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Generalized quadrature formulae for analytic functions ☆

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Dedicated to Prof. Dobrilo Đ. Tošić on the Occasion of his Eightieth Birth Anniversary.

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ABSTRACT

A kind of generalized quadrature formulae of maximal degree of precision for numerical integration of analytic functions is considered. Precisely, a general weighted quadrature of Birkhoff–Young type with $4n+3$ nodes and degree of precision $6n+5$ is studied. Its nodes are characterized by an orthogonality relation and a general numerical method for their computation is given. Special cases and numerical results are also included.

1. Introduction

Recently Acharya et al. [1] have considered numerical approximation of integrals

$$I(w;f) = \int_L f(z)dz, \tag{1.1}$$

over a directed line segment L from the point $z_0 - h$ to the point $z_0 + h$ in the complex plane $\mathbb{C}$, where f is an analytic function in the disk

$$\Omega = \{z \in \mathbb{C} : |z - z_0| \leqslant r,\ r > |h|\}, \tag{1.2}$$

by means of a 7-point quadrature formula of the form

$$Q_7(w;f) = Af(z_0) + B[f(z_0+th) + f(z_0-th)] + C[f(z_0+h) + f(z_0-h)] + D[f(z_0+ih) + f(z_0-ih)],$$

where t is a positive parameter different from 1. Such a formula is exact for all odd degree monomials $f(z) = (z-z_0)^{2k+1}$. In order that the formula is also exact for even monomials $f(z) = (z-z_0)^{2k},\ k = 0,1,2,3$, the authors in [1] determine coefficients in $Q_7(w;f)$ as functions on t,

$$(A,B,C,D) = \frac{h}{105}\left(\frac{8(21t^2-5)}{t^2}, \frac{20}{t^2(1-t^4)}, \frac{2(9-14t^2)}{1-t^2}, \frac{3-7t^2}{1+t^2}\right),$$

and obtain a general formula of degree of precision at least seven for any finite positive value of the real parameter $t \neq 1$. Letting $t \to \infty$, this rule reduces to the well-known five-point Birkhoff–Young formula of fifth degree precision [3], for which

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$A = 8h/5,\ (B = 0),\ C = 4h/15,\ D = -h/15$, and its remainder term $R_5^{BY}(w;f)$ can be estimated as (see [18] or Davis and Rabinowitz [5, p. 136])

$$|R_5^{BY}(w;f)| \leqslant \frac{|h|^7}{1890} \max_{z\in S} |f^{(6)}(z)|,$$

where S denotes the square with vertices $z_0 + i^k h,\ k = 0,1,2,3$. The case of $Q_7(w;f)$ with $h = 1$ was considered in [17]. This kind of Birkhoff–Young quadrature formulae has been investigated by several authors [7,16,15,12–14]. Such quadrature formulas can also be used to integrate real harmonic functions (see [3]).

We mention here that Lyness and Delves [8] and Lyness and Moler [9], and later Lyness [10], developed formulae for numerical integration and numerical differentiation of complex functions.

By an analysis of the remainder term for the general 7-point quadrature formula $Q_7(w;f)$ with respect to the parameter t, it can be obtained a quadrature rule of the maximal precision nine for $t = \sqrt{7/15}$ (see [1]). Some other rules of degree of precision seven can be also derived.

However, with a little modification of $Q_7(w;f)$ we can obtain a modified 7-point quadrature rule $Q_7^M(w;f)$ of degree precision eleven. Furthermore, using such an approach we derive a general $(4n+3)$-point quadrature formula of the maximal degree of precision for a weighted integral.

The paper is organized as follows. In Section 2 we give the modified quadrature formula $Q_7^M(w;f)$. Section 3 is devoted to a general weighted quadrature of Birkhoff–Young type with $4n+3$ nodes and degree of precision $6n+5$. The nodes of such quadratures are characterized by an orthogonality relation. The corresponding weight coefficients of quadratures are given in Section 4. A general numerical method for determining nodes of such quadratures of maximal degree of precision is discussed in Section 5, including numerical results.

2. The modified quadrature formula $Q_7^M(w;f)$

For numerical calculating of the integral (1.1) of an analytic function in the disk (1.2), in this section we consider a modification of the quadrature formula $Q_7(w;f)$ in the following form

$$Q_7^M(w;f) = Af(z_0) + B[f(z_0+th) + f(z_0-th)] + C[f(z_0+\ell h) + f(z_0-\ell h)] + D[f(z_0+i\ell h) + f(z_0-i\ell h)],$$

where t and ℓ are mutually different positive parameters. In this case, from the corresponding system of equations

$$\begin{aligned} &\frac{1}{2}A + B + C + D = h,\\ &Bt^2 + C\ell^2 - D\ell^2 = \frac{h}{3},\\ &Bt^4 + C\ell^4 + D\ell^4 = \frac{h}{5},\\ &Bt^6 + C\ell^6 - D\ell^6 = \frac{h}{7}, \end{aligned}$$

we get

$$\begin{aligned} &A = \frac{2h}{105}\cdot\frac{105t^2\ell^4 - 21t^2 - 35\ell^4 + 15}{t^2\ell^4}, \quad B = \frac{h}{21}\cdot\frac{7\ell^4 - 3}{t^2(\ell^4 - t^4)},\\ &C = \frac{h}{210}\cdot\frac{35t^2\ell^2 + 21t^2 - 21\ell^2 - 15}{\ell^4(t^2 - \ell^2)}, \quad D = \frac{h}{210}\cdot\frac{-35t^2\ell^2 + 21t^2 + 21\ell^2 - 15}{\ell^4(\ell^2 + t^2)}, \end{aligned}$$

where $0 < t,\ \ell < 1,\ t \neq \ell$.

It is pertinent to note that the modified quadrature formula $Q_7^M(w;f)$ boils down to the seventh degree rule due to Acharya et al. [1] for $\ell = 1$, and the modified Birkhoff–Young rule due to Tošić [16] for $\ell = (3/7)^{1/4}$ and $t \to \infty$.

Now, the error-term $R_7^M(w;f) = I(w;f) - Q_7^M(w;f)$ for $f(z) = (z-z_0)^{2k},\ k = 4,5,6$, reduces to

$$\begin{aligned} &R_7((z-z_0)^8) = \frac{2h^9}{315}[21\ell^4(5t^2-3) - 45t^2 + 35],\\ &R_7((z-z_0)^{10}) = \frac{2h^{11}}{231}[11\ell^4(7t^4-3) - 33t^4 + 21),\\ &R_7((z-z_0)^{12}) = \frac{2h^{13}}{1365}\left[91\ell^8(5t^2-3) + 65\ell^4t^2(7t^4-3) - 195t^6 + 105\right]. \end{aligned}$$

Finally, from

$$R_7((z-z_0)^8) = 0, \quad R_7((z-z_0)^{10}) = 0,$$

we obtain two solutions for parameters:

$$\ell^4 = \frac{5}{693}(57 + 4\sqrt{102}), \quad t^2 = \frac{1}{77}(45 - 2\sqrt{102}), \tag{2.1}$$

and

$$\ell^4 = \frac{5}{693}(57 - 4\sqrt{102}), \quad t^2 = \frac{1}{77}(45 + 2\sqrt{102}). \tag{2.2}$$

