Applied Mathematics and Computation 218 (2011) 944–948

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A generalized Birkhoff–Young–Chebyshev quadrature formula for analytic functions ☆

Gradimir V. Milovanović [a,*], Aleksandar S. Cvetković [b], Marija P. Stanić [c]

[a] Faculty of Computer Sciences, Megatrend University, Bulevar Umetnosti 29, 11070 Novi Beograd, Serbia
[b] Department of Mathematics, Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia
[c] Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, P.O. Box 60, 34000 Kragujevac, Serbia

ARTICLE INFO

Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

Keywords:
Quadrature formula
Error term
Chebyshev polynomial
Orthogonality
Analytic function

ABSTRACT

A generalized N-point Birkhoff–Young quadrature of interpolatory type, with the Chebyshev weight, for numerical integration of analytic functions is considered. The nodes of such a quadrature are characterized by an orthogonality relation. Some special cases of this quadrature formula are derived.

1. Introduction

For numerical integration over a line segment in the complex plane, Birkhoff and Young [1] proposed a quadrature formula of the form

$$\int_{z_0-h}^{z_0+h} f(z)\mathrm{d}z \approx \frac{h}{15}\{24f(z_0)+4[f(z_0+h)+f(z_0-h)]-[f(z_0+\mathrm{i}h)+f(z_0-\mathrm{i}h)]\},$$

where $f(z)$ is a complex analytic function in $\Omega=\{z : |z-z_0|\leqslant r\}$ and $|h|\leqslant r$. This five point quadrature formula is exact for all algebraic polynomials of degree at most five, and for its error $R_5^{BY}(f)$ can be proved the following estimate [10] (see also [2, p. 136])

$$|R_5^{BY}(f)| \leqslant \frac{|h|^7}{1890}\max_{z\in S}|f^{(6)}(z)|,$$

where S denotes the square with vertices $z_0+\mathrm{i}^k h$, $k=0,1,2,3$.

Without loss of generality we can consider the integration over $[-1,1]$ for analytic functions in a unit disk $\Omega=\{z : |z|\leqslant 1\}$, so that the previous Birkhoff–Young formula becomes

$$\int_{-1}^{1} f(z)\mathrm{d}z = \frac{8}{5}f(0)+\frac{4}{15}[f(1)+f(-1)]-\frac{1}{15}[f(\mathrm{i})+f(-\mathrm{i})]+R_5(f). \tag{1.1}$$

☆ The authors were supported in part by the Serbian Ministry of Science and Technological Developments (Project: Approximation of integral and differential operators and applications, Grant Number #174015).

* Corresponding author.
E-mail addresses: gvm@megatrend.edu.rs (G.V. Milovanović), acvetkovic@mas.bg.ac.rs (A.S. Cvetković), stanicm@kg.ac.rs (M.P. Stanić).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.02.007

This five point formula can also be used to integrate real harmonic functions (see [1]). We mention here that Lyness and Delves [5] and Lyness and Moler [6], and later Lyness [4], developed formulae for numerical integration and numerical differentiation of complex functions. In 1976 Lether [3] pointed out that the three point Gauss–Legendre quadrature which is also exact for all polynomials of degree at most five is more precise than (1.1) and he recommended it for numerical integration. However, Tošić [9] improved the quadrature (1.1) in the form

$$\int_{-1}^{1} f(z)\mathrm{d}z = Af(0) + B[f(r)+f(-r)] + C[f(\mathrm{i}r)+f(-\mathrm{i}r)] + R_5^T(f;r),$$

where

$$A = 2\left(1-\frac{1}{5r^4}\right),\quad B = \frac{1}{6r^2}+\frac{1}{10r^4},\quad C = -\frac{1}{6r^2}+\frac{1}{10r^4}\quad (0<r<1),$$

and the error-term is given by the expresion

$$R_5^T(f;r) = \left(-\frac{2}{3\cdot 6!}r^4+\frac{2}{7!}\right)f^{(6)}(0) + \left(-\frac{2}{5\cdot 8!}r^4+\frac{2}{9!}\right)f^{(8)}(0)+\cdots \tag{1.2}$$

