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Gauss–Hermite interval quadrature rule ☆

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Abstract

The existence and uniqueness of the Gaussian interval quadrature formula with respect to the Hermite weight function on $\mathbb{R}$ is proved. Similar results have been recently obtained for the Jacobi weight on $[-1, 1]$ and for the generalized Laguerre weight on $[0, +\infty)$. Numerical construction of the Gauss–Hermite interval quadrature rule is also investigated, and a suitable algorithm is proposed. A few numerical examples are included.

Keywords: Interval Gaussian quadrature rule; Hermite weight function; Nodes; Weights

1. Introduction

By the Gaussian interval quadrature formula for the positive weight function w, we assume a quadrature formula of the following form

$$\int_a^b f w \, \mathrm{d}x \approx \sum_{k=1}^{n} \frac{\mu_k}{2h_k} \int_{x_k-h_k}^{x_k+h_k} f w \, \mathrm{d}x, \tag{1.1}$$

which integrates exactly all polynomials of degree less than $2n$, and where the subintervals $(x_k - h_k, x_k + h_k)$, $k = 1, \ldots, n$, do not overlap.

Several results have been published on the existence of the previous kinds of quadratures in the last thirty years (cf. [1–5]).

The question of the existence of bounded a, b was proved in [6] in a much wider context. Suppose that w is a weight function on $[-1, 1]$, i.e., a nonnegative Lebesgue integrable function, such that for $I = (\alpha, \beta) \subset [-1, 1]$, $\alpha \neq \beta$, we have $\int_I w(x) \, \mathrm{d}x \neq 0$. In [6], the following result is proved: *Given the ordered set of odd integers* $\{\nu_1, \ldots, \nu_n\}$, *with the property* $n + \sum_{k=1}^{n} \nu_k = N + 1$, *the Chebyshev system of functions* $\{u_0, \ldots, u_N\}$ *on* $[-1, 1]$, *the Markov system*

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of functions $\{v_0, \dots, v_{m-1}\}$ *on* $[-1, 1]$, *where* $m = \max\{\nu_1, \dots, \nu_n\}$, *and a set of the lengths* $h_1 \geq 0, \dots, h_n \geq 0$, *with* $\sum h_k < 1$, *there exists an interpolatory quadrature formula of the form*

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x \approx \sum_{k=1}^{n} \sum_{\nu=0}^{\nu_k-1} \frac{\mu_{k,\nu}}{2h_k} \int_{I_k} f(x)v_\nu(x)w(x)\,\mathrm{d}x,$$

where intervals $I_k \subset [-1, 1]$, $k = 1, \dots, n$, *are non-overlapping, with the length of* I_k *equals* $2h_k$, *which integrates exactly every element of the linear span* $\{u_0, \dots, u_N\}$.

In [7], it was proved that for the Legendre weight $w(x) = 1$ on $[-1, 1]$, the Gaussian interval quadrature rule is unique. The uniqueness for the corresponding formula with respect to the Jacobi weight on $[-1, 1]$ and its numerical construction were given in [8]. The existence and uniqueness of the Gauss–Lobatto and Gauss–Radau interval quadratures for the Jacobi weight was proved in [9]. For the special case of the Chebyshev weight of the first kind and a special set of lengths, analytic solutions were derived. Recently, Bojanov and Petrov [10] proved the existence and uniqueness of the weighted Gaussian interval quadrature formula for a given system of continuously differentiable functions, which constitute an ET system of order two on a finite interval $[a, b]$.

The case of interval quadratures of the Gaussian type on unbounded intervals was for the first time treated in [11], where the existence and uniqueness of the Gaussian interval quadrature formula with respect to the generalized Laguerre weight function have been presented, including an algorithm for the numerical construction of such a formula.

In this paper, we complete our investigation for all classical weight functions (cf. [12,13]), i.e., we prove the corresponding results for the Gaussian interval quadrature on $\mathbb{R}$ with respect to the Hermite weight function $w(x) = \exp(-x^2)$.

The paper is organized as follows. In Section 2 we introduce some definitions and formulate the main result. In Section 3 we give some preliminary results. The proof of the main result is presented in Section 3. Finally, a few numerical examples are given in Section 4.

2. The main result

Let $\mathbf{h} = (h_1, \dots, h_n)$ and $H, M, \varepsilon_0 > 0$. We denote

$$\mathbf{H}_n^H = \left\{ \mathbf{h} \in \mathbb{R}^n | h_k \geq 0,\ k = 1, \dots, n,\ \sum_{k=1}^{n} h_k < H \right\},$$
$$\mathbf{X}_n(\mathbf{h}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid -\infty < x_1 - h_1 \leq x_1 + h_1 < \cdots < x_n - h_n \leq x_n + h_n < +\infty \right\},$$
$$\widetilde{\mathbf{X}}_n(\mathbf{h}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid -\infty < x_1 - h_1 \leq x_1 + h_1 \leq \cdots \leq x_n - h_n \leq x_n + h_n < +\infty \right\},$$
$$\mathbf{X}_n^{M,\varepsilon_0}(\mathbf{h}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid -M < x_1 - h_1,\ x_{k+1} - h_{k+1} - x_k - h_k > \varepsilon_0,\ k = 1, \dots, n-1,\ x_n + h_n < M \right\},$$

and

$$\mathrm{d}\mu = w(x)\,\mathrm{d}x = \exp(-x^2)\,\mathrm{d}x, \quad \phi = 1,\ \psi = -2x,$$

such that $(\phi w)' = \psi w$. Let $\mathcal{P}_m$ be the set of all algebraic polynomials of degree at most m.

