Estimates for the Maximal Modulus of Rational Functions with Prescribed Poles

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Abstract. In this paper, we obtain certain sharp estimates for the maximal modulus of a rational function with prescribed poles. The proofs of the obtained results are based on the new version of the Schwarz lemma for regular functions which was suggested by Osserman. The obtained results produce several inequalities for polynomials as well.

1. Introduction

For an arbitrary function f, let $\|f\| = \max_{|z|=1} |f(z)|$, the sup-norm of f on the unit circle $|z| = 1$. By $\mathcal{P}_n$, we denote the class of all complex polynomials $P(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ of degree n. The study of polynomial inequalities that relate the norm between polynomials on the disk $|z| = R$, $R > 0$, and their various versions are a classical topic in analysis. Various results of majorization can be found in the comprehensive books of Milovanović et al. [11], Marden [9] and Rahman and Schmeisser [15], where some approaches to obtaining polynomial inequalities are developed on applying the methods and results of the geometric function theory. On the other hand, several papers pertaining to Bernstein-type inequalities for rational functions have appeared in the rational approximation problems. These rational inequalities generalizing the classical polynomial inequalities in approximation theory are of interest in their own right which is witnessed by many recent articles (c.f. [8], [10], [12]–[14]). Let us start with introducing the set of rational functions involved in this article.

For $a_\nu \in \mathbb{C}$, with $\nu = 1, 2, \ldots, n$, let

$$W(z) := \prod_{\nu=1}^{n} (z - a_\nu),$$

and let

$$B(z) := \prod_{\nu=1}^{n} \left(\frac{1 - \bar{a}_\nu z}{z - a_\nu} \right), \quad \mathcal{R}_n := \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \left\{ \frac{P(z)}{W(z)} : P \in \mathcal{P}_n \right\}.$$

Then $\mathcal{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ for $|z| = 1$.

Definition 1.1. (i) For $P \in \mathcal{P}_n$ with $P(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, the conjugate transpose P^* of P is defined as $P^*(z) = z^n \overline{P(1/\overline{z})} = \sum_{\nu=0}^{n} \overline{c}_{n-\nu} z^\nu$.

   (ii) For $r \in \mathcal{R}_n$ with $r(z) := P(z)/W(z)$, the conjugate transpose r^* of r is defined as $r^*(z) = B(z)\overline{r(1/\overline{z})} = P^*(z)/W(z)$.

   If $P \in \mathcal{P}_n$, then concerning the estimate of the maximum modulus of P on the circle $|z| = R \geq 1$, we have

$$\max_{|z|=R} |P(z)| \leq R^n \|P\|, \tag{1}$$

where as, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \|P\|. \tag{2}$$

The inequality (1) is an immediate consequence of the Maximum Modulus Principle (for reference, see [16]), and (2) is due to Ankeny and Rivlin [1]. Equality in (1) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$, and equality in (2) holds for $P(z) = \alpha z^n + \beta$, $|\alpha| = |\beta|$.

   In the past few years, various versions of (1) and (2) have approved in the literature in more generalized forms in which the underlying polynomials are replaced by more general classes of functions. Govil and Mohapatra [7] gave a new perspective to (1) and (2) and extended them to rational functions $r \in \mathcal{R}_n$ with poles outside the unit disk. In fact, they proved that if $r \in \mathcal{R}_n$, then

$$|r(z)| \leq |B(z)| \|r(z)\|, \quad |z| \geq 1, \tag{3}$$

and if $r(z) \neq 0$ in $|z| < 1$, then

$$|r(z)| \leq \left(\frac{|B(z)| + 1}{2} \right) \|r(z)\|, \quad |z| \geq 1. \tag{4}$$

Equality in (3) is attained for $r(z) = \lambda B(z)$, $\lambda \in \mathbb{C}$, and equality in (4) is attained for $r(z) = \alpha B(z) + \beta$, $|\alpha| = |\beta|$.

   Very recently, Milovanović and Mir [10, Remark 5] obtained a refinement of (3), by showing that if $r \in \mathcal{R}_n$, then for $|z| \geq 1$,

$$|r(z)| \leq \left\{ 1 - \frac{(\|r\| - |r^*(0)|)(|z| - 1)}{|r^*(0)| + \|r\| |z|} \right\} |B(z)| \|r\|, \tag{5}$$

provided

$$|r^*(0)|^2 + \|r\| |r^{*\prime}(0)| \leq \|r\|^2, \tag{6}$$

where $r^*(z)$ is given as in Definition 1.1.

   It may be remarked here that the upper bound estimate obtained in the form of (5) sharpens (3) with the additional condition (6), and hence the inequality (5) does not provide a direct refinement of (3).