Thus, there exist two modified 7-point quadratures $Q^M_{7,\nu}(w;f)$, $\nu = 0, 1$, of the maximal degree of precision eleven, with the following parameters:

$$\begin{aligned} Q^M_{7,0}(w;f): \quad & \ell = \sqrt[4]{\frac{5}{693}\left(57 + 4\sqrt{102}\right)} \approx 0.9155808999196944, \\ & t = \sqrt{\frac{1}{77}\left(45 - 2\sqrt{102}\right)} \approx 0.5675304228160498, \\ & A = \frac{256(198 - \sqrt{102})}{77175} = 0.6232915676809758h, \\ & B = \frac{2939400 + 116087\sqrt{102}}{8680644} = 0.4736769794706059h, \\ & C = 0.2151573287932331h, \quad D = -0.0004800921043269324h, \end{aligned}$$

and

$$\begin{aligned} Q^M_{7,1}(w;f): \quad & \ell = \sqrt[4]{\frac{5}{693}\left(57 - 4\sqrt{102}\right)} \approx 0.5883004297385740, \\ & t = \sqrt{\frac{1}{77}\left(45 + 2\sqrt{102}\right)} \approx 0.9201849748878780, \\ & A = \frac{256(198 + \sqrt{102})}{77175} = 0.6902944381499280h, \\ & B = \frac{2939400 - 116087\sqrt{102}}{8680644} = 0.2035538803596094h, \\ & C = 0.4582083249363621h, \quad D = -0.006909424370935494h. \end{aligned}$$

Expanding an analytic function in Taylor series

$$f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \quad z \in \Omega,$$

and having in mind that $Q^M_7(w;f)$ has the maximal degree of precision eleven, the remainder is

$$R^M_7(w;f) = \frac{f^{(12)}(z_0)}{12!} R^M_7((z - z_0)^{12}) + \cdots$$

For the obtained parameters (2.1) and (2.2), its dominant error term reduces to

$$R^M_{7,0}(w;f) \approx \frac{256(516 + 13\sqrt{102})}{3361743} \cdot \frac{h^{13}}{13!} f^{(12)}(z_0) \approx 7.92 \times 10^{-12} h^{13} f^{(12)}(z_0),$$

and

$$R^M_{7,1}(w;f) \approx \frac{256(516 - 13\sqrt{102})}{3361743} \cdot \frac{h^{13}}{13!} f^{(12)}(z_0) \approx 4.70 \times 10^{-12} h^{13} f^{(12)}(z_0),$$

respectively. As we can see the second formula is slightly more accurate than the first one.

3. The generalized weighted quadrature formula $Q^M_{4n+3}(w;f)$

Let $w : (-1, 1) \to \mathbb{R}^+$ be an even positive weight function, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z) dz$, $k = 0, 1, \ldots$, exist. Without loss of generality, in this section we consider a weighted integration over $L = [-1, 1]$ for analytic functions in the unit disk $\Omega = \{z : |z| \leqslant 1\}$ by

$$I(w;f) = \int_{-1}^{1} f(z) w(z) dz = Q_{4n+3}(w;f) + R_{4n+3}(w;f), \tag{3.1}$$

where $Q_{4n+3}(w;f)$ is the $(4n+3)$-point quadrature formula of interpolatory type with nodes at the zeros of a monic polynomial of degree $4n+3$ $(n \in \mathbb{N})$,

$$\Omega_{4n+3}(z) = z(z^2 - r_0)p_n(z^4) = z(z^2 - r_0)\prod_{k=1}^{n}(z^4 - r_k), \tag{3.2}$$

where $0 < r_1 < \cdots < r_n < 1,\ r_0 \in (0,1)$, and $r_0 \neq r_k$ for each $k = 1,\ldots,n$. Here, $R_{4n+3}(w;f)$ is the corresponding remainder term.

According to (3.2) the quadrature formula in (3.1) has the form

$$Q_{4n+3}(w;f) = Af(0) + B[f(x_0) + f(-x_0)] + \sum_{k=1}^{n}\{C_k[f(x_k) + f(-x_k)] + D_k[f(ix_k) + f(-ix_k)]\}, \tag{3.3}$$

where $x_0 = \sqrt{r_0}$ and $x_k = \sqrt[4]{r_k},\ k = 1,\ldots,n$.

Theorem 3.1. *For any $n \in \mathbb{N}$ there exist interpolatory quadratures $Q_{4n+3}(w;f)$ with a maximal degree of precision $d = 6n+5$. The nodes of such quadratures $Q^M_{4n+3}(w;f)$ are characterized by the following orthogonality relation*

$$\int_{-1}^{1} \pi_{2k}(z)z^2(z^2 - r_0)p_n(z^4)w(z)dz = 0, \quad k = 0,1\ldots,n, \tag{3.4}$$

where $\{\pi_\nu\}_{\nu\in\mathbb{N}_0}$ is a sequence of orthogonal polynomials with respect to the weight function on $(-1,1)$.

Proof. Let $\mathcal{P}_d$ denotes the set of algebraic polynomials of degree at most d. For a given $n \in \mathbb{N}$, suppose that $f \in \mathcal{P}_d$, where $d \geqslant 4n+3$. Then, it can be expressed in the form

$$f(z) = u(z)\Omega_{4n+3}(z) + v(z) = u(z)z(z^2 - r_0)p_n(z^4) + v(z),$$

where $u \in \mathcal{P}_{d-4n-3}$ and $v \in \mathcal{P}_{4n+2}$. Applying (3.1), we get

$$I(w;f) = \int_{-1}^{1} u(z)z(z^2 - r_0)p_n(z^4)w(z)dz + I(v).$$

Since this quadrature is of interpolatory type we have that $I(w;v) = Q_{4n+3}(v)$ and also $v(z) = f(z)$ at the zeros of the polynomial Ω_{4n+3}. Therefore, $Q_{4n+3}(w;v) = Q_{4n+3}(w;f)$, so that for each $f \in \mathcal{P}_d$ we have

$$I(w;f) = \int_{-1}^{1} u(z)z(z^2 - r_0)p_n(z^4)w(z)dz + Q_{4n+3}(w;f).$$

It is clear that the quadrature formula $Q_{4n+3}(w;f)$ becomes $Q^M_{4n+3}(w;f)$, i.e., it has a maximal degree of precision, if and only if

$$\int_{-1}^{1} u(z)z(z^2 - r_0)p_n(z^4)w(z)dz = 0 \tag{3.5}$$

for a maximal degree of polynomials $u \in \mathcal{P}_{d-4n-3}$. Evidently, (3.5) is true for every even polynomial. Taking u as an odd polynomial $u(z) = zh(z^2)$, where $h \in \mathcal{P}_n$, the previous "orthogonality conditions" can be represented as

$$\int_{-1}^{1} h(z^2)z^2(z^2 - r_0)p_n(z^4)w(z)dz = 0, \quad h \in \mathcal{P}_n. \tag{3.6}$$

Since the maximal degree of the polynomial $u \in \mathcal{P}_{d-4n-3}$ is

$$d_{\max} - 4n - 3 = 1 + 2n + 1,$$

we conclude that the maximal degree of precision of such a quadrature $Q_{4n+3}(w;f)$ is $d_{\max} = 6n+5$, i.e., $Q_{4n+3}(w;f) = Q^M_{4n+3}(w;f)$.

Introducing the inner product in a usual way as

$$(f,g) = \int_{-1}^{1} f(z)g(z)w(z)dz,$$

the last orthogonality conditions (3.6) can be expressed in terms of orthogonal polynomials $\{\pi_\nu\}_{\nu\in\mathbb{N}_0}$ with respect to this inner product in the form $(z\pi_{2k}, \Omega_{4n+3}) = 0,\ 0 \leqslant k \leqslant n$, i.e., (3.4). □

According to (3.2), the polynomial $(z^4 - rz^2)p_n(z^4)$ can be expressed in the form

$$(z^4 - rz^2)p_n(z^4) = \sum_{j=0}^{n}(-1)^j\sigma_j\left(z^{4(n-j+1)} - rz^{2(2n-2j+1)}\right), \tag{3.7}$$

where σ_j are the so-called *elementary symmetric functions*, defined by

$$\sigma_j = \sum_{(k_1,\ldots,k_j)} r_{k_1}\cdots r_{k_j}, \quad j = 1,\ldots,n,$$

and the summation is performed over all combinations $(k_1,\ldots,k_j)$ of the basic set $\{1,\ldots,n\}$. Thus,

$$\sigma_1 = r_1 + r_2 + \cdots + r_n,\ \sigma_2 = r_1r_2 + \cdots + r_{n-1}r_n,\ldots,\ \sigma_n = r_1r_2\cdots r_n,$$

and for the convenience we put $\sigma_0 = 1$. Also, we put r instead of r_0.

