Multiple Orthogonality and Quadratures of Gaussian Type

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Abstract

In this paper we consider multiple orthogonal polynomials on the real line, defined using orthogonality conditions spread out over r different measures. Such polynomials are generalization of the ordinary orthogonal polynomials (case $r = 1$).

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Using the discretized Stieltjes-Gautschi procedure, we compute recursive coefficients of corresponding recurrence relation for such polynomials, and also, zeros of multiple orthogonal polynomials. We construct and consider the corresponding quadrature formulas of Gaussian type. Specially, we consider quadratures of Gauss–Lobatto and Gauss–Radau type. Some numerical examples are also included.

1 Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy r orthogonality conditions.

Starting with a problem that arise in the evaluation of computer graphics illumination models, Borges [6] has examined the problem of numerically evaluating a set of r definite integrals taken with respect to distinct weight functions but related by a common integrand and interval of integration. The nodes of an optimal set of such quadratures (quadratures of Gaussian type) are the zeros of type II multiple orthogonal polynomials. However, Borges has not used multiple orthogonality. In [12] Milovanović and Stanić have presented effective numerical method for constructing type II multiple orthogonal polynomials, and the corresponding Gaussian quadratures, using *discretized Stieltjes-Gautschi procedure* [7].

In [13] Milovanović and Stanić have investigated multiple orthogonal polynomials on the semicircle, as a generalization of orthogonal polynomials on the semicircle, introduced by Gautschi and Milovanović [10] – complex polynomials orthogonal with respect to the complex-valued inner products $[f,g]_k = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w_k(e^{i\theta})\,d\theta$, for $k = 1,2,\ldots,r$. Also, a numerical method for constructing these polynomials and corresponding Gaussian quadratures has presented.

In this paper we repeat some basic results on multiple orthogonal polynomials, their numerical construction, and consider quadratures of Gauss–Lobatto and Gauss–Radau type, including some numerical examples. The paper is organized as follows. First, some basic facts about the type II multiple orthogonal polynomials are given in Section 2. In Section 3 we give some properties of type II multiple orthogonal polynomials with nearly diagonal multi-indices and numerical procedure for their construction, based on the discretized Stieltjes-Gautschi procedure. In the same section an optimal set of quadrature formulas and the corresponding method for calculating the nodes and weight coefficients of such quadratures are considered. In Section 4 we consider quadrature formulas of Gaussian type with preassigned nodes. Specially, we consider quadratures of Gauss–Lobatto and Gauss–Radau type. Also, some numerical examples are included.

2 Type II Multiple Orthogonal Polynomials

Let $r \geq 1$ be an integer and let $w_1, w_2, \ldots, w_r$ be r weight functions on the real line so that the support of each w_i is a subset of an interval E_i. Let $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a vector of r nonnegative integers, which is called a *multi-index* with the length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$.

There are two types of multiple orthogonal polynomials:

1° *Type I multiple orthogonal polynomials.* Here we want to find a vector of polynomials $(A_{\vec{n},1}, A_{\vec{n},2}, \ldots, A_{\vec{n},r})$ such that each $A_{\vec{n},i}$ is a polynomial of degree $n_i - 1$ and the following orthogonality conditions hold:

$$\sum_{j=1}^{r} \int_{E_j} A_{\vec{n},j}\, x^k w_j(x)dx = 0, \qquad k = 0, 1, \ldots, |\vec{n}| - 2. \tag{2.1}$$

Each $A_{\vec{n},i}$ has n_i coefficients and the type I vector is completely determined if we can find all the $|\vec{n}|$ unknown coefficients. The orthogonality relations (2.1) give $|\vec{n}| - 1$ homogenous linear equations for these $|\vec{n}|$ coefficients. If the matrix of coefficients has full rank, then we can determine the type I vector uniquely up to a multiplicative factor.

For $r = 1$ we have the case of ordinary orthogonal polynomials.

2° *Type II multiple orthogonal polynomials.* The type II multiple orthogonal polynomial is a monic polynomial $\pi_{\vec{n}}$ of degree $|\vec{n}|$ such that the following orthogonality conditions:

$$\int_{E_1} \pi_{\vec{n}}(x)\, x^k w_1(x)dx = 0, \qquad k = 0, 1, \ldots, n_1 - 1, \tag{2.2}$$

$$\int_{E_2} \pi_{\vec{n}}(x)\, x^k w_2(x)dx = 0, \qquad k = 0, 1, \ldots, n_2 - 1, \tag{2.3}$$

$$\vdots$$

$$\int_{E_r} \pi_{\vec{n}}(x)\, x^k w_r(x)dx = 0, \qquad k = 0, 1, \ldots, n_r - 1, \tag{2.4}$$

are satisfied.

