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Some properties of a hypergeometric function which appear in an approximation problem

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Abstract In this paper we consider properties and power expressions of the functions $f:(-1,1)\to\mathbb{R}$ and $f_L:(-1,1)\to\mathbb{R}$, defined by

$$f(x;\gamma)=\frac{1}{\pi}\int_{-1}^{1}\frac{(1+xt)^\gamma}{\sqrt{1-t^2}}\,\mathrm{d}t \quad \text{and} \quad f_L(x;\gamma)=\frac{1}{\pi}\int_{-1}^{1}\frac{(1+xt)^\gamma\log(1+xt)}{\sqrt{1-t^2}}\,\mathrm{d}t,$$

respectively, where γ is a real parameter, as well as some properties of a two parametric real-valued function $D(\,\cdot\,;\alpha,\beta)\colon(-1,1)\to\mathbb{R}$, defined by

$$D(x;\alpha,\beta)=f(x;\beta)f(x;-\alpha-1)-f(x;-\alpha)f(x;\beta-1),\quad \alpha,\beta\in\mathbb{R}.$$

The inequality of Turán type

$$D(x;\alpha,\beta)>0,\quad -1<x<1,$$

for $\alpha+\beta>0$ is proved, as well as an opposite inequality if $\alpha+\beta<0$. Finally, for the partial derivatives of $D(x;\alpha,\beta)$ with respect to α or β, respectively $A(x;\alpha,\beta)$ and $B(x;\alpha,\beta)$, for which $A(x;\alpha,\beta)=B(x;-\beta,-\alpha)$, some results are obtained.

We mention also that some results of this paper have been successfully applied in various problems in the theory of polynomial approximation and some "truncated"

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quadrature formulas of Gaussian type with an exponential weight on the real semiaxis, especially in a computation of Mhaskar–Rahmanov–Saff numbers.

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1 Introduction

In a computation of Mhaskar–Rahmanov–Saff numbers in some problems of polynomial approximation on $(0,+\infty)$ and the corresponding "truncated" quadrature formulas of Gaussian type with respect to the weight function $w(x) = \exp(-x^{-\alpha} - x^{\beta})$, $\alpha > 0, \beta \geq 1$, on the real semiaxis, a one-parametric function $f\colon (-1,1) \to \mathbb{R}$ defined by

$$f(x;\gamma) = \frac{1}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma}}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad \gamma \in \mathbb{R}, \tag{1}$$

is introduced.

In this article we study some properties and power expansions of the function (1), as well as of the function $f_L : (-1,1) \to \mathbb{R}$, defined by

$$f_L(x;\gamma) = \frac{1}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma}\log(1+xt)}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad \gamma \in \mathbb{R}. \tag{2}$$

We prove certain inequalities of Turán type including the function (1) and two real parameters. In 1941 the well-known Hungarian mathematician Paul Turán discovered the following inequality

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) > 0, \quad -1 < x < 1;\ n \in \mathbb{N},$$

where P_n are the classical Legendre polynomials. However, it was published nine years later [19] (see also Szegő [18]). There are several extensions and generalizations of this nice and simple inequality in different ways for several classes of orthogonal polynomials, special functions, etc. L. Alpár [1], who was a Ph.D student of Turán, mentioned that this inequality of Turán had a wide-ranging influence in a number of disciplines (see also a recent paper by Baricz, Jankov, and Pogány [9] that has just been published). Some of these results have been successfully applied in various problems which arise in information theory (cf. [15]), numerical analysis and approximation theory, economic theory, biophysics, etc. For some further recent results one can see the papers [3], [6], [11], [7], [8], [2], [5], [10], among others.

The function (1) can be expressed in terms of the hypergeometric function

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

as follows

$$f(x;\gamma) = {}_2F_1\left(\frac{1-\gamma}{2}, -\frac{\gamma}{2}; 1; x^2\right). \tag{3}$$

Here $(a)_k$ denotes Pochhammer's symbol that is defined by

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad (a)_0 = 1,$$

where Γ is the Euler gamma function.

The function $f(x;\gamma)$ is even, with $f(0;\gamma) = 1$, so that we can consider it only on $[0,1)$.

Thus, for $0 < x < 1$ we consider the following determinant of Turán type

$$D(x;\alpha,\beta) = \begin{vmatrix} f(x;\beta) & f(x;-\alpha) \\ f(x;\beta-1) & f(x;-\alpha-1) \end{vmatrix}, \tag{4}$$

as well as its partial derivatives with respect to the parameters α and β, i.e.,

$$A(x;\alpha,\beta) = \frac{\partial}{\partial\alpha} D(x;\alpha,\beta) \quad \text{and} \quad B(x;\alpha,\beta) = \frac{\partial}{\partial\beta} D(x;\alpha,\beta), \tag{5}$$

respectively. Therefore, we need the corresponding even function $f_L : (-1,1) \to \mathbb{R}$ defined by (2).

Remark 1 In the computation of Mhaskar–Rahmanov–Saff numbers (cf. [14]) for an exponential weight function on $(0,+\infty)$, the following nonlinear equation in x,

$$\beta\left(\frac{\alpha}{\beta}\cdot\frac{f(x;-\alpha-1)}{f(x;\beta-1)}\right)^{\frac{\beta}{\alpha+\beta}} f(x;\beta) - \alpha\left(\frac{\beta}{\alpha}\cdot\frac{f(x;\beta-1)}{f(x;-\alpha-1)}\right)^{\frac{\alpha}{\alpha+\beta}} f(x;-\alpha) = t,$$

must be solved for $\alpha > 0$, $\beta \geq 1$, and a given positive t, where f is defined by (1), i.e., (3). The equation has a unique solution in $(0,1)$.

This paper is organized as follows. Some important properties of functions $f(x;\gamma)$ and $f_L(x;\gamma)$ and their power expansions are given in Section 2. Section 3 is devoted to properties of the determinant $D(x;\alpha,\beta)$, including an inequality of Turán type, as well as a power expansion of $D(x;\alpha,\beta)$. Finally, in Section 4 some properties of $A(x;\alpha,\beta)$ and $B(x;\alpha,\beta)$ are presented.

2 Some Properties of the Functions $f(x;\gamma)$ and $f_L(x;\gamma)$

According to the representation (3), by trivially rewriting the Pochhammer symbols as binomial coefficients,

$$\left(-\frac{\gamma}{2}\right)_k\left(-\frac{\gamma}{2}+\frac{1}{2}\right)_k = 2^{-2k}\gamma(\gamma-1)\cdots(\gamma-(2k-1)) = 2^{-2k}\binom{\gamma}{2k}\binom{2k}{k}(k!)^2,$$

we get the following power expansion of the function (1).

Theorem 1 *For $|x| < 1$ we have*

$$f(x;\gamma) = \sum_{n=0}^{+\infty}\binom{\gamma}{2n}\binom{2n}{n}\left(\frac{x}{2}\right)^{2n}. \tag{6}$$

Remark 2 Since

$$f(x;\gamma) = 1 + \frac{\gamma(\gamma-1)}{4}x^2 + \frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{64}x^4 + \cdots,$$

we have

$$f(x;0) = f(x;1) = 1, \quad f(x;2) = f(x;3) = 1 + \frac{\gamma(\gamma-1)}{4}x^2, \text{ etc.}$$

Also, it is easy to see that

$$f(0;\gamma) = 1, \quad f'(0;\gamma) = 0, \quad f''(0;\gamma) = \frac{\gamma(\gamma-1)}{2}, \text{ etc.}$$

The corresponding expansion of the function $x \mapsto f_L(x;\gamma)$ can be done in a similar way.

Theorem 2 *For* $|x| < 1$ *we have*

$$f_L(x;\gamma) = \sum_{n=1}^{+\infty} \binom{2n}{n} \left\{ \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} \binom{\gamma}{2n-k} \right\} \left(\frac{x}{2}\right)^{2n}. \tag{7}$$

Alternatively, it can be expressed in the form

$$f_L(x;\gamma) = \sum_{n=1}^{+\infty} \frac{\omega_n'(\gamma)}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \tag{8}$$

where $\omega_n(\gamma) = \gamma(\gamma-1)\cdots(\gamma-2n+1)$.