   The main aim of this paper is to strengthen (3) and (4), and our method of proof is different from the method of Govil and Mohapatra. The proofs of our results are based on the new version of the Schwarz lemma for regular functions suggested by Osserman [14], and the essence in the papers of Dubinin ([4], [5]) is the origin of thought for the new inequalities presented in this paper. The obtained results sharpen some inequalities on the maximum modulus of polynomials as well.

2. Main Results

In the sequel, we always assume that all the poles $a_1, a_2, \ldots, a_n$, of the rational $r(z)$ lie in $|z| > 1$. We shall first show that the inequality (5) which is a refinement of (3) holds with out the condition (6).

Theorem 2.1. *If $r \in \mathcal{R}_n$, then for $|z| \geq 1$,*

$$|r(z)| \leq \left\{ 1 - \frac{(\|r\| - |r^*(0)|)(|z| - 1)}{|r^*(0)| + \|r\|\,|z|} \right\} |B(z)|\|r\|. \tag{7}$$

Equality in (7) holds for $r(z) = \lambda B(z)$, $\lambda \in \mathbb{C}$.

Remark 2.1. Since $r^*(z) = B(z)\overline{r\left(1/\overline{z}\right)}$, therefore, $|r(z)| = |r^*(z)|$ for $|z| = 1$. Also, since

$$r^*(z) = \frac{P^*(z)}{W(z)} = \frac{P^*(z)}{\prod\limits_{\nu=1}^{n}(z - a_\nu)},$$

with $|a_\nu| > 1$ for $1 \leq \nu \leq n$, it follows that $r^*(z)$ is analytic in $|z| \leq 1$. Therefore, by the Maximum Modulus Principle, $|r^*(0)| \leq \|r^*\| = \|r\|$. By using this fact, one can easily check that the left hand side of (7) is less than or equal to the left hand side of (3). Thus, Theorem 2.1 yields a sharpening of (3).

Our next result is a refinement of (4).

Theorem 2.2. *If $r \in \mathcal{R}_n$ and all the zeros of $r(z)$ lie in $|z| \geq 1$, then for $|z| \geq 1$,*

$$|r(z)| \leq \left(\frac{|B(z)| + 1}{2} \right) \left\{ 1 - \frac{(|r(0)| - |r^*(0)|)(|z| - 1)}{(|r(0)| + |r^*(0)|)(|z| + 1)} \right\} \|r\|. \tag{8}$$

Equality in (8) holds for $r(z) = \alpha B(z) + \beta$, $|\alpha| = |\beta| \neq 0$.

Remark 2.2. Since $r(z) \neq 0$ in $|z| < 1$, therefore, $r^*(z) = B(z)\overline{r\left(1/\overline{z}\right)}$, has all its zeros in $|z| \leq 1$. This implies that, the function $r^*(z)/r(z)$ is analytic in $|z| \leq 1$. Also $|r(z)| = |r^*(z)|$ for $|z| = 1$, hence by the Maximum Modulus Principle, we have $|r^*(z)| \leq |r(z)|$ for $|z| \leq 1$. This in particular yields $|r^*(0)| \leq |r(0)|$.

Remark 2.3. As shown in Remark 2.2, $|r^*(0)| \leq |r(0)|$, which is equivalent to

$$\frac{|z|\,|r^*(0)| + |r(0)|}{(1 + |z|)(|r(0)| + |r^*(0)|)} \leq \frac{1}{2},$$

for $|z| \geq 1$. This shows that Theorem 2.1 sharpens (4).

Remark 2.4. Since $r \in \mathcal{R}_n$ and $r(z) \neq 0$ for $|z| < 1$. Let $r(z) = P(z)/W(z)$, where $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ and $W(z) = \prod_{\nu=1}^{n}(z - a_\nu)$, with $|a_\nu| > 1$ for $1 \leq \nu \leq n$. Also, $r^*(z) = P^*(z)/W(z)$, where

$$P^*(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)} = \sum_{\nu=0}^{n} \overline{c}_{n-\nu} z^\nu.$$

Further,

$$|r(0)| = \left| \frac{P(0)}{W(0)} \right| = \frac{|c_0|}{\prod\limits_{\nu=1}^{n} |a_\nu|} \quad \text{and} \quad |r^*(0)| = \left| \frac{P^*(0)}{W(0)} \right| = \frac{|c_n|}{\prod\limits_{\nu=1}^{n} |a_\nu|}.$$