Definition 2.1. Given $\mathbf{h} \in \mathbf{H}_n^H$, the Gauss–Hermite interval quadrature rule is an interpolation quadrature rule of the form

$$\int p\,\mathrm{d}\mu = \sum_{k=1}^{n} \frac{\mu_k}{2h_k} \int_{I_k} p\,\mathrm{d}\mu, \quad p \in \mathcal{P}_{2n-1}, \tag{2.1}$$

where $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, and it exists.

Theorem 2.1. *Let w be the Hermite weight and $a = -\infty$, $b = +\infty$. For every $H > 0$, there exist $\varepsilon_0 > 0$ and $M > 0$, such that for every $\mathbf{h} \in \mathbf{H}_n^H$, the Gauss–Hermite quadrature rule (1.1), with the nodes $\mathbf{x} \in \mathbf{X}_n^{\varepsilon_0, M}$ and positive weights μ_k, $k = 1, \ldots, n$, exists uniquely.*

3. Preliminary results

In order to express our results in a more condensed form, we adopt the following definitions for the intervals

$$
\begin{aligned}
&I_k = (x_k - h_k, x_k + h_k), \qquad \overline{I}_k = [x_k - h_k, x_k + h_k], \quad k = 1, \ldots, n, \\
&O_1 = (-\infty, x_1 - h_1), \qquad O_{k+1} = (x_k + h_k, x_{k+1} - h_{k+1}), \quad k = 1, \ldots, n-1, \\
&O_{n+1} = (x_n + h_n, +\infty), \qquad \overline{O}_1 = (+\infty, x_1 - h_1], \qquad \overline{O}_{n+1} = [x_n + h_n, +\infty), \\
&\overline{O}_{k+1} = [x_k + h_k, x_{k+1} - h_{k+1}], \quad k = 1, \ldots, n-1, \\
&I = \bigcup_{k=1}^{n} I_k, \qquad O = \mathbb{R} \setminus I.
\end{aligned}
$$

In order to prove the main theorem, we first present some auxiliary results (for similar results see [7,8,11]).

Lemma 3.1. (i) *Assume $1 \le j_k \le 2$, $k = 1, \ldots, n$, with $\sum_{k=1}^n j_k = N+1$, $\mathbf{h} \in \mathbf{H}_n^H$, $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$ and $f_{m,k}$, $m = 1, j_k$, $k = 1, \ldots, n$, are arbitrary numbers, then the interpolation problem*

$$\frac{1}{2h_k} \int_{I_k} p^{(m-1)}\, \mathrm{d}\mu = f_{m,k}, \quad m = 1, j_k,\ k = 1, \ldots, n, \tag{3.1}$$

has a unique solution in $\mathcal{P}_N$, where for $m = 2$ we take μ as the Lebesgue measure.[1]

(ii) *Assume that $1 \le j_k \le 2$, $k = 1, \ldots, n$, with $\sum_{k=1}^n j_k = N+1$, $\mathbf{h} \in \mathbf{H}_n^H$, $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$; then, for every $c \in \mathbb{C}$, there exists the unique $q_c \in \mathcal{P}_N$, such that $p = cx^{N+1} + q_c$ solves the following interpolation problem*

$$\frac{1}{2h_k} \int_{I_k} p^{(m-1)}\, \mathrm{d}\mu = 0, \quad m = 1, j_k,\ k = 1, \ldots, n,$$

and there holds $q_c = cq_1$, where for $m = 2$ the μ is the Lebesgue measure.

Proof. To prove this lemma, we show that the corresponding homogenous system has only the trivial solution. Note that the conditions can be expressed as a system of linear equations for the coefficients of p. For the first part, we can simply count zeros to see that in every subinterval I_k, there are j_k zeros. So, in total we have $\sum_k j_k = N+1$ zeros, which means that if the solution is not trivial, its degree is at least $N+1$, and therefore it is not a solution in $\mathcal{P}_N$.

For the second part, we can rewrite the interpolation problem in the following form

$$\frac{1}{2h_k} \int_{I_k} q_c^{(m-1)}\, \mathrm{d}\mu = -\frac{c}{2h_k} \int_{I_k} (x^{N+1})^{(m-1)}\, \mathrm{d}\mu, \quad m = 1, j_k,\ k = 1, \ldots, n. \tag{3.2}$$

Now, we can apply the first part of this lemma with

$$f_{m,k} = -\frac{c}{2h_k} \int_{I_k} (x^{N+1})^{(m-1)}\, \mathrm{d}\mu, \quad m = 1, j_k,\ k = 1, \ldots, n,$$

to the interpolation problem (3.2), and denote the unique solution by q_c. Obviously, the linear system of equations which defines q_c, has a free vector multiplied by c, so that $q_c = cq_1$. □

Lemma 3.2. *Suppose $\mathbf{h} \in \mathbf{H}_n^H$ and there exists a Gauss–Hermite quadrature rule with nodes $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$, then $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$.*

[1] This part of the sentence for $m = 2$ is missing in [8,9], and [11].