Proof Using the binomial expansion and the expansion of $\log(1+t)$ in t, we have the following potential series

$$(1+t)^\gamma \log(1+t) = \sum_{n=1}^{+\infty} \left\{ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{\gamma}{n-k} \right\} t^n, \quad |t| < 1,$$

for each $\gamma \in \mathbb{R}$. Then, substituting t by xt and integrating it over $(-1,1)$ with respect to the Chebyshev weight of the first kind, we obtain

$$f_L(x;\gamma) = \sum_{n=1}^{+\infty} \left\{ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{\gamma}{n-k} \right\} \frac{1}{\pi} \int_{-1}^{1} \frac{x^n t^n}{\sqrt{1-t^2}} \,\mathrm{d}t,$$

i.e., (7).

Using the Saalschütz formula [17, p. 11]

$$\sum_{k=0}^{n-1} \binom{z}{k} \frac{x^{n-k}}{n-k} = \sum_{k=1}^{n} \binom{z}{n-k} \frac{x^k}{k} = \sum_{k=1}^{n} \binom{z-k}{n-k} \frac{(x+1)^k - 1}{k},$$

for $x := -1$, $z := \gamma$, and by substituting n by $2n$, we get

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} \binom{\gamma}{2n-k} = \sum_{k=1}^{2n} \binom{\gamma-k}{2n-k} \frac{1}{k} = \frac{\gamma}{2n} \binom{\gamma-1}{2n-1} [\psi(-\gamma) - \psi(2n-\gamma)],$$

where ψ is the digamma function, i.e.,

$$\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\binom{\gamma}{2n-k}=\binom{2n}{n}\binom{\gamma}{2n}\sum_{k=0}^{2n-1}\frac{1}{\gamma-k}=\frac{\omega_n'(\gamma)}{(n!)^2},$$

because of

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}(\log\omega_n(\gamma))=\frac{\omega_n'(\gamma)}{\omega_n(\gamma)}=\sum_{k=0}^{2n-1}\frac{1}{\gamma-k}.$$

The expansion (8) can be also obtained by a formal differentiation of (6) with respect to γ. □

The expansion (7), i.e., (8), gives

$$\begin{aligned}f_L(x;\gamma)&=(2\gamma-1)\left(\frac{x}{2}\right)^2+\frac{1}{2}\left(2\gamma^3-9\gamma^2+11\gamma-3\right)\left(\frac{x}{2}\right)^4\\&+\frac{1}{36}\left(6\gamma^5-75\gamma^4+340\gamma^3-675\gamma^2+548\gamma-120\right)\left(\frac{x}{2}\right)^6\\&+\frac{1}{144}\left(2\gamma^7-49\gamma^6+483\gamma^5-2450\gamma^4+6769\gamma^3-9849\gamma^2+6534\gamma-1260\right)\left(\frac{x}{2}\right)^8\\&+\frac{1}{14400}\left(10\gamma^9-405\gamma^8+6960\gamma^7-66150\gamma^6+379638\gamma^5-1346625\gamma^4\right.\\&\quad\left.+2894720\gamma^3-3518100\gamma^2+2053152\gamma-362880\right)\left(\frac{x}{2}\right)^{10}+O\left(x^{12}\right).\end{aligned}$$

Remark 3 For $\gamma=0$ the expansion (7), i.e., (8), reduces to

$$f_L(x;0)=-\sum_{n=1}^{+\infty}\frac{1}{2n}\binom{2n}{n}\left(\frac{x}{2}\right)^{2n},\quad -1\le x\le 1. \tag{9}$$

Using an equality from [16, §5.2.13.4, p. 711] we get the sum of the series (9) in the form

$$f_L(x;0)=\log\frac{1+\sqrt{1-x^2}}{2}.$$

The expansion (9) is a typical slowly convergent series for x close to 1. For example, for $x=1$, we have

$$f_L(1;0)=-\log 2=-0.69314718055994530941723212145817656807550\ldots,$$

but a direct summation of the first 40000 terms in (9) (for $x=1$) gives only 2-digits accuracy.

For $\gamma=1$, the corresponding expansion is

$$f_L(x;1)=\sum_{n=1}^{+\infty}\frac{1}{2n(2n-1)}\binom{2n}{n}\left(\frac{x}{2}\right)^{2n}=\log\frac{1+\sqrt{1-x^2}}{2}+1-\sqrt{1-x^2}.$$

The monic polynomials $\gamma \mapsto \omega_n(\gamma)$ of degree $2n$ from Theorem 2 has simple zeros at $\gamma = k$, $k = 0, 1, \ldots, 2n-1$, and the zeros of $\omega_n'(\gamma)$, denoted by γ_k, are located in the intervals $\gamma_k^{(n)} \in (k-1, k)$, $k = 1, \ldots, 2n-1$. The sequence of the first zeros $\{\gamma_1^{(n)}\}_{n=1}^{+\infty}$ is a decreasing sequence. Numerical values of this sequence are

$$\{0.5000, 0.3820, 0.3366, 0.3103, 0.2925, 0.2793, 0.2690, 0.2606,\ 0.2536, 0.2477, \ldots\}.$$

It is easy to conclude that $\omega_n'(\gamma) < 0$ for $\gamma < \gamma_1^{(n)}$ and $\omega_n'(\gamma) > 0$ for $\gamma_1^{(n)} < \gamma < \gamma_2^{(n)}$, $n = 2, 3, \ldots$. Since $\gamma = 1/2$ belongs to the last interval, we conclude that all coefficients for $n > 1$ in (8) are positive, so that $f_L(x; 1/2) > 0$. Indeed, for $\gamma = 1/2$ this expansion for $f_L(x; 1/2)$ becomes

$$f_L(x; 1/2) = \frac{x^4}{64} + \frac{31x^6}{3072} + \frac{2689x^8}{393216} + \frac{51719x^{10}}{10485760} + O(x^{12}).$$

The first two derivatives of the function $x \mapsto f(x; \gamma)$ are

$$f'(x;\gamma) = \frac{\gamma}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma-1}}{\sqrt{1-t^2}}\, t \,\mathrm{d}t = \frac{\gamma(\gamma-1)x}{2}\, {}_2F_1\left(\frac{3-\gamma}{2}, \frac{2-\gamma}{2}; 2; x^2\right)$$

and

$$f''(x;\gamma) = \frac{\gamma(\gamma-1)}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma-2}}{\sqrt{1-t^2}} t^2 \,\mathrm{d}t = \frac{\gamma(\gamma-1)x}{2}\, {}_3F_2\left(\frac{3}{2}, \frac{2-\gamma}{2}, \frac{3-\gamma}{2}; \frac{1}{2}, 2; x^2\right).$$

It is also easy to see that

$$x f'(x;\gamma) = \gamma\big(f(x;\gamma) - f(x;\gamma-1)\big). \tag{10}$$

Differentiating $f_L(x;\gamma)$, defined by (2), we get

$$f_L'(x;\gamma) = \frac{1}{\pi}\int_{-1}^{1} \frac{\gamma(1+xt)^{\gamma-1}\log(1+xt) + (1+xt)^{\gamma-1}}{\sqrt{1-t^2}}\, t \,\mathrm{d}t,$$

i.e.,

$$x f_L'(x;\gamma) = \gamma\big(f_L(x;\gamma) - f_L(x;\gamma-1)\big) + f(x;\gamma) - f(x;\gamma-1), \tag{11}$$

from which, combining with (10), we obtain

$$\gamma x f_L'(x;\gamma) = \gamma^2\big(f_L(x;\gamma) - f_L(x;\gamma-1)\big) + x f'(x;\gamma),$$

i.e.,

$$x\big(\gamma f_L'(x;\gamma) - f'(x;\gamma)\big) = \gamma^2\big(f_L(x;\gamma) - f_L(x;\gamma-1)\big). \tag{12}$$

Theorem 3 1° *For* $x > 0$,

$$f'(x;\gamma) > 0 \quad \textit{if} \quad \gamma < 0 \ \vee\ \gamma > 1, \tag{13}$$

and

$$f'(x;\gamma) < 0 \quad \textit{if} \quad 0 < \gamma < 1. \tag{14}$$

2° *For each* $\gamma \in \mathbb{R}$, *we have*

$$f_L(x;\gamma) < f(x;\gamma), \quad -1 < x < 1, \tag{15}$$

and

$$\gamma f_L'(x;\gamma) > f'(x;\gamma), \quad 0 < x < 1. \tag{16}$$

Proof 1° It is easy to see that

$$f'(x;\gamma) = \frac{\gamma}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma-1}t}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{\gamma}{\pi}\int_{0}^{1} \frac{t(1+xt)^{\gamma-1}}{\sqrt{1-t^2}}\left(1-\Big(\frac{1-xt}{1+xt}\Big)^{\gamma-1}\right)\mathrm{d}t.$$

Since $(1-xt)/(1+xt) < 1$ for $0 < x, t < 1$, we conclude that the previous integral is positive for each $\gamma > 1$ or $\gamma < 0$, and negative for $0 < \gamma < 1$.