After substituting these values in (8), we get for $|z| \geq 1$,

$$\left| \frac{P(z)}{\prod_{\nu=1}^{n}(z - a_\nu)} \right| \leq \frac{|z||c_n| + |c_0|}{(1 + |z|)(|c_0| + |c_n|)} \left(1 + \prod_{\nu=1}^{n} \left| \frac{1 - \bar{a}_\nu z}{z - a_\nu} \right| \right) \frac{|P(z_0)|}{\prod_{\nu=1}^{n}|z_0 - a_\nu|},$$

where

$$\|r(z)\| = |r(z_0)| = \frac{|P(z_0)|}{\prod_{\nu=1}^{n}|z_0 - a_\nu|}, \quad |z_0| = 1.$$

This gives for $|z| \geq 1$,

$$|P(z)| \leq |P(z_0)| \prod_{\nu=1}^{n} \left| \frac{z - a_\nu}{z_0 - a_\nu} \right| \left(\left(\frac{|z||c_n| + |c_0|}{(1 + |z|)(|c_0| + |c_n|)} \right) \left(1 + \prod_{\nu=1}^{n} \left| \frac{1 - \bar{a}_\nu z}{z - a_\nu} \right| \right) \right),$$

which on letting $|a_\nu| \to \infty$, $1 \leq \nu \leq n$, gives for $|z| \geq 1$,

$$|P(z)| \leq \left(\frac{|z||c_n| + |c_0|}{(1 + |z|)(|c_0| + |c_n|)} \right) (1 + |z|^n) \|P\|,$$

because $|P(z_0)| \leq \|P\|$.

Thus, from Theorem 2.2, we get the following polynomial inequality which was also proved by Dubinin [3].

Corollary 2.1. *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n such that $P(z) \neq 0$ for $|z| < 1$, then for $\rho \geq 1$,*

$$\max_{|z|=\rho} |P(z)| \leq \left(\frac{(1 + \rho^n)(|c_0| + \rho|c_n|)}{(1 + \rho)(|c_0| + |c_n|)} \right) \|P\|. \tag{9}$$

Equality in (9) holds for $P(z) = z^n + 1$.

In the same way as in Remark 2.4, one can get the following refinement of (1) from Theorem 2.1.

Corollary 2.2. *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n, then for $\rho \geq 1$,*

$$\max_{|z|=\rho} |P(z)| \leq \left(\frac{\|P\| + \rho|c_n|}{\rho\|P\| + |c_n|} \right) \rho^n \|P\|. \tag{10}$$

Equality in (10) holds for $P(z) = \lambda z^n$, $\lambda \in \mathbb{C}$.

Remark 2.5. The inequality (10) was also proved by Govil [6, Lemma 3].

3. Auxiliary Results

We need the following lemmas to prove our theorems.
The next lemma is a new version of the Schwarz lemma for regular function suggested by Osserman [14].

Lemma 3.1. *Let $f(z)$ be analytic in $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$ and $f(0) = 0$. Then for $|z| < 1$,*

$$|f(z)| \leq |z| \frac{|z| + |f'(0)|}{1 + |z||f'(0)|}.$$

Let $r(z) = P(z)/W(z) \in \mathcal{R}_n$ and $W(z) = \prod_{\nu=1}^n (z - a_\nu)$ with $|a_\nu| > 1$, $1 \le \nu \le n$, then $r(z)$ is analytic in $|z| \le 1$. By an application of Lemma 3.1 to the function $f(z) = zr(z)/\|r\|$, we immediately get the following lemma.

Lemma 3.2. *If $r \in \mathcal{R}_n$, then for $|z| \le 1$,*

$$|r(z)| \le \frac{|z|\,\|r\| + |r(0)|}{\|r\| + |z|\,|r(0)|}\|r\|.$$

Lemma 3.3. *If $r \in \mathcal{R}_n$, then for $|z| \ge 1$,*

$$|r(z)| + |r^*(z)| \le (|B(z)| + 1)\|r\|.$$

The above lemma is due to Govil and Mohapatra [7].

4. Proofs of Theorems

Proof of Theorem 2.1. Recall that $r(z) = P(z)/W(z) \in \mathcal{R}_n$, and therefore, the function $r^*(z) = P^*(z)/W(z)$, with $W(z) = \prod_{\nu=1}^n (z - a_\nu)$, $|a_\nu| > 1$ for $1 \le \nu \le n$, is analytic in $|z| \le 1$. Applying Lemma 3.2 to the rational function $r^*(z)$, we get for $|z| \le 1$,

$$|r^*(z)| \le \frac{|z|\,\|r^*\| + |r^*(0)|}{\|r^*\| + |z|\,|r^*(0)|}\|r^*\|. \tag{11}$$

