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EXTREMAL PROBLEMS FOR LORENTZ CLASSES OF NONNEGATIVE POLYNOMIALS IN L^2 METRIC WITH JACOBI WEIGHT

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ABSTRACT. Let L_n be the Lorentz class of nonnegative polynomials on $[-1,1]$. Extremal problems of Markov type, in L^2 norm with Jacobi weight, on the set L_n or on its subset, are investigated.

1. Introduction. In this paper we consider some extremal problems for nonnegative algebraic polynomials on $[-1,1]$ in L^2 metric with Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$ $(\alpha,\beta > -1)$. These problems are related to some previous results due to Varma [**9**–**13**], Milovanović [**6**], Erdös and Varma [**2**], and also to the classical inequalities of A. Markov [**5**], P. Erdös [**1**], G. G. Lorentz [**3**, **4**], and G. Szegö [**8**].

Let L_n be the set of algebraic polynomials of the form

$$P_n(x) = \sum_{k=0}^{n} b_k(1-x)^k(1+x)^{n-k}, \qquad b_k \geq 0 \ (k = 0, 1, \ldots, n). \tag{1.1}$$

These polynomials (transformed to $[0,1]$) were introduced by G. G. Lorentz [**3**] and studied extensively by J. T. Scheick [**7**]. A subset of the Lorentz class L_n for which $P_n^{(i-1)}(-1) = P_n^{(i-1)}(1) = 0$ $(i = 1, \ldots, m)$ will be denoted by $L_n^{(m)}$. Notice that $L_n^{(0)} \supset L_n^{(1)} \supset \cdots$, where $L_n^{(0)} \equiv L_n$. The corresponding representation of a polynomial P_n from $L_n^{(m)}$ is

$$P_n(x) = \sum_{k=m}^{n-m} b_k(1-x)^k(1+x)^{n-k}, \qquad b_k \geq 0 \ (k = m, \ldots, n-m). \tag{1.2}$$

Let $w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, and $\|f\|^2 = (f,f)$, where

$$(f,g) = \int_{-1}^{1} w(x)f(x)g(x)\,dx \qquad (f,g \in L^2(-1,1)).$$

The object of this paper is to determine

$$C_n^{(m)}(\alpha,\beta) = \sup_{P_n \in L_n^{(m)} \setminus \{0\}} \frac{\|P_n'\|^2}{\|P_n\|^2}, \tag{1.3}$$

where $m = 0, 1, \ldots, [n/2]$. The corresponding problem in the class $L_n^{(0)}$ for the uniform norm was considered by G. G. Lorentz [**4**].

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Some auxiliary results, necessary in solving problem (1.3), are presented in §2. The central issue of the paper, the determination of the best constant in (1.3), is given in §3. Some corollaries and special cases of importance are included.

2. Some auxiliary results. We begin this section by proving four lemmas.

LEMMA 2.1. *If $P_n \in L_n$, then for every $x \in [-1,1]$ the inequality*

$$(1-x^2)(P_n'(x)^2 - P_n''(x)P_n(x)) \le nP_n(x)^2 - 2xP_n(x)P_n'(x) \tag{2.1}$$

holds.

PROOF. Putting $t = (1-x)/(1+x)$ and

$$Q_n(t) = \left(\frac{1+t}{2}\right)^n P_n\left(\frac{1-t}{1+t}\right) = \sum_{k=0}^{n} b_k t^k,$$

we obtain

$$\begin{aligned}(1+x)^n Q_n(t) &= P_n(x),\\ 2(1+x)^{n-1}Q_n'(t) &= nP_n(x) - (1+x)P_n'(x),\\ 4(1+x)^{n-2}Q_n''(t) &= n(n-1)P_n(x) - 2(n-1)(1+x)P_n'(x) + (1+x)^2P_n''(x).\end{aligned}$$

Substituting Q_n, Q_n', and Q_n'' from the last three relations in the inequality

$$t[Q_n'(t)^2 - Q_n(t)Q_n''(t)] \le Q_n'(t)Q_n(t), \qquad t \ge 0,$$

which was proved in [**6**], we obtain (2.1).

