Iterative Processes and Integral Equations of the Second Kind

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Abstract. Beside the general theory to operator equations and iterative processes, including existence and uniqueness of solutions, fixed point theory, local properties of iterative processes, main theorems of Newton-Kantorovich method, as well as methods for acceleration of iterative processes, a special attention is dedicated to applications to integral equations of the second kind, including a discretization process by using quadrature formulas. Several kinds of integral equations are considered: nonlinear Volterra-Fredholm integral equations, mixed Volterra-Fredholm integral equations, Volterra integral equations with delayed argument, functional Volterra integral equations and fractional integral equations.

1. Introduction to Operator Equations and Iterative Processes

Let X and Y be two Banach spaces, D be a convex subset of X and $F : D \to Y$ be an operator, in general case, nonlinear. We can consider the operator equation

$$Fu = 0, \tag{1.1}$$

where 0 is zero-vector of the space Y. A large number of problems in science and techniques come down to solving equations of the form (1.1).

A special and important case is when $Y = X$ and $Fu = u - Tu = 0$. We mention a few typical examples.

(a) If $X = Y = \mathbb{R}$, $u = x$, $F = f$, the nonlinear equation $f(x) = x - \cos x = 0$, as well as the algebraic equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0,$$

are of the form (1.1).

(b) If $X = Y = \mathbb{R}^n$, $u = \mathbf{x} = [x_1 \cdots x_n]^T$ and

$$Fu = F(\mathbf{x}) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix},$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ are given functions, the equation (1.1) represents a system of nonlinear equations

$$f_i(x_1, \ldots, x_n) = 0, \qquad i = 1, \ldots, n.$$

If F is a linear operator, for example, $F(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$, where the matrix A and the vector $\mathbf{b}$ are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

respectively, then the equation (1.1) reduces to a system of linear algebraic equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \qquad i = 1, \ldots, n.$$

(c) In the case $X = C^2[a,b]$, $Y = C[a,b] \times \mathbb{R}$, $u \equiv u(t)$,

$$Fu = \begin{bmatrix} f_1(u) \\ f_2(u) \end{bmatrix}$$

and

$$f_1(u)(t) = u''(t) - f(t, u(t), u'(t)) \quad (t \in [a,b]), \quad f_2(u) = g(u(a), u(b)),$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are given functions. Then the operator equation (1.1) represents the boundary value problem

$$u''(t) = f(t, u(t), u'(t)) \quad (t \in [a,b]),$$

$$g(u(a), u(b)) = 0.$$

(d) Let $X = C([a,b])$, $K_1, K_2 \in C([a,b]^2 \times \mathbb{R})$, $g \in C([a,b])$, and the operator $T : X \to X$ be defined by

$$(Tu)(t) = g(t) + \int_a^t K_1(t, x, u(x)) \, dx + \int_a^b K_2(t, x, u(x)) \, dx, \qquad (1.2)$$

where $t \in [a,b]$. In this case we have the operator equation

$$u(t) = (Tu)(t), \quad t \in [a,b],$$

which is, in fact, a nonlinear Volterra-Fredholm integral equations of the second kind.

All mentioned equations, as well as a number of others, can be treated in a unique way. That is why the subject of our consideration in this chapter

is solving the operator equation (1.1), i.e. finding such a point $u \in D$ which satisfies (1.1). Therefore, this unique approach is applied to the equation

$$u = Tu \tag{1.3}$$

where the operator T maps D to D and $Tu = H(u, Fu)$, with an operator $H : D \times Y \to D$. It is clear that for a given equation (1.1), the form (1.3) is not unique, as the following example shows.

A simple equation $f(x) = 0$ can be represented in an equivalent form

$$x = x + \lambda f(x) \tag{1.4}$$

for each λ different from zero, but there are many other equivalent forms different from (1.4).

1.1. Iterative Processes

One of the ways to solve the equation (1.3) as one form of (1.1) is to construct the sequence $\{u_k\}_{k \in \mathbb{N}_0}$ as

$$u_{k+1} = Tu_k, \quad k = 0, 1, \ldots, \tag{1.5}$$

starting from some point $u_0 \in D$. Under certain conditions for the operator T, the sequence $\{u_k\}_{k \in \mathbb{N}_0}$ can converge to the desired solution. The formula (1.5) is known as *an iterative process.*

Remark 1.1. Beside iterative processes of the form (1.5) one can consider more general processes, the so-called *iterative processes with memory*

$$u_{k+1} = S(u_k, u_{k-1}, \ldots, u_{k-m+1}), \quad k = m - 1, m, \ldots,$$

where $S : X^m \to X$. Such a process, with a memory of the length m, needs m starting points $u_0, u_1, \ldots, u_{m-1} \in D$.

1.2. Existence and Uniqueness of Solutions. Fixed Point Theory

Let $(X, \| \cdot \|)$ be a Banach space and (1.5) be an iterative process converging to to $u^* \in X$, so that $u^* = Tu^*$. It means that there exists a point $u^* \in X$ such that

$$\lim_{k \to +\infty} \|u_k - u^*\| = 0.$$

Such $u^* \in X$ is a *fixed point* of the operator T.

The fixed point $u^* \in X$ is a solution of the previous equation (1.3).

To discuss solvability and other properties of the operator equations, let us recall the main results for the *Fixed Point Theory* [5] on a Banach space.

Definition 1.2. Let $(X, \| \cdot \|)$ be a Banach space. A nonlinear operator $T : X \mapsto X$ is a q-contraction if $0 \leq q < 1$ and

$$(\forall u, v \in X) \qquad \|Tu - Tv\| \leq q\|u - v\|.$$

A classical result is the contraction principle on a Banach space.

Theorem 1.3. *Let $(X, \| \cdot \|)$ be a Banach space and $T : X \mapsto X$ be a q-contraction. Then*

(a) *the equation $u = Tu$ has exactly one solution $u^* \in X$;*

(b) *the iterative process $u_{k+1} = Tu_k$, $k = 0, 1, \ldots$, converges to the solution u^* for any arbitrary choice of the initial point $u_0 \in X$;*

(c) *the error estimate*

$$\|u_k - u^*\| \le \frac{q^k}{1 - q} \|Tu_0 - u_0\|$$

holds for each $k \in \mathbb{N}$.

A stronger fixed point result can be formulated in the following form (see Altman [2]).

Theorem 1.4. *Let $(X, \| \cdot \|)$ be a Banach space and $T : X \mapsto X$ be a q-contraction. Let $\{\varepsilon_k\}_{k=0}^{+\infty}$ be a sequence of positive numbers such that $\varepsilon_k \le 1$ and $\sum_{k=0}^{+\infty} \varepsilon_k = +\infty$. Then*

(a) *the equation $u = Tu$ has exactly one solution $u^* \in X$;*

(b) *the iterative process*

$$u_{k+1} = (1 - \varepsilon_k)u_k + \varepsilon_k Tu_k, \quad k = 0, 1, \ldots,$$

converges to u^* for any arbitrary choice of the initial point $u_0 \in X$;

(c) *the error estimate*

$$\|u_k - u^*\| \le \frac{e^{1-q}}{1 - q} \|Tu_0 - u_0\| e^{-(1-q)v_k}$$

holds for each $k \in \mathbb{N}$, where $v_0 = 0$ and $v_k = \sum_{\nu=0}^{k-1} \varepsilon_\nu$, $k \ge 1$.

Remark 1.5. The above results remain valid if instead of the entire space X, we consider any closed subset $Y \subset X$, satisfying $T(Y) \subseteq Y$. Many times, such results are useful if applied on a closed ball $B_\varrho = \{u \in X : \|u - u_0\| \le \varrho\}$, for a suitable point $u_0 \in X$. This issue is addressed in more detail in the next section.

1.3. Local Properties of Iterative Processes

Now we consider some local properties of the iterative process (1.5).

Let $u^* \in X$ be a fixed point of the operator $T : X \to X$ and let U be a convex neighbourhood of the limit point u^*. The iterative process (1.5) is of order r (≥ 1) if

$$\|Tu - u^*\| = O(\|u - u^*\|^r) \quad (u \in U).$$

Theorem 1.6. *If the operator T is r-times differentiable in Fréchet's sense on U, then the iterative process (1.5) is of the order r if and only if the following conditions are satisfied:*

1) $Tu^* = u^*$;

2) $T'_{(u^*)}$, $T''_{(u^*)}$, $\ldots$, $T^{(r-1)}_{(u^*)}$ *are zero operators;*

3) $T^{(r)}_{(u)}$ *is non-zero operator, with a norm bounded on U.*

For the proof of this theorem see, for example, the book by Collatz [9, p. 291].

The most known iterative process of the second order is the *Newton-Kantorovich* method

$$u_{k+1} = u_k - [F'_{(u_k)}]^{-1} F u_k, \quad k = 0, 1, \dots, \tag{1.6}$$

for solving the equation $Fu = 0$. This fundamental extension of the well-known Newton method to functional spaces was given by L.V. Kantorovich[1] in 1948 (see [15] and also the book [16]). Here $F'_{(u_k)}$ is the *Fréchet* derivative of the nonlinear operator F at the point u_k and $[F'_{(u_k)}]^{-1}$ is its inverse. This is one of the fundamental techniques in functional analysis and numerical analysis.

Theorem 1.7 ([15]). *Assume that the operator F is defined and twice continuously differentiable on a ball $B = \{u : \|u - u_0\| \le \varrho\}$, the linear operator $F'_{(u_0)}$ is invertible,*

$$\|[F'_{(u_0)}]^{-1} F u_0\| \le \eta, \quad \|[F'_{(u_0)}]^{-1} F''_{(u)}\| \le K \quad (u \in B),$$

and

$$h = K\eta < \frac{1}{2}, \quad \varrho \ge \frac{1 - \sqrt{1 - 2h}}{h} \eta.$$

Then the equation $Fu = 0$ has a solution $u^* \in B$, the iterative process (1.6) is well defined and converges to u^* with quadratic rate

$$\|u_k - u^*\| \le \frac{\eta}{h 2^k} (2h)^{2^k}.$$

There exist numerous versions of Kantorovich's theorem, which differ in assumptions and results (cf. [29] and reference therein, as well as the books [16], [17], [28]). We mention just one of them, due to Ivan Petrovich Mysovskikh.

Theorem 1.8 ([27]). *Assume that the operator F is defined and twice continuously differentiable on a ball $B = \{u : \|u - u_0\| \le \varrho\}$, the linear operator $F'_{(u_0)}$ is invertible,*

$$\|[F'_{(u)}]^{-1}\| \le \beta, \quad \|F''_{(u)}\| \le K \quad (u \in B), \quad \|Fu_0\| \le \eta,$$

and

$$h = K\beta^2 \eta < 2, \quad \varrho \ge \beta\eta \sum_{\nu=0}^{+\infty} (h/2)^{2^\nu - 1}.$$

Then the equation $Fu = 0$ has a solution $u^* \in B$, the iterative process (1.6) is well defined and converges to u^* with quadratic rate

$$\|u_k - u^*\| \le \frac{\beta\eta(h/2)^{2^k - 1}}{1 - (h/2)^{2^k}}.$$

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[1]**Leonid Vitalyevich Kantorovich** (1912–1986) was a famous Soviet mathematician and economics, the winner of the Nobel Prize for his theory and development of techniques for the optimal allocation of resources in Economic Sciences in 1975.

Taking $F'_{(u_0)}$ instead of $F'_{(u_k)}$ in (1.6), we get the iterative process

$$u_{k+1} = u_k - [F'_{(u_0)}]^{-1} F u_k, \quad k = 0, 1, \dots,$$

of the first order. There are several approaches for modifying the Newton-Kantorovich method in order to achieve global convergence. The simplest way is the so-called *damped Newton-Kantorovich method*

$$u_{k+1} = u_k - \gamma_k [F'_{(u_k)}]^{-1} F u_k, \quad k = 0, 1, \dots,$$

where γ_k $(0 < \gamma_k \le 1)$ is chosen so that $\|F u_k\| < \|F u_{k-1}\|$. This kind of minimization enables a balance between convergence and order of convergence.

Here we mention also the *Levenberg-Marquardt method* (cf. Polyak [29])

$$u_{k+1} = u_k - [\gamma_k I + F'_{(u_k)}]^{-1} F u_k, \quad k = 0, 1, \dots,$$

which reduces to the Newton-Kantorovich method for $\gamma_k = 0$.

1.4. Acceleration of Iterative Processes

Having an iterative process of order k, we can obtain the process of higher order (see [14], [24], [30], [31]).

Theorem 1.9 ([14])**.** *Let* (1.5) *be an iterative method of the order r and the operator T be $(r + 1)$-times differentiable in the sense of Fréchet on U. If we suppose that the inverse operator $\left[I - \dfrac{1}{r} T'_{(u)}\right]^{-1}$ exists for $u \in U$, then the iterative process*

$$u_{k+1} = u_k - \left[I - \frac{1}{r} T'_{(u)}\right]^{-1} (u_k - T u_k)$$

is at least of the order $r + 1$.

Theorem 1.10 ([24])**.** *Let* (1.5) *be an iterative method of the order $r \ge 2$ and the operator T be $(r + 1)$-times differentiable in the sense of Fréchet on U. Then the iterative process*

$$u_{k+1} = T u_k - \frac{1}{r} T'_{(u)}(u_k - T u_k)$$

is at least of the order $r + 1$.

Applying Theorem 1.9 to the iterative process (1.5) of the order $r = 1$ (with a linear convergence), $u_{k+1} = T u_k$, we get the iterative process

$$u_{k+1} = u_k - \left[I - T'_{(u)}\right]^{-1} (u_k - T u_k), \quad k = 0, 1, \dots,$$

of the order at least two (quadratic convergence).

It is exactly the *Newton-Kantorovich* method

$$u_{k+1} = u_k - [F'_{(u_k)}]^{-1} F u_k, \quad k = 0, 1, \dots,$$

where $F u = u - T u = 0$.