2° Consider the difference

$$f(x;\gamma) - f_L(x;\gamma) = \frac{1}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma}}{\sqrt{1-t^2}}\big[1-\log(1+xt)\big]\,\mathrm{d}t.$$

Since $0 < 1+xt < 2 < \mathrm{e}$, we have that $\log[\mathrm{e}/(1+xt)] > 0$, so that the previous integral is positive and the inequality (15) holds.

The inequality (16) follows directly from (12) if we can prove that $f_L(x;\gamma) > f_L(x;\gamma-1)$. Therefore, we consider

$$\begin{aligned} f_L(x;\gamma) - f_L(x;\gamma-1) &= \frac{1}{\pi}\int_{-1}^{1} \frac{\log(1+xt)}{\sqrt{1-t^2}}\big[(1+xt)^{\gamma} - (1+xt)^{\gamma-1}\big]\mathrm{d}t \\ &= \frac{x}{\pi}\int_{-1}^{1} \frac{t(1+xt)^{\gamma-1}\log(1+xt)}{\sqrt{1-t^2}}\,\mathrm{d}t \\ &= \frac{x}{\pi}\int_{0}^{1} \frac{t}{\sqrt{1-t^2}}\,h(xt)\,\mathrm{d}t, \end{aligned}$$

where $h(t) = (1+t)^{\gamma-1}\log(1+t) - (1-t)^{\gamma-1}\log(1-t)$. Evidently, $h(t) > 0$ for $t > 0$, and the last integral is positive for $0 < x < 1$. □

Theorem 4 *For each $\gamma > -1/2$, there exists*

$$f(1;\gamma) = \frac{2^{\gamma}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(\gamma+1)}. \tag{17}$$

If $\gamma < -1/2$, then

$$\lim_{x\to 1}(1-x^2)^{-\gamma-1/2} f(x;\gamma) = \frac{2^{-\gamma-1}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(-\gamma-\frac{1}{2}\right)}{\Gamma(-\gamma)}. \tag{18}$$

Proof According to (1), it is easy to see that for $\gamma > -1/2$ the following integral

$$f(1;\gamma) = \frac{1}{\pi}\int_{-1}^{1} \frac{(1+t)^{\gamma}}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{1}{\pi}\int_{-1}^{1} (1+t)^{\gamma-1/2}(1-t)^{-1/2}\,\mathrm{d}t$$

exists and its value is given by (17).

In order to prove (18) we use the equality [12, §6.8]

$$\lim_{z\to 1}(1-z)^{a+b-c}{}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{19}$$

which is true for each c different from nonpositive integers and $\Re(a+b-c)>0$. Then, setting $a=(1-\gamma))/2$, $b=-\gamma/2$, $c=1$, and $z=x^2$ in the previous equality and using the well-known Legendre duplication formula (cf. [4, p. 22])

$$\Gamma(2t)=\frac{2^{2t-1}}{\sqrt{\pi}}\Gamma(t)\Gamma\left(t+\frac{1}{2}\right),$$

the equality (19) reduces to (18) and it is valid for $a+b-c=-\gamma-1/2<0$, i.e., for $\gamma<-1/2$. □

Remark 4 If γ is a negative integer, we have, for example, $f(x;-1)=(1-x^2)^{-1/2}$,

$$f(x;-2)=\frac{1}{(1-x^2)^{3/2}}\,,\quad f(x;-3)=\frac{2+x^2}{2(1-x^2)^{5/2}}\,,\quad f(x;-4)=\frac{2+3x^2}{2(1-x^2)^{7/2}}\,,\ \text{etc.}$$

According to the previous considerations, we can see that the following statement holds:

Theorem 5 *The function $x\mapsto f(x;\gamma)$ is convex and increasing in $[0,1)$ for $\gamma<0$ or $\gamma>1$, and concave and decreasing for $0<\gamma<1$. In these cases, $f(0;\gamma)=1$ is a minimum and a maximum of this function, respectively. Moreover,*

$$\max_{0\le x\le 1} f(x;\gamma)=f(1;\gamma),\quad \textit{if}\ \ \gamma>1\ \ \textit{or}\ \ -1/2<\gamma<0,$$

and

$$\min_{0\le x\le 1} f(x;\gamma)=f(1;\gamma),\quad \textit{if}\ \ 0<\gamma<1,$$

where $f(1;\gamma)$ is given in Theorem 4.

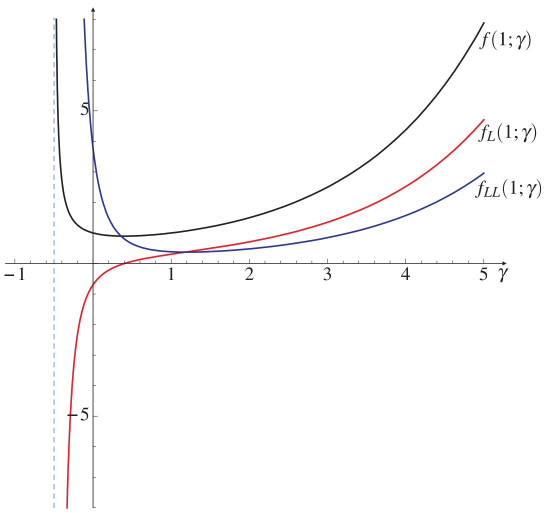

Fig. 1 The graphs of the functions: $\gamma\mapsto f(1;\gamma)$, $\gamma\mapsto f_L(1;\gamma)$, and $\gamma\mapsto f_{LL}(1;\gamma)$ for $-1/2<\gamma\le 5$

For the function $f_L(x;\gamma)$, defined by (2), we can also prove the existence of its value at $x=\pm 1$ for $\gamma>-1/2$.

Theorem 6 *For each $\gamma > -1/2$, there exists*

$$f_L(1;\gamma) = \frac{2^\gamma}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(\gamma+1)} \left[\psi\left(\gamma+\tfrac{1}{2}\right) - \psi(\gamma+1) + \log 2\right], \tag{20}$$

where $\psi(z)$ is the digamma function, defined by $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}(\log\Gamma(z)) = \Gamma'(z)/\Gamma(z)$.

The equation $f_L(1;\gamma) = 0$ has a unique root $\gamma = \widehat{\gamma}$, whose numerical value is

$$\widehat{\gamma} = 0.40044075673535960714733840032575462375799999584117\ldots\,.$$

Proof Using [16, §2.6.10.25, p. 502] (for $a = 1$)

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\log(1-x)\,\mathrm{d}x = B(\alpha,\beta)\left[\psi(\beta) - \psi(\alpha+\beta)\right] \quad (\alpha > -1,\ \beta > 0),$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, after the transformation $x = (1-t)/2$, we obtain an integral over the interval $(-1,1)$,

$$\frac{1}{2^{\alpha+\beta-1}} \int_{-1}^1 (1-t)^{\alpha-1}(1+t)^{\beta-1}\log\frac{1+t}{2}\,\mathrm{d}t = B(\alpha,\beta)\left[\psi(\beta) - \psi(\alpha+\beta)\right],$$

from which, for $\alpha = 1/2$ and $\beta = \gamma + 1/2 > 0$ (i.e., $\gamma > -1/2$), we get

$$\int_{-1}^1 (1+t)^{\gamma-1/2}(1-t)^{-1/2}\log\frac{1+t}{2}\,\mathrm{d}t = 2^\gamma B\Big(\frac{1}{2},\gamma+\frac{1}{2}\Big)\left[\psi\Big(\gamma+\frac{1}{2}\Big) - \psi(\gamma+1)\right].$$

Since (cf. [13, p. 132])

$$\int_{-1}^1 (1+t)^{\gamma-1/2}(1-t)^{-1/2}\,\mathrm{d}t = 2^\gamma B\Big(\frac{1}{2},\gamma+\frac{1}{2}\Big), \quad \gamma > -\frac{1}{2},$$

we have

$$f_L(1;\gamma) = \frac{2^\gamma}{\pi} B\Big(\frac{1}{2},\gamma+\frac{1}{2}\Big)\left[\psi\Big(\gamma+\frac{1}{2}\Big) - \psi(\gamma+1) + \log 2\right],$$

i.e., (20).

