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ON POSITIVE DEFINITENESS OF SOME LINEAR FUNCTIONALS

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. In this paper we investigate the positive definiteness of linear functionals $\mathcal{L}$ defined on the space of all algebraic polynomials $\mathcal{P}$ by

$$\mathcal{L}(p) = \sum_{k\in\mathbb{N}} w_k p(z_k), \quad p \in \mathcal{P}.$$

1. Introduction

Let $\mathcal{P}$ be the space of all algebraic polynomials. In this paper we investigate linear functionals $\mathcal{L}$ defined by

$$\mathcal{L}(p) = \sum_{k\in\mathbb{N}} w_k p(z_k), \quad p \in \mathcal{P}. \tag{1}$$

In general, we investigate functionals for which $w_k, z_k \in \mathbb{C}\backslash\{0\}$, but with the following restrictions. First, we assume that $w_k \neq 0$, $k \in \mathbb{N}$. This condition is rather natural, since, assuming $w_k = 0$, for some $k \in \mathbb{N}$, simply produces a linear functional where summation is performed over $\mathbb{N}\backslash\{k\}$. Additionally, we will not loose any generality if we assume that $z_i \neq z_j$, $i, j \in \mathbb{N}$, since, for example, we may skip summation over j and use $w_i' = w_i + w_j$ at point z_i.

For the set of nodes z_i, $i \in \mathbb{N}$, we introduce the notation $\mathcal{Z} = \{z_k \mid k \in \mathbb{N}\}$.

Second we are going to assume that

$$\lim_{k\to+\infty} z_k = 0 \tag{2}$$

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and, in order to have absolute integrability of all polynomials $p \in \mathcal{P}$, we assume that

$$\sum_{k\in\mathbb{N}} |w_k| \leq M < +\infty. \tag{3}$$

We assume in the sequel that the sequence z_k, $k \in \mathbb{N}$, is ordered in such a way that $|z_{k+1}| \leq |z_k|$, $k \in \mathbb{N}$.

Note that the linear functional $\mathcal{L}$ can be interpreted as the linear functional acting on the space of all bounded complex sequences ℓ_∞. Namely, according to the condition (3) we have that the sequence w_k, $k \in \mathbb{N}$, belongs to the space ℓ_1, the space of all absolutely summable complex sequences (see [3, p. 30], [2, p. 39]). As is known $\ell_1 \subset \ell'_\infty$, where ℓ'_∞ denotes dual of ℓ_∞.

Create now a linear mapping $\mathcal{I} : \mathcal{P} \mapsto \ell_\infty$ in the following way

$$\mathcal{I}(p) = (p(z_1), p(z_2), \ldots, p(z_n), \ldots).$$

The linear space $\mathcal{P}$ can be normed as

$$||p|| = \sup_{k\in\mathbb{N}} |p(z_k)|, \quad p \in \mathcal{P}.$$

Lemma 1.1. *The linear mapping $\mathcal{I} : \mathcal{P} \mapsto \ell_\infty$ is an bounded embedding of $\mathcal{P}$ into ℓ_∞.*

Proof. Given $\mathcal{L}$, any polynomial $p \in \mathcal{P}$ achieves its maximum on the compact set $\overline{\mathcal{Z}}$, hence any sequence $p(z_k)$, $k \in \mathbb{N}$, $p \in \mathcal{P}$, is uniformly bounded in k and belongs to ℓ_∞.

Norm preserving property is easily established. We note that if two polynomials satisfy $\mathcal{I}(p_1 - p_2) = 0$, we have that $p_1 = p_2$, since those are two analytic functions equal on the set $\mathcal{Z}$ which has one accumulation point. Hence, $\mathcal{I}(\mathcal{P}) \subset \ell_\infty$ is an embedding.

It is easily seen that $||\mathcal{I}|| = 1$. $\square$

Now, define the linear functional $\mathcal{L}' : \ell_\infty \mapsto \mathbb{C}$ in the following way

$$\mathcal{L}'(u) = \sum_{k\in\mathbb{N}} w_k u_k, \quad u = (u_1, u_2, \ldots) \in \ell_\infty.$$

Obviously $\mathcal{L}'$ is bounded, since

$$|\mathcal{L}'(u)| \leq \sum_{k\in\mathbb{N}} |w_k||u_k| \leq ||u|| \sum_{k\in\mathbb{N}} |w_k|, \quad u \in \ell_\infty,$$

and $\mathcal{L}' \circ \mathcal{I} = \mathcal{L}$ on $\mathcal{P}$. Hence, for the certain extent we can identify $\mathcal{L}'$ and $\mathcal{L}$ and we may consider $\mathcal{L}'$ as a bounded linear extension of $\mathcal{L}$ to the whole of ℓ_∞.