Applying Theorem 1.10 to the Newton-Kantorovich method of the second order $(r = 2)$,

$$u_{k+1} = u_k - [F'_{(u_k)}]^{-1} F u_k, \quad k = 0, 1, \dots,$$

we obtain an iterative method of the third order $u_{k+1} = \Phi u_k$, $k = 0, 1, \ldots$, where the operator Φ is given by

$$\Phi u = u - [F'_{(u)}]^{-1} F u - \frac{1}{2} [F'_{(u)}]^{-1} F''_{(u)} \left([F'_{(u)}]^{-1} F u, [F'_{(u)}]^{-1} F u \right),$$

supposing the existence of certain higher derivatives of F. Similarly, an application of Theorem 1.9 to the Newton-Kantorovich method gives also a method of the third order (*method of tangent hyperbolas*), considered in 1961 by Altman [1].

2. Applications to Integral Equations of the second Kind – Discretization

Integral equations play a significant role in Applied Mathematics, since they arise in many applications in areas of physics, engineering, biology, hydrodynamics, thermodynamics, elasticity, quantum mechanics, etc. They represent an important tool for modelling the progress of an epidemic and various other biological problems. Also, many reformulations of initial and boundary value problems for partial differential equations can be written as integral equations. As such, finding numerical solutions, approximations of the true solution at a discrete set of points is an important task for researchers.

Iterative methods are particularly suitable, as they not only guarantee the existence of a unique solution (under certain conditions), but they also provide means for finding approximate solutions, via successive approximations. We present several types of integral equations of the second kind and various numerical iterative methods that produce good approximations for their solutions.

2.1. Nonlinear Volterra-Fredholm Integral Equations

We consider nonlinear Volterra-Fredholm integral equations of the second kind

$$u(t) = g(t) + \int_a^t K_1(t, x, u(x)) \, dx + \int_a^b K_2(t, x, u(x)) \, dx, \quad t \in [a, b], \quad (2.1)$$

where $K_1, K_2 \in C([a, b]^2 \times \mathbb{R})$ and $g \in C([a, b])$.

There are several methods for solving this equation, especially for linear equations.

We employ *Fixed Point Theory* for this kind of equations, and therefore, we take $X = C([a, b])$, equipped with the usual norm $\|u\| = \max\limits_{t \in [a,b]} |u(t)|$ and define the integral operator $T : X \to X$ as in (1.2), i.e.,

$$(Tu)(t) = g(t) + \int_a^t K_1(t, x, u(x)) \, dx + \int_a^b K_2(t, x, u(x)) \, dx.$$

In this way we get the operator equation $u(t) = (Tu)(t)$, $t \in [a, b]$.

Several variants of the integral operator T have appeared in papers in the last period. In [23] it was considered just mentioned equation.

The method is based on *Picard iteration* and uses a suitable quadrature formula (composite trapezoidal rule),

$$\int_a^b f(t)\mathrm{d}t = h\Big[\frac{1}{2}f(a) + \sum_{\nu=1}^{n} f(\tau_\nu) + \frac{1}{2}f(b)\Big] + R_{n+2}(f), \qquad (2.2)$$

where $h = (b-a)/(n+1)$, $\tau_\nu = a + h\nu$, $\nu = 0, 1, \ldots, n, n+1$, and the remainder term ($= 0$ for all $f \in \mathcal{P}_1$)

$$R_{n+2}(f) = -\frac{h^2}{12}(b-a)f''(\xi), \quad \xi \in (a,b).$$

The existence and uniqueness of the solution, as well as the error estimates in the approximate solutions, were given under certain conditions, which ensure the application of the fixed point theory.

Basic idea is an approximation of the equation

$$u(t) = (Tu)(t) \quad \text{by} \quad \tilde{u}(t) = (\widetilde{T}\tilde{u})(t) \qquad (t \in [a.b]),$$

usually on a discrete set of points in $[a,b]$, e.g.,

$$a = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = b.$$

Such discretization leads to the determination of a sequence of the vectors at the points $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_n, \tau_{n+1})$,

$$\widetilde{\mathbf{u}}_k = \big(\tilde{u}_k(\tau_0), \tilde{u}_k(\tau_1), \ldots, \tilde{u}_k(\tau_n), \tilde{u}_k(\tau_{n+1})\big), \quad k = 1, 2, \ldots,$$

starting from some $\widetilde{\mathbf{u}}_0$. Here, $\widetilde{\mathbf{u}}_k = \widetilde{\mathbf{u}}_k(\boldsymbol{\tau}) \in \mathbb{R}^{n+2}$.

Then, the iterative process

$$\widetilde{\mathbf{u}}_{k+1} = \widetilde{T}\,\widetilde{\mathbf{u}}_k, \quad k = 0, 1, \ldots,$$

should converge to the solution of the equation $\tilde{u}(\boldsymbol{\tau}) = (\widetilde{T}\tilde{u})(\boldsymbol{\tau})$, denoted by $\widetilde{\mathbf{u}}^* = \widetilde{\mathbf{u}}^*(\boldsymbol{\tau})$, and to be close enough to the solution of the equation $u(t) = (Tu)(t)$ at $(n+2)$ points $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_n, \tau_{n+1})$.

Denote this (discrete) solution by $\mathbf{u}^* = \mathbf{u}^*(\boldsymbol{\tau})$.

Using the uniform norm of vectors in $\mathbb{R}^{n+2}$, we have that

$$\|\widetilde{\mathbf{u}}_k - \mathbf{u}^*\| = \|\widetilde{\mathbf{u}}_k - \widetilde{\mathbf{u}}^* + \widetilde{\mathbf{u}}^* - \mathbf{u}^*\|$$

$$\leq \|\widetilde{\mathbf{u}}_k - \widetilde{\mathbf{u}}^*\| + \|\widetilde{\mathbf{u}}^* - \mathbf{u}^*\|$$

The first term depends on the iterative process and its speed, and the second one depends on the approximation of integrals by the quadrature formulas.

Under conditions that the kernels K_1 and K_2 satisfy Lipschitz's conditions with respect to the third argument, with constants L_1 and L_2 respectively, such that $q = (b-a)(L_1 + L_2) < 1$, and the (weight) coefficients of the quadrature formula A_ν, $\nu = 0, 1, \ldots, n+1$, are such that

$$\gamma = (L_1 + L_2)\sum_{\nu=1}^{n+1} |A_\nu| < 1,$$

then

$$\|\widetilde{\mathbf{u}}_k - \mathbf{u}^*\| \leq \frac{M_1 q^k}{1-q} + \frac{M_2}{1-\gamma}, \tag{2.3}$$

for some positive constants M_1 and M_2.

For the composite trapezoidal formula, in [23] it was proved that

$$\gamma = q = (b-a)(L_1 + L_2), \quad \frac{M_2}{1-\gamma} = O(h^2), \quad h = \frac{b-a}{n+1}, \tag{2.4}$$

and presented a few examples to illustrate her method based on Picard iteration, with this composite rule.

Here we propose the following approximation of the integral equation (2.1): (i) at $t = \tau_0 = a$ by

$$\tilde{u}_{k+1}(\tau_0) = g(\tau_0) + \sum_{\nu=0}^{n+1} A_\nu^{(n+1)} K_2(\tau_0, \tau_\nu, \tilde{u}_k(\tau_\nu)) \tag{2.5}$$

and (ii) at $t = \tau_j$, $j = 1, \ldots, n, n+1$, by

$$\tilde{u}_{k+1}(\tau_j) = g(\tau_j) + \sum_{\nu=0}^{n+1} A_\nu^{(n+1)} K_2(\tau_j, \tau_\nu, \tilde{u}_k(\tau_\nu))$$

$$+ \sum_{\nu=0}^{n+1} A_\nu^{(j)} K_1(\tau_j, \tau_\nu, \tilde{u}_k(\tau_\nu)), \tag{2.6}$$

with starting function $\tilde{u}_0(t) = g(t)$.

These quadrature formulas should be of degree of precision at least $n+1$.

2.2. Construction of Interpolatory Quadrature Formulas for Volterra and Fredholm Parts

Because of simplicity, we construct here *interpolatory quadrature formulas* of *closed type* on the interval $[0, 1]$, with arbitrary n internal nodes,

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = 1.$$

Such formulas are exact for all algebraic polynomials of degree at most $n+1$.

We consider only non-weighted formulas ($w(t) = 1$).

For any other finite interval $[a, b]$, such quadratures can be obtained by a simple linear transformation $t \mapsto a + (b-a)t$.

Thus, we consider

$$\int_0^1 f(t)\mathrm{d}t = \sum_{\nu=0}^{n+1} A_\nu f(\tau_\nu) + R_{n+2}(f) \tag{2.7}$$

and for each $j = 1, \ldots, n$,

$$\int_0^{\tau_j} f(t)\mathrm{d}t = \sum_{\nu=0}^{n+1} A_\nu^{(j)} f(\tau_\nu) + R_{n+2}^{(j)}(f), \tag{2.8}$$

where for $j = n + 1$,

$$A_\nu^{(n+1)} \equiv A_\nu \quad \text{and} \quad R_{n+2}^{(n+1)}(f) = R_{n+2}(f).$$

Here, for each $f \in \mathcal{P}_{n+1}$ we have

$$R_{n+2}^{(n+1)}(f) = 0, \quad j = 1, 2, \ldots, n, n + 1.$$

Remark 2.1. It is easy to transform these formulas to a general interval $[a, b]$ by a linear transformation.

In order to construct these quadrature rules we start with the Lagrange polynomial of degree $\leq n + 1$ at selected nodes

$$L_{n+1}(f; t) = \sum_{\nu=0}^{n+1} f(\tau_\nu) \frac{\omega(t)}{(t - \tau_\nu)\omega'(\tau_\nu)},$$

where ω is the *node polynomial*

$$\omega(t) = (t - \tau_0)(t - \tau_1) \cdots (t - \tau_n)(t - \tau_{n+1})$$

$$= t(t - 1)(t - \tau_1) \cdots (t - \tau_n), \tag{2.9}$$

so that $f(t) = L_{n+1}(f; t) + r(f; t)$ and $r(\mathcal{P}_{n+1}; t) \equiv 0$.

Integrating over $[0, \tau_j]$, $j = 1, \ldots, n, n + 1$, we get

$$\int_0^{\tau_j} f(t)\mathrm{d}t = \int_0^{\tau_j} L_{n+1}(f; t)\mathrm{d}t + R_{n+2}^{(j)}(f),$$

where $R_{n+2}^{(j)}(f) = 0$ for each $f \in \mathcal{P}_{n+1}$.

Taking the expression for the Lagrange polynomial we obtain

$$\int_0^{\tau_j} L_{n+1}(f; t)\mathrm{d}t = \sum_{\nu=0}^{n+1} A_\nu^{(j)} f(\tau_\nu),$$

where

$$A_\nu^{(j)} = \frac{1}{\omega'(t)} \int_0^{\tau_j} \frac{\omega(t)}{t - \tau_\nu} \, \mathrm{d}t, \quad \nu = 0, 1, \ldots, n, n + 1,$$

for each $j = 1, \ldots, n, n + 1$, and $\omega(t)$ is defined in (2.9).

Thus,

$$A_\nu^{(j)} = \int_0^{\tau_j} \prod_{\substack{i=0 \\ i \neq \nu}}^{n+1} \frac{t - \tau_i}{\tau_\nu - \tau_i} \, \mathrm{d}t, \quad \nu = 0, 1, \ldots, n, n + 1.$$

After changing the variables (for fixed $j \in \{1, 2, \ldots, n, n + 1\}$)

$$t = \tau_j \xi \quad \text{and} \quad \tau_\nu = \tau_j \xi_\nu, \quad \nu = 0, 1, \ldots, n, n + 1,$$

so that $\xi_\nu = \tau_\nu / \tau_j$, $\nu = 0, 1, \ldots, n, n + 1$, i.e.,

$$\xi_0 = 0, \quad \ldots, \quad \xi_j = 1, \quad \xi_\nu > 1 \quad (\nu > j),$$

we get

$$A_\nu^{(j)} = \tau_j \int_0^1 \prod_{\substack{i=0 \\ i\neq\nu}}^{n+1} \frac{\xi - \xi_i}{\xi_\nu - \xi_i} \, d\xi, \qquad \begin{aligned} &\nu = 0, 1, \ldots, n, n+1; \\ &j = 1, 2, \ldots, n, n+1, \end{aligned} \qquad (2.10)$$

in the quadrature formulas for each $j = 1, 2, \ldots, n, n+1$,

$$\int_0^{\tau_j} f(t) \, dt = \sum_{\nu=0}^{n+1} A_\nu^{(j)} f(\tau_\nu) + R_{n+2}^{(j)}(f), \quad R_{n+2}^{(j)}(\mathcal{P}_{n+1})] = 0. \qquad (2.11)$$

Using the error in the Lagrange interpolation polynomial and supposing that $f \in C^{n+2}[0,1]$, with $|f^{(n+2)}(t)| \leq M_{n+2}$, we can get

$$|R_{n+2}^{(j)}(f)| \leq \frac{M_{n+2}}{(n+2)!} \int_0^{\tau_j} |\omega(t)| \, dt, \quad j = 1, 2, \ldots, n, n+1, \qquad (2.12)$$

where $\omega(t)$ is defined in (2.9).

Now, we give a few standard sequences for the internal nodes.

2.2.1. Uniform distribution of nodes. We take the internal nodes as in the Newton-Cotes formulas,

$$\tau_\nu = \frac{\nu}{n+1}, \quad \nu = 0, 1, \ldots, n, n+1;$$

Here, for each $j \in \{1, 2, \ldots, n, n+1\}$, we have

$$\xi_0 = 0, \quad \xi_\nu = \frac{\nu}{j}, \quad \nu = 1, \ldots, n, n+1,$$

and

$$A_\nu^{(j)} = \frac{j}{n+1} \int_0^1 \prod_{\substack{i=0 \\ i\neq\nu}}^{n+1} \frac{j\xi - i}{\nu - i} \, d\xi, \quad \nu = 0, 1, \ldots, n, n+1.$$

These coefficients can be calculated very easily in symbolic form.