Using the orthogonality conditions (3.4) and the expansion (3.7) we get the following system of nonlinear equations

$$f_k \equiv \sum_{j=0}^{n}(-1)^j\sigma_j\{s_{k,2n-2j+2} - rs_{k,2n-2j+1}\} = 0, \quad k = 0,1,\ldots,n, \tag{3.8}$$

with respect to unknowns $r,\ \sigma_1,\ldots,\sigma_n$, or equivalently to $r_k,\ k = 0,1,\ldots,n$, where $r = r_0$ and $s_{k,j} = (\pi_{2k}, z^{2j}),\ k,j \geqslant 0$.

Introducing the notations $\boldsymbol{\sigma} = [\sigma_1\ \sigma_2\ \cdots\ \sigma_n]^T$,

$$\mathbf{A} = \mathbf{A}(r) = [a_{k,j}]_{k=0,j=1}^{n,n}, \quad \mathbf{b} = \mathbf{b}(r) = [b_0\ b_1\ \cdots\ b_n]^T,$$

where

$$a_{k,j} = (-1)^{j-1}[s_{k,2n-2j+2} - rs_{k,2n-2j+1}], \quad b_k = s_{k,2n+2} - rs_{k,2n+1}, \tag{3.9}$$

the system of $n+1$ nonlinear equations (3.8) can be written in the matrix form

$$\mathbf{A}\boldsymbol{\sigma} = \mathbf{b}. \tag{3.10}$$

We have seen that the problem for $n = 1$ (and $w(z) = 1$) has two solutions. Numerical experiments show that for an arbitrary n, the number of solutions is $n+1$. This hypothesis can be checked numerically for some reasonable values of n (e.g. $n \leqslant 10$) in the following way.

If we take a fixed value of $r \in (0,1)$, then the overdetermined system of $n+1$ linear equations

$$\mathbf{A}(r)\boldsymbol{\sigma} = \mathbf{b}(r), \tag{3.11}$$

with n unknowns $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\ldots,\sigma_n)$, can be solved as a least squares problem (in the 2-norm)

$$\min_{\boldsymbol{\sigma}} \|\mathbf{A}(r)\boldsymbol{\sigma} - \mathbf{b}(r)\|_2 = \|\mathbf{A}(r)\widehat{\boldsymbol{\sigma}} - \mathbf{b}(r)\|_2,$$

where the vector $\mathbf{A}(r)\widehat{\boldsymbol{\sigma}} - \mathbf{b}(r)$ is the corresponding *least squares residual* and the solution $\widehat{\boldsymbol{\sigma}}$ can be expressed in terms of Moore–Penrose inverse.

Only when for some $r = \widehat{r}_0$, the vector $\mathbf{A}(r)\widehat{\boldsymbol{\sigma}} - \mathbf{b}(r)$ becomes zero, we can identify the existence of a solution $(\widehat{r}_0, \widehat{\boldsymbol{\sigma}}) = (\widehat{r}_0, \widehat{\sigma}_1,\ldots,\widehat{\sigma}_n)$ of our original (nonlinear) system of Eqs. (3.10).

Consider now the case $w(z) = 1$. Following [2, §10.10] we get

$$(P_{2k}, z^{2j}) = k!\binom{j}{k}\frac{\Gamma(j+1/2)}{\Gamma(k+j+3/2)} = k!\binom{j}{k}\prod_{\nu=0}^{k}\frac{2}{2j+2\nu+1}, \tag{3.12}$$

where $P_{2k}(z)$ is the Legendre polynomial of degree $2k$. Taking (3.12) instead of $s_{k,j}$, the corresponding norm $\|\mathbf{A}(r)\widehat{\boldsymbol{\sigma}} - \mathbf{b}(r)\|_2$ as a function of r is presented in Fig. 1 for $1 \leqslant n \leqslant 4$ and $n = 10$.

As we can see, for the weight function $w(z) = 1$ and these values of n, the 2-norm vanishes only at $n+1$ points in $(0,1)$, which means that for a given n, there exist $n+1$ different quadratures $Q^M_{4n+3,\nu},\ \nu = 0,1,\ldots,n$, each of degree of precision $d_{\max} = 6n+5$.

In a general case, if we have a solution of our nonlinear problem, say $\widehat{r}_0$, then in order to construct the corresponding quadrature formula $Q^M_{4n+3}(w;f)$ for such a $r = \widehat{r}_0$, we should solve a system of n linear equations from (3.11) in order to get the values $(\widehat{\sigma}_1,\ldots,\widehat{\sigma}_n)$, and then the zeros $(\widehat{r}_1,\ldots,\widehat{r}_n)$ by solving the equation

$$(z - \widehat{r}_1)\cdots(z - \widehat{r}_n) = z^n - \widehat{\sigma}_1z^{n-1} + \widehat{\sigma}_2z^{n-2} - \cdots + (-1)^n\widehat{\sigma}_n = 0.$$

Then, the nodes in the corresponding quadrature formula (3.3) are

$$x_0 = \sqrt{\widehat{r}_0} \quad \text{and} \quad x_k = \sqrt[4]{\widehat{r}_k}, \quad k = 1,\ldots,n.$$

A determination of the weight coefficients $A,\ B,\ C_k$ and $D_k,\ k = 1,\ldots,n$, in the corresponding quadrature formula (3.3) is a linear problem and it is considered in the next section.

A general numerical method for solving our nonlinear problem (3.8) and finding all solutions for r_0, for a given n and for an arbitrary weight function $w(z)$, is given in Section 5. Also, a general numerical method for calculating the necessary inner products $s_{k,j} = (\pi_{2k}, z^{2j}),\ 0 \leqslant k \leqslant j$, is given. Furthermore, analytic expressions for the generalized Gegenbauer weight are derived.