Again, for $r = 1$ we have the ordinary orthogonal polynomials.

The conditions (2.2)–(2.4) give $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients $a_{k,\vec{n}}$ of the polynomial $\pi_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k,\vec{n}} x^k$, where $a_{|\vec{n}|,\vec{n}} = 1$. But the matrix of coefficients of this system can be singular and we need some additional conditions on the r weight functions to provide the uniqueness of the multiple orthogonal polynomial. If the polynomial $\pi_{\vec{n}}(x)$ is unique, then we say that $\vec{n}$ is a *normal multi-index* and if all multi-indices are normal then we have a *complete system*.

In this paper we consider only the type II multiple orthogonal polynomials. For each of the weight functions w_k, $k = 1, 2, \ldots, r$,

$$(f, g)_k = \int_{E_k} f(x)g(x)w_k(x)dx \tag{2.5}$$

denotes the corresponding inner product of the functions f and g.

Our interest is in systems of r weight functions for which all multi-indices are normal. There are two distinct cases for which the type II multiple orthogonal polynomials are given.

1. *Angelesco systems* for which the intervals E_i, on which the weight functions are supported, are disjoint, i.e., $E_i \cap E_j = \emptyset$ for $1 \le i, j \le r$, $i \ne j$.

2. *AT system* is such that all weight functions are supported on the same interval E and we also require that the $|\vec{n}|$ functions

$$w_1(x),\, xw_1(x),\, \ldots,\, x^{n_1-1}w_1(x),\;\; w_2(x),\, xw_2(x),\, \ldots, x^{n_2-1}w_2(x),$$
$$\ldots,\, w_r(x),\, xw_r(x),\, \ldots,\, x^{n_r-1}w_r(x)$$

form a Chebyshev system on E for each multi-index $\vec{n}$. This means that every linear combination

$$\sum_{j=1}^{r} Q_{n_j-1}(x)w_j(x),$$

where Q_{n_j-1} is a polynomial of degree at most $n_j - 1$, has at most $|\vec{n}| - 1$ zeros on E.

The following two theorems hold (for proof see [16]):

Theorem 2.1 *In an Angelesco system the type II multiple orthogonal polynomial* $\pi_{\vec{n}}(x)$ *factors into* r *polynomials* $\prod_{j=1}^{r} q_{n_j}(x)$*, where each* q_{n_j} *has exactly* n_j *zeros on* E_j*.*

Theorem 2.2 *In an AT system the type II multiple orthogonal polynomial* $\pi_{\vec{n}}(x)$ *has exactly* $|\vec{n}|$ *zeros on* E*.*

3 Type II Multiple Orthogonal Polynomials with Nearly Diagonal Multi-index

Let $n \in \mathbb{N}$ and write it as $n = \ell r + \nu$, with $\ell = [n/r]$ and $0 \leq \nu < r$. The nearly diagonal multi-index $\vec{s}(n)$ corresponding to n is given by

$$\vec{s}(n) = (\underbrace{\ell+1, \ell+1, \ldots, \ell+1}_{\nu \text{ times}}, \underbrace{\ell, \ell, \ldots, \ell}_{r-\nu \text{ times}}).$$

Denote the corresponding type II multiple (monic) orthogonal polynomials by

$$\pi_n(x) = \pi_{\vec{s}(n)}(x).$$

Type II multiple orthogonal polynomials with nearly diagonal multi-index satisfy the following recurrence relation of order $r+1$[1]

$$x\pi_m(x) = \pi_{m+1}(x) + \sum_{i=0}^{r} \alpha_{m,r-i}\pi_{m-i}(x)\,, \quad m \geq 0, \tag{3.1}$$

with initial conditions $\pi_0(x) = 1$ and $\pi_i(x) = 0$ for $i = -1, -2, \ldots, -r$ (see [15]).