Proof. Let $\mathbf{h} \in \mathbf{H}_n^H$ and $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$, but $\mathbf{x} \notin \mathbf{X}_n(\mathbf{h})$. Then at least one of the equalities

$$x_k + h_k = x_{k+1} - h_{k+1}, \quad k = 1, \ldots, n-1,$$

holds. Suppose we have $x_k + h_k = x_{k+1} - h_{k+1}$; then according to the interpolation Lemma 3.1, there exists a monic polynomial $p \in \mathcal{P}_{2n-2}$, such that

$$\frac{1}{2h_k}\int_{I_k} p\,\mathrm{d}\mu = \frac{1}{2h_{k+1}}\int_{I_{k+1}} p\,\mathrm{d}\mu = 0,$$
$$\frac{1}{2h_\nu}\int_{I_\nu} p^{(m-1)}\,\mathrm{d}\mu = 0, \quad m = 1, 2,\ \nu = 1, \ldots,\ k-1, k+2, \ldots, n.$$

Obviously such a p annihilates the Gauss–Hermite interval quadrature sum, and it has a constant sign on O. This means that $\int_O p\,\mathrm{d}\mu = \int p\,\mathrm{d}\mu \neq 0$, which is a contradiction. □

The following lemma is crucial, since it allows us to treat the problem for an unbounded interval with tools which are designed for compact supporting sets.

Lemma 3.3. *There exists an $M > 0$, such that for every $\mathbf{h} \in \mathbf{H}_n^H$ and nodes $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ of the corresponding Gauss–Hermite quadrature rule (2.1), there holds*

$$|x_k| < M, \quad k = 1, \ldots, n.$$

Proof. Suppose it is not the case. Then, for every $M > 0$, there exists $\mathbf{h}^M \in \mathbf{H}_n^H$ and a respective set of nodes $\mathbf{x}^M \in \mathbf{X}_n(\mathbf{h})$, such that we have

$$|x_k^M| > M$$

at least for one $k \in \{1, \ldots, n\}$.

Let us assume that we have k_- and k_+ negative and positive nodes, respectively, that are not bounded as M increases. Then, in total we have $n - k_- - k_+ \geq 0$ nodes which remain bounded as M increases. Suppose those nodes are bounded by some L, i.e., $|x_k^M| < L$, $k = k_- + 1, \ldots, n - k_+$. We can always choose L sufficiently large such that

$$\sum_{k=k_-+1}^{n-k_++1} h_k^M < L,$$

since all other nodes are unbounded; actually it is enough to choose L bigger than H. Since the nodes x_k^M and the lengths h_k^M, $k = k_- + 1, \ldots, n - k_+$, are bounded, we can always extract the convergent sequences. Denote by x_k^j and h_k^j, $k = 1, \ldots, n$, the sequences of nodes and lengths such that the nodes x_k^j, $k = 1, \ldots, k_-$ and $k = n - k_+ + 1, \ldots, n$, are unbounded and that the nodes x_k^j and the lengths h_k^j, $k = k_- + 1, \ldots, n - k_+$, $j \in \mathbb{N}_0$, are convergent. Now consider the sets

$$(-L, L) \setminus \left(\bigcup_{k=k_-+1}^{n-k_+} I_k^j\right), \quad j \in \mathbb{N}_0,$$

for every $j \in \mathbb{N}_0$, which are not empty. Moreover, since $H < L$ and the sequences of the nodes x_k^j and the lengths h_k^j, $k = k_- + 1, \ldots, n - k_+$, are convergent, there exists an interval (a, b) of positive length, such that

$$(a, b) \subset (-L, L) \setminus \left(\bigcup_{k=k_-+1}^{n-k_++1} I_k^j\right), \quad j > j_0,$$

for a j_0 sufficiently large.

Now, consider a polynomial p_j of degree $2n-1$, given by

$$\frac{1}{2h_k^j}\int_{I_k^j} p_j^{(m-1)}\,\mathrm{d}\mu = 0, \quad m = 1, 2,\ k = 1, \ldots, n-1, \qquad \frac{1}{2h_n^j}\int_{I_n^j} p_j\,\mathrm{d}\mu = 0,$$

with the leading coefficient -1, which exists, according to Lemma 3.1, part (ii). Obviously, this polynomial p_j annihilates the quadrature formula: since it is of degree $2n-1$ it must be $\int p_j \,\mathrm{d}\mu = \int_O p_j \,\mathrm{d}\mu = 0$. p_j has positive sign on the set $O^j \setminus O^j_{n+1}$, and negative on the set O^j_{n+1}, so that, it must be

$$\int_{O\setminus O_{n+1}} p_j \,\mathrm{d}\mu - \int_{O_{n+1}} (-p_j) \,\mathrm{d}\mu = 0.$$

We can give the following estimation for the first integral

$$\int_{O\setminus O_{n+1}} p_j \,\mathrm{d}\mu \geq \int_a^b (x-a)^{n_1}(b-x)^{n_2} \,\mathrm{d}\mu = J_1 > 0,$$

where we used the fact that $(a,b) \subset O$, and that polynomial p_j has n_1 zeros smaller than a and n_2 zeros bigger than b. As we can see quantity J_1 does not depend on j, and it is constant. On the other hand, for the second integral we have the bound

$$\int_{O_{n+1}} (-p_j) \,\mathrm{d}\mu \leq \int_{x_n^j + h_n^j}^{+\infty} (x - x_1^j + h_1^j)^{2n-1} \,\mathrm{d}\mu(x) = J_2^j > 0$$

so that we have

$$\int_{O\setminus O_{n+1}} p_j \,\mathrm{d}\mu - \int_{O_{n+1}} (-p_j) \,\mathrm{d}\mu \geq J_1 - J_2^j,$$

where J_2^j depends on j, and J_1 does not.