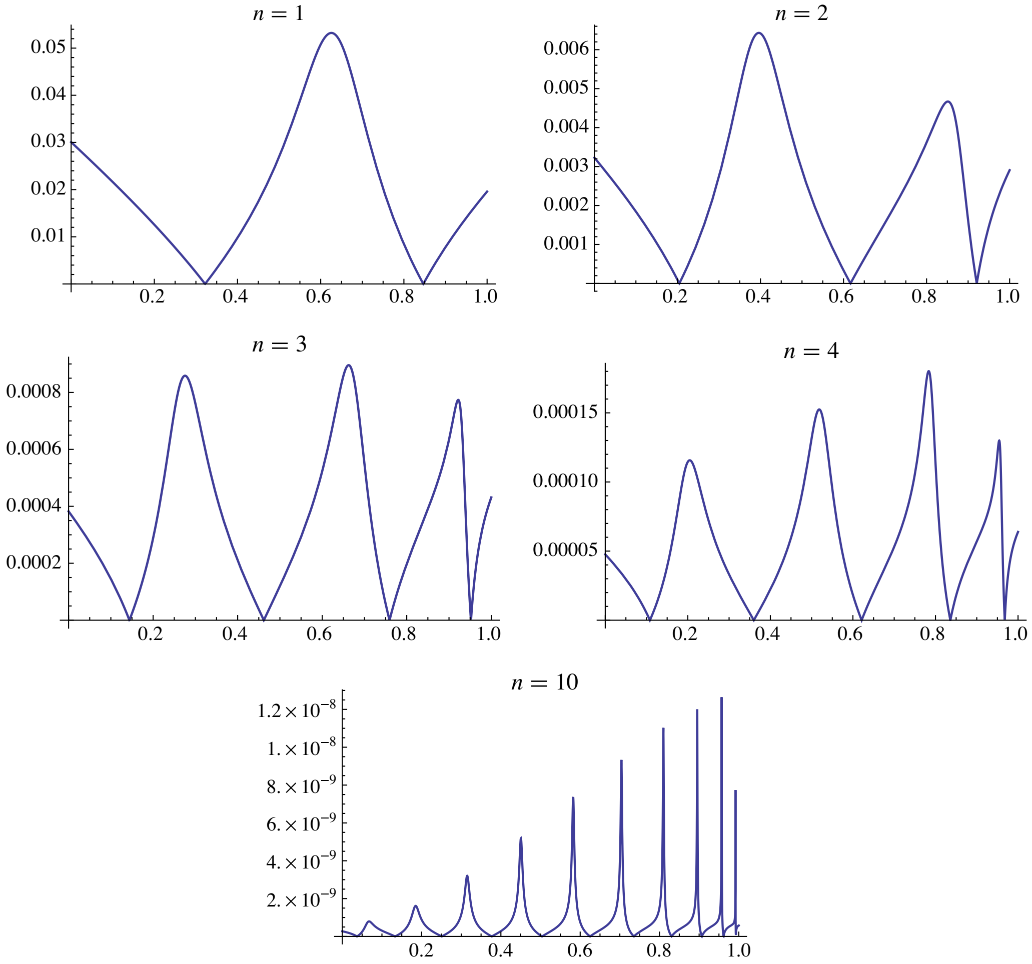

Fig. 1. The norm $\|A(r)\widehat{\boldsymbol{\sigma}} - b(r)\|_2$ as a function of r for $1 \leqslant n \leqslant 4$ and $n = 10$.

4. The weight coefficients in quadrature formula $Q^{M}_{4n+3}(w;f)$

In order to determine the weight coefficients W_ν in a quadrature formula of interpolatory type

$$Q_{4n+3}(w;f) = \sum_{z_\nu \in Z} W_\nu f(z_\nu),$$

we use the Lagrange polynomial constructed at the set of simple nodes $Z = \{z_\nu\}$. In our case,

$$Z = \{0, \pm x_0, \pm x_k, \pm i x_k, k = 1, \ldots, n\},$$

where $x_0 = \sqrt{r_0}$ and $x_k = \sqrt[4]{r_k}$, $k = 1, \ldots, n$, and the node polynomial is

$$\Omega_{4n+3}(z) = z(z^2 - r_0)\prod_{k=1}^{n}(z^4 - r_k).$$

The corresponding Lagrange polynomial is

$$L_{4n+3}(f;z) = \sum_{z_\nu \in Z} \frac{\Omega_{4n+3}(z)}{(z - z_\nu)\Omega'_{4n+3}(z_\nu)} f(z_\nu),$$

so that

$$W_\nu = \frac{1}{\Omega'_{4n+3}(z_\nu)} \int_{-1}^{1} \frac{\Omega_{4n+3}(z)w(z)}{z - z_\nu}\,dz. \tag{4.1}$$

Theorem 4.1. *Let r_0 and $r_k,\ k = 1,\dots,n$, be determined according to Theorem 3.1. Then the weight coefficients in the quadrature formula $Q^M_{4n+3}(w;f)$ with the maximal degree of precision $d = 6n+5$ are given by*

$$\begin{aligned}
A &= \frac{-1}{r_0 p_n(0)} \int_{-1}^{1} (z^2 - r_0) p_n(z^4) w(z) dz,\\
B &= \frac{1}{2 r_0 p_n(r_0^2)} \int_{-1}^{1} z^2 p_n(z^4) w(z) dz,\\
C_k &= \frac{1}{4 r_k(\sqrt{r_k} - r_0) p_n'(r_k)} \int_{-1}^{1} \frac{z^2(z^2 - r_0) p_n(z^4)}{z^2 - \sqrt{r_k}} w(z) dz, \quad k = 1,\dots,n,\\
D_k &= \frac{-1}{4 r_k(\sqrt{r_k} + r_0) p_n'(r_k)} \int_{-1}^{1} \frac{z^2(z^2 - r_0) p_n(z^4)}{z^2 + \sqrt{r_k}} w(z) dz, \quad k = 1,\dots,n,
\end{aligned}$$

where $p_n(z) = \prod_{\nu=1}^{n}(z - r_\nu)$.

Proof. Let $x_0 = \sqrt{r_0}$ and $x_k = \sqrt[4]{r_k}$, $k = 1,\dots,n$. According to

$$\Omega'_{4n+3}(z) = (3z^2 - r_0)\prod_{\nu=1}^{n}(z^4 - r_\nu) + 4z^4(z^2 - r_0)\sum_{j=1}^{n}\prod_{\nu\neq j}(z^4 - r_\nu),$$

we have

$$\begin{aligned}
\Omega'_{4n+3}(0) &= -r_0 \prod_{\nu=1}^{n}(-r_\nu) = -r_0 p_n(0),\\
\Omega'_{4n+3}(\pm x_0) &= 2r_0 \prod_{\nu=1}^{n}(r_0^2 - r_\nu) = 2 r_0 p_n(r_0^2),\\
\Omega'_{4n+3}(\pm x_k) &= 4r_k(\sqrt{r_k} + r_0)\prod_{\nu\neq k}(r_k - r_\nu) = 4 r_k(\sqrt{r_k} - r_0) p_n'(r_k),\\
\Omega'_{4n+3}(\pm i x_k) &= 4r_k(-\sqrt{r_k} - r_0)\prod_{\nu\neq k}(r_k - r_\nu) = -4 r_k(\sqrt{r_k} + r_0) p_n'(r_k),
\end{aligned}$$

where $k = 1,\dots,n$. Now, applying (4.1) and using notations for coefficients as in (3.3), we get desired results. □

For analytic functions we can give an explicit formula for the interpolation error $E_{4n+3}(z) = f(z) - L_{4n+3}(f;z)$ in the form (see [11, pp. 55–56])

$$E_{4n+3}(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{\Omega_{4n+3}(z)}{\Omega_{4n+3}(\zeta)} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (z \in \operatorname{int}\Gamma),$$

where Γ is a simple closed contour in $\mathbb{C}$, such that all interpolation nodes belong to $\operatorname{int}\Gamma$. Then, the remainder term in (3.1) can be expressed in the integral form

$$R_{4n+3}(w;f) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\Omega_{4n+3}(\zeta)} \left(\int_{-1}^{1} \frac{\Omega_{4n+3}(z) w(z)}{\zeta - z} dz\right) d\zeta. \tag{4.2}$$

Some estimate of (4.2) will be given elsewhere.

5. Numerical methods for constructing quadratures

In the sequel we need the inner products $s_{k,j} = (\pi_{2k}, z^{2j}),\ 0 \leqslant k \leqslant j$. For $k = 0$ these products reduce to the moments of the weight function w, i.e.,

$$s_{0,j} = (1, z^{2j}) = \int_{-1}^{1} z^{2j} w(z) dz = \mu_{2j}, \quad j = 0, 1, \dots. \tag{5.1}$$

First, we give an analytic expression of $s_{k,j}$ for a wide class of weight functions, and after that we introduce a general numerical method for easy calculation of $s_{k,j}$ for every even weight function.