Setting $m = 0, 1, \ldots, n-1$ in (3.1), we get

$$x \begin{bmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1}(x) \end{bmatrix} = H_n \begin{bmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1}(x) \end{bmatrix} + \pi_n(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

[1] It is known that ordinary orthogonal polynomials on the real line always satisfy a three-term recurrence relation.

i.e.,

$$H_n \mathbf{p}_n(x) = x\,\mathbf{p}_n(x) - \pi_n(x)\mathbf{e}_n, \tag{3.2}$$

where

$$\mathbf{p}_n(x) = \big[\pi_0(x) \;\; \pi_1(x) \;\; \ldots \;\; \pi_{n-1}(x)\big]^T, \qquad \mathbf{e}_n = [0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1]^T,$$

and H_n is the following lower (banded) Hessenberg matrix[2] of order n

$$H_n = \begin{bmatrix} \alpha_{0,r} & 1 & & & & & & & \\ \alpha_{1,r-1} & \alpha_{1,r} & 1 & & & & & & \\ \vdots & \ddots & \ddots & \ddots & & & & & \\ \alpha_{r,0} & \cdots & \alpha_{r,r-1} & \alpha_{r,r} & 1 & & & & \\ & \alpha_{r+1,0} & \cdots & \alpha_{r+1,r-1} & \alpha_{r+1,r} & 1 & & & \\ & & \ddots & & \ddots & \ddots & \ddots & & \\ & & & \alpha_{n-2,0} & \cdots & \alpha_{n-2,r-1} & \alpha_{n-2,r} & 1 \\ & & & & \alpha_{n-1,0} & \cdots & \alpha_{n-1,r-1} & \alpha_{n-1,r} \end{bmatrix}.$$

Let $\xi_\nu \equiv \xi_\nu^{(n)}$ $(\nu = 1, \ldots, n)$ be zeros of $\pi_n(x)$. Then (3.2) reduces to the eigenvalue problem

$$\xi_\nu \mathbf{p}_n(\xi_\nu) = H_n \mathbf{p}_n(\xi_\nu).$$

Thus, ξ_ν are eigenvalues of the matrix H_n and $\mathbf{p}_n(\xi_\nu)$ are the corresponding eigenvectors.

For computing zeros of $\pi_n(x)$ as the eigenvalues of the matrix H_n we use the EISPACK routine COMQR [3, pp. 277–284]. Notice that this routine needs an upper Hessenberg matrix, i.e., H_n^T. Also, the MATLAB or MATHEMATICA can be used.

Our aim here is to compute the recurrence coefficients in (3.1), i.e., the elements of the Hessenberg matrix H_n. Only for the simplest case of multiple orthogonality, i.e., when $r = 2$, for some classical weight functions

[2]This kind of matrix was obtained also in construction of orthogonal polynomials on the radial rays in the complex plane (see [11]).

(Jacobi, Laguerre, Hermite) one can find explicit formulas for the recurrence coefficients (see [14], [16]). In [12] the elements of H_n for arbitrary r have been calculated numerically, using the discretized Stieltjes-Gautschi procedure [7].

At first, we express the elements of H_n in terms of the inner products (2.5), and then we use the corresponding Gaussian formulas to discretize these inner products. Of course, we suppose that the type II multiple orthogonal polynomials exist with respect to the inner products $(\,\cdot\,,\,\cdot\,)_k$, $k = 1, 2, \ldots, r$, given by (2.5).

Taking that for inner products $(\,\cdot\,,\,\cdot\,)_{j+mr} = (\,\cdot\,,\,\cdot\,)_j$ $(m \in \mathbb{Z})$, the following result holds:

Theorem 3.1 *The type II multiple monic orthogonal polynomials $\{\pi_n\}$, with nearly diagonal multi-index, satisfy the recurrence relation*

$$\pi_{n+1}(x) = (x - \alpha_{n,r})\pi_n(x) - \sum_{k=0}^{r-1} \alpha_{n,k}\pi_{n-r+k}(x), \quad n \geq 0, \tag{3.3}$$

where

$$\alpha_{n,0} = \frac{\big(x\pi_n, \pi_{[(n-r)/r]}\big)_{\nu+1}}{\big(\pi_{n-r}, \pi_{[(n-r)/r]}\big)_{\nu+1}}$$

and

$$\alpha_{n,k} = \frac{\left(x\pi_n - \sum\limits_{i=0}^{k-1} \alpha_{n,i}\pi_{n-r+i}, \pi_{[(n-r+k)/r]}\right)_{\nu+k+1}}{\big(\pi_{n-r+k}, \pi_{[(n-r+k)/r]}\big)_{\nu+k+1}}, \quad k = 1, 2, \ldots, r$$

Here, we put $n = \ell r + \nu$, where $\ell = [n/r]$ and $\nu \in \{0, 1, \ldots, r-1\}$ ($[t]$ is integer part of t).