The function $\gamma \mapsto f_L(1;\gamma)$ is increasing in $(-\frac{1}{2},+\infty)$ and has a unique zero $\widehat{\gamma}$ near 0.4. Its numerical value with 50 decimal digits was obtained by Newton's method.

□

Notice that $f_L(1;\gamma) = f(1;\gamma)\left[\psi\left(\gamma+\frac{1}{2}\right) - \psi(\gamma+1) + \log 2\right]$ for $\gamma > -1/2$. Otherwise, for $\gamma \le -1/2$, $\lim_{x\to 1-} f_L(x;\gamma) = -\infty$.

The graphs of $\gamma \mapsto f(1;\gamma)$ and $\gamma \mapsto f_L(1;\gamma)$ are displayed in Figure 1. Evidently, they illustrate the inequality (15) at $x = 1$.

Now, we consider some basic properties of the function $x \mapsto f_L(x;\gamma)$. Since

$$f_L(0;\gamma) = f_L'(0;\gamma) = 0,\ f_L''(0;\gamma) = \gamma - \frac{1}{2},\ f_L'''(0;\gamma) = 0,\ f_L^{(4)}(0;\gamma) = \frac{3}{4}(2\gamma-3)(\gamma^2-3\gamma+1),$$

it is clear that the function $x \mapsto f_L(x;\gamma)$ has a local extremum at $x=0$. For $\gamma \geq 1/2$ it is a local minimum, and for $\gamma < 1/2$ this function has a local maximum. Notice also that a positive function $x \mapsto f_{LL}(x;\gamma)$, defined by

$$f_{LL}(x;\gamma) = \frac{\partial}{\partial\gamma} f_L(x;\gamma) = \frac{1}{\pi}\int_{-1}^{1} \frac{(1+xt)^{\gamma}\log^2(1+xt)}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad \gamma \in \mathbb{R}, \tag{21}$$

has the value at $x=\pm 1$ if $\gamma > -1/2$. The proof of such a result is similar to the proof of Theorem 6, where in addition we need the equality [16, §2.6.10.24, p. 502] (for $a=1$)

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\log^2(1-x)\,\mathrm{d}x = \frac{\partial^2}{\partial\beta^2}B(\alpha,\beta) \quad (\alpha > -2,\ \beta > 0).$$

Theorem 7 *For each $\gamma > -1/2$, there exists*

$$f_{LL}(1;\gamma) = f(1;\gamma)\left[\psi'\left(\gamma+\tfrac{1}{2}\right) - \psi'(\gamma+1) + \left(\psi\left(\gamma+\tfrac{1}{2}\right) - \psi(\gamma+1) + \log 2\right)^2\right],$$

where $f(1;\gamma)$ is given by (17).

The graph of $\gamma \mapsto f_{LL}(1;\gamma)$ is also displayed in Figure 1.

According to Theorems 4 and 6, as well as the recurrence relations (10) and (11), we obtain the following result:

Corollary 1 *For each $\gamma > 1/2$, there exist the values*

$$f'(1;\gamma) = (\gamma-1)f(1;\gamma-1) \quad \textit{and} \quad f_L'(1;\gamma) = (\gamma-1)f_L(1;\gamma-1) + f(1;\gamma-1).$$

Theorem 8 *Let $0 \leq x \leq 1$.*

1° *For $\gamma \geq \frac{1}{2}$, $x \mapsto f_L(x;\gamma)$ is a positive increasing function on $(0,1)$, with*

$$\min_{x\in[0,1]} f_L(x;\gamma) = f_L(0;\gamma) = 0 \quad \textit{and} \quad \max_{x\in[0,1]} f_L(x;\gamma) = f_L(1;\gamma),$$

where $f_L(1;\gamma)$ is given by (20).

2° *For $\gamma < 0$, $x \mapsto f_L(x;\gamma)$ is a negative decreasing function on $(0,1)$, with*

$$\max_{x\in[0,1]} f_L(x;\gamma) = f_L(0;\gamma) = 0$$

and

$$\min_{x\in[0,1]} f_L(x;\gamma) = f_L(1;\gamma), \quad -\tfrac{1}{2} < \gamma < 0.$$

For $\gamma \leq -\frac{1}{2}$, we have $\lim_{x\to 1} f_L(x;\gamma) = -\infty$.

3° *For $0 < \gamma < \frac{1}{2}$, there exists $x_0 \in (0,1)$ such that $f_L'(x_0;\gamma) = 0$ and*

$$\min_{x\in[0,1]} f_L(x;\gamma) = f_L(x_0;\gamma) < 0. \tag{22}$$

If $0 < \gamma < \widehat{\gamma}$*, where* $\widehat{\gamma}$ *is given in Theorem* 6*, the inequality* $f_L(x;\gamma) < 0$ *holds, with*

$$\max_{x\in[0,1]} f_L(x;\gamma) = f_L(0;\gamma) = 0,$$

and for $\widehat{\gamma} < \gamma < \frac{1}{2}$ *there exists* $\widehat{x} \in (0,1)$ *such that* $f_L(\widehat{x};\gamma) = 0$*, as well as* $f_L(x;\gamma) < 0$ *for* $0 < x < \widehat{x}$ *and* $f_L(x;\gamma) > 0$ *for* $\widehat{x} < x \le 1$*, with*

$$\max_{x\in[0,1]} f_L(x;\gamma) = f_L(1;\gamma).$$

Proof 1° According to (13) and (16), for $\gamma > 1$ we have $f_L'(x;\gamma) > f'(x;\gamma) > 0$, so that $x \mapsto f_L(x;\gamma)$ is a positive increasing function on $(0,1]$, with $\min_{x\in[0,1]} f_L(x;\gamma) = 0$ and $\max_{x\in[0,1]} f_L(x;\gamma) = f_L(1;\gamma)$, where this value is given in Theorem 6. This holds also for each $\gamma \ge \frac{1}{2}$, because of positivity of $f_{LL}(x;\gamma)$, defined by (21), and the positivity of $f_L(x;\frac{1}{2})$ (see comments after Theorem 2). Namely, for an arbitrary fixed x, the function $\gamma \mapsto f_L(x;\gamma)$ increases for each $\gamma > \frac{1}{2}$, so that $f_L(x;\gamma) > f_L(x;\frac{1}{2}) > 0$.

2° If $\gamma < 0$, then, again by (13) and (16), we conclude that $f_L'(x;\gamma) < \gamma^{-1} f'(x;\gamma) < 0$, i.e., $x \mapsto f_L(x;\gamma)$ is a negative decreasing function on $[0,1)$, with the maximum at $x = 0$, i.e.,

$$\max_{0\le x<1} f_L(x;\gamma) = f_L(0;\gamma) = 0.$$

3° Let $0 < \gamma < \frac{1}{2}$. Since $f_L(x;\gamma)$ and $f_L'(x;\gamma)$ are negative for a sufficiently small positive x and $\lim_{x\to 1-} f_L'(x;\gamma) = +\infty$, we conclude that the continuous function $x \mapsto f_L'(x;\gamma)$ in $(0,1)$ changes its sign and therefore there exists $x_0 \in (0,1)$ such that $f_L'(x_0;\gamma) = 0$, as well as (22).

Suppose now that $\widehat{\gamma}$ (≈ 0.4) is defined as a zero of the function $\gamma \mapsto f_L(1;\gamma)$. Thus, we have $f_L(0;\widehat{\gamma}) = f_L(1;\widehat{\gamma}) = 0$ and $f_L(x;\widehat{\gamma}) < 0$ for each $x \in (0,1)$. Due to the positivity of the function $f_{LL}(x;\gamma)$, defined by (21), we conclude that $f_L(x;\gamma) < f_L(x;\widehat{\gamma}) < 0$ for $\gamma < \widehat{\gamma}$ and $\max_{x\in[0,1]} f_L(x;\gamma) = f_L(0;\gamma) = 0$.