For $n = 5$, i.e., for the nodes $\{\tau_\nu\}_{\nu=0}^6 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$, we obtain the weight coefficients given by the following matrix:

$$\begin{bmatrix} \frac{19087}{362880} & \frac{2713}{15120} & -\frac{15487}{120960} & \frac{293}{2835} & -\frac{6737}{120960} & \frac{263}{15120} & -\frac{863}{362880} \\[6pt] \frac{1139}{22680} & \frac{47}{189} & \frac{11}{7560} & \frac{166}{2835} & -\frac{269}{7560} & \frac{11}{945} & -\frac{37}{22680} \\[6pt] \frac{137}{2688} & \frac{27}{112} & \frac{387}{4480} & \frac{17}{105} & -\frac{243}{4480} & \frac{9}{560} & -\frac{29}{13440} \\[6pt] \frac{143}{2835} & \frac{232}{945} & \frac{64}{945} & \frac{752}{2835} & \frac{29}{945} & \frac{8}{945} & -\frac{4}{2835} \\[6pt] \frac{3715}{72576} & \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{3875}{24192} & \frac{235}{3024} & -\frac{275}{72576} \\[6pt] \frac{41}{840} & \frac{9}{35} & \frac{9}{280} & \frac{34}{105} & \frac{9}{280} & \frac{9}{35} & \frac{41}{840} \end{bmatrix}.$$

The coefficients $A_\nu^{(j)}$, $\nu = 0, 1, \ldots, n+1$ (here $n = 5$), in the quadrature formula (2.8) for $j = 1, \ldots, n$ are given in the j-th row of the previous matrix, while in the last row ($j = n+1$) these elements are weight coefficients are in

the quadrature formula (2.7). The weight coefficients corresponding to nodes in $[0, \tau_j]$ are marked in bold.

Remark 2.2. Quadrature formulas with nodes outside the interval of integration can be found in the literature, e.g., quadratures of *Birkhoff-Young type* (see [6], [25]). The so-called *extended Simpson rule* [32, p. 124] has also two nodes outside the interval of integration,

$$\int_{c-h}^{c+h} f(t)\mathrm{d}t = \frac{h}{90}\Big\{114f(c) + 34[f(c+h) + f(c-h)]$$

$$- [f(c+2h) + f(c-2h]\Big\} + R_5^{\mathrm{ES}}(f),$$

where

$$|R_5^{\mathrm{ES}}(f)| \sim \frac{|h|^7}{756}|f^{(6)}(\xi)|, \quad c - 2h < \xi < c + 2h,$$

supposing that $f \in C^6[c - 2h, c + 2h]$.

According to (2.12) the bounds for $|R_{n+2}^{(j)}(f)|$, when $n = 5$ and $j \in \{1, 2, \ldots, 6\}$ are respectively

$$\frac{M_7}{7!}\left\{\frac{1375}{40310784}, \frac{863}{20155392}, \frac{71}{1492992}, \frac{527}{10077696}, \frac{2459}{40310784}, \frac{71}{746496}\right\},$$

where $M_7 = \max_{0 \le t \le 1}|f^{(7)}(t)|$.

2.2.2. Nodes of Lobatto formula. We take the internal nodes as zeros of the polynomials $\pi_n(t)$, which are *orthogonal* on $(0, 1)$ with respect to the weight function $t \mapsto w(t) = t(1 - t)$ (see Mastroianni and Milovanović [18, p. 330])

Using the moments

$$m_k = \int_0^1 w(t)t^k \, \mathrm{d}t = \frac{1}{(k+2)(k+3)}, \quad k = 0, 1, \ldots,$$

we can obtain the coefficients α_k and β_k in the three-term recurrence relation for orthogonal polynomials $\pi_k(t)$,

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 1, 2, \ldots,$$

where $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$.

In this case, using MATHEMATICA package `OrthogonalPolynomials` (see [10], [26]) we obtain (by routine `aChebyshevAlgorithma` in symbolic mode)

$$\alpha_k = \frac{1}{2} \ (k \ge 0); \quad \beta_0 = \frac{1}{6}, \ \beta_k = \frac{k(k+2)}{4(2k+1)(2k+3)} \ (k \ge 1).$$

The quadrature nodes are: $\tau_0 = 0$, $\tau_{n+1} = 1$, as well as the zeros of $\pi_n(t)$: $\tau_1, \ldots, \tau_n$. These *internal nodes* $\tau_1, \ldots, \tau_n$ are also eigenvalues of the

Jacobi matrix (cf. [18, p. 326])

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

and they can be determined very easy (with arbitrary precision) using the mentioned package.

For example, in the case $n = 5$ we have the nodes

$$\{\tau_\nu\}_{\nu=0}^6 = \left\{0, \frac{1}{66}\left(33 - \sqrt{495 + 66\sqrt{15}}\right), \frac{1}{66}\left(33 - \sqrt{495 - 66\sqrt{15}}\right), \frac{1}{2},\right.$$

$$\left.\frac{1}{66}\left(33 + \sqrt{495 - 66\sqrt{15}}\right), \frac{1}{66}\left(33 + \sqrt{495 + 66\sqrt{15}}\right), 1\right\},$$

i.e., $\{\tau_\nu\}_{\nu=0}^6 = \{0, 0.084888052, 0.26557560, 0.5, 0.73442440, 0.91511195, 1\}$.

The weight coefficients (with only ≤ 4 decimal digits to save space) are given by the following matrix:

$$\begin{bmatrix} 0.03285 & 0.0593 & -0.0108 & 0.0056 & -0.0035 & 0.0022 & -0.00084 \\ 0.01800 & 0.1577 & 0.1024 & -0.0185 & 0.0096 & -0.0057 & 0.00210 \\ 0.02753 & 0.1278 & 0.2375 & 0.1219 & -0.0216 & 0.0106 & -0.00372 \\ 0.02171 & 0.1441 & 0.2063 & 0.2623 & 0.1135 & -0.0193 & 0.00581 \\ 0.02465 & 0.1362 & 0.2194 & 0.2382 & 0.2266 & 0.0791 & -0.00904 \\ 0.02381 & 0.1384 & 0.2159 & 0.2438 & 0.2159 & 0.1384 & 0.02381 \end{bmatrix}.$$

In this case the values of integrals $\int_0^{\tau_j} |\omega(t)| \, dt$, for $j = 1, 2, \ldots, n, n+1$ $(n = 5)$ are

6.102×10^{-6}, 2.447×10^{-5}, 5.154×10^{-5}, 7.860×10^{-5}, 9.697×10^{-5}, 1.031×10^{-4},

respectively.

2.2.3. Use of Chebyshev polynomials of the first kind. Chebyshev polynomials of the first kind defined by $T_n(x) = \cos(n \arccos x)$, $n = 0, 1, \ldots$, are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$.

We use here their transformed version $T_n(2t - 1)$ on $[0, 1]$. The graphics for $n = 5, 6, 7$ are displayed in Figure 1.

As internal nodes we can use the zeros of $T_n(2t - 1)$ (cf. [18, p. 12])

$$\tau_\nu = \frac{1}{2}\left(1 - \cos\frac{(2\nu - 1)\pi}{2n}\right) = \sin^2\frac{(2\nu - 1)\pi}{4n}, \quad \nu = 1, \ldots, n,$$

adding then two bounds $\tau_0 = 0$ and $\tau_{n+1} = 1$. The case $n = 5$ is shown in Figure 2.

Similarly, we can use extremal points of $T_n(2t-1)$ (cf. [18, p. 12])

$$\tau_\nu = \frac{1}{2}\left(1 - \cos\frac{\nu\pi}{n+1}\right) = \sin^2\frac{\nu\pi}{2(n+1)}, \quad \nu = 0,1,\ldots,n,n+1.$$

The case $n=5$ is presented in Figure 3.

Remark 2.3. Standard Newton-Cotes formulas with zeros of the Chebyshev polynomials of the first and second kind are known as *Fejér's rules* (cf. Dahlquist and Björk [11, pp. 538–539]).

2.2.4. Nodes of the Clenshaw-Curtis formula. This is an interesting choice of the nodes induced by the Clenshaw-Curtis formula [8] (cf. also Trefethen [33]). The nodes are extremal points of $T_{n+1}(2t-1)$, which are *orthogonal* on $(0,1)$ with respect to the weight function $t \mapsto w(t) = 1/\sqrt{t(1-t)}$.

Since the extremal points of $T_{n+1}(t)$ are given by $-\cos(\nu\pi/(n+1))$, $\nu = 0,1,\ldots,n,n+1$, we have

$$\tau_\nu = \sin^2\frac{\nu\pi}{2(n+1)}, \quad \nu = 0,1,\ldots,n,n+1. \tag{2.13}$$

Note that $\tau_0 = 0$ and $\tau_{n+1} = 1$.

In the case $n=5$ the nodes are

$$\left\{\tau_\nu\right\}_{\nu=0}^6 = \left\{0, \frac{1}{4}\left(2-\sqrt{3}\right), \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}\left(2+\sqrt{3}\right), 1\right\},$$

i.e., $\left\{\tau_\nu\right\}_{\nu=0}^6 = \left\{0, 0.06698729810778, 0.25, 0.5, 0.75, 0.9330127018922, 1\right\}.$
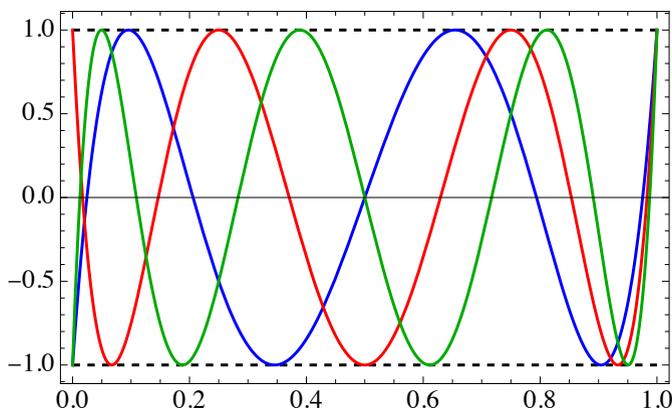
FIGURE 1. Chebyshev polynomials $T_n(2t-1)$ transformed to $[0,1]$ for $n=5$ (blue), $n=6$ (red), and $n=7$ (green)
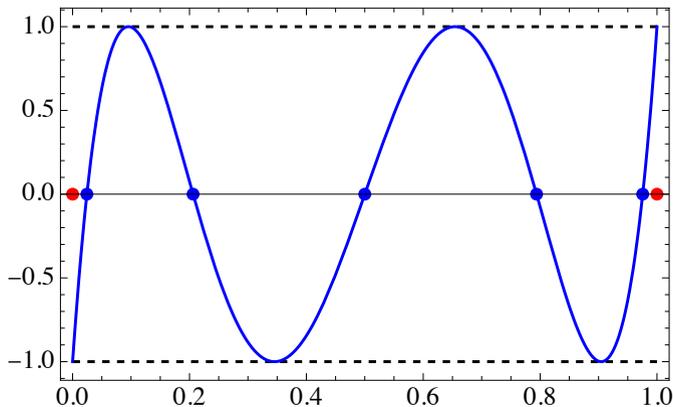
FIGURE 2. Zeros of $T_n(2t - 1)$ for $n = 5$ as internal nodes and additional zeros at 0 and 1
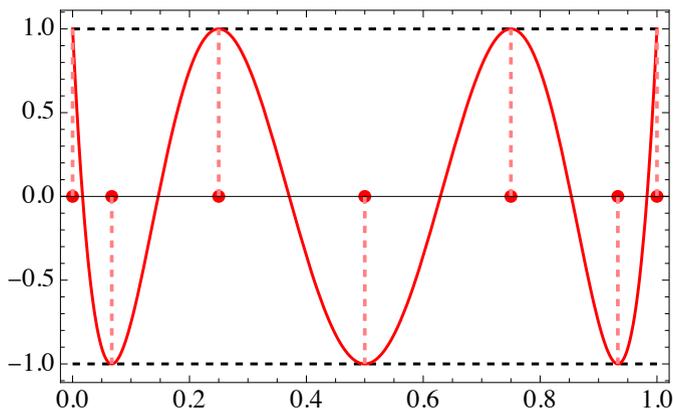
FIGURE 3. Extremal points of $T_{n+1}(2t - 1)$ for $n = 5$

The weight coefficients (with only ≤ 4 decimal digits to save space) are given by the following matrix:

$$
\begin{bmatrix}
0.02722 & 0.0433 & -0.0050 & 0.0023 & -0.0015 & 0.0012 & -0.00056 \\
0.00357 & 0.1546 & 0.1040 & -0.0183 & 0.0103 & -0.0078 & 0.00357 \\
0.02103 & 0.1116 & 0.2532 & 0.1302 & -0.0246 & 0.0154 & -0.00675 \\
0.01071 & 0.1348 & 0.2183 & 0.2786 & 0.1246 & -0.0276 & 0.01071 \\
0.01485 & 0.1258 & 0.2301 & 0.2580 & 0.2336 & 0.0837 & -0.01293 \\
0.01429 & 0.1270 & 0.2286 & 0.2603 & 0.2286 & 0.1270 & 0.01429
\end{bmatrix}
.
$$

In this case the values of integrals $\int_0^{\tau_j} |\omega(t)| \, dt$, for $j = 1, 2, \ldots, n, n+1$ ($n = 5$) are

$$\frac{5}{1572864}, \ \frac{37}{1572864}, \ \frac{1}{16384}, \ \frac{155}{1572864}, \ \frac{187}{1572864}, \ \frac{1}{8192},$$

respectively.

These quadrature will be considered in detail somewhere.

In this subsection we present numerical results for iterations with the composite trapezoidal formula (2.2) like in [23], as well as for the method based on approximation of integral equation (2.1) at the selected nodes $a = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = b$, using the quadrature formulas (2.7) and (2.8), especially by the Clenshaw-Curtis nodes.