Evidently, for $r = 1$ this formula reduces to (1.1) and for $r = \sqrt{3/5}$ to the Gauss–Legendre formula (then $C = 0$). Moreover, for $r = \sqrt[4]{3/7}$ the first term on the right-hand side in (1.2) vanishes and the formula reduces to the modified Birkhoff–Young quadrature of the maximum accuracy (named *MF* in [9]), with the coefficients

$$A = \frac{16}{15},\quad B = \frac{1}{6}\left(\frac{7}{5}+\sqrt{\frac{7}{3}}\right),\quad C = \frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right),$$

and with the error-term

$$R_5^{MF}(f) = R_5^T\left(f;\sqrt[4]{3/7}\right) = \frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots$$

This formula was extended by Milovanović and Đorđević [8] to the following quadrature formula of interpolatory type:

$$\int_{-1}^{1} f(z)\mathrm{d}z = Af(0)+C_{11}[f(r_1)+f(-r_1)]+C_{12}[f(\mathrm{i}r_1)+f(-\mathrm{i}r_1)]+C_{21}[f(r_2)+f(-r_2)]+C_{22}[f(\mathrm{i}r_2)+f(-\mathrm{i}r_2)]+R_9(f;r_1,r_2),$$

where $0 < r_1 < r_2 < 1$. They proved that for

$$r_1 = r_1^* = \sqrt[4]{\frac{63-4\sqrt{114}}{143}}\quad\text{and}\quad r_2 = r_2^* = \sqrt[4]{\frac{63+4\sqrt{114}}{143}},$$

this formula has the algebraic precision $p = 13$, with the error-term

$$R_9(f;r_1^*,r_2^*) = \frac{1}{28122661066500}f^{(14)}(0)+\cdots \approx 3.56\cdot 10^{-14}f^{(14)}(0).$$

In this paper we consider a generalized quadrature with respect to the Chebyshev weight.

2. Generalized Birkhoff–Young–Chebyshev quadrtaure formula

For analytic functions in the unit disk $\Omega = \{z : |z| \leqslant 1\}$, we consider numerical integration

$$I(f) := \int_{-1}^{1}\frac{f(z)}{\sqrt{1-z^2}}\mathrm{d}z = Q_N(f)+R_N(f), \tag{2.1}$$

where Q_N is the N-point quadrature formula of interpolatory type with nodes at the zeros of a monic polynomial of degree N,

$$\omega_N(z) = z^\nu p_{n,\nu}(z^4) = z^\nu\prod_{k=1}^{n}(z^4-r_k),\quad 0<r_1<\cdots<r_n<1, \tag{2.2}$$

where $n = [N/4]$ and $\nu = N - 4[N/4]$, i.e., $N = 4n+\nu,\ n\in\mathbb{N},\ \nu\in\{0,1,2,3\}$, and $R_N(f)$ is the corresponding remainder term.

According to (2.2) the quadrature formula in (2.1) has the form

$$Q_N(f) = \sum_{j=0}^{\nu-1} C_j f^{(j)}(0) + \sum_{k=1}^{n}\{A_k[f(x_k)+f(-x_k)]+B_k[f(\mathrm{i}x_k)+f(-\mathrm{i}x_k)]\},$$

where $x_k = \sqrt[4]{r_k},\ k=1,\ldots,n$. For $\nu = 0$, the first sum in $Q_N(f)$ is empty. Also, in order to have $Q_N(f) = I(f) = 0$ for $f(z) = z$, it must be $C_1 = 0$, so that $Q_{4n+1}(f) \equiv Q_{4n+2}(f)$.

Theorem 2.1. *For any $N \in \mathbb{N}$ there exists a unique interpolatory quadrature $Q_N(f)$ with a maximal degree of precision $d = 6n + s$, where $n = [N/4]$, $\nu = N - 4[N/4] \in \{0,1,2,3\}$, and*

$$s = \begin{cases} \nu - 1, & \nu = 0, 2, \\ \nu, & \nu = 1, 3. \end{cases} \tag{2.3}$$

The nodes of such a quadrature are characterized by the following orthogonality relation:

$$\left(T_{2k}, z^{s+1} p_{n,\nu}(z^4)\right) = \int_{-1}^{1} \frac{T_{2k}(z) z^{s+1} p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz = 0, \quad k = 0, 1 \ldots, n-1, \tag{2.4}$$

where T_k is the Chebyshev polynomials of the first kind of degree k.

Proof. Let $\mathcal{P}_d$ denote the set of algebraic polynomials of degree at most d.

