Methods for Slowly Convergent Series and Applications to Special Functions

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Methods for summation of slowly convergent series are described and their application to calculating some special functions and mathematical constants are presented.

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Introduction

Slowly convergent series appear very often in calculation of some special functions and important constants (e.g., the Euler-Mascheroni constant γ, Apéry's constant $\zeta(3)$, Theodorus constant [3], Erdős-Borwein constant, etc.), but also in many problems in applied and computational sciences.

There are several numerical methods based on linear and nonlinear transformations. In general, starting from the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of the slowly convergent series S $(= \lim_{n\to\infty} S_n)$, these transformations give other sequences with faster convergence to the same limit S. In other words, these so-called *accelerating transformations* $\{S_n\}_{n=1}^{\infty} \to \{T_n\}_{n=1}^{\infty}$ must be *limit-preserving*, i.e.,

$$\lim_{n\to\infty} \frac{T_n - S}{S_n - S} = 0.$$

We mention here some well-known transformations as Euler's transformation, Aitken's Δ^2-process, Shanks's transformation, etc. (For more details see [2], [8], [14]).

Some alternative methods of summation of slowly convergent series are based on integral representations of series and an application of the Gaussian quadratures. Such summation/integration procedures for slowly convergent series have been developed in [6] (Laplace transform method), [9, 10] (Contour integration method), and [12] (Modified contour integration method). In [11] we derived a method for fast summation of trigonometric series.

Under certain conditions (see [12]) we can prove that

$$T_m = \sum_{k=m}^{+\infty} f(k) = \frac{\pi}{4} \sum_{\nu=1}^{n} A_\nu^{(n)} \Phi\left(m - \frac{1}{2}, \frac{\sqrt{\xi_\nu^{(n)}}}{2}\right) + R_n(\Phi), \tag{1}$$

where F is an integral of f, $\Phi(x,y) = -\frac{1}{2}\left[F(x+\mathrm{i}y) + F(x-\mathrm{i}y)\right]$, $\left(A_\nu^{(n)}, \xi_\nu^{(n)}\right)$, $\nu = 1,\ldots,n$, are the parameters (weights and nodes) of the n-point Gaussian quadrature rule

$$\int_0^{+\infty} \frac{g(x)}{\sqrt{x}\cosh^2 \frac{\pi\sqrt{x}}{2}} \mathrm{d}x = \sum_{\nu=1}^{n} A_\nu^{(n)} g\big(\xi_\nu^{(n)}\big) + R_n(g), \tag{2}$$

and $R_n(g)$ is the corresponding error term.

Some Special Functions and Constants

In this section we only give short account on calculation some special functions and mathematical constants defined by slowly convergent series. A rich treasury of significant numbers, mathematical constants, can be found in a two-volume book written by Finch [4, 5]. We use only a few of them.

Mathieu series

We consider the famous infinite functional series so-called Mathieu series of the form [7]

$$S(r) = \sum_{k=1} \frac{2k}{(k^2+r^2)^2}, \quad \widetilde{S}(r) = \sum_{k=1} (-1)^{k-1} \frac{2k}{(k^2+r^2)^2}, \quad r > 0.$$

The last alternating version of Mathieu series was introduced and investigated by Pogány et al. [16, p. 72]. Using the approach from [9], Milovanović and Pogány [13] obtained the integral representations for these series $S(r)$ and $\widetilde{S}(r)$,

$$S(r) = \pi \int_0^\infty \frac{r^2 - x^2 + \frac{1}{4}}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \frac{\mathrm{d}x}{\cosh^2(\pi x)},$$

$$\widetilde{S}(r) = \pi \int_0^\infty \frac{x}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \frac{\sinh(\pi x)\,\mathrm{d}x}{\cosh^2(\pi x)},$$