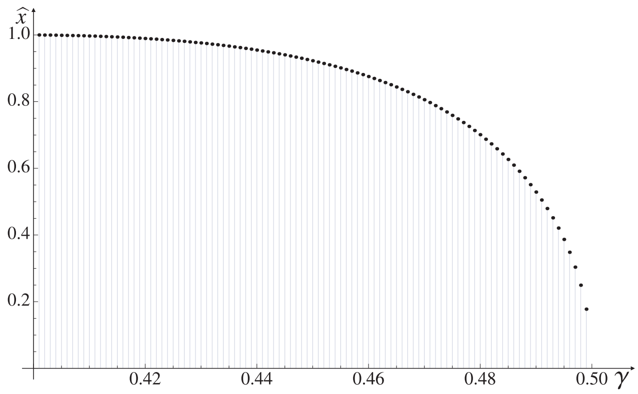

Fig. 2 The root $\widehat{x}$ of the equation $f_L(\widehat{x};\gamma) = 0$ as a function of $\gamma \in (\widehat{\gamma}, 1/2)$

If $\widehat{\gamma} < \gamma < \frac{1}{2}$, by the same argument, we claim that $f_L(x;\gamma) > f_L(x;\widehat{\gamma})$. Since, for such a γ we have $f_L(1;\gamma) > 0$, it follows that the continuous function $x \mapsto f_L(x;\gamma)$ in $[0,1]$ must be equal to zero at some point $\widehat{x} \in (0,1)$, i.e., $f_L(\widehat{x};\gamma) = 0$. The rest of the proof is obvious. The points $\widehat{x}$ for different values of $\gamma \in (\widehat{\gamma}, 1/2)$ are displayed in Figure 2. □

Remark 5 The graphs of $x \mapsto f_L(x;\gamma)$, $0 < x < 1$, for some typical values of $\gamma \in (0,1/2)$ are displayed in Figure 3.

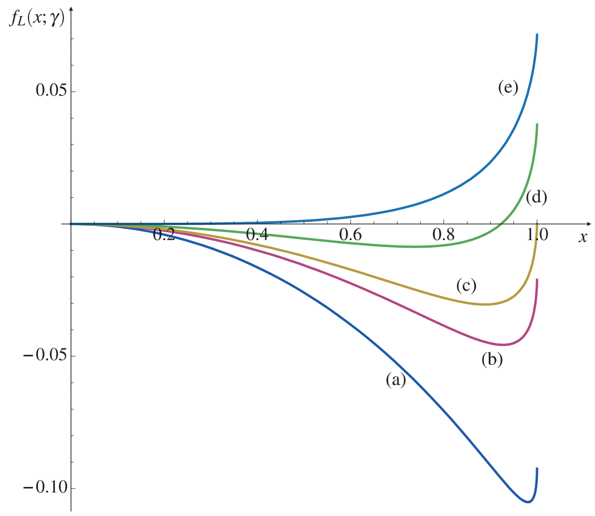

Fig. 3 The graphs of $x \mapsto f_L(x;\gamma)$, $0 < x < 1$, for (a) $\gamma = 0.3$; (b) $\gamma = 0.375$; (c) $\gamma = \widehat{\gamma} \approx 0.40$; (d) $\gamma = 0.45$; and (e) $\gamma = 0.5$

3 Properties of the function $D(x;\alpha,\beta)$

In this section we consider some properties of the determinant (4), i.e.,

$$D(x;\alpha,\beta) = f(x;\beta)f(x;-\alpha-1) - f(x;\beta-1)f(x;\alpha),$$

where $f(x;\gamma)$ is defined by (1), i.e., (3).

By applying (1) and (4), we can express $D(x;\alpha,\beta)$ as a double integral over the square $S = \{(u,v) : -1 < u < 1, -1 < v < 1\}$ in the representation

$$D(x;\alpha,\beta) = \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{(1+xu)^{\beta}(1+xv)^{-\alpha-1} - (1+xu)^{\beta-1}(1+xv)^{-\alpha}}{\sqrt{1-u^2}\sqrt{1-v^2}}\,\mathrm{d}u\mathrm{d}v,$$

i.e.,

$$D(x;\alpha,\beta) = \frac{x}{\pi^2}\iint\limits_{S} \frac{(1+xu)^{\beta-1}(1+xv)^{-\alpha-1}}{\sqrt{1-u^2}\sqrt{1-v^2}}\,(u-v)\,\mathrm{d}u\mathrm{d}v. \tag{23}$$

The following properties of the function $D(x;\alpha,\beta)$ and its partial derivatives with respect to α and β are satisfied:

Lemma 1 *For each $\alpha,\beta\in\mathbb{R}$, one has $D(x;-\beta,-\alpha)=-D(x;\alpha,\beta)$.*

The proof follows immediately from the definition of $D(x;\alpha,\beta)$ given in (4).

Evidently, $D(x;\alpha,\beta)=0$ when $\alpha=-\beta$. An interesting property of $D(x;\alpha,\beta)$ is the following:

Theorem 9 *For each $\alpha,\beta\in\mathbb{R}$, we have*

$$D(x;\alpha,\beta)=(1-x^2)^{\beta-\alpha}D(x;\beta,\alpha).$$

Proof By Euler's hypergeometric transformation formula (cf. [4, p. 68])

$${}_2F_1(a,b;c;z)=(1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z),$$

one has

$${}_2F_1\left(\frac{1-\gamma}{2},-\frac{\gamma}{2};1;x^2\right)=(1-x^2)^{1/2+\gamma}{}_2F_1\left(\frac{1+\gamma}{2},\frac{2+\gamma}{2};1;x^2\right).$$

By substituting γ by β, $-\alpha-1$, $-\alpha$, and $\beta-1$, successively in the previous formula, one obtains

$$\begin{aligned}
D(x;\alpha,\beta) &= f(x;\beta)f(x;-\alpha-1)-f(x;-\alpha)f(x;\beta-1)\\
&= {}_2F_1\left(\frac{1-\beta}{2},-\frac{\beta}{2};1;x^2\right){}_2F_1\left(\frac{2+\alpha}{2},\frac{1+\alpha}{2};1;x^2\right)\\
&\quad -{}_2F_1\left(\frac{1+\alpha}{2},\frac{\alpha}{2};1;x^2\right){}_2F_1\left(\frac{2-\beta}{2},\frac{1-\beta}{2};1;x^2\right)\\
&= (1-x^2)^{\beta-\alpha}\left\{{}_2F_1\left(\frac{1+\beta}{2},\frac{2+\beta}{2};1;x^2\right){}_2F_1\left(-\frac{\alpha}{2},\frac{1-\alpha}{2};1;x^2\right)\right.\\
&\quad \left.-{}_2F_1\left(\frac{1-\alpha}{2},\frac{2-\alpha}{2};1;x^2\right){}_2F_1\left(\frac{\beta}{2},\frac{1+\beta}{2};1;x^2\right)\right\},
\end{aligned}$$

i.e.,

$$D(x;\alpha,\beta)=(1-x^2)^{\beta-\alpha}D(x;\beta,\alpha),$$

which completes the proof of the result. □

According to Theorem 4 we can prove the corresponding result for the function $x\mapsto D(x;\alpha,\beta)$ when $(\alpha,\beta)\in\mathbb{R}^2$. In this regard we first have to identify six domains