Proof of the previous theorem one can find in [12].

We use an alternatively recurrence relation and give formulas for coefficients. Knowing π_0 we compute $\alpha_{0,r}$, then knowing $\alpha_{0,r}$ we compute π_1, and then again $\alpha_{1,r}$ and $\alpha_{1,r-1}$, etc. Continuing in this manner, we can generate as many polynomials, and therefore as many of the recurrence coefficients as are desired.

All of the necessary inner products can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulas with respect to the corresponding weight function w_i, $i = 1, 2, \ldots r$.

3.1 Optimal Set of Quadratures

Denote with $W = \{w_1, w_2, \ldots, w_r\}$ an AT system.

Following [6, Definition 3] we introduced the following definition (see [12]):

Definition 3.2 *Let W be an AT system (the weight functions w_i, $i = 1, 2, \ldots, r$ are supported on the interval E), $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index, and $n = |\vec{n}|$. A set of quadrature formulas of the form:*

$$\int_E f(x)\, w_m(x)dx \approx \sum_{i=1}^{n} A_{m,i} f(x_i), \qquad m = 1, 2, \ldots, r \tag{3.4}$$

will be called an optimal set (quadratures of Gaussian type) with respect to $(W, \vec{n})$ if and only if the weight coefficients, $A_{m,i}$, and the nodes, x_i, satisfy the following equations:

$$\begin{array}{lcl} \displaystyle\sum_{i=1}^{n} A_{m,i} & = & \displaystyle\int_E w_m(x)dx \\ \displaystyle\sum_{i=1}^{n} A_{m,i}\, x_i & = & \displaystyle\int_E x\, w_m(x)dx \\ & \vdots & \\ \displaystyle\sum_{i=1}^{n} A_{m,i}\, x_i^{n+n_m-1} & = & \displaystyle\int_E x^{n+n_m-1}\, w_m(x)dx \end{array} \tag{3.5}$$

for $m = 1, 2, \ldots, r$.

For these optimal set of quadratures the next generalization of fundamental theorem of Gauss-Christoffel quadrature formulas holds (see [12]):

Theorem 3.3 *Let W be an AT system, $\vec{n} = (n_1, n_2, \ldots, n_r)$, $n = |\vec{n}|$. Consider the quadrature formulas:*

$$\int_E f(x)\, w_m(x)dx \approx \sum_{i=1}^{n} A_{m,i} f(x_i) \tag{3.6}$$

where $m = 1, 2, \ldots, r$.

These formulas form an optimal set with respect to $(W,\vec{n})$ if and only if:

1° *They are exact for all polynomials of degree $\leq n-1$.*

2° *The polynomial $q(x) = \prod_{i=1}^{n}(x-x_i)$ is the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to W.*

For the case of the nearly diagonal multi-indices $\vec{s}(n)$ the nodes x_i, $i = 1,2,\ldots,n$, of the Gaussian type quadrature formulas can be computed as eigenvalues of the corresponding banded Hessenberg matrix H_n.

The weight coefficients $A_{m,i}$ can be computed by requiring that each rule correctly generate the first n modified moments. Denote by $V_n = \begin{bmatrix}\mathbf{p}_n(x_1) & \mathbf{p}_n(x_2) & \ldots & \mathbf{p}_n(x_n)\end{bmatrix}$ the matrix of the eigenvectors of matrix H_n, each normalized so that the first component is equal to 1. Then, the weight coefficients $A_{m,i}$ can be found by solving

$$V_n \cdot \begin{bmatrix} A_{m,1} \\ A_{m,2} \\ \vdots \\ A_{m,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{*(m)} \\ \mu_1^{*(m)} \\ \vdots \\ \mu_{n-1}^{*(m)} \end{bmatrix}, \quad m = 1,2,\ldots,r\,, \tag{3.7}$$

where

$$\mu_i^{*(m)} = \int_E \pi_i(x)\, w_m(x)dx\,, \quad m = 1,2,\ldots,r, \quad i = 0,1,\ldots,n-1\,,$$

are modified moments and $\pi_i = \pi_{\vec{s}(i)}$.