Define $\mathcal{P}_+$ to be the set of all polynomials $p \in \mathcal{P}\backslash\{0\}$ which are nonnegative on the real line and denote by $\mathcal{P}_\mathbb{R}$ the set of all real algebraic polynomials.

We recall that linear functional $\mathcal{L} : \mathcal{P} \mapsto \mathbb{C}$ is called positive definite provided for every polynomial $p \in \mathcal{P}_+$ we have $\mathcal{L}(p) > 0$ (see [1, p. 13]). As a direct consequence of positive definiteness we have:

Lemma 1.2. *If the linear functional $\mathcal{L} : \mathcal{P} \mapsto \mathbb{C}$ is positive definite, then*

$$\mathcal{L}(x^{2n}) > 0, \quad \mathcal{L}(x^{2n+1}) \in \mathbb{R}, \quad \mathcal{L}(p) \in \mathbb{R}, \quad p \in \mathcal{P}_\mathbb{R}, \quad n \in \mathbb{N}_0. \tag{4}$$

Proof. Since $x^{2n} \in \mathcal{P}_+$, we have directly $\mathcal{L}(x^{2n}) > 0$. For the odd powers we have

$$\mathcal{L}(x-1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{2n-k} \mathcal{L}(x^k) > 0,$$

and using induction over $n \in 2\mathbb{N}$, we have

$$\mathcal{L}(x^{2n-1}) < \frac{1}{2n} \sum_{k=0, k\neq 2n-1}^{2n} \binom{2n}{k} (-1)^k \mathcal{L}(x^k).$$

Finally, we have according to linearity of $\mathcal{L}$ the last statement. $\square$

The question we answer is summarized in the following theorem.

Theorem 1.1. *A linear functional $\mathcal{L}$ given by* (1) *is positive definite if and only if $w_k > 0$ and $z_k \in \mathbb{R}$, $k \in \mathbb{N}$.*

Finally, we introduce the following notation

$$e_n = (0, \ldots, 0, 1, 0, \ldots) \in \ell_\infty, \quad n \in \mathbb{N},$$

where number 1 occupies n-th position with zeros on all other positions.

2. **Auxiliary results**

We give first, the following auxiliary lemmas.

Lemma 2.1. *Choose $z_n \in \mathcal{Z}$ and assume that $\overline{z}_n \notin \mathcal{Z}$. Then there exists $p^n \in \mathbb{C}$, $|p^n| = 1$, such that for every $r^n \in \mathcal{P}_{\mathbb{R}}$ we have $p^n r^n(z_n) e_n \in \overline{\mathcal{I}(\mathcal{P}_{\mathbb{R}})}$. If $z_n \in \mathbb{R}\backslash\{0\}$ then $p^n = 1$.*

Proof. We are going to construct the sequence $p_k^n \in \mathcal{P}_+$, $k \in \mathbb{N}$, $n \in \mathbb{N}$, such that $\lim\limits_{k\to+\infty} \mathcal{I}(p_k^n) = \alpha_n e_n$ for some complex number $\alpha_n \in \mathbb{C}\backslash\{0\}$.

Choose some fixed $z_n \in \mathcal{Z}$ and assume that $\overline{z}_n \notin \mathcal{Z}$. Then choose some polynomial $r^n \in \mathcal{P}_{\mathbb{R}}$. We define

$$p_k^n(z) = r^n(z) \prod_{i=1, i\neq n}^{k} \frac{(z-z_i)(z-\overline{z}_i)}{\lambda_i^n}, \quad k \in \mathbb{N},$$

where we denote

$$\lambda_i^n = |z_n - z_i||z_n - \overline{z}_i|, \quad i \neq n.$$

Obviously we have $p_k^n \in \mathcal{P}_+$, $k, n \in \mathbb{N}$.