5.1. Analytic expression of $s_{k,j}$ for the generalized Gegenbauer weight

We consider the so-called generalized Gegenbauer weight function defined by $w(z) = |z|^\gamma (1 - z^2)^\alpha,\ \gamma,\ \alpha > -1$, on $(-1, 1)$. The monic polynomials $W_\nu^{(\alpha,\beta)}(z),\ \nu = 0, 1, \dots$, orthogonal with respect to this weight function, where $\beta = (\gamma - 1)/2$, were introduced by Laščenov [6] (see, also, Chihara [4, pp. 155–156] and Mastroianni and Milovanović [11, pp. 147–148]). These polynomials can be expressed in terms of the Jacobi polynomials $P_\nu^{(\alpha,\beta)}(z),\ \nu = 0, 1, \ldots$, which are orthogonal on $(-1, 1)$ with respect to the weight function $w^{(\alpha,\beta)}(z) = (1-z)^\alpha(1+z)^\beta,\ \alpha, \beta > -1$. Namely,

$$\begin{aligned} W_{2k}^{(\alpha,\beta)}(z) &= \frac{k!}{(k+\alpha+\beta+1)_k} P_k^{(\alpha,\beta)}(2z^2-1), \\ W_{2k+1}^{(\alpha,\beta)}(z) &= \frac{k!}{(k+\alpha+\beta+2)_k} z P_k^{(\alpha,\beta+1)}(2z^2-1). \end{aligned} \tag{5.2}$$

Notice that $W_{2k+1}^{(\alpha,\beta)}(z) = zW_{2k}^{(\alpha,\beta+1)}(z)$. These polynomials satisfy the following three-term recurrence relation

$$\begin{aligned} &W_{\nu+1}^{(\alpha,\beta)}(z) = zW_\nu^{(\alpha,\beta)}(z) - \beta_\nu W_{\nu-1}^{(\alpha,\beta)}(z), \quad \nu = 0, 1, \ldots, \\ &W_{-1}^{(\alpha,\beta)}(z) = 0, \quad W_0^{(\alpha,\beta)}(z) = 1, \end{aligned}$$

where

$$\beta_{2k} = \frac{k(k+\alpha)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}, \quad \beta_{2k-1} = \frac{(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)},$$

for $k = 1, 2, \ldots$, except when $\alpha + \beta = -1$; then $\beta_1 = (\beta+1)/(\alpha+\beta+2)$.

Now, we want to find an explicit expression for the products

$$s_{k,j} = (W_{2k}^{(\alpha,\beta)}, z^{2j}) = \int_{-1}^{1} W_{2k}^{(\alpha,\beta)}(z) z^{2j} |z|^{2\beta+1}(1-z^2)^\alpha dz, \quad 0 \leqslant k \leqslant j. \tag{5.3}$$

Lemma 5.1. *Let $\alpha, \beta > -1$. Then the products defined in (5.3) are*

$$s_{k,j} = \frac{k!}{(k+\alpha+\beta+1)_k} \binom{j}{k} \frac{\Gamma(k+\alpha+1)\Gamma(j+\beta+1)}{\Gamma(k+j+\alpha+\beta+2)} \tag{5.4}$$

for $0 \leqslant k \leqslant j$. Otherwise, $s_{k,j} = 0$.

In order to prove this lemma we need an auxiliary result.

Lemma 5.2. *For $\alpha, \beta > -1$ and $j \in \mathbb{N}_0$ the expansion in Jacobi polynomials $P_\nu^{(\alpha,\beta)}(t), \nu = 0, 1, \ldots, j$,*

$$(1+t)^j = 2^j\Gamma(j+\beta+1) \sum_{\nu=0}^{j} \nu! \binom{j}{k} \frac{(2\nu+\alpha+\beta+1)\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\beta+1)\Gamma(\nu+j+\alpha+\beta+2)} P_\nu^{(\alpha,\beta)}(t)$$

holds.

The proof of this expansion can be given by induction in j, having in mind that

$$\begin{aligned} (1+t)P_\nu^{(\alpha,\beta)}(t) = {} & \frac{2(\nu+1)(\nu+\alpha+\beta+1)}{(2\nu+\alpha+\beta+1)(2\nu+\alpha+\beta+2)} P_{\nu+1}^{(\alpha,\beta)}(t) + \left(1 + \frac{\beta^2-\alpha^2}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+1)}\right) P_\nu^{(\alpha,\beta)}(t) \\ & + \frac{2(\nu+\alpha)(\nu+\beta)}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+1)} P_{\nu-1}^{(\alpha,\beta)}(t). \end{aligned}$$

We mention that the corresponding expansion in Bateman and Erdélyi [2, pp. 212] has a mistake.

Proof of Lemma 5.2. According to (5.3) and (5.2) we have

$$s_{k,j} = \frac{2k!}{(k+\alpha+\beta+1)_k} \int_0^1 P_k^{(\alpha,\beta)}(2z^2-1) z^{2j+2\beta+1}(1-z^2)^\alpha dz.$$

Changing the variables $t = 2z^2 - 1$ it reduces to

$$s_{k,j} = \frac{k!}{2^{\alpha+\beta+j+1}(k+\alpha+\beta+1)_k} \int_{-1}^{1} P_k^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^{\beta+j} dt.$$

Now, using the expansion from Lemma 5.2 and the orthogonality of Jacobi polynomials,

$$(P_k^{(\alpha,\beta)}, P_\nu^{(\alpha,\beta)}) = \|P_k^{(\alpha,\beta)}\|^2 \delta_{k,\nu} = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)} \delta_{k,\nu},$$

where $\delta_{k,\nu}$ is Kronecker's delta, we obtain

$$s_{k,j} = \frac{k! 2^j \Gamma(j+\beta+1) k! \binom{j}{k}}{2^{\alpha+\beta+j+1}(k+\alpha+\beta+1)_k} \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(k+j+\alpha+\beta+2)} \|P_k^{(\alpha,\beta)}\|^2,$$

i.e., (5.4). For $k > j$, because of orthogonality, $s_{k,j} = 0$ □

We mention some special cases:

1° $w(z) = 1$, i.e., $\alpha = 0,\ \gamma = 0\ (\beta = -1/2)$: In this Legendre case we have

$$s_{k,j} = \frac{(k!)^2}{(k+1/2)_k} \binom{j}{k} \frac{\Gamma(j+1/2)}{\Gamma(k+j+3/2)}.$$

Compared with (3.12), the additional factor $k!/(k+1/2)_k\ (= 1/a_{2k})$ comes from the leading coefficient in the Legendre polynomial $P_{2k}(z) = a_{2k}z^{2k} +$ terms of lower degree.

2° $w(z) = 1/\sqrt{1-z^2}$, i.e., $\alpha = -1/2,\ \gamma = 0\ (\beta = -1/2)$: In the Chebyshev case of the first kind, the inner product (5.4) reduces to

$$s_{k,j} = \frac{\pi}{2^{2j+2k-1}} \binom{2j}{j-k}.$$

3° $w(z) = \sqrt{1-z^2}$, i.e., $\alpha = 1/2,\ \gamma = 0\ (\beta = -1/2)$: In the Chebyshev case of the second kind, it becomes

$$s_{k,j} = \frac{\pi}{2^{2j+2k+1}} \frac{2k+1}{2j+1} \binom{2j+1}{j-k}.$$

4° $w(z) = (1-z^2)^\alpha$, i.e., $\alpha > -1,\ \gamma = 0\ (\beta = -1/2)$: In this Gegenbauer case, we have

$$s_{k,j} = \frac{k!}{(k+\alpha+1)_k} \binom{j}{k} \frac{\Gamma(k+\alpha+1)\Gamma(j+1/2)}{\Gamma(k+j+\alpha+3/2)}.$$

5° $w(z) = |z|$, i.e., $\alpha = 0,\ \gamma = 1\ (\beta = 0)$: In this case

$$s_{k,j} = \frac{1}{k+j+1} \frac{\binom{2j}{j-k}}{\binom{2k}{k}\binom{2j}{j}}.$$

5.2. General numerical method for calculating $s_{k,j}$

It is well-known that monic polynomials $\{\pi_\nu\}_{\nu\in\mathbb{N}_0}$ orthogonal with respect to an even weight function satisfy the three-term recurrence relation of the form

$$\pi_{\nu+1}(z) = z\pi_\nu(z) - \beta_\nu \pi_{\nu-1}(z), \quad \nu = 0, 1, \ldots, \tag{5.5}$$

with $\pi_0(z) = 1$ and $\pi_{-1}(z) = 0$. It is convenient to put $\beta_0 = \mu_0$.