All of the modified moments can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulas with respect to the corresponding weight function w_m, $m = 1,2,\ldots r$.

4 Quadrature Formulae of Gaussian Type with Preassigned Nodes

Let $W = \{w_1, w_2, \ldots, w_r\}$ be an AT system.

Following Definition 3.2 and ordinary quadrature formulas of Gaussian type with preassigned abscissas (see for example [1, Subsection 2.2.1]) we introduce the following definition:

Definition 4.1 *Let W be an AT system (the weight functions w_i, $i = 1, 2, \ldots, r$ are supported on the interval E), $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index, $n = |\vec{n}|$. A set of quadrature formulas of the form:*

$$\int_E f(x)w_m(x) \approx \sum_{i=1}^{k} a_{m,i} f(y_i) + \sum_{i=1}^{n} A_{m,i} f(x_i), \quad m = 1, 2, \ldots, r, \tag{4.1}$$

where the nodes $y_i \in E$, $i = 1, 2, \ldots, k$ are fixed and prescribed in advance, will be called an optimal set with preassigned nodes $\{y_i\}_{i=1}^k$ with respect to $(W, \vec{n})$ if and only if the weight coefficients, $a_{m,i}, A_{m,i}$, and the nodes, x_i, satisfy the following equations:

$$\begin{array}{lcl} \sum\limits_{i=1}^{k} a_{m,i} + \sum\limits_{i=1}^{n} A_{m,i} & = & \int\limits_E w_m(x)dx \\ \sum\limits_{i=1}^{k} a_{m,i}\, y_i + \sum\limits_{i=1}^{n} A_{m,i}\, x_i & = & \int\limits_E x\, w_m(x)dx \\ & \vdots & \\ \sum\limits_{i=1}^{k} a_{m,i}\, y_i^{n+n_m+k-1} + \sum\limits_{i=1}^{n} A_{m,i}\, x_i^{n+n_m+k-1} & = & \int\limits_E x^{n+n_m+k-1}\, w_m(x)dx \end{array} \tag{4.2}$$

for $m = 1, 2, \ldots, r$.

For a set of preassigned nodes $\{y_i\}_{i=1}^k$ we introduce $s(x)$ as a polynomial of degree k, with zeros at y_i, $i = 1, 2, \ldots, k$.

Denote

$$\widetilde{W} = \{\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_r\}, \quad \widetilde{w}_m(x) = s(x)w_m(x), \;\; m = 1, 2, \ldots, r. \tag{4.3}$$

Theorem 4.2 *Let W be an AT system, $\vec{n} = (n_1, n_2, \ldots, n_r)$, $n = |\vec{n}|$. Suppose that for preassigned nodes, $\{y_i\}_{i=1}^k$, $\widetilde{W}$ is also AT system. The set of quadrature formulas* (4.1) *form the optimal set with preassigned nodes $\{y_i\}_{i=1}^k$ with respect to $(W, \vec{n})$ if and only if:*

1° *They are exact for all polynomials of degree $\leq n + k - 1$.*

2° *The polynomial $q(x) = \prod\limits_{i=1}^{n} (x - x_i)$ is the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to $\widetilde{W}$.*

Proof. Suppose first that the quadrature formulas (4.1) form the optimal set with preassigned nodes $\{y_i\}_{i=1}^k$ with respect to $(W, \vec{n})$.

In order to prove 1° we note that for each $m = 1, 2, \ldots, r$, the corresponding quadrature formula (4.1) is exact for all polynomials of degree $\leq n + n_m + k - 1$ and then it is exact for those of degree $\leq n + k - 1$.

To prove 2°, for $m = 1, 2, \ldots r$, assume that $p_m(x)$ is a polynomial of degree $\leq n_m - 1$. Then the polynomial $q(x)p_m(x)s(x)$ has degree $\leq n + n_m + k - 1$. Since the corresponding quadrature formula is exact for all such polynomials, it follows that

$$\begin{aligned}\int_E q(x)p_m(x)\, s(x)w_m(x)dx \; &= \sum_{i=1}^{k} a_{m,i} q(y_i)p_m(y_i)s(y_i) \\ &+ \sum_{i=1}^{n} A_{m,i} q(x_i)p_m(x_i)s(x_i).\end{aligned} \tag{4.4}$$

Since $s(y_i) = 0$ for $i = 1, 2, \ldots, k$ and $q(x_i) = 0$ for $i = 1, 2, \ldots, n$, the both sums on the right hand side in (4.4) are identically zero. Thus, we have

$$\int_E q(x)p_m(x)\, s(x)w_m(x)dx = 0, \quad m = 0, 1, \ldots, r$$

and 2° follows.