Since r^n is an algebraic polynomial it is uniformly bounded on the compact set $\overline{\mathcal{Z}}$. Hence, for some $M > 0$ we have $|r^n(z_\nu)| < M$, $\nu \in \mathbb{N}$.

According to the property (2), we can choose some $i_{01} \in \mathbb{N}$ such that

$$|z_n|/2 < |z_\nu - z_i|, \quad |z_n|/2 < |z_\nu - \overline{z}_i|, \quad i > i_{01}, \quad \nu = 1, \ldots, n.$$

Fix some $q \in (0,1)$. We can choose some $i_{02} \in \mathbb{N}$ such that

$$|z_i| < |z_n|q/4, \quad i > i_{02}.$$

Now, define $i_0 = \max\{i_{01}, i_{02}\}$. For $k > i_0$ and $\nu > k$, we have

$$\begin{aligned}
|p_k^n(z_\nu)| &= |r^n(z_\nu)| \prod_{i=1,i\neq n}^{i_0} \frac{|z_\nu - z_i||z_\nu - \overline{z}_i|}{\lambda_i^n} \prod_{i=i_0+1}^{k} \frac{|z_\nu - z_i||z_\nu - \overline{z}_i|}{|z_n - z_i||z_n - \overline{z}_i|} \\
&\leq M \prod_{i=1,i\neq n}^{i_0} \frac{|z_\nu - z_i||z_\nu - \overline{z}_i|}{\lambda_i^n} \prod_{i=i_0+1}^{k} \frac{|z_n|q/2|z_n|q/2}{|z_n|/2|z_n|/2} \\
&\leq M \left(\frac{2|z_1|}{m}\right)^{2i_0-2} q^{2(k-i_0)},
\end{aligned}$$

where $m = \min_{i=1,\ldots,i_0,\ i\neq n}\{|z_n - z_i|, |z_n - \overline{z}_i|\} > 0$. We note that $p_k^n(z_\nu) = 0$ for $\nu < k$, $\nu \neq n$. From here it can be easily seen that we have uniform convergence in $\nu \neq n$ of $p_k^n(z_\nu)$ to zero for $k \to +\infty$, i.e., given $\varepsilon > 0$, for

$$k > k_{01} = i_0 + \frac{1}{2\log q} \log \frac{\varepsilon}{M} \left(\frac{m}{2|z_1|}\right)^{2i_0-2},$$

we have $|p_k^n(z_\nu) - 0| < \varepsilon$, $\nu \in \mathbb{N}\backslash\{n\}$.

Now, we consider $p_k^n(z_n)$, we have

$$|p_k^n(z_n)| = |r^n(z_n)|,$$

according to the definition of λ_i^n. This means that $p_k^n(z_n)$ has constant norm as $k \to +\infty$.

The product

$$\prod_{i=1,i\neq n}^{k} \frac{(z_n - z_i)(z_n - \overline{z}_i)}{\lambda_i^n}, \quad k \in \mathbb{N},$$

is just product of the complex numbers having modulus 1, hence, represent the sequence on the unit circle in the complex plane. According to the compactness of the unit circle in $\mathbb{C}$, we easily conclude that there exists some subsequence of the products which converge to some p^n which norm is one.

Denote set of indices for convergent subsequence as N_1. Then according to the convergence, given $\varepsilon > 0$, we can choose some $k_{02} \in N_1$, such that for $k > k_{02}$, $k \in N_1$, we have

$$|p_k^n(z_n) - r^n(z_n)p^n| < \varepsilon.$$

Now consider the vector $r^n(z_n)p^n e_n$, we have

$$||\mathcal{I}(p_k^n) - r^n(z_n)p^n e_n|| = \sup_{\nu\in\mathbb{N}} |p_k^n(z_\nu) - r^n(z_n)p^n e_n| < \varepsilon,$$

for $k > \max\{k_{01}, k_{02}\}$, $k \in N_1$.

Hence, if we enumerate, again the sequence p_k^n using only indexes $k \in N_1$, we have the sequence $p_k^n \in \mathcal{P}_{\mathbb{R}}$, such that

$$\lim_{k\to+\infty} \mathcal{I}(p_k^n) = p^n e_n.$$

Finally, if $z_n \in \mathbb{R}\backslash\{0\}$ we see that since $r^n \in \mathcal{P}_{\mathbb{R}}$, we have $r^n(z_n) \in \mathbb{R}$ and

$$p_k^n(z_n) = r^n(z_n) \prod_{i=1, i\neq n}^{k} \frac{|(z_n - z_i)(z_n - \overline{z}_i)|}{\lambda_i^n} \in \mathbb{R}$$

and also the terms of the product are simply equal to 1, hence, $p^n = 1$.