Lemma 5.3. *Let $\beta_\nu,\ \nu \geqslant 0$, be recursion coefficients in (5.5) for polynomials orthogonal with respect to the even weight function w on $(-1,1)$, with the moments $\mu_\nu = \int_{-1}^{1} z^\nu w(z)dz,\ \nu \geqslant 0$. For the inner products $s_{k,j} = (\pi_{2k}, z^{2j})$ the following recurrence relation*

$$s_{k,j+1} = s_{k+1,j} + (\beta_{2k} + \beta_{2k+1})s_{k,j} + \beta_{2k}\beta_{2k-1}s_{k-1,j} \tag{5.6}$$

holds, with $s_{0,j} = \mu_{2j},\ j = 0, 1, \ldots$, and $s_{k,j} = 0$ for $k > j$.

Remark 5.4. Coefficients from the relation (5.6) appear in the recurrence relation for polynomials $\{\pi_{2k}(\sqrt{t})\}_{k\in\mathbb{N}_0}$ orthogonal with respect to the weight function $w(\sqrt{t})/\sqrt{t}$ on $(0,1)$ (see [11, pp. 101–103]).

Proof of Lemma 5.3. Because of orthogonality, it is clear that $s_{k,j} = (\pi_{2k}, z^{2j}) = 0$ for $k > j$. When $k = 0$, for the boundary values $s_{0,j}$ we have (5.1). For diagonal elements we have

$$s_{j,j} = (\pi_{2j}, z^{2j}) = \|\pi_{2j}\|^2 = \prod_{\nu=0}^{2j} \beta_\nu.$$

In order to get (5.6), we start with the recurrence relation (5.5). Thus,

$$\begin{aligned} s_{k+1,j} &= \int_{-1}^{1} \pi_{2k+2}(z)w(z)dz = \int_{-1}^{1} [z\pi_{2k+1}(z) - \beta_{2k+1}\pi_{2k}(z)]z^{2j}w(z)dz \\ &= \int_{-1}^{1} \pi_{2k+1}(z)z^{2j+1}w(z)dz - \beta_{2k+1}\int_{-1}^{1} \pi_{2k}(z)z^{2j}w(z)dz = \int_{-1}^{1} [z\pi_{2k}(z) - \beta_{2k}\pi_{2k-1}(z)]z^{2j+1}w(z)dz - \beta_{2k+1}s_{k,j} \\ &= s_{k,j+1} - \beta_{2k}\int_{-1}^{1} z\pi_{2k-1}(z)z^{2j}w(z)dz - \beta_{2k+1}s_{k,j}. \end{aligned}$$

Expanding $z\pi_{2k-1}(z)$ as a linear combination of $\pi_{2k}(z)$ and $\pi_{2k-2}(z)$, we get

$$s_{k,j+1} = s_{k+1,j} + \beta_{2k}\int_{-1}^{1} [\pi_{2k}(z) + \beta_{2k-1}\pi_{2k-2}(z)]z^{2j}w(z)dz + \beta_{2k+1}s_{k,j} = s_{k+1,j} + \beta_{2k}s_{k,j} + \beta_{2k}\beta_{2k-1}s_{k-1,j} + \beta_{2k+1}s_{k,j},$$

i.e., (5.6). □

Fig. 2 displays the triangular array of the inner products $s_{k,j} = (\pi_{2k}, z^{2j}),\ 0 \leqslant k \leqslant j$, and the computing stencil showing that the circled entry is computed in terms of the three other entries. The entries in the boxes are the known boundary values

$$s_{0,j} = \mu_{2j}, \quad s_{j,j} = \beta_0\beta_1\cdots\beta_{2j}, \quad s_{j+1,j} = 0, \quad j = 0, 1, \ldots, 2n+2.$$

Zero entries $s_{j+1,j}$ are displayed as white circles (in the boxes).

As we can see, for generating the system of Eqs. (3.8), i.e., (3.10), we use only entries $s_{k,j}$ for $k \leqslant n$. Thus, we need the following matrix of the type $(n+1) \times (2n+3)$,

$$\mathbf{S} = \begin{bmatrix} s_{0,1} & s_{0,1} & \ldots & s_{0,n} & \ldots & s_{0,2n+1} & s_{0,2n+2} \\ & s_{1,1} & \ldots & s_{1,n} & \ldots & s_{1,2n+1} & s_{1,2n+2} \\ & & \ddots & & & & \\ & & & s_{n,n} & \ldots & s_{n,2n+1} & s_{n,2n+2} \end{bmatrix}. \tag{5.7}$$

5.3. Method for calculating the nodes

Consider again the system of $n+1$ nonlinear equations (3.8). Taking only the last n equations of (3.8), we get

$$\mathbf{C}\boldsymbol{\sigma} = \mathbf{d}, \tag{5.8}$$

where $\mathbf{C} = \mathbf{C}(r) = [a_{k,j}]_{k=1,j=1}^{n,n}$ and $\mathbf{d} = \mathbf{d}(r) = [b_1\ \cdots\ b_n]^T$. According to (3.9), the elements $a_{k,j}$ and b_k are expressed in terms of the elements of the matrix $\mathbf{S}$ given by (5.7). The determinant of the matrix $\mathbf{C}$ has the form

$$\det\mathbf{C} = (-1)^{n(n-1)/2}\begin{vmatrix} s_{1,2n} - rs_{1,2n-1} & s_{1,2n-2} - rs_{1,2n-3} & \ldots & s_{1,2} - rs_{1,1} \\ s_{2,2n} - rs_{2,2n-1} & s_{2,2n-2} - rs_{2,2n-3} & \ldots & s_{2,2} - rs_{2,1} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n,2n} - rs_{n,2n-1} & s_{n,2n-2} - rs_{n,2n-3} & \ldots & s_{n,2} - rs_{n,1} \end{vmatrix},$$

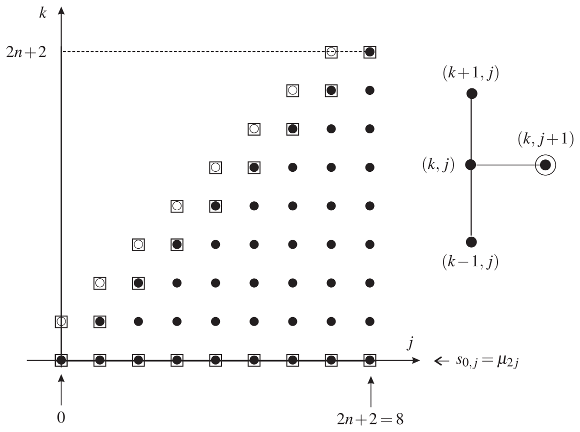

Fig. 2. The scheme for calculating the inner products $s_{k,j} = (\pi_{2k}, z^{2j})$ for $n = 3$.

where its elements $s_{k,2n-2j+2} - rs_{k,2n-2j+1}$ are equal to zero for each $k,\ j \in \{1,2,\ldots,n\}$ such that $k+2j > 2n+2$, because of $s_{k,j} = 0$ for $k > j$. It is easy to see that such zero-elements are the last $n-2$ elements in the last nth column, the last $n-4$ elements in the $(n-1)$st column, etc. Otherwise, $\det \mathbf{C} = \Delta_n(r)$ is a polynomial of degree n. Since the vector $\mathbf{d}$ on the right side in (5.8) is given by

$$\mathbf{d} = \begin{bmatrix} s_{1,2n+2} - rs_{1,2n+1} \\ s_{2,2n+2} - rs_{2,2n+1} \\ \vdots \\ s_{n,2n+2} - rs_{n,2n+1} \end{bmatrix},$$

the corresponding determinants in Cramer's rule $\Delta_n^{(j)}(r), j = 1,\ldots,n$, are also polynomials of degree n. Thus, for a given r, such that $\Delta_n(r) \neq 0$, the unique solution of (5.8) is given by

$$\widehat{\sigma}_j = \widehat{\sigma}_j(r) = \frac{\Delta_n^{(j)}(r)}{\Delta_n(r)}, \quad j = 1,\ldots,n.$$

Using MATHEMATICA package it can be obtained in a symbolic form as rational functions in r. Substituting $\widehat{\sigma}_j(r),\ j = 1,\ldots,n$, in

$$f_0 = \sum_{j=0}^{n} (-1)^j \widehat{\sigma}_j \{s_{0,2n-2j+2} - rs_{0,2n-2j+1}\} = 0,$$

we obtain the following algebraic equation of degree $n+1$,

$$\Phi_{n+1}(r) \equiv \sum_{j=0}^{n} (-1)^j \{s_{0,2n-2j+2} - rs_{0,2n-2j+1}\} \Delta_n^{(j)}(r) = 0,$$

where $\Delta_n^{(0)}(r) \equiv \Delta_n(r)$.