For a given $N \in \mathbb{N}$, suppose that $f \in \mathcal{P}_d$, where $d \geqslant N = 4n + \nu$ ($n = [N/4]$, $\nu = N - 4[N/4]$). Then, it can be expressed in the form

$$f(z) = u(z)\omega_N(z) + v(z) = u(z) z^\nu p_{n,\nu}(z^4) + v(z), \quad u \in \mathcal{P}_{d-N}, \; v \in \mathcal{P}_{N-1},$$

from which, applying (2.1), we get

$$I(f) = \int_{-1}^{1} \frac{u(z) z^\nu p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz + I(v).$$

Note that $I(v) = Q_N(v)$ (interpolatory quadrature!) and $v(z) = f(z)$ at the zeros of ω_N, and therefore $Q_N(v) = Q_N(f)$, so that for each $f \in \mathcal{P}_d$ we have

$$I(f) = \int_{-1}^{1} \frac{u(z) z^\nu p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz + Q_N(f).$$

It is clear that the quadrature formula $Q_N(f)$ has a maximal degree of precision if and only if

$$\int_{-1}^{1} \frac{u(z) z^\nu p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz = 0$$

for a maximal degree of the polynomial $u \in \mathcal{P}_{d-N}$.

According to the values of ν, the previous "orthogonality condition" can be considered as

$$\int_{-1}^{1} \frac{h(z^2) z^\nu p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz = 0 \quad \text{and} \quad \int_{-1}^{1} \frac{z h(z^2) z^\nu p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz = 0$$

for $\nu = 0$, 2 and $\nu = 1$, 3, respectively, where $h \in \mathcal{P}_{n-1}$. It can be represented in a compact form

$$\int_{-1}^{1} \frac{h(z^2) z^{s+1} p_{n,\nu}(z^4)}{\sqrt{1-z^2}} dz = 0, \quad h \in \mathcal{P}_{n-1}. \tag{2.5}$$

Thus, the maximal degree of the polynomial $u \in \mathcal{P}_{d-N}$ is

$$d_{\max} - N = \begin{cases} 2n - 1, & \nu = 0, 2, \\ 2n, & \nu = 1, 3, \end{cases}$$

i.e., $d_{\max} = 6n + s$, where s is defined by (2.3). Notice that $s + 1 \in \{0,2,4\}$.

The orthogonality conditions (2.5) can be expressed in terms of Chebyshev polynomials of the first kind, i.e., in the form (2.4), where the inner product is defined in a usual way as $(f,g) = \int_{-1}^{1} f(z)g(z)(1-z^2)^{-1/2} dz$. □

According to (2.2), the polynomial $z^{s+1} p_{n,\nu}(z^4)$ can be expressed in the form

$$z^{s+1} p_{n,\nu}(z^4) = \sum_{j=0}^{n} (-1)^j \sigma_j z^{4(n-j)+s+1}, \tag{2.6}$$

where σ_j are the so-called *elementary symmetric functions*, defined by

$$\sigma_j = \sum_{(k_1,\ldots,k_j)} r_{k_1} \cdots r_{k_j}, \quad j = 1, \ldots, n,$$

and the summation is performed over all combinations $(k_1, \ldots, k_j)$ of the basic set $\{1, \ldots, n\}$. Thus,

$$\sigma_1 = r_1 + r_2 + \cdots + r_n, \quad \sigma_2 = r_1 r_2 + \cdots + r_{n-1} r_n, \quad \ldots, \quad \sigma_n = r_1 r_2 \cdots r_n,$$

and for the convenience we put $\sigma_0 = 1$.

In the sequel we need the inner product (T_{2k}, z^{2m}), $0 \leqslant k \leqslant m$. Using the formula (cf. [7, p. 105])

$$x^k = \frac{1}{2^{k-1}} \sum_{i=0}^{[k/2]} \binom{k}{i} \frac{T_{k-2i}(x)}{1+\delta_{k,2i}},$$

where $\delta_{k,2i}$ is Kronecker's delta, we get

$$(T_{2k}, z^{2m}) = \frac{\pi}{2^{2m}} \binom{2m}{m-k}, \quad 0 \leqslant k \leqslant m.$$