Consider now the following monic polynomial

$$\frac{1}{2h_1^j}\int_{I_1^j} p_j \,\mathrm{d}\mu = 0, \qquad \frac{1}{2h_k^j}\int_{I_k^j} p_j^{(m-1)} \,\mathrm{d}\mu = 0, \quad m = 1,2,\ k = 2,\ldots,n,$$

which exists according to the Lemma 3.1, part (ii), it is of degree $2n-1$, and it annihilates quadrature formula, so it must be $\int p_j \,\mathrm{d}\mu = \int_O p_j \,\mathrm{d}\mu = 0$. p_j is of positive sign on $O^j \setminus O_1^j$ and negative on O_1^j, so that it must be

$$-\int_{O_1^j} (-p_j) \,\mathrm{d}\mu + \int_{O^j\setminus O_1^j} p_j \,\mathrm{d}\mu = 0,$$

using the same reasoning we can conclude that

$$\int_{O^j\setminus O_1^j} p_j \,\mathrm{d}\mu \geq \int_a^b (x-a)^{n_3}(b-x)^{n_4} \,\mathrm{d}\mu = J_3 > 0,$$

which does not depend on j, and

$$\int_{O_1^j} (-p_j) \,\mathrm{d}\mu \leq \int_{-\infty}^{x_1^j - h_1^j} (x_n^j + h_n^j - x)^{2n-1} \,\mathrm{d}\mu = J_4^j > 0,$$

so that we have

$$-\int_{O_1^j} (-p_j) \,\mathrm{d}\mu + \int_{O^j\setminus O_1^j} p_j \,\mathrm{d}\mu \geq J_3 - J_4^j.$$

We are going to prove that at least one of the quantities J_3 and J_4^j tends to zero as $x_1^j - h_1^j$ and $x_n^j + h_n^j$ tend to $-\infty$ and $+\infty$, respectively. According to this fact, we have that at least one of the quantities

$$J_1 - J_2^j \quad \text{or} \quad J_3 - J_4^j,$$

is positive, which produces a contradiction. For reasons of brevity, we introduce the following notation $x_1^j - h_1^j = -M_-^j$ and $x_n^j + h_n^j = M_+^j$.

At first, we see that for $w(x) = e^{-x^2}$, $m \in \mathbb{N}$, and $M \geq \sqrt{m}$,

$$\sup_{|x|\geq M} |x^m w^{1/2}(x)| = M^m w^{1/2}(M), \quad |x| \geq M,$$

as well as

$$\int_M^{+\infty} x^m w(x)\,\mathrm{d}x \leq M^m w^{1/2}(M)\varepsilon(M), \quad \varepsilon(M) = \int_M^{+\infty} w^{1/2}(x)\,\mathrm{d}x,$$

where obviously $\varepsilon(M) \to 0$ as M tends to $+\infty$.

According to these facts, for M_+^j and M_-^j sufficiently large, we have

$$\begin{aligned} J_2^j &= \int_{M_+^j}^{+\infty} (x + M_-^j)^{2n-1} w(x)\,\mathrm{d}x \\ &= \sum_{\nu=0}^{2n-1} \binom{2n-1}{\nu} (M_-^j)^{2n-1-\nu} \int_{M_+^j}^{+\infty} x^\nu w(x)\,\mathrm{d}x \\ &\leq (1 + M_-^j)^{2n-1} \int_{M_+^j}^{+\infty} x^{2n-1} w(x)\,\mathrm{d}x \\ &\leq (M_+^j(1 + M_-^j))^{2n-1} w^{1/2}(M_+^j) \int_{M_+^j}^{+\infty} w^{1/2}(x)\,\mathrm{d}x \\ &\leq (2M_+^j M_-^j)^{2n-1} w^{1/2}(M_+^j)\varepsilon(M_+^j). \end{aligned}$$

Also, we get

$$J_4^j = \int_{-\infty}^{-M_-^j} (M_+^j - x)^{2n-1} w(x)\,\mathrm{d}x = \int_{M_-^j}^{+\infty} (x + M_+^j)^{2n-1} w(x)\,\mathrm{d}x,$$

i.e.,

$$J_4^j \leq (2M_-^j M_+^j)^{2n-1} w^{1/2}(M_-^j)\varepsilon(M_-^j).$$

Now suppose that $(2M_+^j M_-^j)^{2n-1} w^{1/2}(M_+^j)$ tends to some $C > 0$ as M_-^j and M_+^j tend to infinity. Then

$$M_-^j \sim \frac{C^{1/(2n-1)}}{2M_+^j w^{1/(2(2n-1))}(M_+^j)}.$$

Using the fact that $\lim_{|x|\to+\infty} |x|^\mu w^\lambda(x) = 0$, for each $\lambda, \mu > 0$, we conclude that

$$\begin{aligned} \frac{J_4^j}{\varepsilon(M_-^j)} &\leq \left((M_-^j)^{2n-1} w^{1/4}(M_-^j)\right) \left(\frac{C^{1/(2n-1)}}{2M_+^j w^{1/(2(2n-1))}(M_+^j)} w^{1/4}\left(\frac{C^{1/(2n-1)}}{2M_+^j w^{1/(2(2n-1))}(M_+^j)}\right)\right) \\ &\quad \times \left((M_+^j)^{2n} w^{1/(2(2n-1))}(M_+^j)\right) \times \frac{2^{2n}}{C^{1/(2n-1)}} \to 0. \end{aligned}$$

In the case that $(2M_+^j M_-^j)^{2n-1} w^{1/2}(M_+^j)$ tends to $+\infty$, using some similar reasoning we get $J_4^j \to 0$. □

Lemma 3.4. *In the Gauss–Hermite interval quadrature rule (2.1), we have $\mu_k \neq 0$, $k = 1, \ldots, n$.*

Proof. Suppose there is some $k \in \{1, \ldots, n\}$, such that $\mu_k = 0$. According to the interpolation Lemma 3.1, there exists $p \in \mathcal{P}_{2n-2}$ such that

$$\frac{1}{2h_\nu} \int_{I_\nu} p^{(m-1)}\,\mathrm{d}\mu = 0, \quad m = 1, 2, \ \nu = 1, \ldots, k-1, k+1, \ldots, n.$$