REMARK 2.1. Inequality (2.1) can be represented in the form

$$\frac{d}{dx}\left\{(x^2-1)\frac{P_n'(x)}{P_n(x)}\right\} \le n.$$

Now, we define the following integrals:

$$I_{n,i}(\alpha,\beta) = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n(x) P_n^{(i)}(x)\,dx \qquad (i = 0,1,2),$$

$$J_n(\alpha,\beta) = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n'(x)^2\,dx.$$

LEMMA 2.2. *If $P_n \in L_n$, then for $\alpha, \beta > 0$ the following recurrence relations are valid*:

$$\begin{aligned}I_{n,2}(\alpha,\beta) &= \alpha I_{n,1}(\alpha-1,\beta) - \beta I_{n,1}(\alpha,\beta-1) - J_n(\alpha,\beta),\\ 2I_{n,1}(\alpha,\beta) &= \alpha I_{n,0}(\alpha-1,\beta) - \beta I_{n,0}(\alpha,\beta-1).\end{aligned}$$

However, if $P_n \in L_n^{(1)}$, then the first relation holds for $\alpha, \beta > -1$, and the second of them for $\alpha, \beta > -2$.

The proof of this lemma is a simple application of integration by parts and will be omitted.

Integrating (2.1) we obtain

LEMMA 2.3. *If $P_n \in L_n$ (or $L_n^{(1)}$), then for $\alpha, \beta > 0$ (or > -1),*

$$J_n(\alpha,\beta) \le nI_{n,0}(\alpha-1,\beta-1) + I_{n,2}(\alpha,\beta) - I_{n,1}(\alpha-1,\beta) + I_{n,1}(\alpha,\beta-1).$$

From Lemmas 2.2 and 2.3 there immediately follows

LEMMA 2.4. *If $P_n \in L_n$ (or $L_n^{(1)}$), then for $\alpha, \beta > 1$ (or > -1), the inequality*

$$\begin{aligned} 4J_n(\alpha,\beta) \le{}& (\alpha-1)^2 I_{n,0}(\alpha-2,\beta) + (\beta-1)^2 I_{n,0}(\alpha,\beta-2) \\ &+ [2n+\alpha+\beta-2\alpha\beta] I_{n,0}(\alpha-1,\beta-1) \end{aligned}$$

holds.

REMARK 2.2. If $P_n \in L_n$ and $\alpha = \beta = 1$, the above inequality is also valid. Namely, we then have

$$J_n(1,1) \le \frac{n}{2} I_{n,0}(0,0).$$

Also, when $\alpha = 1$ and $\beta > 1$, we have

$$4J_n(1,\beta) \le (\beta-1)^2 I_{n,0}(1,\beta-2) + (2n+1-\beta) I_{n,0}(0,\beta-1).$$

A symmetric result holds for $\alpha > 1$ and $\beta = 1$.

Now let $n \in N$, $m = 0, 1, \ldots, [n/2]$, and $\Delta_{n,m} = [2m, 2n-2m]$. We define the rational function $f\colon \Delta_{n,m} \to R$ by

$$\begin{aligned} f(x) ={}& \frac{(\alpha-1)^2}{(x+\alpha-1)(x+\alpha)} + \frac{(\beta-1)^2}{(2n-x+\beta-1)(2n-x+\beta)} \\ &+ \frac{2n+\alpha+\beta-2\alpha\beta}{(x+\alpha)(2n-x+\beta)}. \end{aligned} \tag{2.2}$$

The parameters α and β can take the values

(a) $\alpha, \beta \ge 1$ if $m = 0$;

(b) $\alpha, \beta > -1$ if $m \ge 1$.