Numerical experiments show that the (monic) polynomial $\Phi_{n+1}(r)$ has $n+1$ different real zeros located in $(0,1)$.

In the case $n = 1$ we have seen in Section 2 that two different solutions exist for r and they give two quadratures of the same precision eleven.

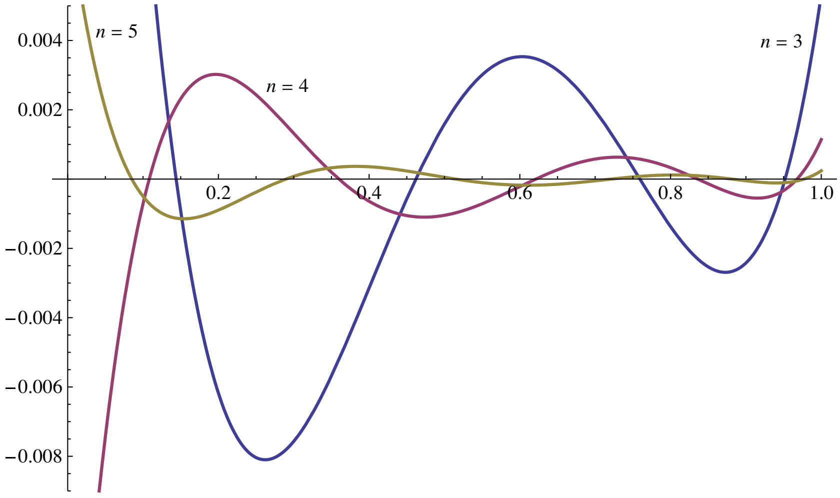

Fig. 3. Graphs of $\Phi_{n+1}(r),\ n = 3,4,5$, for $w(z) = 1$.

Table 1
Different solutions of $\Phi_{n+1}(r) = 0$ for $n = 2(1)5$ and $w(z) = 1$.

$n = 2$	$n = 3$	$n = 4$	$n = 5$
0.2044987378293505	0.1439216162367618	0.1081897446669971	0.08510161904718037
0.6167356745407912	0.4619273121368076	0.3598672165580655	0.2897653961037322
0.9208470355936592	0.7593055545829755	0.6211046569905429	0.5148988061113188
	0.9519663824480733	0.8360221823612692	0.7211387868476094
		0.9678238003414767	0.8814830739148880
			0.9769659264002607

The graphs of monic polynomials $\Phi_{n+1}(r)$ on $[0,1]$ for $n = 3,4,5$ are displayed in Fig. 3.

In Table 1 we present the corresponding numerical values of the zeros of the polynomials $\Phi_{n+1}(r)$ for $n = 2,3,4,5$.

Similar results can be obtained for other weight functions. The cases with the Chebyshev weight function of the first kind $w(z) = (1-z^2)^{-1/2}$ and the weight function $w(z) = |z|$ are displayed in Figs. 4 and 5, respectively.

The previous method for finding all solutions for $\widehat{r}_0$ works for some reasonable values of n, e.g. $n \leqslant 50$. For example, for the Chebyshev weight of the first kind and $n = 20$ we obtain 21 solutions for $\widehat{r}_0$ (given to 30 decimal digits):

```
Out[7]= {0.0150485864753572668744527330521,
  0.0557532669882252298665743917182, 0.108187398514305913909380337895,
  0.167824679058336195589651992843, 0.232108607836412242442640888453,
  0.299251018982542294927095298222, 0.367850351163304408081805272186,
  0.436729680690725174227220507435, 0.504858032671624316488143131170,
  0.571309464731063706091020367257, 0.635241286837459266216610097877,
  0.695882703617760968552811397194, 0.752529402435763252982400288756,
  0.804541615955365152530647156549, 0.851344214028606784515849170776,
  0.892427938359564164814715995310, 0.927351213720803665019429733552,
  0.955742161852690778718938880015, 0.977300564840333779616628490742,
  0.991799603894642762751920785079, 0.999087253927436018093044192512}
```

In the case $w(z) = 1$ and $n = 50$, the following sequence of 51 solutions for $\widehat{r}_0$ is given with 20 decimal digits:

```
Out[11]= {0.0040059914529631981612, 0.015198148896652530762, 0.030153553927246011838,
  0.047786461968069102119, 0.067502176479469289610, 0.088904351714626293601,
  0.11170137135198798583, 0.13566492220893164063, 0.16060826297875345727,
  0.18637354088540880140, 0.21282382747632203945, 0.23983784595920233875,
  0.26730633917663595361, 0.29512949030801918880, 0.32321504715488414291,
  0.35147693246320767511, 0.37983419928314226360, 0.40821023692094646313,
  0.43653216243154454449, 0.46473035175990477588, 0.49273807747585617047,
  0.52049122885459292527, 0.54792809622616881572, 0.57498920592171805202,
  0.60161719533987947594, 0.62775672001012940416, 0.65335438628580607294,
  0.67835870462602991647, 0.70272005943876842745, 0.72639069223892979455,
  0.74932469548418699756, 0.77147801492965236242, 0.79280845872161408075,
  0.81327571175331934504, 0.83284135404945289269, 0.85146888214350868901,
  0.86912373257358626880, 0.88577330675487417323, 0.90138699659712564769,
  0.91593621032745143452, 0.92939439805653787567, 0.94173707669323433174,
  0.95294185387170192266, 0.96298845061167595446, 0.97185872249560409663,
  0.97953667924407666148, 0.98600850280069667575, 0.99126256482379910043,
  0.99528944820940483897, 0.99808200354293588811, 0.99963584456149960413}
```

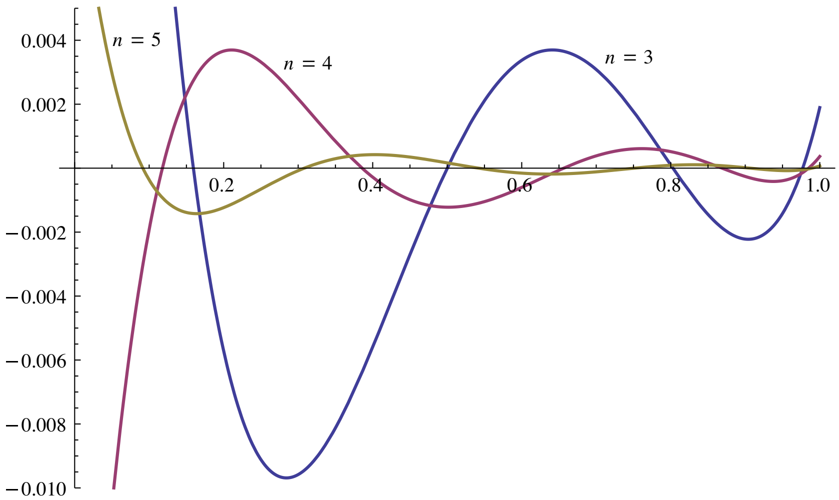

Fig. 4. Graphs of $\Phi_{n+1}(r)$, $n = 3,4,5$, for $w(z) = (1-z^2)^{-1/2}$.