This p annihilates the Gauss–Hermite interval quadrature sum, but it has a constant sign on O, such that we have $\int_O p\,\mathrm{d}\mu = \int p\,\mathrm{d}\mu \neq 0$, which is a contradiction. □

Definition 3.1. We denote

$$\Omega = \prod_{k=1}^{n}(x - x_k - h_k)(x - x_k + h_k),$$

$$\Omega_k = \frac{\Omega}{(x - x_k - h_k)(x - x_k + h_k)}, \quad k = 1, \dots, n,$$

and

$$\Delta_k(\Omega_k \phi w) = \frac{(\Omega_k \phi w)(x_k + h_k) - (\Omega_k \phi w)(x_k - h_k)}{2h_k}, \quad h_k \neq 0,$$

and

$$\Delta_k(\Omega_k \phi w) = \partial_{x_k}[(\Omega_k \phi w)(x_k)], \quad h_k = 0,$$

for $k = 1, \dots, n$, where we use the short notation $\partial_{x_k} = \partial/\partial x_k$.

Theorem 3.1. *For every* $\mathbf{h} \in \mathbf{H}_n^H$*, the nodes of the quadrature rule* (2.1) *satisfy the system of equations*

$$\Delta_k(\Omega_k \phi w) = 0, \quad k = 1, \dots, n. \tag{3.3}$$

For $\mathbf{h} \in \mathbf{H}_n^H$*, every solution* $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ *of the system* (3.3) *defines the nodes for the Gauss–Hermite quadrature rule* (2.1).

Proof. Applying the Gauss–Hermite interval quadrature rule (2.1) to the polynomial $(\Omega_\nu \phi w)'/w$ of degree $2n - 1$, we get

$$0 = \int \frac{(\Omega_\nu \phi w)'}{w} \, \mathrm{d}\mu = \sum_{k=1}^{n} \frac{\mu_k}{2h_k} \int_{I_k} \frac{(\Omega_\nu \phi w)'}{w} \, \mathrm{d}\mu = \mu_\nu \Delta_\nu(\Omega_\nu \phi w),$$

i.e., if $\mathbf{x}$ are nodes of the Gauss–Hermite quadrature rule they must satisfy (3.3), since according to Lemma 3.4, we have $\mu_\nu \neq 0$, $\nu = 1, \dots, n$.

For any $p \in \mathcal{P}_{2n-2}$, we have

$$\sum_{k=1}^{n} \left(\frac{p(x_k + h_k)}{\Omega'(x_k + h_k)} + \frac{p(x_k - h_k)}{\Omega'(x_k - h_k)} \right) = 0. \tag{3.4}$$

This can be proved by applying the Cauchy Residue Theorem to the rational function p/Ω, over the contour $\Lambda = \{x \mid |x| = R\}$ in the complex x-plane, where R is sufficiently large. Note that for $p \in \mathcal{P}_{2n-2}$, we have $p/\Omega \sim x^{-2}$ as $x \to \infty$, which gives

$$\oint_\Lambda \frac{p(x)}{\Omega(x)} \, \mathrm{d}x = 0.$$

Now, suppose that for $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, we have

$$\frac{1}{(\Omega' \phi w)(x_k + h_k)} + \frac{1}{(\Omega' \phi w)(x_k - h_k)} = 0. \tag{3.5}$$

Then obviously, according to (3.4), we have for any $p \in \mathcal{P}_{2n-2}$

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \frac{1}{(\Omega' \phi w)(x_k + h_k)} ((p\phi w)(x_k + h_k) - (p\phi w)(x_k - h_k)) \\ &= \sum_{k=1}^{n} \frac{1}{(\Omega' \phi w)(x_k + h_k)} \int_{I_k} (p\phi w)' \, \mathrm{d}x. \end{aligned}$$

But also

$$\int \frac{(p\phi w)'}{w} \, \mathrm{d}\mu = (p\phi w) \,|_{-\infty}^{+\infty} = 0,$$

so that for every $r \in \mathcal{P}_{2n-1}$ of the form $r = p'\phi + p\psi$, $p \in \mathcal{P}_{2n-2}$, we have

$$\int r\,\mathrm{d}\mu = C \sum_{k=1}^{n} \frac{1}{(\Omega'\phi w)(x_k + h_k)} \int_{I_k} r\,\mathrm{d}\mu = 0,$$

for an arbitrary C. All we need to do is to choose C, such that the formula is exact for all $r \in \mathcal{P}_{2n-1}$. Thus, an element of $\mathcal{P}_{2n-1}$, which is not of the form $p'\phi + p\psi$, $p \in \mathcal{P}_{2n-1}$ is the constant polynomial, so that we can choose

$$C = \frac{m_0}{\sum_{k=1}^{n} \frac{2h_k m_{0,k}}{(\Omega'\phi w)(x_k+h_k)}}, \qquad m_0 = \mu(\mathbb{R}), m_{0,k} = \frac{\mu(I_k)}{2h_k}.$$

The system of equations (3.5) defines the Gauss–Hermite quadrature rule, and it is equivalent to (3.3). It is enough to note that

$$\Omega'(x_k \pm h_k) = \pm 2h_k \Omega_k(x_k \pm h_k), \quad k = 1, \ldots, n.$$

Using these equations, we can conclude by definition of Ω_k, that

$$(\Omega'\phi w)(x_k + h_k) = 2h_k(\Omega_k \phi w)(x_k + h_k) > 0,$$

for $\mathbf{h} \in \mathbf{H}_n^H$ and $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, which gives $C > 0$. Thus, all weights in the constructed quadrature rule are positive.