Suppose now that for quadrature formulas (4.1) 1° and 2° hold.

For $m = 1, 2, \ldots, r$, let $t_m(x)$ be a polynomial of degree $\leq n + n_m + k - 1$. We can write $t_m(x) = u_m(x) \cdot q(x)s(x) + v(x)$, where $u_m(x)$ is a polynomial of degree $\leq n_m - 1$ and $v(x)$ is a polynomial of degree $\leq n + k - 1$. It is easy to see that

$$\begin{aligned} t_m(y_i) &= v(y_i), & i &= 1, 2, \ldots, k, \\ t_m(x_i) &= v(x_i), & i &= 1, 2, \ldots, n. \end{aligned} \tag{4.5}$$

Then, we obtain

$$\begin{aligned}\int_E t_m(x)\, w_m(x)dx \; &= \int_E [u_m(x)q(x)s(x) + v(x)]\, w_m(x)dx \\ &= \int_E q(x)u_m(x)\, s(x)w_m(x)dx + \int_E v(x)\, w_m(x)dx\,.\end{aligned}$$

According to 2° we have $\int_E q(x)u_m(x)\,s(x)w_m(x)dx = 0$ and therefore we obtain

$$\int_E t_m(x)\,w_m(x)dx = \int_E v(x)\,w_m(x)dx\,.$$

Since $v(x)$ is a polynomial of degree $\leq n+k-1$, it follows from 1° that

$$\int_E v(x)\,w_m(x)dx = \sum_{i=1}^{k} a_{m,i}v(y_i) + \sum_{i=1}^{n} A_{m,i}v(x_i)$$

and hence using (4.5) we obtain

$$\begin{aligned}\int_E t_m(x)\,w_m(x)dx &= \sum_{i=1}^{k} a_{m,i}v(y_i) + \sum_{i=1}^{n} A_{m,i}v(x_i)\\ &= \sum_{i=1}^{k} a_{m,i}t_m(y_i) + \sum_{i=1}^{n} A_{m,i}t_m(x_i).\end{aligned}$$

This proves that for each $m = 1, 2, \ldots, r$, the corresponding quadrature formula is exact for all polynomials of degree $\leq n + n_m + k - 1$. ■

According to Theorem 4.2, the nodes x_i, $i = 1, 2, \ldots, n$, of the optimal set of quadrature formulas (4.1) are the zeros of the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to the AT system $\widetilde{W}$. For computing these zeros in the case of nearly diagonal multi-index we use the discretized Stieltjes–Gautschi procedure (see [12, 13]). When we find the nodes, then for $m = 1, 2, \ldots, r$ we can choose the weight coefficients $a_{m,i}$, $i = 1, 2, \ldots, k$ and $A_{m,i}$, $i = 1, 2, \ldots, n$, so that they satisfy the following Vandermonde system of equations

$$V(y_1, \ldots, y_k, x_1, \ldots, x_n)\begin{bmatrix} a_{m,1}\\ \vdots\\ a_{m,k}\\ A_{m,1}\\ \vdots\\ A_{m,n}\end{bmatrix} = \begin{bmatrix} \mu_0^{(m)}\\ \mu_1^{(m)}\\ \vdots\\ \mu_{n+k-1}^{(m)}\end{bmatrix}, \quad m = 1, 2, \ldots, r, \tag{4.6}$$

where

$$\mu_i^{(m)} = \int_E x^i \, w_m(x) dx, \quad m = 1, 2, \ldots, r, \quad i = 0, 1, \ldots, n+k-1,$$

are moments which can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulas with respect to the corresponding weight function w_m, $m = 1, 2, \ldots r$.

Each of Vandermonde systems (4.6) has unique solution if all of preassigned nodes are distinct from the zeros of type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to $\widetilde{W}$. This is always satisfied in cases when the preassigned nodes are at the end points of the interval E.