Example. In this example we consider the integral equation [23]

$$u(t) = g(t) + \frac{1}{12} \int_0^t \sin x \, u(x)^2 \, dx + \frac{1}{36} \int_0^{\pi/2} \cos t (1 + \cos^2 t) u(x) \, dx \quad (2.14)$$

on $t \in [0, \pi/2]$, where

$$g(t) = \frac{1}{36}(35 \cos t - 1) \quad \text{and} \quad u^*(t) = \cos t$$

is the exact solution.

The equation (2.14) is of the form (2.1), with

$$K_1(t, x, u) = \frac{1}{12} \sin x \, u^2, \quad K_2(t, x, u) = \frac{1}{36} \cos t (1 + \cos^2 t) u, \quad t, x \in [0, \pi/2],$$

where $x \mapsto u(x)$.

(1) First we use the composite trapezoidal rule (2.2), with the nodes $\tau_\nu = \dfrac{\nu \pi}{2(n+1)}$, $\nu = 0, 1, \ldots, n, n+1$, $h = \dfrac{\pi}{2(n+1)}$, so that we have at $t = \tau_0 = a$

$$\tilde{u}_{k+1}(\tau_0) = g(\tau_0) + h \sum_{\nu=0}^{n+1} {}'' K_2(\tau_0, \tau_\nu, \tilde{u}_k(\tau_\nu)) \quad (2.15)$$

and at $t = \tau_j$, $j = 1, \ldots, n, n+1$, by

$$\tilde{u}_{k+1}(\tau_j) = g(\tau_j) + h \sum_{\nu=0}^{n+1} {}'' K_2(\tau_j, \tau_\nu, \tilde{u}_k(\tau_\nu)) + h \sum_{\nu=0}^{j} {}'' K_1(\tau_j, \tau_\nu, \tilde{u}_k(\tau_\nu)),$$

$$(2.16)$$

where the double prime on the sum means that the first and last terms should be halved.

We present absolute errors in iterations $|\tilde{u}_k(\tau_\nu) - u^*(\tau_\nu)|$, finding $\tilde{u}_k(\tau_\nu)$, $\nu = 0, 1, \ldots, n+1$ in iterations $k = 1, 2, \ldots$, starting by $u_0(\tau_\nu) = g(\tau_\nu)$, $\nu = 0, 1, \ldots, n+1$. We give graphics $t \mapsto E_{n+2}(t)$ joined the obtained $n + 2$ points in each of iterations into a line (as a first order spline interpolation). Also, we calculate

$$\|\tilde{u}_k(\tau_\nu) - u^*(\tau_\nu)\|_\infty = \max_{0 \le \nu \le n+1} |\tilde{u}_k(\tau_\nu) - u^*(\tau_\nu)|, \quad k = 1, 2, \ldots .$$
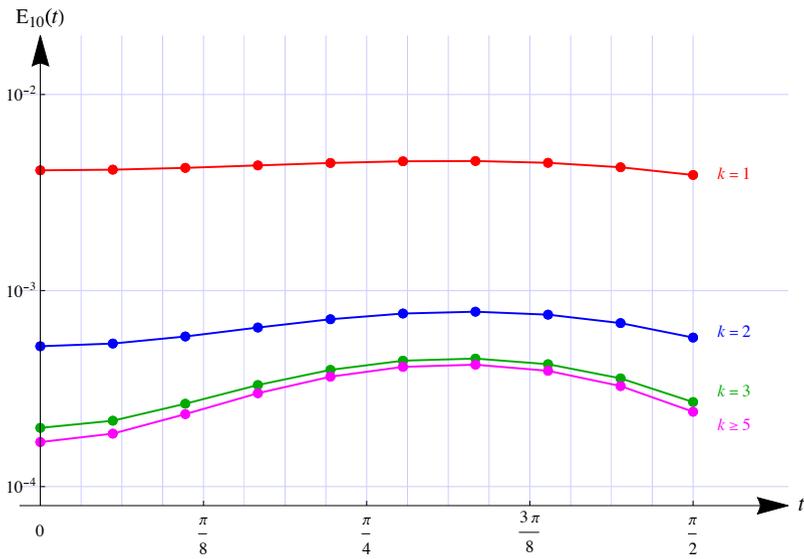
FIGURE 4. Errors at $n + 2 = 10$ equidistant nodes for the trapezoidal rule and $k = 1, 2, 3$ and $k \geq 5$ iterations
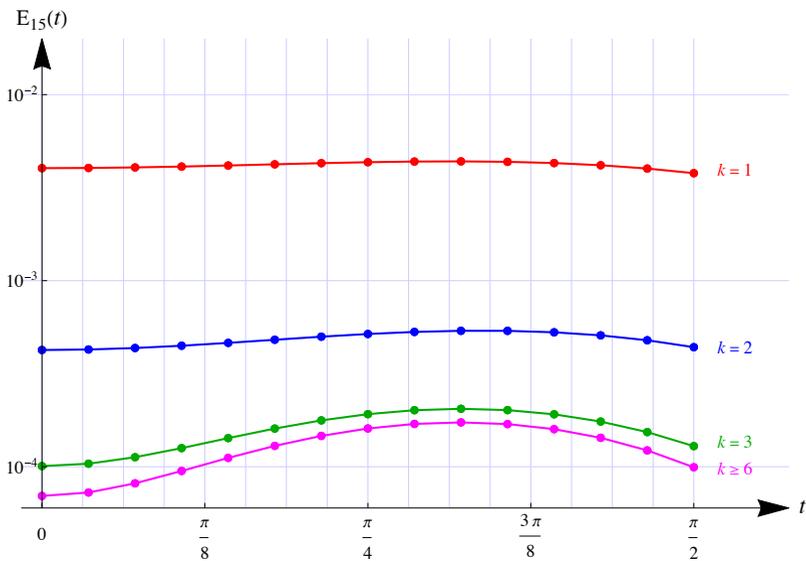
FIGURE 5. Errors at $n + 2 = 15$ equidistant nodes for the trapezoidal rule and $k = 1, 2, 3$ and $k \geq 6$ iterations
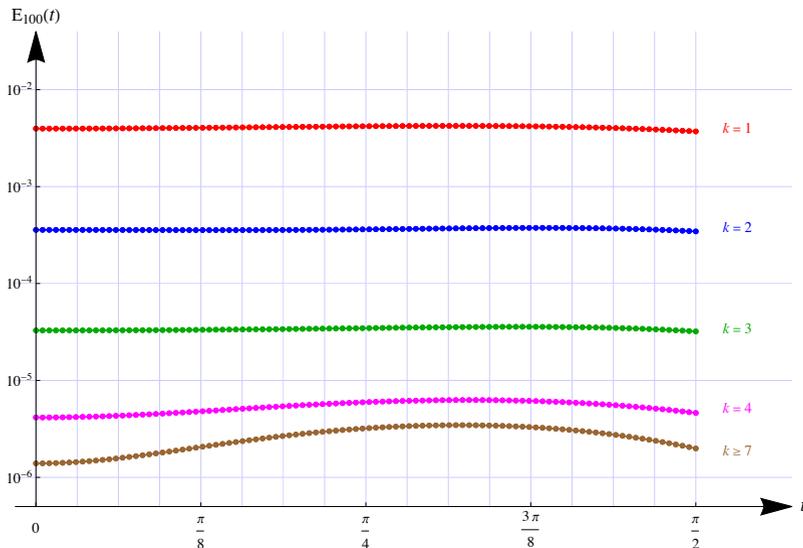
FIGURE 6. Errors at $n + 2 = 100$ equidistant nodes for the trapezoidal rule and $k = 1, 2, 3, 4$ and $k \geq 7$ iterations

In this case, the uniform (maximum) norm of the function $t \mapsto E_{n+2}(t)$ in k-th iteration is exactly

$$\|E_{n+2}\|_\infty := \|E_{n+2}^{[k]}\|_\infty = \max_{0 \leq t \leq \pi/2} |E_{n+2}^{[k]}(t)| = \|\tilde{u}_k(\tau_\nu) - u^*(\tau_\nu)\|_\infty.$$

Graphics of the functions $t \mapsto E_{n+2}(t)$ on $[0, \pi/2]$, when $n + 2 = 10$, 15, and 100 are presented in Figures 4, 5, and 6, respectively, after k iterations.

We mention again that the integral equation (2.14) is approximated by the discrete model, given by

$$\tilde{u}(\tau_j) = g(\tau_j) + h \sum_{\nu=0}^{n+1} {}'' K_2(\tau_j, \tau_\nu, \tilde{u}(\tau_\nu)) + h \sum_{\nu=0}^{j} {}'' K_1(\tau_j, \tau_\nu, \tilde{u}(\tau_\nu)),$$

for $j = 0, 1, \ldots, n+1$, where for $j = 0$ the last sum vanishes.

Using the iterative process (2.15)–(2.16), the iteration $\widetilde{\mathbf{u}}_k$, for sufficiently large k, approximates the exact solution of this discrete model $\widetilde{\mathbf{u}}^*$ up to machine precision (see the first term on the right hand side in the inequality (2.3)). However, the second term in (2.3), $M_2/(1 - \gamma) = O(h^2)$ (see (2.4)) depends on the quadrature rule and it determines the main part of the error $\|\widetilde{\mathbf{u}}_k - \mathbf{u}^*\|$.

This shows that we have a limitation in obtaining a satisfactory approximation of the exact solution u^* of the integral equation (2.14), depending of the quadrature rule. For example, with the rule with $n + 2 = 10$ nodes (see Figure 4) the minimal absolute error is achieved with $k = 5$

iterations $\|E_{10}^{[k]}\|_\infty = 4.18 \times 10^{-4}$, so that the further iterations do not reduce this error. For the rule with $n + 2 = 15$ nodes, this minimal error is 1.73×10^{-4} for $k \geq 6$. Furthermore, the 100-point rule gives the minimal error $\|E_{100}^{[k]}\|_\infty = 3.46 \times 10^{-6}$ for $k \geq 7$.

(2) Now we consider interpolatory quadrature process (2.11), with the Clenshaw-Curtis nodes τ_ν and Cotes numbers $A_\nu^{(j)}$, $\nu = 0, 1, \ldots, n, n+1$, $j = 0, 1, \ldots, n, n+1$, which are given by (2.13) and (2.10) (transformed to $[0, \pi/2]$ or $[a, b]$, in general), respectively. According to (2.11) these interpolatory formulas are exact for all polynomials of degree at most $n + 1$.

Then, the corresponding discrete approximation of the integral equation (2.14) is given by

$$\tilde{u}_k(\tau_j) = g(\tau_j) + \sum_{\nu=0}^{n+1} A_\nu^{(j)} K_1(\tau_j, \tau_\nu, \tilde{u}_k(\tau_\nu)) + \sum_{\nu=0}^{n+1} A_\nu^{(n+1)} K_2(\tau_j, \tau_\nu, \tilde{u}_k(\tau_\nu)),$$

for $j = 0, 1, \ldots, n + 1$, where for $j = 0$ the first sum on the right hand side vanishes, because $A_\nu^{(0)} = 0$ for each $\nu = 0, 1, \ldots, n+1$. However, for $j = n+1$ these Cotes numbers $A_\nu^{(n+1)} = A_\nu$ are just coefficients of the Clenshaw-Curtis formula [8] (cf. also Trefethen [33]).
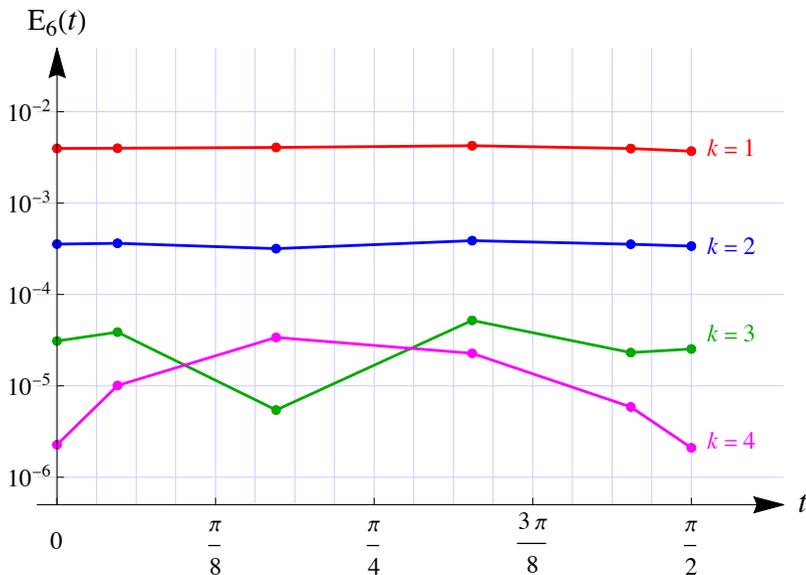
FIGURE 7. Errors at the Clenshaw-Curtis $n + 2 = 6$ nodes for the rules and $k = 1, 2, \ldots, 4$ iterations