To be completely fair, we need to give an explanation for the case $h_k = 0$ for some k. Then, the corresponding term of (3.4) is given by

$$\frac{p'\Omega_k - p\Omega_k'}{\Omega_k^2}(x_k),$$

and it can be transformed to the form

$$\begin{aligned}\frac{p'\Omega_k - p\Omega_k'}{\Omega_k^2} &= \frac{p'\Omega_k\phi w + p\Omega_k(\phi w)' - p\Omega_k(\phi w)' - p\Omega_k'\phi w}{\Omega_k^2\phi w} \\ &= \frac{(p\phi w)'\Omega_k - p(\Omega_k\phi w)'}{\Omega_k^2\phi w}.\end{aligned}$$

We require that the term with p vanish, so that we have

$$(\Omega_k\phi w)'(x_k) = \partial_{x_k}[(\Omega_k\phi w)(x_k)] = 0.$$

This is exactly what equation of the system (3.3) is when $h_k = 0$. □

Lemma 3.5. *The weights in the Gauss–Hermite interval quadrature rule (2.1) are positive.*

Proof. Actually, we have

$$\mu_k = \frac{m_0}{\sum_{\nu=1}^{n} \frac{m_{0,\nu}}{(\Omega_\nu\phi w)(x_\nu+h_\nu)}} \frac{1}{(\Omega_k\phi w)(x_k + h_k)}, \quad k = 1, \ldots, n, \tag{3.6}$$

where $m_0 = \mu(\mathbb{R})$, $m_{0,k} = \mu(I_k)/(2h_k)$, $k = 1, \ldots, n$.

Since $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, all terms are positive. □

Lemma 3.6. *There exist $\varepsilon_0 > 0$ and $M > 0$, such that for all $\mathbf{h} \in \mathbf{H}_n^H$ and all nodes $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ of the Gauss–Hermite quadrature rule (2.1), we have $\mathbf{x} \in \mathbf{X}_n^{\varepsilon_0, M}$.*

Proof. The existence of M is already proved, so that we prove now only the existence of ε_0. Assume the contrary; then for every $\varepsilon_0 > 0$, there exists $\mathbf{h}^{\varepsilon_0} \in \mathbf{H}_n^H$ and the corresponding set of nodes $\mathbf{x}^{\varepsilon_0} \in \mathbf{X}_n(\mathbf{h}^{\varepsilon_0})$, for which the interpolation quadrature rule (2.1) is Gaussian (i.e., exact on $\mathcal{P}_{2n-1}$), with the property that at least one of the following equalities holds:

$$x_k^{\varepsilon_0} + h_k^{\varepsilon_0} + \varepsilon_0 = x_{k+1}^{\varepsilon_0} - h_{k+1}^{\varepsilon_0}, \quad k = 1, \ldots, n-1,$$

Since the sets $\mathbf{h}^{\varepsilon_0}$ and $\mathbf{x}^{\varepsilon_0}$ are bounded, there are convergent sequences $\mathbf{h}^k$, $\mathbf{x}^k$, $k \in \mathbb{N}$, with limits $\mathbf{h}^0$ and $\mathbf{x}^0$, such that at least one of the equalities

$$x_k^0 + h_k^0 = x_{k+1}^0 - h_{k+1}^0, \quad k = 1, \ldots, n-1,$$

holds. Since weights μ_ν, $\nu = 1, \ldots, n$, are continuous functions of $\mathbf{h}$ and $\mathbf{x}$, according to (3.6), for $\mathbf{h}^{(0)}$ and respective set of nodes $\mathbf{x}^{(0)}$, we have that the rule

$$\sum_{k=1}^{n} \frac{\mu_k^0}{2h_k^0} \int_{I_k^0} p \,\mathrm{d}\mu,$$

is exact for $p \in \mathcal{P}_{2n-1}$, because of continuity. Since, for this Gauss–Hermite interval quadrature rule, we have at least two intervals which have the boundary point in common, we can apply the same argument as in the proof of Lemma 3.2 to produce a contradiction. This means that the statement of Lemma 3.6 is correct. □

4. Proof of the main result

To prove the main result, we are going to need the following topological result, which can be found in [14,15] and [16].

Assume that D is a bounded open set in $\mathbb{R}^n$, with the closure $\overline{D}$ and the boundary ∂D, and $\Phi : \overline{D} \to \mathbb{R}^n$ is a continuous mapping. By $\deg(\Phi, D, \mathbf{c})$, we denote the topological degree of Φ with respect to D and $\mathbf{c} \notin \Phi(\partial D)$.

Lemma 4.1. (i) *If* $\deg(\Phi, D, \mathbf{c}) \neq 0$*, the equation* $\Phi(\mathbf{x}) = \mathbf{c}$ *has a solution in D.*

(ii) *Let* $\Phi(\mathbf{x}, \lambda)$ *be a continuous map* $\Phi : \overline{D} \times [0, 1] \to \mathbb{R}^n$*, such that* $\mathbf{c} \notin \Phi(\partial D, [0, 1])$*, then* $\deg(\Phi(\mathbf{x}, \lambda), D, \mathbf{c})$ *is a constant independent of* λ.