4.1 Quadrature Formulae of Gauss–Radau and Gauss–Lobatto Type

a) Jacobi weight functions

In the case of Gauss–Radau type quadratures we have only one preassigned node, $y_1 = -1$, so that $s(x) = x + 1$. For each AT system W consisting of Jacobi weight functions [12]

$$w_m(x) = (1-x)^\alpha (1+x)^{\beta_m}, \;\; m = 1, 2, \ldots, r,$$

$\alpha, \beta_m > -1$, $m = 1, 2, \ldots, r$; $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, we have

$$\widetilde{w}_m(x) = (1-x)^\alpha (1+x)^{\beta_m + 1}, \;\; m = 1, 2, \ldots, r,$$

and corresponding set $\widetilde{W}$ is also AT system. The Vandermonde system (4.6) (for $n+1$ unknown weights) has a unique solution for each $m = 1, 2, \ldots, r$.

In Table 1 the nodes and weights for quadrature formulas of Gauss–Radau type with respect to an AT system of Jacobi weights and nearly diagonal multi-index are given. Numbers in parentheses denote decimal exponents.

In the case of Gauss–Lobatto type quadratures we have two preassigned nodes, $y_1 = -1$ and $y_2 = 1$; $s(x) = 1 - x^2$, and for AT system W of Jacobi weight functions set $\widetilde{W}$ with elements

$$\widetilde{w}_m(x) = (1-x)^{\alpha+1} (1+x)^{\beta_m + 1}, \;\; m = 1, 2, \ldots, r,$$

Table 1: Quadrature formulas of Gauss–Radau type, $r = 3$, $\alpha = -1/4$, $\beta_1 = -1/2$, $\beta_2 = 1/4$, $\beta_3 = 1$, when $n = 10$ and $n = 16$

	y_1, x_i	$a_{1,1}$, $A_{1,i}$	$a_{2,1}$, $A_{2,i}$	$a_{3,1}$, $A_{3,i}$
1	−1.	1.007145484059894(−1)	2.062876155609072(−4)	−9.562972005938697(−7)
1	−9.833138304264925(−1)	2.435730645050421(−1)	1.129361273403368(−2)	5.241035783221104(−4)
2	−9.164273880324528(−1)	2.936452002423003(−1)	4.564271303762693(−2)	7.094582228332064(−3)
3	−7.794038086868831(−1)	3.223986760309847(−1)	1.037748068284181(−1)	3.340336785072431(−2)
4	−5.678600313784354(−1)	3.348502881280588(−1)	1.784715467514467(−1)	9.512338968694678(−2)
5	−2.928754974215839(−1)	3.340611167810139(−1)	2.576011782190145(−1)	1.986413967159547(−1)
6	2.129858755086223(−2)	3.214765372195719(−1)	3.265982323756700(−1)	3.318015253350430(−1)
7	3.412119085055776(−1)	2.974582873741706(−1)	3.707226525059520(−1)	4.620321265761330(−1)
8	6.294703272423117(−1)	2.612115619838483(−1)	3.767274261216885(−1)	5.433279924982002(−1)
9	8.505734897833458(−1)	2.096384020761912(−1)	3.326217108650106(−1)	5.277525560393169(−1)
10	9.765092384227134(−1)	1.306461010903117(−1)	2.177813020250095(−1)	3.630318479902930(−1)
1	−1.	4.955072915498704(−2)	3.270184865300507(−5)	−4.351164761828841(−8)
1	−9.957694114109361(−1)	1.262846817355126(−1)	2.093519371442199(−3)	3.471023381962368(−5)
2	−9.773102983214776(−1)	1.601763480957459(−1)	9.364154511359857(−3)	5.474482568814665(−4)
3	−9.360947441653007(−1)	1.838016263570097(−1)	2.336155953750918(−2)	2.969301411091863(−3)
4	−8.662104158604537(−1)	1.998820166290791(−1)	4.421711974927511(−2)	9.781538692113400(−3)
5	−7.646982637559852(−1)	2.102958318380690(−1)	7.104761664033177(−2)	2.400315683002052(−2)
6	−6.315299779035927(−1)	2.161684249080054(−1)	1.022336528091070(−1)	4.834989093124742(−2)
7	−4.694526660368208(−1)	2.181821415446974(−1)	1.356320427214107(−1)	8.431510884534577(−2)
8	−2.837170944030709(−1)	2.167503150578853(−1)	1.687614224963532(−1)	1.313973532900836(−1)
9	−8.168730183681652(−2)	2.121093566498412(−1)	1.989769497308692(−1)	1.866576144939541(−1)
10	1.276618526350826(−1)	2.043629625254307(−1)	2.236332403317656(−1)	2.447205969383093(−1)
11	3.343397209281050(−1)	1.934927571945595(−1)	2.402229655222859(−1)	2.982389315292906(−1)
12	5.279967697041811(−1)	1.793346219930818(−1)	2.464654052473528(−1)	3.387254246200834(−1)
13	6.986047395245323(−1)	1.614985143761679(−1)	2.402911524197785(−1)	3.575255051372864(−1)
14	8.371173741032776(−1)	1.391503514761979(−1)	2.195770701687902(−1)	3.464891696815435(−1)
15	9.360696233793310(−1)	1.103159913709379(−1)	1.810627867925640(−1)	2.971802397248579(−1)
16	9.900928342242115(−1)	6.831711355306033(−2)	1.144681096278244(−1)	1.917959855196322(−1)