In a recent joint paper with Parmar and T. K. Pogány [15] we have proved the following series expansions for all $r > 0$,

$$S(r) = \frac{1}{r} \sum_{n=0}^{\infty} s \left\{ \mathrm{e}^{-rs} \Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{rs} \Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] \right\}\Big|_{s=2\pi(n+1)},$$

$$\widetilde{S}(r) = \frac{1}{r} \sum_{n=0}^{\infty} s \left\{ \mathrm{e}^{rs} \Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs} \Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] \right\}\Big|_{s=\pi(2n+1)},$$

where $E_1(z) = \int_z^\infty x^{-1}\mathrm{e}^{-x}\,\mathrm{d}x$ ($|\arg(z)| < \pi$) is the exponential integral of the first order [1, p. 228, Eq. 5.1.1] and $\Re[z]$ denotes the real part of $z \in \mathbb{C}$.

For calculating these last slowly convergent series $S(r)$ and $\widetilde{S}(r)$, we used the well-known Euler-Abel transformation very successfully (see [15] for details).

Riemann zeta function

The Riemann zeta function $s \mapsto \zeta(s)$ is defined by $\zeta(s) = \sum_{k=1}^{+\infty} k^{-s}$ for $\Re s > 1$. The series converges for any s with $\Re s > 1$, uniformly, for any fixed $\sigma > 1$, in any subset of $\Re s \geq \sigma$, which establishes that $\zeta(s)$ is an analytic function in $\Re s > 1$. The function $\zeta(s)$ admits analytic continuation to $\mathbb{C}$, where it satisfies the functional equation $\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s)$. Thus, by means of analytic continuation,

$\zeta(s)$ is analytic function for any complex s, except for $s = 1$, which is a simple pole of $\zeta(s)$ with residue 1.

Using our approach we get an integral representation of $\zeta(s+1)$. Since $f(z) = 1/z^{s+1}$ and $F(z) = -1/(sz^s)$, using (1) and (2) we obtain

$$\zeta(s+1) = \sum_{k=1}^{m-1} \frac{1}{k^{s+1}} + \frac{\pi}{4s\left(m-\frac{1}{2}\right)^s} \sum_{\nu=1}^{n} A_\nu^{(n)} g\big(\xi_\nu^{(n)}\big) + E_{n,m}(s),$$

where

$$g(t;s) = \exp\Big(-\frac{s}{2}\log(1+t^2)\Big)\cos\big(s\arctan t\big), \quad c_m = \frac{1}{2m-1},$$

and $E_{n,m}(s)$ is the corresponding error term.

Euler-Mascheroni constant

The Euler-Mascheroni constant γ is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721566490153286060651209008\ldots\ .$$

This constant can be expressed as the following slowly convergent series (cf. [4, p. 30])

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \log\left(1+\frac{1}{k}\right)\right).$$

Applying our method to summation of this series we use (1), with

$$f(z) = \frac{1}{z} - \log\left(1+\frac{1}{z}\right) \quad \text{and} \quad F(z) = 1-(z+1)\log\left(\frac{z+1}{z}\right),$$

where $\lim_{z\to\infty} F(z) = 0$.

The corresponding relative errors $\mathrm{err}_{n,m} = \big|\ (Q_{n,m}(f)-\gamma)/\gamma\ \big|$ for number of nodes n in the quadrature formula (2) and $m = 1, 2, 3, 6, 11$ and 16 are presented in Table 1.

n	$m=1$	$m=2$	$m=3$	$m=6$	$m=11$	$m=16$
10	2.73(−4)	2.03(−9)	2.12(−13)	4.50(−22)	2.39(−31)	1.34(−37)
20	5.92(−5)	2.79(−11)	1.90(−16)	1.49(−28)	5.53(−43)	1.32(−53)
30	2.43(−5)	2.26(−12)	3.06(−18)	2.00(−32)	3.53(−50)	6.92(−64)
100	1.77(−6)	1.30(−15)	1.43(−23)	5.23(−44)	4.60(−72)	7.33(−96)

Table 1: Relative errors $\mathrm{err}_{n,m}$ in of Gaussian approximations $Q_{n,m}(f)$ of $T_1(10)$ for $n = 10, 20, 30$ and 100 and for some selected values of m

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