(iii) *Suppose* $\Phi \in C^1(D)$*,* $\mathbf{c} \notin \Phi(\partial D)$ *and* $\det(\Phi'(\mathbf{x})) \neq 0$ *for any* $\mathbf{x} \in D$ *such that* $\Phi(\mathbf{x}) = \mathbf{c}$. *Then, the equation* $\Phi(\mathbf{x}) = \mathbf{c}$ *has only finitely many solutions* $\mathbf{x}^\nu$ *in D, and there holds*

$$\deg(\Phi, D, \mathbf{c}) = \sum_{\mathbf{x}^\nu} \operatorname{sgn}\left(\det(\Phi'(\mathbf{x}^\nu))\right).$$

Now we are ready to prove the main result.

Proof of Theorem 2.1. We are solving the system of equations

$$\Psi_k = -\Delta_k(\Omega_k \phi w) = 0, \quad k = 1, \ldots, n. \tag{4.1}$$

Suppose $\mathbf{x} \in \mathbf{X}_n^{\varepsilon_0, M}(\mathbf{h})$ is a solution of (4.1). Then we have

$$\begin{aligned}\partial_{x_k} \Psi_k &= (\Omega_k \phi w)(x_k + h_k) \left(\sum_{\nu \neq k} \frac{1}{(x_k + h_k - x_\nu - h_\nu)(x_k - h_k - x_\nu - h_\nu)} \right. \\ &\qquad \left. + \frac{1}{(x_k + h_k - x_\nu + h_\nu)(x_k - h_k - x_\nu + h_\nu)} + 2 \right) > 0,\end{aligned}$$

and the inequality is obvious. Similarly,

$$\begin{aligned}\partial_{x_m} \Psi_k &= -(\Omega_k \phi w)(x_k + h_k) \left(\frac{1}{(x_k + h_k - x_m - h_m)(x_k + h_k - x_m + h_m)} \right. \\ &\qquad \left. + \frac{1}{(x_k - h_k - x_m - h_m)(x_k - h_k - x_m + h_m)} \right) < 0,\end{aligned}$$

from where the inequality is obvious.

Also it is clear that

$$\partial_{x_k} \Psi_k + \sum_{m \neq k} \partial_{x_m} \Psi_k = 2(\Omega_k \phi w)(x_k + h_k) > 0,$$

which gives

$$\partial_{x_k} \Psi_k > -\sum_{m\neq k} \partial_{x_m} \Psi_k = \sum_{m\neq k} |\partial_{x_m} \Psi_k|.$$

This means that the Jacobian is diagonally dominant, with positive elements on the main diagonal and negative elsewhere. Thus,

$$\operatorname{sgn}(|\partial_{x_m} \Psi_k|_{m,k=1,\dots,n}) = 1.$$

The rest of the proof goes exactly as it is given in [7]. The proof has N steps, where N is defined by $h = (N+\eta)\frac{\varepsilon_0}{4}$, $0 < \eta \leq 1$, with $h = \max\{h_1, \dots, h_n\}$. At the j-th step, uniqueness is proved for the $\mathbf{h}^{(j)} = (j+\eta)\frac{\varepsilon_0}{4h}\mathbf{h}$, $j = 0, 1, \dots, N$.

In the first step, the mappings

$$\Phi^{(0)}(\mathbf{x}, \lambda) = \left(\mathbf{\Psi_1}(\mathbf{x}, \lambda\mathbf{h}^{(0)}), \dots, \mathbf{\Psi_n}(\mathbf{x}, \lambda\mathbf{h}^{(0)})\right)$$

are considered on $\mathbf{X}_n^{\varepsilon_0,M}(\mathbf{0})$ for each $0 \leq \lambda \leq 1$. It is obvious $\Phi^{(0)}(\mathbf{x}, 0) = 0$ has a solution for $\lambda = 0$, and that solution is unique. Namely, this solution is, really, the classical Gauss–Hermite quadrature rule. Since the sign of the determinant of the Jacobian is positive, using Lemma 4.1, we conclude that

$$\deg(\Phi^{(0)}(\mathbf{x}, 0), \mathbf{X}_n^{\varepsilon_0,M}(\mathbf{0}), \mathbf{0}) = 1.$$

For $\mathbf{x} \in \mathbf{X}_n^{\varepsilon_0,M}(\mathbf{0})$ and $0 \leq \lambda \leq 1$, we have

$$0 < x_1 - \lambda h_1^1, \qquad x_k + \lambda h_k^1 < x_{k+1} - \lambda h_{k+1}^1, \quad k = 1, \dots, n-1.$$

Then, for any solution $\mathbf{x}$ of the system $\Phi^{(0)}(\mathbf{x}, \lambda) = 0$, we have that the sign of $\det(J(\mathbf{x}, \lambda\mathbf{h}^{(0)}))$ is positive. Hence, according to Lemma 4.1 part (ii), we have

$$\deg(\Phi^{(0)}(\mathbf{x}, \lambda), \mathbf{X}_n^{\varepsilon_0,M}(\mathbf{0}), \mathbf{0}) = 1,$$

for all $\lambda \in [0, 1]$, and in particular for $\lambda = 1$. This means that the system $\Phi^{(0)}(\mathbf{x}, 1) = 0$ has a unique solution in $\mathbf{X}_n^{\varepsilon_0,M}(\mathbf{0})$. It is also the unique solution on the smaller set $\mathbf{X}_n^{\varepsilon_0,M}(\mathbf{h}^{(0)})$, according to Lemma 3.6.

In the case $N \neq 0$, we proceed with the same arguments to the mappings

$$\Phi^{(1)}(\mathbf{x}, \lambda) = (\mathbf{\Psi_1}(\mathbf{x}, \lambda\mathbf{h^{(1)}} + (\mathbf{1}-\lambda)\mathbf{h^{(0)}}), \dots, \mathbf{\Psi_n}(\mathbf{x}, \lambda\mathbf{h^{(1)}} + (1-\lambda)\mathbf{h^{(0)}})),$$

to prove that there is the unique solution in $\mathbf{X}_n^{\varepsilon_0,M}(\mathbf{h}^{(0)})$, which is also unique in the set $\mathbf{X}_n^{\varepsilon_0,M}(\mathbf{h}^{(0)})$, according to Lemma 3.6.