In order to find the maximum of $f(x)$ on the interval $\Delta_{n,m}$, we investigate the derivative

$$f'(x) = \frac{R(x)}{[(x+\alpha-1)(x+\alpha)(2n-x+\beta-1)(2n-x+\beta)]^2},$$

where R is a polynomial of degree five and whose coefficients depend on α, β, and n. It is easy to see that

1° $R(0) < 0, R(2n) > 0$;

2° R has the unique zero ξ in $(0, 2n)$.

Based on the above we can conclude that

$$\max_{x \in \Delta_{n,m}} f(x) = \max(f(2m), f(2n-2m)). \tag{2.3}$$

Now, we consider two cases

(a) $m = 0$ $(\alpha, \beta \ge 1)$. Since

$$f(2n) - f(0) = \frac{(\beta-\alpha)(\beta+\alpha+4n-1)}{(2n+\alpha-1)(2n+\alpha)(2n+\beta-1)(2n+\beta)},$$

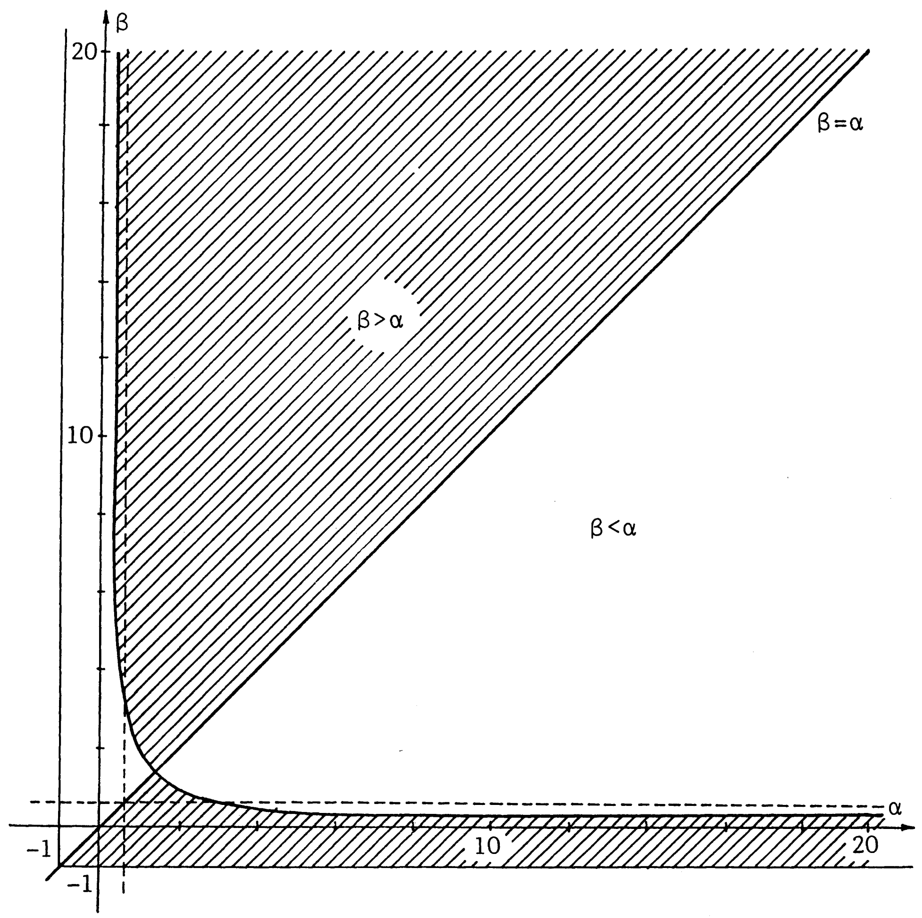

FIGURE 1

we have $\operatorname{sgn}(f(2n) - f(0)) = \operatorname{sgn}(\beta - \alpha)$, and then we find

$$(2.4) \qquad \max_{x\in\Delta_{n,0}} f(x) = \frac{4n^2}{(2n+\lambda-1)(2n+\lambda)}, \qquad \lambda = \min(\alpha, \beta).$$