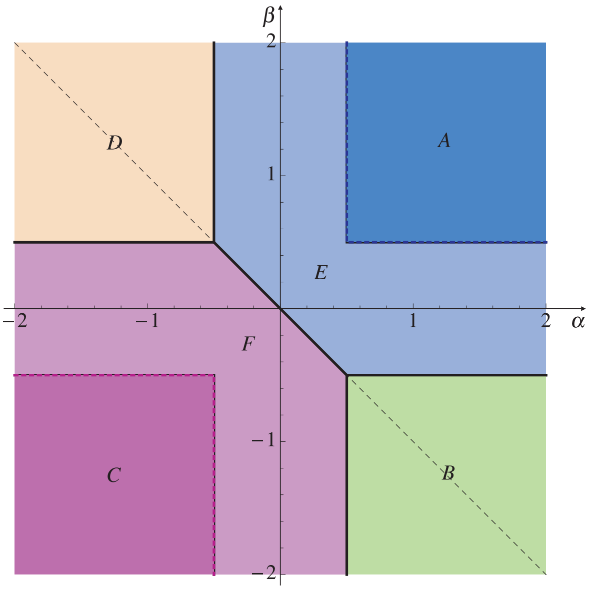

Fig. 4 Different domains for $(\alpha,\beta)\in\mathbb{R}^2$, defined by (24)

in $\mathbb{R}^2$ as presented in Figure 4, i.e.,

$$\left.\begin{aligned}
A &= \left\{(\alpha,\beta)\;\middle|\;\alpha>\frac{1}{2}\wedge\beta>\frac{1}{2}\right\},\\
B &= \left\{(\alpha,\beta)\;\middle|\;\alpha>\frac{1}{2}\wedge\beta<-\frac{1}{2}\right\},\\
C &= \left\{(\alpha,\beta)\;\middle|\;\alpha<-\frac{1}{2}\wedge\beta<-\frac{1}{2}\right\},\\
D &= \left\{(\alpha,\beta)\;\middle|\;\alpha<-\frac{1}{2}\wedge\beta>\frac{1}{2}\right\},\\
E &= \left\{(\alpha,\beta)\;\middle|\;\left(|\alpha|<\frac{1}{2}\vee|\beta|<\frac{1}{2}\right)\wedge\alpha+\beta>0\right\},\\
F &= \left\{(\alpha,\beta)\;\middle|\;\left(|\alpha|<\frac{1}{2}\vee|\beta|<\frac{1}{2}\right)\wedge\alpha+\beta<0\right\}.
\end{aligned}\right\}\tag{24}$$

Theorem 10 *Let A,B,C,D,E,F be domains in $\mathbb{R}^2$ defined by* (24).

1° *For each $(\alpha,\beta)\in D$, there exists the finite value of the function $x\mapsto D(x;\alpha,\beta)$ at $x=\pm1$, given by*

$$D(1;\alpha,\beta)=\frac{2^{\beta-\alpha-2}(\alpha+\beta)}{\sqrt{\pi}}\cdot\frac{\Gamma\left(-\alpha-\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\right)}{\Gamma(-\alpha+1)\Gamma(\beta+1)}.\tag{25}$$

2° *For* $(\alpha,\beta)\in A\cup E$,

$$\lim_{x\to 1}(1-x^2)^{\alpha+1/2}D(x;\alpha,\beta)=\frac{3^{\alpha+\beta}}{\pi}\cdot\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(\alpha+1)\Gamma(\beta+1)}; \tag{26}$$

3° *For* $(\alpha,\beta)\in B$,

$$\lim_{x\to 1}(1-x^2)^{\alpha-\beta}D(x;\alpha,\beta)=\frac{2^{\alpha-\beta-2}(\alpha+\beta)}{\pi}\cdot\frac{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(-\beta-\frac{1}{2}\right)}{\Gamma(\alpha+1)\Gamma(-\beta+1)}; \tag{27}$$

4° *For* $(\alpha,\beta)\in C\cup F$,

$$\lim_{x\to 1}(1-x^2)^{-\beta+1/2}D(x;\alpha,\beta)=-\frac{2^{-(\alpha+\beta)}}{\pi}\cdot\frac{\Gamma\left(-\alpha+\frac{1}{2}\right)\Gamma\left(-\beta+\frac{1}{2}\right)}{\Gamma(-\alpha+1)\Gamma(-\beta+1)}. \tag{28}$$

Proof 1° For $(\alpha,\beta)\in D$, i.e., when $\alpha<-1/2$ and $\beta>1/2$, all four functions which appear on the right hand side of (4) are defined at ± 1 so that

$$D(1;\alpha,\beta)=f(1;\beta)f(1;-\alpha-1)-f(1;\beta-1)f(1;-\alpha).$$

Using Theorem 4, we get

$$D(1;\alpha,\beta)=\frac{2^{\beta}}{\sqrt{\pi}}\frac{\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(\beta+1)}\cdot\frac{2^{-\alpha-1}}{\sqrt{\pi}}\frac{\Gamma\left(-\alpha-\frac{1}{2}\right)}{\Gamma(-\alpha)}-\frac{2^{\beta-1}}{\sqrt{\pi}}\frac{\Gamma\left(\beta-\frac{1}{2}\right)}{\Gamma(\beta)}\cdot\frac{2^{-\alpha}}{\sqrt{\pi}}\frac{\Gamma\left(-\alpha+\frac{1}{2}\right)}{\Gamma(-\alpha+1)},$$

i.e.,

$$D(1;\alpha,\beta)=\frac{2^{\beta-\alpha-1}}{\pi}\frac{\Gamma\left(-\alpha-\frac{1}{2}\right)}{\Gamma(-\alpha+1)}\frac{\Gamma\left(\beta-\frac{1}{2}\right)}{\Gamma(\beta+1)}\left[\left(\beta-\frac{1}{2}\right)(-\alpha)-\beta\left(-\alpha-\frac{1}{2}\right)\right],$$

which is equivalent to (25).

2° In order to prove (26) for $(\alpha,\beta)\in A\cup E$, we first prove it for $(\alpha,\beta)\in A$, i.e., when $\alpha,\beta>1/2$. Since

$$\begin{aligned}(1-x^2)^{\alpha+1/2}D(x;\alpha,\beta) &= f(x;\beta)\left[(1-x^2)^{\alpha+1/2}f(x;-\alpha-1)\right]\\ &\quad -f(x;\beta-1)\left[(1-x^2)^{\alpha-1/2}f(x;-\alpha)\right](1-x^2),\end{aligned}$$

by Theorem 4, we get (26), where the contribution to this value is given only by the first term on the right hand side in the previous equality, because the second term tends to zero when $x\to 1$.

Consider now a subset of E, when $\beta>\frac{1}{2}\ \wedge\ |\alpha|<\frac{1}{2}$. Then the values of $f(1;\beta)$, $f(1;\beta-1)$, and $f(1;-\alpha)$ exist, and

$$L(-\alpha-1)=\lim_{x\to 1}(1-x^2)^{\alpha+1/2}f(x;-\alpha-1)=\frac{2^{\alpha}}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}.$$

Therefore,

$$\lim_{x\to 1}(1-x^2)^{\alpha+1/2}D(x;\alpha,\beta)=f(1;\beta)L(-\alpha-1)-f(1;\beta-1)f(1;-\alpha)\lim_{x\to 1}(1-x^2)^{\alpha+1/2}$$

reduces to (26).

In the part of E, where $\alpha > \frac{1}{2} \wedge |\beta| < \frac{1}{2}$, the value of $f(1;\beta)$ exists, and

$$L(\beta-1) = \lim_{x\to 1}(1-x^2)^{-\beta+1/2} f(x;\beta-1) = \frac{2^{-\beta}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(-\beta+\frac{1}{2}\right)}{\Gamma(-\beta+1)},$$

$$L(-\alpha) = \lim_{x\to 1}(1-x^2)^{\alpha-1/2} f(x;-\alpha) = \frac{2^{\alpha-1}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)},$$

as well as the limit $L(-\alpha-1)$. Then, we have

$$\lim_{x\to 1}(1-x^2)^{\alpha+1/2} D(x;\alpha,\beta) = f(1;\beta)L(-\alpha-1) - L(\beta-1)L(-\alpha)\lim_{x\to 1}(1-x^2)^{\beta+1/2},$$

i.e., (26).

Finally, in the part of E, where $|\alpha| < \frac{1}{2} \wedge |\beta| < \frac{1}{2} \wedge \alpha+\beta > 0$, the values of $f(1;\beta)$ and $f(1;-\alpha)$ exist, as well as the limits $L(-\alpha-1)$ and $L(\beta-1)$. Therefore,

$$\lim_{x\to 1}(1-x^2)^{\alpha+1/2} D(x;\alpha,\beta) = f(1;\beta)L(-\alpha-1) - f(1;-\alpha)L(\beta-1)\lim_{x\to 1}(1-x^2)^{\alpha+\beta},$$

i.e., (26) holds.