We can repeat construction for every $n \in \mathbb{N}$, i.e., every point $z_n \in \mathcal{Z}$ for which $\overline{z}_n \notin \mathcal{Z}$. $\square$

In the case $r^n \in \mathcal{P}_+$, we easily see that the sequence p_k^n also belongs to $\mathcal{P}_+$, so that we have the following result.

Lemma 2.2. *Assume that $\overline{z}_n \notin \mathcal{Z}$. Then there exists $p^n \in \mathbb{C}$, $|p^n| = 1$, such that for every $r^n \in \mathcal{P}_+$ we have $p^n r^n(z_n) e_n \in \overline{\mathcal{I}(\mathcal{P}_+)}$. If $z_n \in \mathbb{R}\backslash\{0\}$ we have $p^n = 1$.*

Next we consider the case when $\overline{z}_n \in \mathcal{Z}$. Without loss of generality we may assume that $z_{n+1} = \overline{z}_n$, since this can be achieved by the simple renumeration of the sequence z_n, $n \in \mathbb{N}$.

Lemma 2.3. *Let $z_{n+1} = \overline{z}_n$ for some $n \in \mathbb{N}$. Then there exist some $p^n \in \mathbb{C}$, $|p^n| = 1$, such that for every $r^n \in \mathcal{P}_{\mathbb{R}}$ we have*

$$p^n r^n(z_n) e_n + \overline{p^n r^n(z_n)} e_{n+1} \in \overline{\mathcal{I}(\mathcal{P}_{\mathbb{R}})}.$$

Proof. We consider the sequence of the polynomials

$$p_k^n(z) = r^n(z) \prod_{i=1, i\neq n, n+1}^{k} \frac{(z - z_i)(z - \overline{z}_i)}{\lambda_i^n},$$

where all notation is from the proof of Lemma 2.1. The only problem is definition of the sequence λ_i^n, but luckily we have

$$|z_n - z_i||z_n - \overline{z}_i| = |z_{n+1} - z_i||z_{n+1} - \overline{z}_i|,$$

since $z_{n+1} = \overline{z}_n$. Hence, we can apply safely the same definition.

It can proved using the same arguments that

$$|p_k^n(z_\nu) - 0| < \varepsilon,$$

provided

$$k > k_{01} = i_0 + \frac{1}{2\log q} \log \frac{\varepsilon}{M} \left(\frac{m}{2|z_1|}\right)^{2i_0-4}.$$

Also we have $p_k^n(z_n) = \overline{p_k^n(z_{n+1})}$, which gives the convergence for some sequence of $k \in N_1$ to mutually conjugated values. $\square$

It is clear that we may choose $r^n \in \mathcal{P}_+$ to get the following immediate consequence.

Lemma 2.4. *Let $z_{n+1} = \overline{z}_n$ for some $n \in \mathbb{N}$. Then there exist some $p^n \in \mathbb{C}$, $|p^n| = 1$, such that for every $r^n \in \mathcal{P}_+$ we have*

$$p^n r^n(z^n)e_n + \overline{p^n r^n(z_n)}e_{n+1} \in \overline{\mathcal{I}(\mathcal{P}_+)}.$$

3. **Proof of the main result**

Now we are ready to prove the main result.

Proof of Theorem 1.1. It can be easily seen that if $w_n > 0$ and $z_n \in \mathbb{R}$, $n \in \mathbb{N}$, for some $p \in \mathcal{P}_+$, we have

$$\mathcal{L}(p) = \sum_{k\in\mathbb{N}} w_k p(z_k) > 0,$$

according to the simple fact that $p(z_k) \geq 0$, $k \in \mathbb{N}$.