(b) $m \geq 1$ $(\alpha, \beta > -1)$. The maximum of $f(x)$ on $\Delta_{n,m}$ is given by (2.3). The domination of the value $f(2m)$ with respect to $f(2n-2m)$, and conversely, changes in the points of the $\alpha 0 \beta$ plane for which

$$g(\alpha, \beta) \equiv f(2n-2m) - f(2m) = 0$$

is valid. It is easy to show that

$$g(\alpha, \beta) = (\beta - \alpha) \sum_{k,j=0}^{3} q_{kj} \alpha^k \beta^j,$$

where $q_{kj} = q_{jk}$, that is, $g(\alpha, \beta) = g(\beta, \alpha)$ (a symmetry with respect to the straight line $\beta = \alpha$). The curve $g(\alpha, \beta) = 0$, where m and n are the parameters, has three branches, one of them is, obviously, the straight line $\beta = \alpha$. The region

$\alpha, \beta > -1$ contains the branch which has the horizontal asymptote $\alpha = a(m,n)$ and the vertical asymptote $\beta = a(m,n)$ (because of a symmetry in reference to the straight line $\beta = \alpha$), where

$$a(m,n) = \frac{4m^2 + n^2 - 4mn + \sqrt{16m^2n^2 + 16m^4 + n^2 - 32m^3n}}{2n} > 0.$$

To illustrate graphically the regions of domination and the corresponding bounds, the case when $m = 1$ and $n = 8$ is displayed in Figure 1. The horizontal and vertical asymptotes are given by $\alpha = a(1,8) \cong 0.57$ and $\beta = a(1,8) \cong 0.57$. In the shaded region the inequality $f(2) < f(14)$ holds.

3. Main results. In this section we give the results related to problem (1.3). We begin with the following assertion.

THEOREM 3.1. *If $P_n \in L_n$ and $\alpha, \beta \geq 1$, then the best constant $C_n^{(0)}(\alpha,\beta)$, defined in (1.3), is given by*

$$C_n^{(0)}(\alpha,\beta) = \frac{n^2(2n+\alpha+\beta)(2n+\alpha+\beta+1)}{4(2n+\lambda)(2n+\lambda-1)}, \tag{3.1}$$

where $\lambda = \min(\alpha,\beta)$.

PROOF. We suppose that $P_n \in L_n$, i.e. that P_n is given by (1.1). Then

$$P_n(x)^2 = \sum_{k=0}^{2n} a_k(1-x)^k(1+x)^{2n-k}, \qquad a_k \geq 0,$$

and

$$\|P_n\|^2 = I_{n,0}(\alpha,\beta) = \sum_{k=0}^{2n} s_k^{(n)}(\alpha,\beta),$$

where $s_k^{(n)}(\alpha,\beta) = 2^{2n+\alpha+\beta+1}a_kB(k+\alpha+1, 2n-k+\beta+1)$ and B is the beta function. Using Lemma 2.4 we obtain

$$16J_n(\alpha,\beta) \leq (2n+\alpha+\beta)(2n+\alpha+\beta+1)\sum_{k=0}^{2n} s_k^{(n)}(\alpha,\beta)H_k(\alpha,\beta).$$

H_k is defined by means of the function f, given by (2.2), namely, $H_k(\alpha,\beta) \equiv f(k)$, $k = 0, 1, \dots, 2n$. From the last inequality it follows that

$$\|P_n'\|^2 \leq \frac{1}{16}(2n+\alpha+\beta)(2n+\alpha+\beta+1)\left(\max_{0\leq k\leq 2n} H_k(\alpha,\beta)\right)\|P_n\|^2.$$

Thus, we have

$$C_n^{(0)}(\alpha,\beta) \leq \frac{1}{16}(2n+\alpha+\beta)(2n+\alpha+\beta+1)\left(\max_{0\leq k\leq 2n} H_k(\alpha,\beta)\right), \tag{3.2}$$

where the maximum on the right-hand side is given by (2.4).