3° Let $(\alpha,\beta)\in B$, i.e., $\alpha > \frac{1}{2} \wedge \beta < -\frac{1}{2}$. Then, the values of $f(1;\beta)$, $f(1;\beta-1)$, $f(1;-\alpha)$, and $f(1;-\alpha-1)$ do not exist, but the corresponding limits $L(\beta)$, $L(\beta-1)$, $L(-\alpha)$, and $L(-\alpha-1)$ exist, so that we have

$$\begin{aligned}\lim_{x\to 1}(1-x^2)^{\alpha-\beta} D(x;\alpha,\beta) &= L(\beta)L(-\alpha-1) - L(\beta-1)L(-\alpha)\\ &= \frac{2^{\alpha-\beta-1}}{\pi} \cdot \frac{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(-\beta-\frac{1}{2}\right)}{\Gamma(\alpha+1)\Gamma(-\beta+1)}\left(\left(\alpha-\tfrac{1}{2}\right)(-\beta) - \left(-\beta-\tfrac{1}{2}\right)\alpha\right),\end{aligned}$$

i.e., (27).

4° If $(\alpha,\beta)\in C\cup F$ then $(-\beta,-\alpha)\in A\cup E$. The result (28) follows directly from 2°, because of the property $D(x;\alpha,\beta) = -D(x;-\beta,-\alpha)$ (see Lemma 1). □

Now, we prove an inequality of Turán type for the function $x \mapsto D(x;\alpha,\beta)$.

Theorem 11 *For each $\alpha,\beta\in\mathbb{R}$ such that $\alpha+\beta>0$, the following inequality*

$$D(x;\alpha,\beta) > 0 \tag{29}$$

holds. If $\alpha+\beta<0$, then the opposite inequality holds.

Proof Interchanging variables $u \leftrightarrow v$ in (23) one can express $D(x;\alpha,\beta)$ in the form of the following arithmetic mean

$$\begin{aligned}D(x;\alpha,\beta) = \frac{x}{2\pi^2}\Bigg\{&\iint_S \frac{(1+xu)^{\beta-1}(1+xv)^{-\alpha-1}}{\sqrt{1-u^2}\sqrt{1-v^2}}(u-v)\,\mathrm{d}u\mathrm{d}v\\ &+\iint_S \frac{(1+xv)^{\beta-1}(1+xu)^{-\alpha-1}}{\sqrt{1-v^2}\sqrt{1-u^2}}(v-u)\,\mathrm{d}v\mathrm{d}u\Bigg\},\end{aligned}$$

i.e.,

$$D(x;\alpha,\beta)=\frac{x}{2\pi^2}\iint\limits_S \frac{(u-v)\left[(1+xu)^{\alpha+\beta}-(1+xv)^{\alpha+\beta}\right]}{\sqrt{1-u^2}\sqrt{1-v^2}(1+xu)^{\alpha+1}(1+xv)^{\alpha+1}}\,\mathrm{d}v\mathrm{d}u. \tag{30}$$

Now we define a function $\phi:(-1,1)\to\mathbb{R}$ by $\phi(t)=x(1+xt)^{\alpha+\beta}$. Since $\phi'(t)=x^2(\alpha+\beta)(1+xt)^{\alpha+\beta-1}>0$ for $\alpha+\beta>0$, $1+xu>0$, and $1+xv>0$, it follows that

$$(u-v)\left[x(1+xu)^{\alpha+\beta}-x(1+xv)^{\alpha+\beta}\right]>0,$$

which implies that the integrand of (30) is positive, and therefore inequality (29) holds.

If $\alpha+\beta<0$, according to Lemma 1, we conclude that the opposite inequality of (29) holds.

Remark 6 In some special cases, when α and β take integer values, one can express $D(x;\alpha,\beta)$ in an explicit form. In fact, for specific integer values of α and β, one has

$$\begin{aligned}
D(x;0,1)&=\frac{1-(1-x^2)^{1/2}}{(1-x^2)^{1/2}},\ D(x;0,2)=\frac{x^2+2-2(1-x^2)^{1/2}}{2(1-x^2)^{1/2}},\\
D(x;1,1)&=\frac{x^2}{(1-x^2)^{3/2}},\ D(x;1,2)=\frac{3x^2}{2(1-x^2)^{3/2}},\ D(x;1,3)=\frac{x^2\left(x^2+4\right)}{2(1-x^2)^{3/2}},\\
D(x;2,2)&=\frac{x^2\left(x^2+8\right)}{4(1-x^2)^{5/2}},\ D(x;2,3)=\frac{5x^2\left(x^2+2\right)}{4(1-x^2)^{5/2}},\\
D(x;2,4)&=\frac{3x^2\left(x^4+18x^2+16\right)}{16(1-x^2)^{5/2}},\ D(x;3,3)=\frac{x^2\left(x^4+12x^2+12\right)}{4(1-x^2)^{7/2}},\\
D(x;3,4)&=\frac{7x^2\left(x^2+4\right)\left(3x^2+2\right)}{16(1-x^2)^{7/2}},\ D(x;4,4)=\frac{x^2\left(9x^6+288x^4+672x^2+256\right)}{64(1-x^2)^{9/2}},\\
D(x;5,5)&=\frac{x^2\left(9x^8+360x^6+1680x^4+1600x^2+320\right)}{64(1-x^2)^{11/2}}.
\end{aligned}$$

Using the series (6) we can directly obtain the following power expansion for the function $D(x;\alpha,\beta)$:

Theorem 12 *The function $D:(0,1)\to\mathbb{R}$ can be expressed in the power series*

$$D(x;\alpha,\beta)=\sum_{n=1}^{+\infty}d_n\left(\frac{x}{2}\right)^{2n},\quad -1<x<1, \tag{31}$$

where the coefficients $d_n=d_n(\alpha,\beta)$ are given by

$$d_n=\sum_{k=1}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{\beta}{2k}\binom{-\alpha}{2n-2k}\left(\frac{2n-2k}{\alpha}+\frac{2k}{\beta}\right).$$

Remark 7 Taking $\alpha = -\beta$ in the expression of d_n it is easy to see that $d_n(-\beta, \beta) = 0$, from which we conclude that $(\alpha + \beta) \,|\, d_n(\alpha, \beta)$.

The first four coefficients in (31) are

$$
\begin{aligned}
d_1 &= 2(\alpha+\beta), \\
d_2 &= (\alpha+\beta)\big(\alpha^2+\alpha\beta+4\alpha+\beta^2-4\beta+9\big), \\
d_3 &= \frac{1}{6}(\alpha+\beta)\big[\alpha^4+2\alpha^3(\beta+6)+\alpha^2\left(4\beta^2+67\right)+2\alpha\left(\beta^3-2\beta+78\right) \\
&\qquad +\beta^4-12\beta^3+67\beta^2-156\beta+220\big], \\
d_4 &= \frac{1}{72}(\alpha+\beta)\big[\alpha^6+3\alpha^5(\beta+8)+\alpha^4\left(9\beta^2+24\beta+262\right)+3\alpha^3\left(3\beta^3+16\beta^2+32\beta+504\right) \\
&\qquad +\alpha^2\left(9\beta^4-48\beta^3+474\beta^2-552\beta+5221\right) \\
&\qquad +3\alpha\big(\beta^5-8\beta^4+32\beta^3+184\beta^2-669\beta+3136\big) \\
&\qquad +\beta^6-24\beta^5+262\beta^4-1512\beta^3+5221\beta^2-9408\beta+10500\big].
\end{aligned}
$$

4 Some properties of the functions $A(x;\alpha,\beta)$ and $B(x;\alpha,\beta)$

In this section we consider some properties of the function $A(x;\alpha,\beta)$ (as well as $B(x;\alpha,\beta)$), defined in (5). First we give a power expansion of the function

$$
A(x;\alpha,\beta) = \frac{\partial}{\partial\alpha} D(x;\alpha,\beta) = f(x;\beta-1) f_L(x;-\alpha) - f(x;\beta) f_L(x;-\alpha-1), \quad (32)
$$

using results of Theorems 1 and 2.