Now, assume that $\mathcal{L}$ is positive definite. Choose some $n \in \mathbb{N}$ and suppose that $\overline{z}_n \notin \mathcal{Z}$. Then, according to Lemma (2.1), we have

$$\lim_{k\to+\infty} \mathcal{L}(p_k^n) = \lim_{k\to+\infty} (\mathcal{L}' \circ \mathcal{I})(p_k^n) = \mathcal{L}'(p^n r^n(z_n)e_n),$$

where we have used the fact that $\mathcal{L}'$ is continuous on ℓ_∞. But then

$$\mathcal{L}'(p^n r^n(z_n)e_n) = w_n r^n(z_n)p^n.$$

Choose $r^n(z) = 1$, $r^n \in \mathcal{P}_+$, and $r^n(z) = z$, $r^n \in \mathcal{P}_\mathbb{R}$, to get

$$\mathcal{L}'(p^n e_n) = w_n p^n \geq 0 \quad \text{and} \quad \mathcal{L}'(p^n z_n e_n) = w_n z_n p^n \in \mathbb{R}.$$

Since $z_n \neq 0$ and according to the construction $p^n \neq 0$, we have that $\mathcal{L}'(p^n e^n) = w_n p^n > 0$. Then we have

$$z_n = \frac{\mathcal{L}'(p^n z_n e_n)}{\mathcal{L}'(p^n e_n)} \in \mathbb{R}$$

and also $w_n > 0$, according to the fact that $p^n = 1$ for $z_n \in \mathbb{R}\backslash\{0\}$.

Now let $\overline{z}_n = z_{n+1}$. Note that in this case we cannot have $z_n \in \mathbb{R}$, since in that case we would have $z_n = z_{n+1}$, which is impossible according to the conditions imposed on the set $\mathcal{Z}$. Then, according to Lemma (2.3) and positive definiteness of $\mathcal{L}$ for $r^n(z) = 1$ and $r^n(z) = z$, we have

$$\mathcal{L}'(p^n e_n + \overline{p^n} e_{n+1}) = w_n p^n + w_{n+1}\overline{p^n} = \alpha \geq 0$$

and

$$\mathcal{L}'(p^n z_n e_n + \overline{p^n z_n} e_{n+1}) = w_n z_n p^n + w_{n+1}\overline{z}_n = \beta \in \mathbb{R}.$$

We can rewrite these equations as the linear system in p^n and $\overline{p^n}$, which has the unique solution

$$p^n = \frac{\alpha \overline{z}_n - \beta}{w_n(\overline{z}_n - z_n)}, \quad \overline{p^n} = \frac{\alpha z_n - \beta}{w_{n+1}(z_n - \overline{z}_n)}.$$

Using these expressions we readily get $w_{n+1} = \overline{w}_n$ and also we see that we cannot have $\alpha^2 + \beta^2 = 0$, since it would imply $p^n = 0$, which is impossible.

Now, choose $r^n(z) = z^{2\nu}$, $\nu \in \mathbb{N}$. We have

$$\mathcal{L}'(p^n z_n^{2\nu} e_n + \overline{p^n z_n^{2\nu}} e_{n+1}) = w_n z_n^{2\nu} p^n + \overline{w_n z_n^{2\nu} p^n} = \mathrm{Re}(w_n z_n^{2\nu} p^n) \geq 0, \quad \nu \in \mathbb{N}_0.$$

If we denote $\alpha_n = \arg(w_n)$, $\beta_n = \arg(p^n)$ and $\varphi_n = \arg(z_n)$, where $\varphi_n \neq 0$ and $\varphi_n \neq \pi$, we get

$$|w_n z_n^{2\nu} p^n| \cos(\alpha_n + \beta_n + 2\nu\varphi_n) \geq 0, \quad \nu \in \mathbb{N}_0.$$

We want to show that there exist some $\nu \in \mathbb{N}_0$ such that cos function is negative which will produce a contradiction.

The cos-function is negative provided 2ν is an element of some interval

$$J_k = \left(\frac{(4k+1)\pi - 2(\alpha_n + \beta_n)}{2\varphi_n}, \frac{(4k+3)\pi - 2(\alpha_n + \beta_n)}{2\varphi_n}\right), \quad k \in \mathbb{Z}.$$

The interval J_k has length $\pi/|\varphi_n| > 1$, hence, there is at least one integer inside every interval J_k. If $\pi/|\varphi_n| > 2$ then there are at least two consecutive integers inside every J_k and at least one of them is even. Choosing 2ν to be equal to such an integer produces a contradiction. So, we assume $\pi/|\varphi_n| \leq 2$.