is again AT system and Vandermonde system (4.6) (for $n+2$ unknown weights) also has unique solution for each $m = 1, 2, \ldots, r$.

In Table 2 the nodes and weights for quadrature formulas of Gauss–Lobatto type with respect to an AT system of Jacobi weights and nearly diagonal multi-index are given.

Table 2: Quadrature formulas of Gauss–Lobato type, $r = 2$, $\alpha = -1/4$, $\beta_1 = 1$, $\beta_2 = -1/2$, $n = 18$

	y_i, x_i	$a_{1,i}$, $A_{1,i}$	$a_{2,i}$, $A_{2,i}$
1	−1.	−5.427654059531051(−7)	6.278044907035962(−2)
2	1.	5.253978709241791(−2)	1.857561986737788(−2)
1	−9.940683226494066(−1)	6.258461212382826(−5)	1.366066177674554(−1)
2	−9.973295322435293(−1)	6.724998794425119(−4)	1.541068169804305(−1)
3	−9.330999859656411(−1)	2.894341625676589(−3)	1.672669872578609(−1)
4	−8.706256860528897(−1)	8.198883072026320(−3)	1.761905550567668(−1)
5	−7.847743707278070(−1)	1.815730482233846(−2)	1.818487840801168(−1)
6	−6.759923125940263(−1)	3.409994333062372(−2)	1.848930926177310(−1)
7	−5.460968834029435(−1)	5.679958534303609(−2)	1.857378515147516(−1)
8	−3.981042546631459(−1)	8.622406731661059(−2)	1.846488449675086(−1)
9	−2.360418319851534(−1)	1.213912667316848(−1)	1.817954776087856(−1)
10	−6.474035626481439(−2)	1.603454558780371(−1)	1.772794371553346(−1)
11	1.103930065314497(−1)	2.002575860755584(−1)	1.711488163863528(−1)
12	2.836154392458143(−1)	2.376344935881942(−1)	1.634018925424208(−1)
13	4.490957313876524(−1)	2.686044097870690(−1)	1.539811371776487(−1)
14	6.011595130523827(−1)	2.892279000156788(−1)	1.427539994158158(−1)
15	7.345236460917255(−1)	2.957584275197416(−1)	1.294692974446239(−1)
16	8.445112388464046(−1)	2.847195061026688(−1)	1.136567577648302(−1)
17	9.272392748240476(−1)	2.524575707052816(−1)	9.435935045249755(−2)
18	9.797689999534666(−1)	1.926868609416974(−1)	6.917199832716925(−2)

b) Laguerre weight functions

For Laguerre weights we can construct Gauss–Radau type quadratures. The preassigned node is $y_1 = 0$, and for each AT system of generalized Laguerre weight functions [12]

$$w_m(x) = x^{s_m} e^{-x}, \quad m = 1, 2, \ldots, r$$

$s_m > -1$, $m = 1, 2, \ldots, r$; $s_i - s_j \notin \mathbb{Z}$ whenever $i \neq j$, set $\widetilde{W}$ with weights

$$\widetilde{w}_m(x) = x^{s_m+1} e^{-x}, \quad m = 1, 2, \ldots, r$$

is also AT system.

The Vandermonde system (4.6) (for $n+1$ unknown weights) has a unique solution for each $m = 1, 2, \dots, r$.

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