In order to show that $C_n^{(0)}(\alpha,\beta)$, defined in (3.1), is the best possible, i.e. that (3.2) reduces to an equality, we consider the polynomials $p_{n,0}(x) = (1+x)^n$ and $p_{n,n}(x) = (1-x)^n$. Since

$$\|p_{n,0}'\|^2 = C_n^{(0)}(\alpha,\beta)\|p_{n,0}\|^2, \qquad \beta \leq \alpha,$$

and

$$\|p'_{n,n}\|^2 = C_n^{(0)}(\alpha,\beta)\|p_{n,n}\|^2, \qquad \beta \geq \alpha,$$

we conclude that $p_{n,0}(x)$ is an extremal polynomial for $\beta \leq \alpha$, and $p_{n,n}(x)$ for $\beta \geq \alpha$.

COROLLARY 3.2. *If $P \in L_n$, then*

$$C_n^{(0)}(1,1) = \frac{n(n+1)(2n+3)}{4(2n+1)}.$$

This result was obtained by P. Erdös and A. K. Varma [**2**] (see, also, Varma [**11**]).

Using a consideration similar to the previous one, we can prove the following assertion for the class of polynomials $L_n^{(m)}$ $(1 \leq m \leq [n/2])$.

THEOREM 3.3. *If $P \in L_n^{(m)}$ $(1 \leq m \leq [n/2])$, $\alpha, \beta > -1$, we have*

$$C_n^{(m)}(\alpha,\beta) = \frac{1}{16}(2n+\alpha+\beta)(2n+\alpha+\beta+1)\max(H_{2m}(\alpha,\beta), H_{2n-2m}(\alpha,\beta)),$$

where $H_k(\alpha,\beta) \equiv f(k)$ and f is given by (2.2).

Especially interesting cases appear when $\alpha = \beta$. Then we have

THEOREM 3.4. *If $P \in L_n^{(m)}$, $m \geq 1$, $\alpha = \beta > -1$, then*

$$C_n^{(m)}(\alpha,\beta) = \frac{(n+\alpha)(2n+2\alpha+1)[\alpha(\alpha-1)n^2 + 2m(n-m)(n-1+3\alpha-2\alpha^2)]}{2(2m+\alpha-1)(2m+\alpha)(2n-2m+\alpha-1)(2n-2m+\alpha)}.$$

In the special cases when $\alpha = 0$ (Legendre case), $\alpha = -1/2$ (Chebyshev case), and $\alpha = 1$, we have

COROLLARY 3.5. *If $P \in L_n^{(m)}$, $m \geq 1$, then*

$$C_n^{(m)}(0,0) = \frac{n(n-1)(2n+1)}{4(2m-1)(2n-2m-1)}, \tag{3.3}$$

$$C_n^{(m)}\left(-\frac{1}{2},-\frac{1}{2}\right) = \frac{2n(2n-1)[3n^2+8m(n-m)(n-3)]}{(4m-3)(4m-1)(4n-4m-3)(4n-4m-1)}, \tag{3.4}$$

$$C_n^{(m)}(1,1) = \frac{n(n+1)(2n+3)}{4(2m+1)(2n-2m+1)}. \tag{3.5}$$

REMARK 3.1. From Corollary 3.2 we see that (3.5) holds and for $m = 0$ too.
REMARK 3.2. For $m = 1$, the best constants (3.3) and (3.4) reduce to

$$C_n^{(1)}(0,0) = \frac{n(n-1)(2n+1)}{4(2n-3)} \tag{3.6}$$

and

$$C_n^{(1)}\left(-\frac{1}{2},-\frac{1}{2}\right) = \frac{2n(2n-1)(11n^2-32n+24)}{3(4n-5)(4n-7)}. \tag{3.7}$$

REMARK 3.3. It is of interest to note that Erdös and Varma [**2**] proved that the best constant in the Lorentz class L_n $(n \geq 2)$ for $\alpha = \beta = 0$ is the same one as that in (3.6), i.e. $C_n^{(0)}(0,0) = C_n^{(1)}(0,0)$.

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