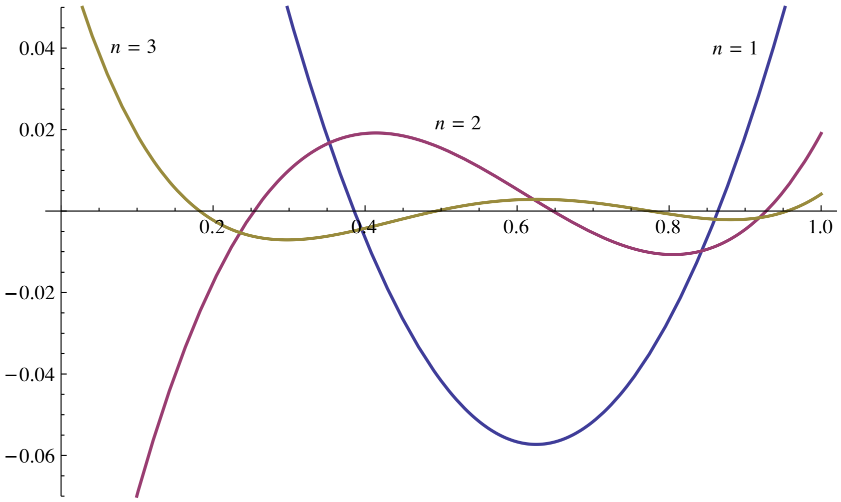

Fig. 5. Graphs of $\Phi_{n+1}(r),\ n = 1, 2, 3$, for $w(z) = |z|$.

From the theoretical point of view it would be nice to prove (or disprove) the following conjecture:

Conjecture 5.5. For a given weight function $w(z)$ and each $n \in \mathbb{N}$, all zeros of $\Phi_{n+1}(r)$ are real and distinct and are located in $(0, 1)$. Zeros of $\Phi_{n+1}(r)$ and $\Phi_{n+2}(r)$ interlace.

5.4. Numerical results

For quadratures $Q^M_{4n+3}(w;f)$ of degree of precision $d = 6n + 5$, in our MATHEMATICA procedure we obtain complete parameters in the form

$$\{\mathtt{x0}, \mathtt{xk}, \mathtt{A}, \mathtt{B}, \mathtt{CK}, \mathtt{DK}\} = \{x_0, \{x_1, \ldots, x_n\}, A, B, \{C_1, \ldots, C_n\}, \{D_1, \ldots, D_n\}\}.$$

For example, let $w(z) = 1$ and $n = 2$. Then we have three quadratures $Q^M_{11}(1;f)$ of degree of precision $d = 17$ (see first column in Table 1). For these values of r_0, i.e.,

```
{0.2044987378293505, 0.6167356745407912, 0.9208470355936592}
```

we get the following parameters:

```
{0.4522153666444237,{0.7754684395027309,0.9570916645968834},
0.4880467095490914,0.3936044844812900,{0.2527549012554169,0.1099114468981711},
{−0.0003299665322107021,0.00003577912278703132}},

{0.7853252030469869,{0.4741479794169331,0.9589531260262328},0.5473979047003460,
0.2416521097533237,{0.3846699903497127,0.1051207091442720},
{-0.005150757567968953,8.995970487465769 ∗ 10ˆ −6}},

{0.9596077509032840,{0.4802111190778518,0.7885463525798828},
0.5616568463150571,0.1034616930531016,{0.3835087691311978,0.2383909330938098},
{−0.006285348161458679, 0.00009552972582096275}},
```

respectively.

In the case of the Chebyshev weight $w(z) = 1/\sqrt{1 - z^2}$ and $n = 2$, for the corresponding values of r_0,

```
{0.2321837439443931, 0.6740206457250330, 0.9609831103305740},
```

the quadrature parameters are respectively:

```
{0.4818544842007731,{0.8124087172755511,0.9790447658917281},0.5249337433901672,
0.4705720970208580,{0.4268191199148742,0.4112815014700504},
{-0.0003991705374755941,0.00005590723150597970}},
```

Table 2
Relative errors in quadrature sums for $w(z) = 1$ and $w(z) = 1/\sqrt{1-z^2}$.

QF	N	$d_{\max}$	$w(z) = 1$	$w(z) = (1-z^2)^{-1/2}$
$Q^M_{11,0}(w;f)$	4	17	4.44(−5)	7.72(−7)
$Q^M_{11,1}(w;f)$	4	17	5.31(−6)	4.55(−5)
$Q^M_{11,2}(w;f)$	4	17	7.01(−6)	5.79(−5)
$Q^G_7(w;f)$	4	13	2.48(−4)	3.29 (−4)
$Q^G_8(w;f)$	4	15	5.73(−6)	3.06 (−5)
$Q^G_9(w;f)$	5	17	2.36(−5)	3.67 (−5)

```
{0.8209876038802492,{0.5034904974569647,0.9799958361402526},0.5871100876953411,
0.4159965436705901,{0.4649125011919272,0.4017086864355387},
{-0.005389674191177676,0.00001322584034774685}},

{0.9802974601265546,{0.5090243926812192,0.8235455809669467},0.6010918915307779,
0.3986693558288102,{0.4647957620089833,0.4131917386857352},
{-0.006515105833929775,0.0001086303399087344}}.
```

At the end of this paper we give a numerical example. The formula (3.3) could be interesting for real functions of the form $f(z) = g(z^4)$. According to (3.3), in that case each of quadratures $Q^M_{4n+3,\nu}(w;f),\ \nu = 0,1,\ldots,n$ (for $n+1$ different values of $\widehat{r}_0$), becomes a quadrature formula of the following form with $N\ (= n+2)$ nodes,

$$I(w;f) = \int_{-1}^{1} f(x)w(x)dx \approx Af(0) + 2Bf(x_0) + \sum_{k=1}^{n} W_k f(x_k), \tag{5.9}$$

where $W_k = 2(C_k + D_k),\ k = 1,\ldots,n.$

Let $f(x) = 1/(1+x^8)$. We consider two integrals

$$\int_{-1}^{1} \frac{dx}{1+x^8} = \frac{1}{4}\left\{\sin\frac{\pi}{8}\left[\pi + 2\tanh^{-1}\left(\sin\frac{\pi}{8}\right)\right] + \cos\frac{\pi}{8}\left[\pi + 2\tanh^{-1}\left(\cos\frac{\pi}{8}\right)\right]\right\} \approx 1.849303411551076,$$

and

$$\int_{-1}^{1} \frac{1}{1+x^8}\frac{dx}{\sqrt{1-x^2}} = \pi\,{}_4F_3\left(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8};\frac{1}{4},\frac{1}{2},\frac{3}{4};-1\right) \approx 2.626270969212133.$$

For their calculation we apply all quadratures $Q^M_{11,\nu}(w;f),\ \nu = 0,1,2$, whose parameters are given before. Here $n = 2,\ N = 4$, and $d_{\max} = 17$. The corresponding relative errors

$$\left|\frac{Q^M_{11,\nu}(w;f) - I(w;f)}{I(w;f)}\right|, \quad \nu = 0,1,2,$$

are given in the last two columns in Table 2. Numbers in parentheses indicate decimal exponents.

We compare these results with corresponding ones obtained by m-point Gaussian quadratures (for even functions), with respect to Legendre and Chebyshev weights,

$$I(w;f) \approx Q^G_m(w;f) = \sum_{\nu=1}^{m} A_\nu f(\tau_\nu) = 2\sum_{\nu=1}^{[m/2]} A_\nu f(\tau_\nu) + \begin{cases} A_{(m+1)/2}f(0), & \text{if } m \text{ is odd},\\ 0, & \text{if } m \text{ is even}, \end{cases}$$

where A_ν and $\tau_\nu,\ \nu = 1,\ldots,m$, are Christoffel numbers and nodes, respectively (see [11, pp. 324–325]), such that $1 > \tau_1 > \ldots > \tau_m > -1$. If we take $m = 7,8,9$, the number of nodes in the corresponding quadrature are $N = 4,4,5$, respectively.

As we can see, the quadratures $Q^M_{11,\nu}(w;f),\ \nu = 0,1,2$, have a higher degree of precision than quadratures $Q^G_m(w;f)$, $m = 7,8$ (with the same number of nodes), as well as that they give a better accuracy.

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