{4.1}$$

where

$$\|P\| = \left(\int_{0}^{+\infty} |P(x)|^2 \, d\sigma(x) \right)^{1/2}.$$

The inequality (4.1) *reduces to an equality if and only if*

$$P(x) = A\Big(L_n^s(x) - (s+1)(n-1)!\sum_{\nu=0}^{n-1}\frac{1}{\nu!}L_\nu^s(x)\Big),$$

where L_ν^s *is the generalized Laguerre polynomial of degree* ν *and* $A = \text{const}$.

It is interesting to mention that we can obtain the inequality (4.1) considering the non-restricted class of polynomials of degree $n-1$, with the measure $d\sigma(x) = x^{s+2}e^{-x}dx$ on $(0,+\infty)$.

5. Hermite Case

Now, we have $d\sigma(x) = e^{-x^2}dx$ on $(-\infty,+\infty)$.

Using the Hermite polynomials H_n, which satisfy the three-term recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad n = 1,2,\dots,$$

with $H_0(x) = 1$ and $H_1(x) = 2x$, and $\|H_n\|^2 = 2^n n!\sqrt{\pi}$, we obtain the coefficients in relations (2.13) and (2.14):

$$\lambda_n = \sqrt{\frac{n}{2}}, \qquad \tau_n = 0, \qquad b_n = \sqrt{\frac{2^n}{n!\sqrt{\pi}}}\,.$$

It is interesting to consider the case when $\alpha = 0$. Since $H_n'(x) = 2nH_{n-1}(x)$ and

$$H_{2m}(0) = (-1)^m\frac{(2m)!}{m!}, \qquad H_{2m+1}(0) = 0,$$

we obtain the following result:

Corollary 5.1. *Let* $P(x) = \sum_{\nu=0}^n a_\nu x^\nu$ *and*

$$\|P\| = \left(\int_{-\infty}^{+\infty}|P(x)|^2e^{-x^2}dx\right)^{1/2}.$$

1° *If* n *is an even number, then*

$$|a_n| \le \sqrt{\frac{2^n}{n!\sqrt{\pi}}\cdot\frac{n}{n+1}}\;\|P\|$$

and

$$|a_{n-1}| \le \sqrt{\frac{2^{n-1}}{(n-1)!\sqrt{\pi}}}\;\|P\|\,. \tag{5.1}$$

2° *If n is an odd number, then*

$$|a_n| \le \sqrt{\frac{2^n}{n!\sqrt{\pi}}}\ \|P\| \tag{5.2}$$

and

$$|a_{n-1}| \le \sqrt{\frac{2^{n-1}}{(n-1)!\sqrt{\pi}} \cdot \frac{n-1}{n}}\ \|P\|\,.$$

Notice that we have not got any improvements in the inequalities (5.1) and (5.2) in comparison with the corresponding inequalities in the class of all polynomials of degree at most n, although the restriction $P(0) = 0$ has been used.

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EKSTREMALNI PROBLEMI ZA KOEFICIJENTE ALGEBARSKIH POLINOMA

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U radu se razmatraju ekstremalni problemi za koeficijente polinoma $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$. Drugim rečima, pod izvesnim restrikcijama klase svih polinoma stepena od n, dobijaju se gornje granice za $|P^{(k)}(0)|$, koje uključuju L^2-normu polinoma P u odnosu na neku nenegativnu meru na realnoj pravoj.