Using iterative process (2.5) at $t = \tau_0 = 0$ and (2.6) at $t = \tau_j$, $j = 1, \ldots, n, n + 1$, we obtain the iterations $\tilde{u}_k(\tau_j)$, $j = 0, 1, \ldots, n + 1$, as in the previous case with the composite trapezoidal rule.
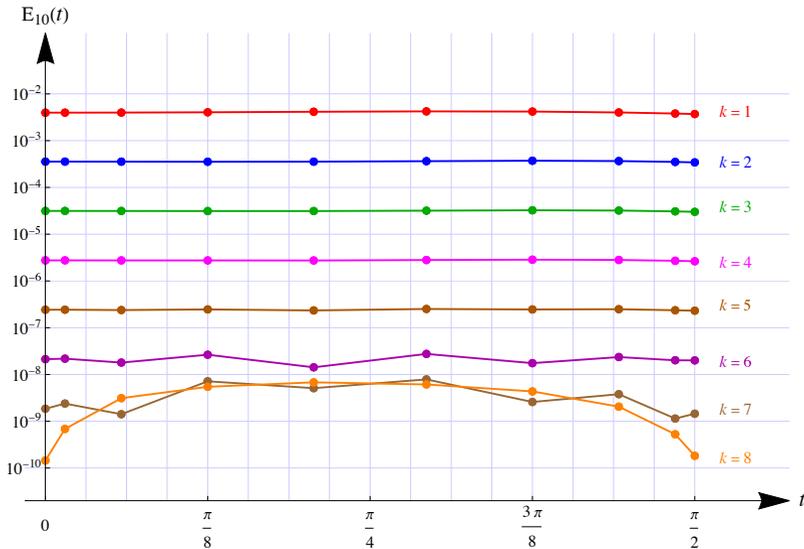
FIGURE 8. Errors at the Clenshaw-Curtis $n + 2 = 10$ nodes for the rules and $k = 1, 2, \ldots, 8$ iterations
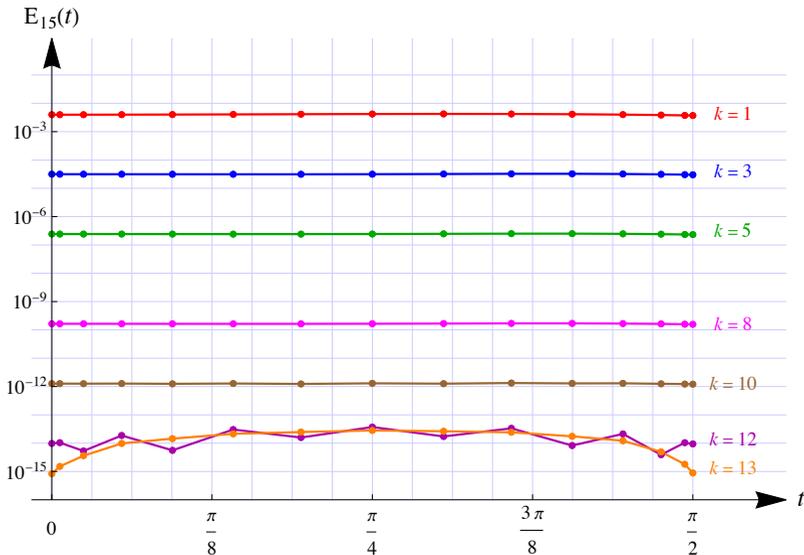
FIGURE 9. Errors at the Clenshaw-Curtis $n + 2 = 15$ nodes for the rules and $k = 1, 3, 5, 8, 10, 12, 13$ iterations

The corresponding graphics for the rules with $n + 2 = 6$, 10 and 15 Clenshaw-Curtis nodes are presented in Figures 7, 8 and 9, respectively.

We note that in the case $n + 2 = 6$, the maximal error after 2, 3, and 4 iterations are 3.88×10^{-4}, 5.20×10^{-5}, and 3.39×10^{-5}, respectively.

In the case, with $n + 2 = 10$ Clenshaw-Curtis nodes, the maximal error after 7 iterations is 7.85×10^{-9}, and further increasing the number of iterations does not contribute to further reducing the norm $\|E_{10}^{[k]}\|_\infty$ ($k \geq 7$).

In the case ($n + 2 = 15$) after 10 iterations, the maximal error is 1.33×10^{-12}, while for 12 iterations it is 3.71×10^{-14}. This is also the maximum number of iterations for this kind of quadrature rules with 15 nodes (see Figure 9).

Remark 2.4. The Newton-Kantorovich method for solving a system of 2×2 nonlinear Volterra integral equations, where the unknown function is in logarithmic form, was considered in [13]. We also mention a paper by Ezquerro et al. [12], where the authors use high order iterative methods for solving nonlinear integral equations of Fredholm type.

2.3. Mixed Volterra-Fredholm Integral Equations

Consider integral equations of the form ([19])

$$u(t, x) = \int_0^t \int_a^b K\big(t, x, \tau, y, u(\tau, y)\big) \, \mathrm{d}y \, \mathrm{d}\tau + g(t, x), \qquad (2.17)$$

$(t, x) \in D = [0, T] \times [a, b]$, where $K \in C\left(D^2 \times \mathbb{R}\right)$ and $g \in C(D)$. Such integrals arise in integral reformulations of the heat equation with Dirichlet, Neumann, or mixed boundary conditions.

Let $X = C(D)$ be equipped with the (uniform) Chebyshev norm $\|u\| = \max\limits_{(t,x) \in D} |u(t, x)|$, consider a closed ball $B_\varrho := \{u \in C(D) : \|u - g\| \leq \varrho\}$, $\varrho > 0$, and define the integral operator $F : X \to X$ by

$$Fu(t, x) := \int_0^t \int_a^b K\big(t, x, \tau, y, u(\tau, y)\big) \, \mathrm{d}y \, \mathrm{d}\tau + g(t, x).$$

The method described in [19] uses Theorem 1.4 on B_ϱ with $\varepsilon_k = 1/(k+1)$. For the first part of the approximation (the iterative process), the following result holds.

Let $K \in C\left(D^2 \times \mathbb{R}\right)$, $g \in C(D)$ and

$$\varrho_1 = \min_{(t,x) \in D} g(t, x), \quad \varrho_2 = \max_{(t,x) \in D} g(t, x).$$

Assume that there exists a constant $L > 0$ such that

$$\big|K(t, x, \tau, y, u) - K(t, x, \tau, y, v)\big| \leq L \|u - v\|, \qquad (2.18)$$

for all $(t, x), (\tau, y) \in D$ and all $u, v \in [\varrho_1 - \varrho, \varrho_2 + \varrho]$. In addition, assume that

$$q := LT(b - a) < 1 \qquad (2.19)$$

and

$$M_K T(b - a) \leq \varrho, \qquad (2.20)$$

where $M_K := \max |K(t, x, \tau, y, u)|$ over all $(t, x), (\tau, y) \in D$ and all $u, v \in [\varrho_1 - \varrho, \varrho_2 + \varrho]$. Then the equation (2.17) has a unique solution $u^* \in B_\varrho$

and that solution can be found as the limit of the sequence of successive approximations

$$u_{k+1} = \left(1 - \frac{1}{k+1}\right) u_k + \frac{1}{k+1} F u_k, \quad k = 0, 1, \ldots, \tag{2.21}$$

starting with any initial point $u_0 \in B_\varrho$. In addition, the error estimate

$$\|u_k - u^*\| \le \frac{\mathrm{e}^{1-q}}{1-q} \, \mathrm{e}^{-(1-q)z_k} \, \|u_0 - u_1\|$$

holds for every $k \in \mathbb{N}$, where the sequence $\{z_k\}$ is given by

$$z_0 = 0, \quad z_k = \sum_{i=0}^{k-1} \frac{1}{i+1}, \quad k \ge 1. \tag{2.22}$$

It is noteworthy to mention the fact that the Lipschitz and contraction conditions (2.18)–(2.19) could restrict the area of applicability of this method if they would have to be satisfied on the entire space. That is why only this local existence and uniqueness result is used on B_ϱ, for some $\varrho > 0$.

For the second part of the method (the approximation of the iterates in (2.21), consider a numerical integration scheme

$$\int_a^b \int_c^d \varphi(x, w) \, \mathrm{d}w \, \mathrm{d}x = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{ij} \varphi(x_i, w_j) + R_\varphi, \tag{2.23}$$

with nodes $a = x_0 < x_1 < \ldots < x_{n_1} = b$, $c = w_0 < w_1 < \cdots < w_{n_2} = d$, coefficients $a_{ij} \in \mathbb{R}$, $i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$, such that there exists $M > 0$ with $|R_\varphi| \le M$, where $M \to 0$ as $n_1, n_2 \to \infty$.

For our approximations, let $0 = t_0 < t_1 < \cdots < t_{n_1} = T$ and $a = x_0 < x_1 < \cdots < x_{n_2} = b$ be partitions of $[0, T]$ and $[a, b]$, respectively, and let $u_0 = \tilde{u}_0 \equiv g$ be the initial approximation. We use the successive iterations (2.21) and the numerical integration formula (2.23) to approximate the values $u_k(t_l, x_\nu)$ by $\tilde{u}_k(t_l, x_\nu)$, for $l = \overline{0, n_1}$, $\nu = \overline{0, n_2}$ and $k = 0, 1, \ldots$, where

$$\tilde{u}_k(t_l, x_\nu) = \left(1 - \frac{1}{k}\right) \tilde{u}_{k-1}(t_l, x_\nu) \tag{2.24}$$

$$+ \frac{1}{k} \left(\sum_{i=0}^{l} \sum_{j=0}^{n_2} a_{ij} \nu\big(t_l, x_\nu, t_i, x_j, \tilde{u}_{k-1}(t_i, x_j)\big) + g(t_l, x_\nu) \right),$$

By an inductive argument, we get

$$\mathrm{err}(u_k, \tilde{u}_k) = \max_{(t_l, x_\nu) \in D} |u_k(t_l, x_\nu) - \tilde{u}_k(t_l, x_\nu)|$$

$$\le M\big(1 + \gamma + \cdots + \gamma^{n-1}\big),$$

where $\gamma := L \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} |a_{ij}|$.

Thus, under the conditions (2.18)–(2.20) and the extra assumption

$$\gamma = L \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} |a_{ij}| < 1,$$

the error estimate

$$\text{err}(u^*, \tilde{u}_k) \leq \frac{e^{1-q}}{1-q} e^{-(1-q)z_k} \|u_0 - Fu_0\| + \frac{M}{1-\gamma} \tag{2.25}$$

holds for every $k \in \mathbb{N}$, where u^* is the true solution of equation (2.17) and the sequence $\{z_k\}_{k \in \mathbb{N}}$ is defined in (2.22).

Let us use the two-dimensional trapezoidal rule (as in [19]):

$$\int_a^b \int_c^d \varphi(\tau, y) \, dy \, d\tau = \frac{(b-a)(d-c)}{4n_1 n_2} \Bigg[\varphi(a,c) + \varphi(b,c) + \varphi(a,d)$$

$$+ \varphi(b,d) + 2 \sum_{i=1}^{n_1 - 1} \big(\varphi(\tau_i, c) + \varphi(\tau_i, d) \big)$$

$$+ 2 \sum_{j=1}^{n_2 - 1} \big(\varphi(a, y_j) + \varphi(b, y_j) \big)$$

$$+ 4 \sum_{i=1}^{n_1 - 1} \sum_{j=1}^{n_2 - 1} \varphi(\tau_i, y_j) \Bigg) \Bigg] + R_\varphi$$

with nodes

$$x_i = a + \frac{b-a}{n_1} i, \quad w_j = c + \frac{d-c}{n_2} j, \quad i = \overline{0, n_1}, \quad j = \overline{0, n_2}.$$

The remainder is given by

$$R_\varphi = -\Bigg[\frac{(b-a)^3(d-c)}{12n_1^2 n_2} \varphi^{(2,0)}(\xi, \eta_1) + \frac{(b-a)(d-c)^3}{12n_1 n_2^2} \varphi^{(0,2)}(\xi_1, \eta)$$

$$+ \frac{(b-a)^3(d-c)^3}{144n_1^2 n_2^2} \varphi^{(2,2)}(\xi, \eta) \Bigg], \quad \xi, \xi_1 \in (a,b), \ \eta, \eta_1 \in (c,d),$$

where we used the notation $\varphi^{(\alpha,\beta)}(t,x) = \dfrac{\partial^{\alpha+\beta}\varphi}{\partial t^\alpha \partial x^\beta}(t,x).$

For our integrals, we get

$$\int_0^{t_l} \int_a^b K\big(t_l, x_\nu, \tau, y, u_k(\tau, y)\big) \, \mathrm{d}y \, \mathrm{d}\tau = \frac{t_l(b-a)}{4ln_2} \bigg[K_{l,\nu,0,0} + K_{l,\nu,l,0} + K_{l,\nu,0,n_2}$$

$$+ K_{l,\nu,l,n_2} + 2 \sum_{i=0}^{l-1} \big(K_{l,\nu,i,0} + K_{l,\nu,i,n_2} \big)$$

$$+ 2 \sum_{j=0}^{n_2-1} \big(K_{l,\nu,0,j} + K_{l,\nu,l,j} \big)$$

$$+ 4 \sum_{i=0}^{l-1} \sum_{j=0}^{n_2-1} K_{l,\nu,i,j} \bigg] + R_K,$$

with nodes

$$t_l = \frac{T}{n_1} l, \quad x_\nu = a + \frac{b-a}{n_2} \nu, \quad l = \overline{0, n_1}, \quad \nu = \overline{0, n_2}$$

and the simplifying notation $K_{l,\nu,i,j} = K\big(t_l, x_\nu, t_i, x_j, u_k(t_i, x_j)\big)$. Since $t_l/l = T/n_1$, in this case, $\gamma \leq LT(b-a) = q$, which is already assumed to be strictly less than 1 by (2.19).

Now, we focus to the remainder. It is clear that if $K^{(2,0)}\big(\tau, y, u_k(\tau, y)\big)$, $K^{(0,2)}\big(\tau, y, u_k(\tau, y)\big)$ and $K^{(2,2)}\big(\tau, y, u_k(\tau, y)\big)$ are bounded, then the remainder R_K is of the form $\mathcal{O}\big(1/n_1^2\big) + \mathcal{O}\big(1/n_2^2\big)$. So, if K and g are C^4 functions with bounded fourth order partial derivatives, then there exists $M > 0$, independent of k, such that $|R_K| \leq M$, with $M \to 0$ as $n_1, n_2 \to \infty$. Then, we have the error estimate (2.25).

Example. We now illustrate the applicability of the above method on a numerical example. Consider the nonlinear mixed Volterra-Fredholm integral equation

$$u(t,x) = 2 \int_0^t \int_0^1 x^2 y \tau \mathrm{e}^{-\tau} \mathrm{e}^{u(\tau,y)} \, \mathrm{d}y \, \mathrm{d}\tau + x^2(1 - \mathrm{e}^{-t}), \qquad (2.26)$$

for $t \in [0, 1/4]$, whose exact solution is $u^*(t,x) = tx^2$. The theoretical assumptions are satisfied for $\varrho = 1$.