Theorem 13 *The function $A : (0,1) \to \mathbb{R}$ can be expressed in the power series*

$$
A(x;\alpha,\beta) = \sum_{n=1}^{+\infty} a_n \left(\frac{x}{2}\right)^{2n}, \quad -1 < x < 1, \quad (33)
$$

where the coefficients $a_n = a_n(\alpha,\beta)$ are given by

$$
a_n = \sum_{j=1}^{n} \binom{2j}{j} \binom{2n-2j}{n-j} \binom{\beta}{2n-2j} \sum_{\nu=1}^{2j} \frac{(\alpha)_{2j-\nu}}{\nu(2j-\nu)!} \left(\frac{2n-2j}{\beta} + \frac{2j-\nu}{\alpha}\right).
$$

The first five coefficients in (33) are

$$
\begin{aligned}
a_1 &= 2, \\
a_2 &= 3\alpha^2+4\alpha(\beta+2)+2\beta^2+9, \\
a_3 &= \frac{1}{6}\big[5\alpha^4+12\alpha^3(\beta+4)+3\alpha^2(6\beta^2+12\beta+67)+6\alpha\left(2\beta^3+21\beta+52\right) \\
&\qquad +3\beta^2(\beta^2-4\beta+21)+220\big],
\end{aligned}
$$

$$a_4 = \frac{1}{72}\big[7\alpha^6+24\alpha^5(\beta+6)+10\alpha^4(6\beta^2+24\beta+131)+8\alpha^3(9\beta^3+36\beta^2+179\beta+756)$$
$$+3\alpha^2(18\beta^4+570\beta^2+960\beta+5221)+4\alpha(6\beta^5-36\beta^4+285\beta^3+1607\beta+4704)$$
$$+2(2\beta^6-24\beta^5+179\beta^4-480\beta^3+1607\beta^2+5250)\big],$$

$$a_5 = \frac{1}{1440}\big[9\alpha^8+40\alpha^7(\beta+8)+70\alpha^6(2\beta^2+12\beta+73)+60\alpha^5(4\beta^3+32\beta^2+149\beta+760)$$
$$+5\alpha^4(60\beta^4+240\beta^3+3500\beta^2+9600\beta+49873)$$
$$+20\alpha^3(12\beta^5+620\beta^3+3040\beta^2+8617\beta+41576)$$
$$+60\alpha^2(2\beta^6-12\beta^5+155\beta^4+3372\beta^2+4626\beta+28041)$$
$$+40\alpha(\beta^7-16\beta^6+175\beta^5-760\beta^4+3372\beta^3+12783\beta+43080)$$
$$+5\beta^8-120\beta^7+1490\beta^6-9600\beta^5+43085\beta^4-92520\beta^3+255660\beta^2+828576\big].$$

Evidently, for each $(\alpha,\beta)\in\mathbb{R}^2$ we have $A(0;\alpha,\beta)=0$. Moreover, from Theorem 13 we conclude the following result:

Corollary 2 *For each $(\alpha,\beta)\in\mathbb{R}^2$ there is a neighbourhood of the point $x=0$, in notation $(-r,r)$, $0<r<1$, such that $A(x;\alpha,\beta)>0$ for each $x\in(-r,r)$, except at the point $x=0$, and*

$$\min_{-r<x<r} A(x;\alpha,\beta)=A(0;\alpha,\beta)=0.$$

Thus, the function $x\mapsto A(x;\alpha,\beta)$, defined by (32), has a local minimum at $x=0$ for each $(\alpha,\beta)\in\mathbb{R}^2$.

Using the same approach as in Section 3, we have the following integral representations:

$$A(x;\alpha,\beta)=-\frac{x}{\pi^2}\iint\limits_S \frac{(1+xu)^{\beta-1}(1+xv)^{-\alpha-1}\log(1+xv)}{\sqrt{1-u^2}\sqrt{1-v^2}}(u-v)\,\mathrm{d}u\mathrm{d}v \tag{34}$$

and

$$B(x;\alpha,\beta)=\frac{x}{\pi^2}\iint\limits_S \frac{(1+xu)^{\beta-1}(1+xv)^{-\alpha-1}\log(1+xu)}{\sqrt{1-u^2}\sqrt{1-v^2}}(u-v)\,\mathrm{d}u\mathrm{d}v,$$

where $S=\{(u,v): -1<u<1, -1<v<1\}$, from which we conclude that the following result holds:

Lemma 2 *For each $\alpha,\beta\in\mathbb{R}$, one has $B(x;\alpha,\beta)=A(x;-\beta,-\alpha)$.*

Using the same procedure as in the proof of Theorem 11 one can express $A(x;\alpha,\beta)$, defined in (34), in the form of the following arithmetic mean

$$A(x;\alpha,\beta)=\frac{x}{2\pi^2}\iint\limits_S \frac{(u-v)\big[(1+xu)^{-\alpha-\beta}\log(1+xu)-(1+xv)^{-\alpha-\beta}\log(1+xv)\big]}{\sqrt{1-u^2}\sqrt{1-v^2}(1+xu)^{-\beta+1}(1+xv)^{-\beta+1}}\,\mathrm{d}v\mathrm{d}u.$$

In a similar way, as before, one can see that the function

$$\psi(t) = x(1+xt)^{-\alpha-\beta}\log(1+xt)$$

is increasing on $(-1,1)$ when

$$0 \leq \alpha+\beta \leq \frac{1}{\log 2} \approx 1.4427. \quad (35)$$

Indeed, in this case, $\psi'(t) = x^2(1+xt)^{-\alpha-\beta-1}[-(\alpha+\beta)\log(1+xt)+1] > 0$, because of the fact that $\log(1+xt) < \log 2$.

Thus, we obtain the following result:

Proposition 1 *Under conditions* (35)*, the inequality* $A(x;\alpha,\beta) > 0$ *holds.*

Example 1 Let $\alpha = -1$ and $\beta = 1$. Since $f(x;0) = f(x;1) = 1$, according to Remark 3, we have

$$A(x;-1,1) = f_L(x;1) - f_L(x;0) = \sum_{n=1}^{+\infty} \frac{1}{2n-1}\binom{2n}{n}\left(\frac{x}{2}\right)^{2n}, \quad -1 \leq x \leq 1,$$

i.e.,

$$A(x;-1,1) = 1 - \sqrt{1-x^2} > 0, \quad 0 < x \leq 1.$$

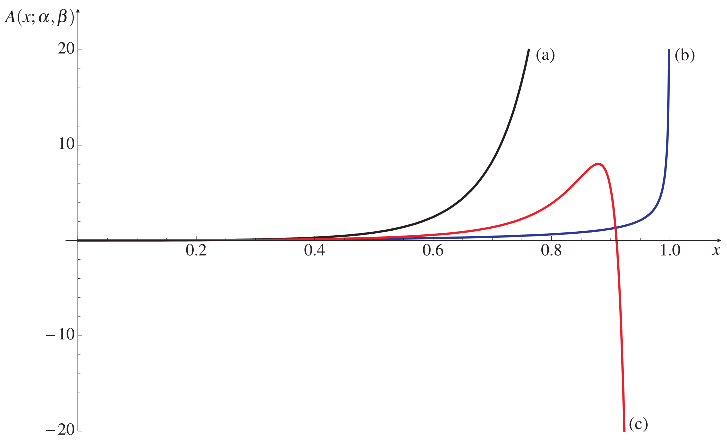

Fig. 5 Graphs of $x \mapsto A(x;\alpha,\beta)$ for (a) $\alpha = -17/4$, $\beta = -7/3$ (black line); (b) $\alpha = -19/10$, $\beta = -1/10$ (blue line); (c) $\alpha = 12/10$, $\beta = -21/10$ (red line)

Numerical experiments show that the inequality of Turán type,

$$A(x;\alpha,\beta) > 0, \quad 0 < x < 1, \quad (36)$$

is also true in a wider domain than the strip (35), but not in whole $\mathbb{R}^2$. The graphs of $x \mapsto A(x;\alpha,\beta)$ on $(0,1)$ for some specific parameters α and β are presented in Figure 5. All computations are performed in MATHEMATICA Package. As we can see $A(x;12/10,-21/10)$ changes its sign near $x = 0.91$. It could be interesting to find the exact domains in $\mathbb{R}^2$ where the inequality (36) is true.

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