The intervals

$$G_k = \left[\frac{(4k+3)\pi - 2(\alpha_n+\beta_n)}{2\varphi_n}, \frac{(4k+5)\pi - 2(\alpha_n+\beta_n)}{2\varphi_n}\right], \quad k \in \mathbb{Z},$$

we are going to call gaps, obviously $\mathbb{R} = \cup_{k\in\mathbb{Z}}(J_k \cup G_k)$.

If $\pi/|\varphi_n| = 2$, we have $\varphi_n = \pm\pi/2$, which means that if

$$\cos(\alpha_n + \beta_n \pm 2\cdot 0\cdot \pi/2) > 0,$$

we have

$$\cos(\alpha_n + \beta_n \pm 2\cdot 1\cdot \pi/2) = -\cos(\alpha + \beta_n) < 0,$$

which produces a contradiction. If $\cos(\alpha_n + \beta_n \pm 2\cdot 0\cdot \pi/2) = 0$, then we have

$$\mathrm{Re}(w_n z_n^{2\nu} p^n) = |w_n z_n^{2\nu} p^n| \cos(\alpha_n + \beta_n \pm \nu\pi) = 0, \quad \nu \in \mathbb{N},$$

and, choosing $r^n(z) = z^{2\nu+1}$, $\nu \in \mathbb{N}_0$, we have

$$\begin{aligned}\mathrm{Re}(w_n z^{2\nu+1} p^n) &= |w_n z_n^{2\nu+1} p^n| \cos(\alpha_n + \beta_n \pm (2\nu+1)\pi/2)\\ &= \pm(-1)^\nu |w_n z_n^{2\nu+1} p^n| \sin(\alpha_n+\beta_n) \neq 0, \quad \nu \in \mathbb{N}_0.\end{aligned}$$

According to the fact $\cos(\alpha_n + \beta_n) = 0$, we have $\sin(\alpha_n + \beta_n) = \pm 1$, therefore, the expression cannot be equal zero. Consider now polynomials $r^n(z) = z^{2\nu}(z-1)^2$, $\nu \in \mathbb{N}_0$. Obviously $r^n \in \mathcal{P}_+$, so that it must be

$$\begin{aligned}\lim_{k\to+\infty} \mathcal{L}(p_k^n) &= \lim_{k\to+\infty} (\mathcal{L}' \circ \mathcal{I})(p_k^n) = w_n z_n^{2\nu}(z_n-1)^2 p^n + \overline{w_n z_n^{2\nu}(z_n-1)^2 p^n}\\ &= \mathrm{Re}(w_n z_n^{2\nu}(z_n-1)^2 p^n) \geq 0.\end{aligned}$$

According to linearity we must have

$$\begin{aligned}\mathrm{Re}(w_n z_n^{2\nu}(z_n-1)^2 p^n) &= \mathrm{Re}(-2 w_n z_n^{2\nu+1} p^n)\\ &= \mp 2(-1)^\nu |w_n z_n^{2\nu+1} p^n| \sin(\alpha_n+\beta_n) > 0, \quad \nu \in \mathbb{N}_0.\end{aligned}$$

This is, of course, a contradiction.

Finally, it must be $1 < \pi/|\varphi_n| < 2$. Assume that in some interval J_k we have an integer $2m+1$. Then we can always choose some $\nu \in \mathbb{N}$, such that

$$\frac{\pi}{|\varphi_n|} > \frac{2\nu - 2m - 1}{2\nu - 2m - 2} > 1.$$

Then counting from $2m+1$ and finishing with 2ν there are exactly $2\nu-2m$ integers and those are covered with

$$2\nu - 2m - 2 + 1 > \frac{2\nu - 2m - 1}{\pi/|\varphi_n|} + 1,$$

intervals and gaps. Since we are starting and ending with an interval there are $\nu-m-1$ gaps and $\nu-m$ intervals. According to pigeon-hole principle there is at least one either interval or gap which contains at least two consecutive integers. If some interval contains two consecutive integers we are done. So assume that it is some gap. If gap contains even and odd integer, then next interval holds an even integer and we are done. If gap holds odd and even integer, then interval in front of it holds an even integer, and we are done.

We conclude that it cannot be $z_n, \overline{z}_n \in \mathcal{Z}$. We have seen also that if $z_n \in \mathcal{Z}$, then $z_n \in \mathbb{R}$ and $w_n > 0$, which finishes the proof. $\square$

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