We use the trapezoidal rule with $n_1 = n_2 = 18$ and nodes

$$t_i = \frac{i}{4n_1}, \quad i = \overline{0, n_1}, \quad x_j = \frac{j}{n_2}, \quad j = \overline{0, n_2}.$$

The numerical implementation of (2.24) is done in MATLAB, in double precision arithmetic. The errors $\mathrm{err}(u^*, \tilde{u}_k)$ are given in the table below, with initial approximation $u_0(t,x) = g(t,x) = x^2(1 - \mathrm{e}^{-t})$.

2.4. Volterra Integral Equations with Delayed Argument

Next, let us consider Volterra integral equations of the form ([22])

$$u(t) = \begin{cases} \varphi(0) + g(t) + \int_0^t K\big(t, x, u(x), u(x - \delta)\big) \mathrm{d}x, & t \in [0, b], \\ \\ \varphi(t), & t \in [-\delta, 0], \end{cases} \qquad (2.27)$$

k	$\mathrm{err}(u^*, \tilde{u}_k)$
1	$2.034743e - 01$
5	$9.354733e - 04$
10	$3.077314e - 05$

TABLE 1. Errors for Eq. (2.26), $n_1 = n_2 = 18$

where $\delta > 0$, $K \in C\left([0, b] \times [0, b] \times \mathbb{R}^2\right)$, $\varphi \in C[-\delta, 0]$, $g \in C[0, b]$ and $g(0) = 0$.

These delayed argument equations are used to model dynamical systems, such as population growth or decay, or the evolution of an epidemic.

Consider the space $X = C[-\delta, b]$ endowed with the Bielecki norm

$$\|u\|_\tau := \max_{t \in [-\delta, b]} |u(t)| \, e^{-\tau t}, \quad u \in X,$$

for some suitable $\tau > 0$. Then $(X, \|\cdot\|_\tau)$ is a Banach space on which the theoretical results in Section 1 hold. Let us remark that, when employing such fixed point results, the use of the Bielecki norm has sometimes a major advantage over the usual max norm: the Lipschitz or contraction-type conditions that the operator has to satisfy can be fulfilled by a convenient choice of the parameter τ, *without* imposing extra restrictions on the kernel function.

We define the operator $T : X \to X$ by

$$Tu(t) = \begin{cases} \varphi(0) + g(t) + \int_0^t K\big(t, x, u(x), u(x - \delta)\big)\mathrm{d}x, & t \in [0, b], \\ \varphi(t), & t \in [-\delta, 0]. \end{cases} \quad (2.28)$$

Again, we use a local fixed point result. Let $B_\varrho \subset X$ be the closed ball $B_\varrho = \{u \in X \,:\, \|u - \widetilde{\varphi}\| \leq \varrho\}$, where

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in [-\delta, 0], \\ \varphi(0) + g(t), & t \in [0, b] \end{cases}$$

and $\|\cdot\|$ denotes the Chebyshev norm on X. Applying Theorem 1.4 with $\varepsilon_k = 1/(k + 1)$ to the operator T from (2.28), we have the following result.

Assume that there exist constants $L_1, L_2 > 0$ such that

$$|K(t, x, u_1, v_1) - K(t, x, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|,$$

for all $t, x \in [0, b]$ and all $u_1, u_2, v_1, v_2 \in [\varrho_1 - \varrho, \varrho_2 + \varrho]$, where

$$\varrho_1 := \min_{t \in [-\delta, b]} \widetilde{\varphi}(t), \quad \varrho_2 := \max_{t \in [-\delta, b]} \widetilde{\varphi}(t).$$

Further assume that

$$bM \leq \varrho,$$

where $M := \max |K(t, x, u, v)|$ over all $t, x \in [0, b]$ and all $u, v \in [\varrho_1 - \varrho, \varrho_2 + \varrho]$. Then the integral equation (2.27) has a unique solution $u^* \in B_\varrho$ and the

sequence defined by

$$u_{k+1} = \left(1 - \frac{1}{k+1}\right) u_k + \frac{1}{k+1} Tu_k, \quad k = 0, 1, \ldots, \qquad (2.29)$$

converges to the solution u^* for any initial point $u_0 \in B_\varrho$. Moreover, for every $k \in \mathbb{N}$, the following error estimate

$$\|u_k - u^*\|_\tau \leq \frac{\mathrm{e}^{1-q}}{1-q} \|u_0 - Tu_0\|_\tau \, \mathrm{e}^{-(1-q)z_k}, \qquad (2.30)$$

holds, where

$$z_0 = 0, \quad z_k = \sum_{i=0}^{k-1} \frac{1}{i+1} \ \ (k \geq 1) \quad \text{and} \quad q = \frac{L_1 + L_2}{\tau} < 1$$

(meaning that the constant τ is chosen so that this condition is satisfied).

To approximate numerically the integrals in (2.29), we have to carefully choose the quadrature nodes t_ν. Because of the delayed argument $x - \delta$, we have to make sure that at each iteration, the discrete values of the solution are available both at t_ν and at $t_\nu - \delta$. We will use the composite trapezoidal rule on $[-\delta, b]$ with the $n+1$ nodes $t_\nu = -\delta + \frac{b+\delta}{n}\nu$, $\nu = \overline{0, n}$, where n is taken so that one of the nodes is 0, say $t_{\nu_0} = 0$ for some $\nu_0 \in \{0, 1, \ldots, n\}$, i.e., $n = \frac{b+\delta}{\delta}\nu_0$. Then for $\nu = \overline{0, \nu_0}$,

$$u_{k+1}(t_\nu) = \varphi(t_\nu), \quad k = 0, 1, \ldots,$$

and for $l = \overline{1, n - \nu_0}$ (i.e., $\nu = \nu_0 + l \in \{\nu_0 + 1, \ldots, n\}$), we approximate

$$u_{k+1}(t_{\nu_0+l}) = \left(1 - \frac{1}{k+1}\right) u_k(t_{\nu_0+l}) + \frac{1}{k+1}\left(\varphi(0) + g(t_{\nu_0+l})\right.$$
$$\left. + \int_0^{t_{\nu_0+l}} K\left(t_{\nu_0+l}, x, u_k(x), u_k(x-\delta)\right) \mathrm{d}x\right),$$

using the trapezoidal rule with the initial approximation $u_0(t) = \tilde\varphi(t)$. Since for each $l = \overline{1, n - \nu_0}$ and each $j = \overline{0, l}$, $t_{\nu_0+j} - \delta = t_{\nu_0+j} - t_0 = t_j$, we have

$$u_{k+1}(t_{\nu_0+l}) = \left(1 - \frac{1}{k+1}\right) u_k(t_{\nu_0+l})$$
$$+ \frac{1}{k+1}\left[\varphi(0) + g(t_{\nu_0+l}) + \frac{b+\delta}{2m}\left(K\left(t_{\nu_0+l}, 0, u_k(0), u_k(-\delta)\right)\right.\right.$$
$$+ 2\sum_{j=1}^{l-1} K\left(t_{\nu_0+l}, t_{\nu_0+j}, u_k(t_{\nu_0+j}), u_k(t_j)\right)$$
$$\left.\left. + K\left(t_{\nu_0+l}, t_{\nu_0+l}, u_k(t_{\nu_0+l}), u_k(t_l)\right)\right) + R_{k+1,\nu_0+l}\right], \quad k \in \mathbb{N}.$$

Then we approximate $u_{k+1}(t_{\nu_0+l})$ by $\tilde{u}_{k+1}(t_{\nu_0+l})$, where

$$\tilde{u}_0(t_{\nu_0+l}) = u_0(t_{\nu_0+l}),$$

$$\tilde{u}_{k+1}(t_{\nu_0+l}) = \left(1 - \frac{1}{k+1}\right)\tilde{u}_k(t_{\nu_0+l})$$

$$+ \frac{1}{k+1}\Big[\varphi(0) + g(t_{\nu_0+l}) + \frac{b+\delta}{2n}\Big(K\left(t_{\nu_0+l}, 0, \tilde{u}_k(0), \tilde{u}_k(-\delta)\right)$$

$$+ 2\sum_{j=1}^{l-1} K\left(t_{\nu_0+l}, t_{\nu_0+j}, \tilde{u}_k(t_{\nu_0+j}), \tilde{u}_k(t_j)\right)$$

$$+ K\left(t_{\nu_0+l}, t_{\nu_0+l}, \tilde{u}_k(t_{\nu_0+l}), \tilde{u}_k(t_l)\right)\Big)\Big], \quad k \in \mathbb{N}.$$

In order to bound the errors in the approximations above, we need to make some considerations on the smoothness of the iterations u_k. To this end, let $X_0 = \left\{u \in C[-\delta, b] \;\middle|\; u\big|_{[-\delta, 0]} = 0\right\}$. Then:

(1) $T(X_0) \subseteq X_0$;
(2) if $K, g \in C^1$ and $K(0, 0, 0, 0) = g'(0) = 0$, then

$$T\left(X_0 \cap C^1[-\delta, b]\right) \subseteq X_0 \cap C^1[-\delta, b];$$

(3) if $K, g \in C^2$ and

$$K(0, 0, 0, 0) = \frac{\partial K}{\partial t}(0, 0, 0, 0) = \frac{\partial K}{\partial x}(0, 0, 0, 0) = g'(0) = g''(0) = 0,$$

then $T\left(X_0 \cap C^2[-\delta, b]\right) \subseteq X_0 \cap C^2[-\delta, b]$.

It follows that for all $\nu = 0, 1, \ldots, n$, we have

$$\left|u_k(t_\nu) - \tilde{u}_k(t_\nu)\right| \le \frac{b^3}{12n^2}M, \tag{2.31}$$

where $M > 0$ depends on the bounds of the derivatives of the functions K, g and φ, but *not* on k.

Under the assumptions above, we can choose $u_0 \in X_0 \cap C^2[-\delta, b] \cap B_\varrho$, such that the sequence defined in (2.29) has the following properties:

(a) $u_k \in X_0 \cap C^2[-\delta, b] \cap B_\varrho, \;\; \|u_k - g\| \le \varrho$;

(b) $\{u_k'\}$ and $\{u_k''\}$ are bounded sequences.

Combining the errors in (2.30) and (2.31), we get the composite error

$$
\begin{aligned}
\left|u^*(t_\nu) - \tilde{u}_k(t_\nu)\right| &\le \left|u^*(t_\nu) - u_k(t_\nu)\right| + \left|u_k(t_\nu) - \tilde{u}_k(t_\nu)\right| \\
&= \left|u^*(t_\nu) - u_k(t_\nu)\right|e^{-\tau t}e^{\tau t} + \left|u_k(t_\nu) - \tilde{u}_n(t_\nu)\right| \\
&\le \|u_k - u^*\|_\tau e^{\tau b} + \left|u_k(t_\nu) - \tilde{u}_k(t_\nu)\right| \\
&\le \frac{e^{\tau b + 1 - q}}{1 - q}\|u_0 - Tu_0\|_\tau \, e^{-(1-q)z_k} + \frac{b^3}{12n^2}M,
\end{aligned}
$$

at each node t_ν, $\nu = \overline{0, n}$.

Example. Consider the integral equation with delayed argument ([22])

$$u(t) = \begin{cases} g(t) + \dfrac{35}{34} \displaystyle\int_0^t t^2\Big(u(x) - 1\Big)\Big(u(x-1)+1\Big)\mathrm{d}x, & t \in [0,2], \\ 0, & t \in [-1,0], \end{cases} \quad (2.32)$$

where

$$g(t) = -\frac{35^2}{34}\left[35\Big(\frac{1}{7}t^9 - \frac{1}{2}t^8 - \frac{3}{5}t^7 - \frac{1}{4}t^6\Big) + t^5 - \frac{3}{2}t^4\right].$$
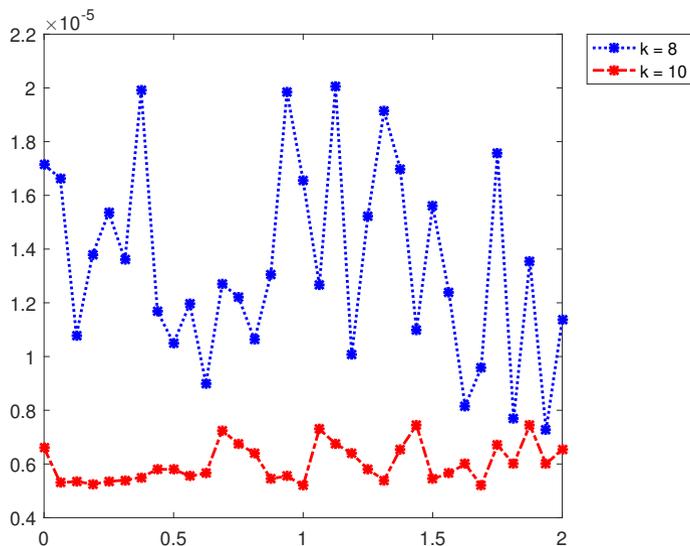
FIGURE 10. Errors at the non-negative nodes for $k = 8$ and $k = 10$ iterations

The exact solution of equation (2.32) is

$$u^*(t) = \begin{cases} 0, & t \in [-1,0], \\ 35t^3, & t \in [0,2]. \end{cases}$$

We take $n = 48$ and the nodes $t_\nu = -1 + \nu/16$, $\nu = \overline{0,48}$. Notice that $t_{17} = 0$. The initial approximations are $u_0(t_\nu) = 0$, for $\nu = \overline{0,17}$ and $u_0(t_\nu) = g(t_\nu)$, for $\nu = \overline{18,48}$.

The graph of the errors at the non-negative nodes is given in Figure 10 for $k = 8$ and 10 iterations (since, for the negative nodes, the approximation is exact).

2.5. Functional Volterra Integral Equations

Functional integral equations have many application in radiative transfer, control theory, mechanical engineering, etc.