Now, since $\|r\| = \|r^*\|$ and noting that $r^*(z) = B(z)\overline{r(1/\bar{z})}$, we get from (11) that for $|z| \le 1$,

$$\left|\overline{r\left(\frac{1}{\bar{z}}\right)}\right| \le \frac{|z|\,\|r\| + |r^*(0)|}{\|r\| + |z|\,|r^*(0)|} \frac{\|r\|}{|B(z)|}.$$

This gives by replacing z by $1/\bar{z}$ that for $|z| \ge 1$,

$$|r(z)| \le \frac{\|r\| + |z|\,|r^*(0)|}{|z|\,\|r\| + |r^*(0)|} \frac{\|r\|}{|B(\frac{1}{\bar{z}})|}. \tag{12}$$

Since, as is easy to verify that $|B(1/z)| = 1/|B(z)|$, which on using in (12) gives after simplification the inequality (7).

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Since $r(z) = P(z)/W(z) \in \mathcal{R}_n$, where $W(z) = \prod_{\nu=1}^n (z - a_\nu)$, with $|a_\nu| > 1$, $\nu = 1, 2, \ldots, n$. Also $r(z) \ne 0$ for $|z| < 1$, therefore,

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)} = \frac{P^*(z)}{W(z)},$$

where $P^*(z) = z^n\overline{P(1/\bar{z})}$, will have all its zeros in $|z| \le 1$. Thus the function $g(z) = zr^*(z)/r(z)$ is analytic in $|z| < 1$ with $g(0) = 0$.

Applying Lemma 3.1 to $g(z)$, we get for $|z| < 1$,

$$|g(z)| \le |z|\frac{|z| + |g'(0)|}{1 + |z|\,|g'(0)|},$$

which is equivalent to

$$|r^*(z)| \le \frac{|z|\,|r(0)| + |r^*(0)|}{|r(0)| + |z|\,|r^*(0)|}|r(z)|. \tag{13}$$

First note that, the inequality (13) is trivially true for all z on $|z| = 1$ by Remark 2.2. Again, since $r^*(z) = B(z)\overline{r(1/\overline{z})}$, we get from (13) that for $|z| \leq 1$,

$$\left| \overline{r\left(\frac{1}{\overline{z}}\right)} \right| \leq |z| \frac{|z| |r(0)| + |r^*(0)|}{|r(0)| + |z| |r^*(0)|} \frac{|r(z)|}{|B(z)|}. \tag{14}$$

Replacing z by $1/\overline{z}$ in (14) and noting that $|B(1/\overline{z})| = 1/|B(z)|$, we get $|z| \geq 1$,

$$|r(z)| \leq \frac{|r(0)| + |z| |r^*(0)|}{|z| |r(0)| + |r^*(0)|} |r^*(z)|. \tag{15}$$

The inequality (15) when combined with Lemma 3.3 gives for $|z| \geq 1$,

$$\left(1 + \frac{|z| |r(0)| + |r^*(0)|}{|r(0)| + |z| |r^*(0)|} \right) |r(z)| \leq |r(z)| + |r^*(z)| \leq (|B(z)| + 1)\|r\|,$$

which after simplification gives (8).

This completes the proof of Theorem 2.2. $\square$

We end this section by obtaining an inequality concerning the minimum modulus of a rational function with prescribed poles. Let $r \in \mathcal{R}_n$, and suppose $r(z)$ has all its zeros in $|z| \leq 1$. First suppose that $r(z)$ has no zeros on $|z| = 1$, then $r^*(z)$ does not vanish in $|z| \leq 1$ and $|r(z)| = |r^*(z)|$ for $|z| = 1$, and

$$m = \min_{|z|=1} |r(z)| = \min_{|z|=1} |r^*(z)|.$$

Clearly $m \leq |r(z)| = |r^*(z)|$ for $|z| = 1$. Therefore, the function $m/r^*(z)$ is analytic in $|z| \leq 1$ and $|m/r^*(z)| \leq 1$ for $|z| = 1$. Hence, by the Maximum Modulus Principle, it follows that $m \leq |r^*(z)|$ for $|z| \leq 1$. Replace z by $1/\overline{z}$ and noting that $r^*(z) = B(z)\overline{r(1/\overline{z})}$, we get

$$m \leq \left| B\left(\frac{1}{\overline{z}}\right) \right| |r(z)| \quad \text{for} \ |z| \geq 1.$$

Equivalently

$$|r(z)| \geq m|B(z)| \quad \text{for} \ |z| \geq 1,$$

thereby, giving a rational analogue of an inequality due to Aziz and Dawood [2, Theorem 1]. The above inequality obviously holds when $r(z)$ has some zeros on $|z| = 1$.

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