After that, the same arguments are iterated to the mappings

$$\Phi^{(j)}(\mathbf{x}, \lambda) = (\mathbf{\Psi_1}(\mathbf{x}, \lambda\mathbf{h^{(j)}} + (\mathbf{1}-\lambda)\mathbf{h^{(j-1)}}), \dots, \mathbf{\Psi_n}(\mathbf{x}, \lambda\mathbf{h^{(j)}} + (1-\lambda)\mathbf{h^{(j-1)}})),$$

until j reaches N. $\square$

Note that we have proved existence and uniqueness.

5. Numerical examples

In this section, we present some numerical results.

At first it is clear that formula (3.6) is perfect for the numerical construction of the weight coefficients μ_k, $k = 1, \dots, n$, since all terms involved are positive. The only problem which may occur is that the points $x_k + h_k$ and $x_{k+1} - h_{k+1}$ can be very close. In that case, we can calculate Ω_k only with a limited precision.

For the calculation of nodes x_k, $k = 1, \dots, n$, we find an inspiration in the proof of Theorem 2.1. It is clear that system of equations (3.3) defines nodes x_k, $k = 1, \dots, n$, as the implicit functions of $\mathbf{h} \in \mathbf{H}_n^H$, and those functions are continuous. This means that we can start with nodes of the classical Gauss–Hermite quadrature rule, for which $\mathbf{h} = \mathbf{0}$, and then increase $\mathbf{h}$ for small amounts, and hope that Newton–Kantorovich method for the system (3.3) will converge. It might happen that $x_k + h_k > x_{k+1} - h_{k+1}$, in which case we should restart the Newton–Kantorovich process with a smaller increment of $\mathbf{h}$.

Table 5.1
Nodes and weights in (2.1) for $n = 50$ and $\mathbf{h} = (2^{-3}, \dots, 2^{-3})$

k	x_k	μ_k
1	±1.576219292408715(−1)	0.3152685888372449
2	±4.730143650629980(−1)	0.3155660693561908
3	±7.888544948727665(−1)	0.3161649659270913
4	±1.105447391349437	0.3170732907729500
5	±1.423108293084983	0.3183034260669310
6	±1.742167192894129	0.3198725925669350
7	±2.062973997782141	0.3218035322507192
8	±2.385904502191648	0.3241254642343076
9	±2.711367489208992	0.3268754026383889
10	±3.039813383373453	0.3300999686436088
11	±3.371745041949398	0.3338578958553713
12	±3.707731519424494	0.3382235340057477
13	±4.048426024275854	0.3432918290620441
14	±4.394589898163156	0.3491855501412883
15	±4.747125449705993	0.3560660461057651
16	±5.107122177920594	0.3641497517936693
17	±5.475923939196887	0.3737344631965771
18	±5.855230237570730	0.3852430613897511
19	±6.247255952466629	0.3993003475680983
20	±6.654997543368066	0.4168776133953174
21	±7.082709257137268	0.4395897647977475
22	±7.536839431011300	0.4703830517143040
23	±8.028134941383717	0.5154222060249561
24	±8.577463734918571	0.5908700751553940
25	±9.239535296868326	0.7640455731144232

Table 5.2
Nodes and weights in (2.1) for $n = 10$ and $\mathbf{h} = (3/4, 3/4, 3/4, 3/4, 3/4, 1/8, 1/8, 1/8, 1/8, 1/8)$

k	x_k	μ_k
1	−4.816271238792111	1.500798253926158
2	−3.316159513032665	1.500001550137709
3	−1.816159149501863	1.500000055825165
4	−3.161591097606944(−1)	1.500000048097458
5	1.183841237100445	1.500001723011623
6	2.065454440706280	2.745979117439452(−1)
7	2.390271706952510	3.863445772988279(−1)
8	2.843090364402305	5.177514569029990(−1)
9	3.426577766353976	6.538002450601201(−1)
10	4.178920908762311	8.846879309648761(−1)

It is interesting, since there is no bound on the real line for the placement of nodes, that we can increase $\mathbf{h}$ for quite large amounts. For example, in Table 5.1, we present nodes and weights for an interval quadrature (2.1), when $n = 50$ and $\mathbf{h} = (2^{-3}, \dots, 2^{-3})$. Starting with nodes for the ordinary Gauss–Hermite quadrature rule, using the *Mathematica* package OrthogonalPolynomials [17], we need just 8 iterations in the numerical construction. This is quite an amazing performance, since in similar situations for the Laguerre and Jacobi measures, we have much worse performance, due to the boundness of the support for the Laguerre and Jacobi measures (see [8,11]).

Table 5.2 presents nodes and weights for the Gauss–Hermite interval quadrature rule (2.1) for $n = 10$ and $\mathbf{h} = (3/4, 3/4, 3/4, 3/4, 3/4, 1/8, 1/8, 1/8, 1/8, 1/8)$. The construction is performed first for the vector of lengths $\mathbf{h} = (1/8, \dots, 1/8)$, using as starting values nodes for the ordinary Gauss–Hermite quadrature rule ($\mathbf{h} = \mathbf{0}$). Then the first five components of $\mathbf{h}$ are increased by the amounts 0.05. In all computations, we needed at most 10 iterations. However, we have 25 intermediate steps, which makes it a really painful process.

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