We consider a Volterra functional integral equation of the type ([21])

$$u(t) = \lambda \int_{-t}^{t} K\big(t, x, u(x)\big)\mathrm{d}x + g(t), \quad t \in [-T, T], \tag{2.33}$$

for $T > 0$ and $\lambda \in \mathbb{R}$, with $K \in C\big([-T,T]^2 \times \mathbb{R}\big)$ and $g \in C[-T,T]$. Let $X = C[-T,T]$ be endowed with the uniform norm $\|u\| = \max\limits_{t \in [-T,T]} |x(t)|$ and consider the closed ball $B_\varrho = \{u \in X \mid \|u - g\| \leq \varrho\}, \varrho > 0$. We define the integral operator $F : X \to X$ associated with equation (2.33) by

$$Fu(t) := \lambda \int_{-t}^{t} K\big(t, x, u(x)\big)\mathrm{d}x + g(t).$$

Using the contraction principle (Theorem 1.3) on the ball B_ϱ, we have the following result.

We assume that there exists a function $L : [-T, T] \to \mathbb{R}_+^*$ such that

$$\big|K(t, x, u) - K(t, x, v)\big| \leq L(x)|u - v|,$$

for all $t, x \in [-T, T]$ and all $u, v \in [\varrho_1 - \varrho, \varrho_2 + \varrho]$, where $\varrho_1 := \min\limits_{t \in [-T,T]} g(t)$, $\varrho_2 := \max\limits_{t \in [-T,T]} g(t)$. Also, assume that

$$q := |\lambda| \int_{-T}^{T} L(x)\mathrm{d}x < 1$$

and that

$$2|\lambda|M_K T \leq \varrho,$$

where $M_K = \max |K(t, x, u)|$ over $t, x \in [-T, T]$ and $u, v \in [\varrho_1 - \varrho, \varrho_2 + \varrho]$. Then equation (2.33) has exactly one solution $u^* \in B_\varrho$, which is the limit of the sequence given by

$$u_{k+1} = Fu_k, \quad k = 0, 1, \ldots, \tag{2.34}$$

with any arbitrary initial point $u_0 \in B_\varrho$ and we have the error estimate

$$\|u_k - u^*\| \leq \frac{q^k}{1 - q}\|u_1 - u_0\|,$$

for every $k \in \mathbb{N}$.

To approximate numerically the integrals in (2.34), we consider a symmetric quadrature formula

$$\int_{-b}^{b} \varphi(x)\,\mathrm{d}x = \sum_{i=-j}^{j} a_i\varphi(x_i) + R_{\varphi,j},$$

for any $0 < b \le T$, with nodes $0 = x_0 < x_1 < \cdots < x_j = b$, $x_{-i} = -x_i$, $i = 0, 1, \ldots j$, coefficients $a_i \in \mathbb{R}$, $i = -j, \ldots, j$ and for which the remainder satisfies

$$R_{\varphi,j} \to 0 \quad \text{as} \quad j \to \infty.$$

Let $n \ge 1$ be fixed and let $0 = t_0 < t_1 < \cdots < t_n = T$, $t_{-i} = -t_i$. Then, for $\nu = -n, \ldots, n$, we have

$$u_0(t_\nu) = g(t_\nu),$$

$$u_{k+1}(t_\nu) = \lambda \int_{-t_\nu}^{t_\nu} K\big(t_\nu, x, u_k(x)\big) \mathrm{d}x + g(t_\nu), \quad k = 0, 1, \ldots. \qquad (2.35)$$

In addition, we assume that when the quadrature formula is applied on the interval $[-t_\nu, t_\nu]$, $\nu = 0, 1, \ldots, n$, the remainders $R_{K,\nu}$ satisfy

$$|R_{K,\nu}| \le M,$$

where M depends on the fixed number n and $M \to 0$ as $n \to \infty$.

Now we apply the quadrature scheme to our integrals above. To simplify the writing in (2.35), we make the following notations:

$$K_{\nu,i,k} := K\big(t_\nu, t_i, u_k(t_i)\big)$$
$$\widetilde{K}_{\nu,i,k} := K\big(t_\nu, t_i, \tilde{u}_{k,n}(t_i)\big), \quad \text{for} \quad \nu, i = -n, \ldots, n; \quad k = 1, 2, \ldots,$$

where

$$\tilde{u}_{1,n}(t_\nu) \;=\; \lambda \sum_{i=-\nu}^{\nu} a_i K\big(t_\nu, t_i, g(t_i)\big) + g(t_\nu)$$

$$\;=\; \lambda \sum_{i=-\nu}^{\nu} a_i K_{\nu,i,0} + g(t_\nu),$$

$$\tilde{u}_{k,n}(t_\nu) \;=\; \lambda \sum_{i=-\nu}^{\nu} a_i \widetilde{K}_{\nu,i,k-1} + g(t_\nu).$$

For a fixed n, we approximate $u_k(t_\nu)$ by $\tilde{u}_{k,n}(t_\nu)$, the following way:

$$u_1(t_\nu) \;=\; \lambda \int_{-t_\nu}^{t_\nu} K\big(t_\nu, x, g(x)\big) \mathrm{d}x + f(t_\nu)$$

$$\;=\; \lambda\left(\sum_{i=-\nu}^{\nu} a_i K\big(t_\nu, t_i, g(t_i)\big) + R_{K,\nu}\right) + g(t_\nu)$$

$$\;=\; \tilde{u}_{1,n}(t_\nu) + \widetilde{R}_{1,\nu}$$

Then, denoting by

$$\|u_k - \tilde{u}_{k,n}\| := \max_{t_\nu \in [-T,T]} |u_k(t_\nu) - \tilde{u}_{k,n}(t_\nu)|$$

and

$$\widetilde{R}_k := \max_{-n \le \nu \le n} |\widetilde{R}_{k,\nu}|,$$

we have
$$\|u_1 - \tilde{u}_{1,n}\| \leq \widetilde{R}_1 \leq |\lambda|M.$$
Proceeding further, in a similar fashion, we get

$$
\begin{aligned}
u_2(t_\nu) &= \lambda \int_{-t_\nu}^{t_\nu} K\big(t_\nu, x, u_1(x)\big)\mathrm{d}x + g(t_\nu) \\
&= \lambda \left(\sum_{i=-\nu}^{\nu} a_i K\big(t_\nu, t_i, u_1(t_i)\big) + R_{K,\nu} \right) + g(t_\nu) \\
&+ \lambda \left(\sum_{i=-\nu}^{\nu} a_i K\big(t_\nu, t_i, \tilde{u}_{1,n}(t_i) + \tilde{R}_{1,i}\big) + R_{K,\nu} \right) + g(t_\nu) \\
&= \tilde{u}_{2,n}(t_\nu) + \widetilde{R}_{2,\nu}.
\end{aligned}
$$

To estimate the error $\widetilde{R}_2$, denote by $\gamma = |\lambda| \sum_{i=-n}^{n} |a_i| L(t_i)$. Then,

$$
\begin{aligned}
\|u_2 - \tilde{u}_{2,n}\| &\leq \widetilde{R}_2 \\
&\leq |\lambda| \sum_{i=-\nu}^{\nu} |a_i| L(t_i) \widetilde{R}_1 + |\lambda| \big| R_{K,\nu} \big| \\
&\leq |\lambda| \sum_{i=-n}^{n} |a_i| L(t_i) |\lambda| M + |\lambda| M \\
&= |\lambda| M (1 + \gamma)
\end{aligned}
$$

and, by induction, we get

$$
\begin{aligned}
\|u_k - \tilde{u}_{k,n}\| &\leq \widetilde{R}_k \\
&\leq \widetilde{R}_{k-1}\gamma + |\lambda|M \\
&= |\lambda|M \big(\gamma(1 + \gamma + \cdots + \gamma^{k-2}) + 1\big) \\
&= |\lambda|M \big(1 + \gamma + \cdots + \gamma^{k-1}\big).
\end{aligned}
$$

Thus, under all the conditions assumed so far, if $\gamma < 1$, then the error estimate

$$\|\tilde{u}_{k,n} - u^*\| \leq \frac{q^k}{1-q}\|u_1 - u_0\| + \frac{|\lambda|M}{1-\gamma}$$

holds for every $k \in \mathbb{N}$. Thus, as $k, n \to \infty$, $\tilde{u}_{k,n} \to u^*$.

In particular, let us consider the trapezoidal rule for approximating integrals over symmetric intervals:

$$\int_{-b}^{b} \varphi(x)\mathrm{d}x = \frac{b}{2n}\left[\varphi(-b) + 2\sum_{j=-n+1}^{n-1} \varphi(x_j) + \varphi(b)\right] + R_{\varphi,n},$$

where the $2n+1$ nodes are $x_j = -b + bj/n$, $j = \overline{0, 2n}$, and the remainder is given by

$$R_{\varphi,n} = -\frac{b^3}{6n^2}\varphi''(\eta), \quad \eta \in (-b, b).$$

For $\nu = \overline{0, n}$, let $t_\nu = T\nu/n$ and $t_{-\nu} = -t_\nu$. We have, for $\nu = \overline{0, n}$ (i.e., for $t_\nu \geq 0$),

$$\int_{-t_\nu}^{t_\nu} K(t_\nu, x, u_k(x)) \mathrm{d}x = \frac{2t_\nu}{4\nu} \left(K_{\nu, -\nu, k} + 2 \sum_{j=-\nu+1}^{\nu-1} K_{\nu, j, k} + K_{\nu, \nu, k} \right) + R_{K, \nu},$$

$$\int_{t_\nu}^{-t_\nu} K(-t_\nu, x, u_k(x)) \mathrm{d}x = -\frac{2t_\nu}{4\nu} \left(K_{-\nu, -\nu, k} + 2 \sum_{j=-\nu+1}^{\nu-1} K_{-\nu, j, k} + K_{-\nu, \nu, k} \right) - R_{K, \nu}.$$

Notice that $\dfrac{2t_\nu}{4\nu} = \dfrac{T}{2n}$, so, in this case, $\gamma \leq \dfrac{|\lambda| T}{n} \sum\limits_{i=-n}^{n} L(t_i)$, which will be assumed to be less than 1. For the remainder, notice that for all $\nu = 0, 1, \ldots, n$, we have

$$|R_{K, \nu}| = \frac{t_\nu^3}{6\nu^2} \left| K(t_\nu, \eta_\nu, u_k(\eta_\nu)) \big|_x'' \right|$$

$$= \frac{T^3 \nu^3}{6n^3 \nu^2} \left| K(t_\nu, \eta_\nu, u_k(\eta_\nu)) \big|_x'' \right|$$

$$\leq \frac{T^3}{6n^2} \left| K(t_\nu, \eta_\nu, u_k(\eta_\nu)) \big|_x'' \right|.$$

Thus, if K and g are C^2 functions with bounded second order (partial) derivatives and $\dfrac{|\lambda| T}{n} \sum\limits_{i=-n}^{n} L(t_i) < 1$, then

$$|R_{K, \nu}| \leq \frac{T^3}{6n^2} M_0 =: M,$$

for some $M_0 > 0$ that does not depend on k or ν. Hence, from all the work above, we get

$$\|\tilde{u}_{k, n} - u^*\| \leq \frac{q^k}{1 - q} \|x_1 - x_0\| + \frac{T^3}{6n^2} \frac{M_0}{1 - \gamma},$$

for all $k = 1, 2, \ldots$.

Example. Let us consider the nonlinear functional integral equation ([21])

$$u(t) = \frac{1}{32} \int_{-t}^{t} \cos(x) u^2(x) \, \mathrm{d}x + \sin(t) - \frac{1}{48} \sin^3(t), \qquad (2.36)$$

for $t \in [-\pi/2, \pi/2]$. The exact solution of equation (2.36) is $u^*(t) = \sin(t)$.

In this case

$$\lambda = \frac{1}{32}, \quad K(t, x, u) = \cos(x) u^2, \quad g(t) = \sin(t) - \frac{1}{48} \sin^3(t).$$

Let $\varrho = 1$. Notice that g is an increasing function on $[-\pi/2, \pi/2]$, so

$$\varrho_1 = g\left(-\frac{\pi}{2}\right) = -\frac{47}{48}, \quad \varrho_2 = g\left(\frac{\pi}{2}\right) = \frac{47}{48}.$$

We have

$$M_K = (\varrho_2 + \varrho)^2 = \left(\frac{95}{48}\right)^2$$

and so that

$$2|\lambda|M_K T = \frac{\pi}{32}\left(\frac{95}{48}\right)^2 < 1 = \varrho.$$
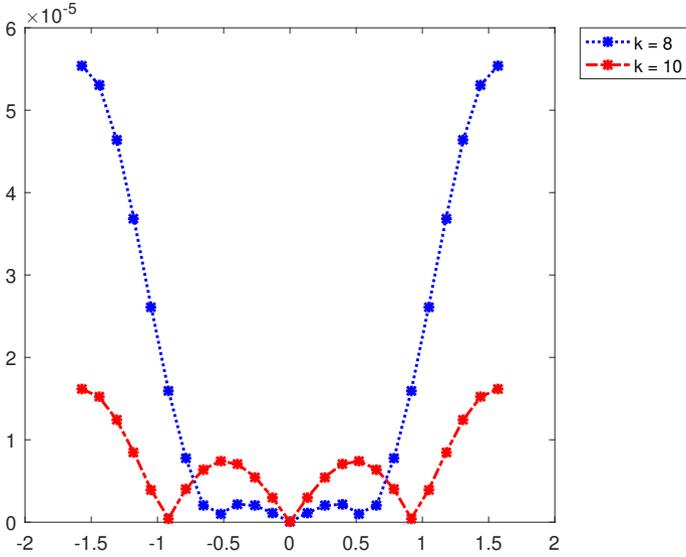
FIGURE 11. Errors at the nodes for $k = 8$ and $k = 10$ iterations

Also, on $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right] \times \left[-\frac{95}{48}, \frac{95}{48}\right]$, we have

$$\left|K(t, x, u) - K(t, x, v)\right| \leq \cos(x)|u + v| \cdot |u - v| \leq \frac{95}{24}\cos(x)|u - v|.$$

Let $L(x) = \frac{95}{24}\cos(x)$. Then

$$q = |\lambda| \int_{-\pi/2}^{\pi/2} L(x)\mathrm{d}x = \frac{95}{384} < 1.$$

For the second part, note that

$$\gamma \leq \frac{1}{32} \cdot \frac{\pi}{2n} \cdot \frac{95}{24} \sum_{i=-n}^{n} \cos(t_i) \leq \frac{95\pi}{768} \cdot \frac{2n + 1}{2n} \leq \frac{95\pi}{768} \cdot \frac{3}{2} < 1,$$

for any $n \geq 1$.

Thus, all our theoretical assumptions are satisfied.

We use the trapezoidal rule with $n = 12$ and the 25 nodes $t_\nu = \dfrac{\pi}{24}\nu$, $t_{-\nu} = -t_\nu$, $\nu = \overline{0, 12}$.

In Figure 11 we present the graph of the errors at the nodes for $k = 8$ and 10 iterations.

2.6. Fractional Integral Equations

In recent years, Fractional Calculus has been studied extensively, as more and more applications have developed in various fields from physics and engineering where domains are fractal curves (continuous, but non-differentiable functions), where ideas and methods from classical Calculus cannot be used.

Here, we consider the following fractional order integral equation ([20])

$$u(t) = a(t)J^\alpha[b(t)u(t)] + g(t),$$

i.e.,

$$u(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(x)(t - x)^{\alpha-1}u(x)\,\mathrm{d}x + g(t), \quad t \in [0, T], \qquad (2.37)$$

where $0 < \alpha < 1$ and $a, b, g : [0, T] \to \mathbb{R}$ are continuous functions. The term

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - x)^{\alpha-1}f(x)\,\mathrm{d}x$$

is called the *fractional integral of f of order α* and

$$\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-x}x^{\alpha-1}\,\mathrm{d}x, \quad \alpha > 0$$

is Euler's Gamma function.

On the space $X = C[0, T]$ we consider again the Bielecki norm $\|u\|_\tau = \max_{t\in[0,T]} |u(t)|\,\mathrm{e}^{-\tau t}$ for some $\tau > 0$ and the ball $B_\varrho := \{u \in X : \|u - g\|_\tau \le \varrho\}$, for some $\varrho > 0$. We define the fractional integral operator

$$Fu(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(x)(t - x)^{\alpha-1}u(x)\,\mathrm{d}x + g(t).$$

For continuous functions a, b, g and u on $[0, T]$, it can be shown that Fu is also continuous on $[0, T]$ (see e.g. [3]), so $F : X \to X$ is well defined.

We choose the constant τ such that

$$\tau \ge (2\|a\| \cdot \|b\|)^{1/\alpha},$$

where $\|\cdot\|$ denotes the Chebyshev norm and the radius ϱ so that

$$\varrho \ge \max\{-\varrho_1, \varrho_2\},$$

where $\varrho_1 := \min_{t\in[0,T]} g(t)$ and $\varrho_2 := \max_{t\in[0,T]} g(t)$.

These conditions will ensure the fact that $F(B_\varrho) \subseteq B_\varrho$ and that $F : B_\varrho \to B_\varrho$ is a contraction with constant

$$q = \frac{\|a\| \cdot \|b\|}{\tau^\alpha} < 1$$

(see [20] for details).

Then by the contraction principle, equation (2.37) has a unique solution $u^* \in B_\varrho$, which is the limit of the iterative process $u_{k+1} = Fu_k$, $k = 0, 1, \ldots$, starting with any initial value $u_0 \in B_\varrho$ and the error of the approximation of u^* by u_k is of order $\dfrac{q^k}{1-q}\|u_1 - u_0\|$. Moreover, if $a, b, g \in C^2[0, T]$, then it can be shown that $u^* \in C^2[0, T]$, also (see e.g. [7]).

Since the integrals in the iterative process $u_{k+1} = Fu_k$ are singular, we use product integration for their numerical approximation We mention here just the basic ideas, for details, see [4].

For $\varphi \in C^2[0, d]$, the weight function $w(x) = (d-x)^{\alpha-1}$ and the equidistant nodes $x_\nu = \nu h = \nu d/n$, $\nu = 0, 1, \ldots, n$, the integral

$$I(\varphi) = \int_0^d \varphi(x)(d - x)^{\alpha-1}\mathrm{d}x$$

is approximated by the sequence

$$I_n = \int_0^d \varphi_n(x)(d - x)^{\alpha-1}\,\mathrm{d}x,$$

where

$$\varphi_n(x) = \frac{1}{h}\big[(x_j - x)\varphi(x_{j-1}) + (x - x_{j-1})\varphi(x_j)\big] \quad \text{for} \ \ x \in [x_{j-1}, x_j]$$

is the linear interpolation polynomial of the function φ on $[x_{j-1}, x_j]$. We have

$$I(\varphi) = \sum_{\nu=1}^{n}\int_{x_{\nu-1}}^{x_\nu} \varphi(x)w(x)\,\mathrm{d}x \approx \sum_{\nu=1}^{n}\int_{x_{\nu-1}}^{x_\nu} \varphi_\nu(x)w(x)\,\mathrm{d}x = \sum_{\nu=0}^{n} w_\nu\varphi(x_\nu).$$

Then

$$\|\varphi - \varphi_n\| \le \frac{h^2}{8}\|\varphi''\| \quad \text{and} \quad \|I(\varphi) - I_n(\varphi)\| \le \frac{h^2}{8}\frac{d^\alpha}{\alpha}\|\varphi''\|.$$

The coefficients w_ν above are given by

$$w_0 = \frac{1}{h}\int_{x_0}^{x_1}(x_1 - x)w(x)\,\mathrm{d}x, \qquad w_n = \frac{1}{h}\int_{x_{n-1}}^{x_n}(x - x_{n-1})w(x)\,\mathrm{d}x,$$

$$w_j = \frac{1}{h}\left[\int_{x_{j-1}}^{x_j}(x - x_{j-1})w(x)\,\mathrm{d}x + \int_{x_j}^{x_{j+1}}(x_{j+1} - x)w(x)\,\mathrm{d}x\right], \quad j = \overline{1, n-1}.$$

With the change of variables $x - x_{j-1} = yh$, $0 \le y \le 1$ and the notations

$$\psi_1(i) = \int_0^1 y\big(d - (i + y)h\big)^{\alpha-1}\mathrm{d}y, \quad \psi_2(i) = \int_0^1 (1 - y)\big(d - (i + y)h\big)^{\alpha-1}\mathrm{d}y,$$

$i = 0, 1, \ldots$, the coefficients w_ν can be written as

$$w_0 = h\psi_2(0), \quad w_n = h\psi_1(n - 1),$$

$$w_j = h\psi_1(j - 1) + h\psi_2(j), \quad j = \overline{1, n-1}.$$

We apply these formulas to the integrals in the iterates Fu_k. For a fixed n, let $h = T/n$ and $t_\nu = \nu h$, $\nu = 0, 1, \ldots, n$. For a fixed $\nu \in \{0, 1, \ldots, n\}$, let

$w^{(\nu)}(x) = (t_\nu - x)^{\alpha-1}$. On each interval $[0, t_\nu]$, we use the nodes $\{t_0, t_1, \ldots, t_\nu\}$. Note that $t_\nu/\nu = \nu h/\nu = h$. With adapted notations, we have

$$\int_0^{t_\nu} b(x)u_k(x)(t_\nu - x)^{\alpha-1}\,\mathrm{d}x = \int_0^{t_\nu} b(x)u_k(x)w^{(\nu)}(x)\,\mathrm{d}x$$

$$= \sum_{j=0}^{\nu} w_{j,\nu} b(t_j)u_k(t_j) + \widehat{R}_{k,\nu}.$$

and

$$\psi_{1,\nu}(i) = \int_0^1 y\Big(t_\nu - (i+y)h\Big)^{\alpha-1}\,\mathrm{d}y = h^{\alpha-1}\int_0^1 y(\nu - i - y)^{\alpha-1}\,\mathrm{d}y,$$

$$\psi_{2,\nu}(i) = \int_0^1 (1-y)\Big(t_\nu - (i+y)h\Big)^{\alpha-1}\,\mathrm{d}y$$

$$= h^{\alpha-1}\int_0^1 (1-y)(\nu - i - y)^{\alpha-1}\,\mathrm{d}y, \quad i = 0, 1, \ldots\ .$$

By a simple computation, we get

$$\psi_{1,\nu}(i) + \psi_{2,\nu}(i) = \frac{h^{\alpha-1}}{\alpha}\Big[(\nu - i)^\alpha - (\nu - i - 1)^\alpha\Big]$$

and, consequently,

$$\sum_{j=0}^{\nu} w_{j,\nu} \le h\sum_{j=0}^{\nu}\Big(\psi_{1,\nu}(j) + \psi_{2,\nu}(j)\Big) \le \frac{T^\alpha}{\alpha}.$$

For the remainder term $\widehat{R}_{k,\nu}$, we have that

$$\big|\widehat{R}_{k,\nu}\big| \le \frac{h^2}{8}\frac{T^\alpha}{\alpha}\|(bu_k)''\|,$$

and that it does not depend on ν, so we can write $\widehat{R}_k$. If we choose $u_0 \in B_\varrho \cap C^2[0,T]$, so that $u_k \in B_\varrho \cap C^2[0,T]$, the sequences $\{u_k\}$, $\{u_k'\}$ and $\{u_k''\}$ will be uniformly bounded. Since a, b and g also have bounded second order derivatives, we can find $M > 0$ such that

$$|\widehat{R}| = |\widehat{R}_k| \le \frac{T^2}{8n^2}\cdot\frac{T^\alpha}{\alpha}M,$$

where the constant M depends on a, b, g, ϱ and τ, but not on n, ν or k.

As before, we approximate the values $u_k(t_\nu)$ by $\tilde{u}_k(t_\nu)$ given by

$$\tilde{u}_0(t_\nu) = g(t_\nu),$$

$$\tilde{u}_{k+1}(t_\nu) = \frac{1}{\Gamma(\alpha)}a(t_\nu)\sum_{j=0}^{\nu} w_{j,\nu}b(t_j)\tilde{u}_k(t_j) + g(t_\nu).$$

Denoting by

$$\gamma = \frac{T^\alpha}{\Gamma(\alpha)}M_a M,$$

where $M_a = \max\{\|a\|, \|a'\|, \|a''\|\}$, by computations similar to the ones in the previous section (details can be found in [20]), we get, inductively, that

$$\|u_k - \tilde{u}_k\| \quad := \quad \max_{t_\nu \in [0,T]} |u_k(t_\nu) - \tilde{u}_k(t_\nu)|$$

$$\leq \quad \frac{T^2}{8n^2}\gamma\big(1 + \gamma + \cdots + \gamma^{n-1}\big).$$

So, if we assume $\gamma < 1$, we have the error estimate

$$\|\tilde{u}_k - u^*\| \leq \frac{q^k}{1-q}\|u_1 - u_0\| + \frac{T^2}{8n^2}\frac{\gamma}{1-\gamma},$$

for every $k \in \mathbb{N}$.

Example. Now, consider the fractional integral equation

$$u(t) = \frac{0.01}{\Gamma(1/2)}t^{5/2}\int_0^t (t-x)^{-1/2}u(x)\,\mathrm{d}x + \sqrt{\pi}(1+t)^{-3/2} - 0.02\frac{t^3}{1+t},$$

for $t \in [0, 1]$, whose exact solution is $u^*(t) = \sqrt{\pi}\,(1+t)^{-3/2}$.

Here, we have $\alpha = 1/2$, $a(t) = 0.01t^{5/2}$ and $b(t) \equiv 1$. Then $\|a\| = 0.01$, $\|b\| = 1$, so we can take $\tau = 1$, satisfying our theoretical requirements.
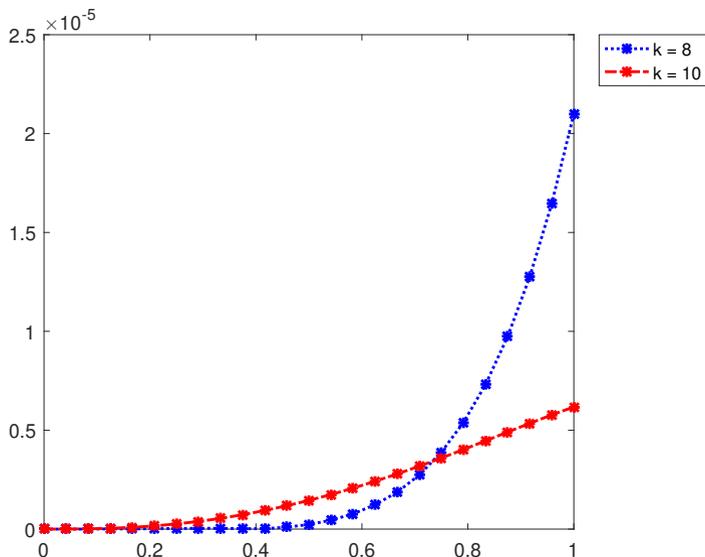
FIGURE 12. Errors at the nodes for $k = 8$ and $k = 10$ iterations

Since

$$g(t) = \sqrt{\pi}(1+t)^{-3/2} - 0.02\frac{t^3}{1+t},$$

we have $\varrho_1 = \sqrt{\pi} \cdot 2^{-3/2}$, $\varrho_2 = \sqrt{\pi}$, so we can choose $\varrho = 2$. Then $u^* \in B_\varrho$ and since $M_a = 0.06$, $M = \dfrac{15}{4}\sqrt{\pi}$, we have $\gamma \approx 0.281 < 1$.

We use the iterative scheme described above with $n = 24$, so corresponding nodes $t_\nu = \nu/24$, $\nu = \overline{0, 24}$, and the initial approximation $u_0(t) = g(t)$. The errors $|\tilde{u}_k(t_\nu) - u^*(t_\nu)|$ at the nodes, for $k = 8$ and $k = 10$ iterations are illustrated in Figure 12.

Acknowledgment

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