Then, using (2.6), the orthogonality conditions (2.4) give the following system of linear equations:

$$\sum_{j=1}^{n} (-1)^{j-1} (T_{2k}, z^{4(n-j)+s+1}) \sigma_j = (T_{2k}, z^{4n+s+1}), \quad k = 0, 1, \ldots, n-1,$$

i.e., $\mathbf{A}\sigma = \mathbf{b}$, where $\mathbf{A} = [a_{kj}]_{k,j=1}^n$, $\sigma = [\,\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_n\,]^T$, $\mathbf{b} = [\,b_1 \quad b_2 \quad \cdots \quad b_n\,]^T$, and

$$a_{kj} = (-1)^{j-1} 2^{4j} \binom{4(n-j)+s+1}{2(n-j)+(s+1)/2-k+1}, \quad k,j = 1, \ldots, n,$$

$$b_k = \binom{4n+s+1}{2n+(s+1)/2-k+1}, \quad k = 1, \ldots, n.$$

3. Special cases

In order to calculate parameters of quadrature formula (2.1), first we calculate values of σ_k, $k = 1, \ldots, n$. Knowing values of σ_k, $k = 1, \ldots, n$, we calculate r_k, $k = 1, \ldots, n$. In Table 3.1 we give values of σ_k, $k = 1, \ldots, n$, for $n = 1, 2, 3, 4$.

The parameters of the quadrature formula (2.1) as well as the corresponding maximal degree of exactness $d = 6n + s$ are presented in Table 3.2 for $n = 1$, $\nu = 0, 1, 2, 3$, and in Table 3.3 for $n = 2$, $\nu = 0, 1, 2, 3$.

Table 3.1
The values of σ_k, $k = 1, \ldots, n$, for $n = 1, 2, 3, 4$.

n	ν	σ_1	σ_2	σ_3	σ_4
1	0	3/8			
	1, 2	5/8			
	3	35/48			
2	0	7/8	7/128		
	1, 2	21/20	21/128		
	3	33/28	33/128		
3	0	297/224	99/256	33/4096	
	1, 2	143/96	143/256	143/2096	
	3	13/8	1287/1794	143/512	
4	0	39/22	117/128	65/4096	39/32768
	1, 2	85/44	221/192	221/1024	221/32768
	3	323/156	969/704	323/1024	1615/98304

Table 3.2
Parameters and the maximal degree of exactness of the generalized Birkhoff–Young–Chebyshev quadrature formula for $n = 1$, $\nu = 0, 1, 2, 3$.

ν	x_1	A_1	B_1	C_0	C_2	d
0	$\sqrt[4]{\frac{3}{8}}$	$\frac{\pi}{2}\left(\frac{1}{2}+\frac{1}{\sqrt{6}}\right)$	$\frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{\sqrt{6}}\right)$			5
1, 2	$\sqrt[4]{\frac{5}{8}}$	$\frac{3+\sqrt{10}}{20}\pi$	$\frac{3-\sqrt{10}}{20}\pi$	$\frac{2\pi}{5}$		7
3	$\frac{1}{2}\sqrt[4]{\frac{35}{3}}$	$\frac{3(21+2\sqrt{105})}{490}\pi$	$\frac{3(21-2\sqrt{105})}{490}\pi$	$\frac{17\pi}{35}$	$\frac{\pi}{28}$	9

Table 3.3
Parameters and the maximal degree of exactness of the generalized Birkhoff–Young–Chebyshev quadrature formula for $n = 2$, $\nu = 0, 1, 2, 3$.

ν	0	1,2	3
x_1	$\frac{\sqrt[4]{7-\sqrt{35}}}{2}$	$\frac{1}{2}\sqrt[4]{\frac{42-\sqrt{714}}{5}}$	$\frac{1}{2}\sqrt[4]{\frac{66-\sqrt{1122}}{7}}$
x_2	$\frac{\sqrt[4]{7+\sqrt{35}}}{2}$	$\frac{1}{2}\sqrt[4]{\frac{42+\sqrt{714}}{5}}$	$\frac{1}{2}\sqrt[4]{\frac{66+\sqrt{1122}}{7}}$
A_1	$\frac{\left(35+\sqrt{35}+2\sqrt{5\left(49+\sqrt{35}\right)}\right)\pi}{280}$	$\frac{\left(1309+2\sqrt{714}+\sqrt{170\left(10626+53\sqrt{714}\right)}\right)\pi}{14280}$	$\frac{\left(77\left(2567+4\sqrt{1122}\right)+\sqrt{2618\left(14659854+36115\sqrt{1122}\right)}\right)\pi}{2419032}$
B_1	$\frac{\left(35+\sqrt{35}-2\sqrt{5\left(49+\sqrt{35}\right)}\right)\pi}{280}$	$\frac{\left(1309+2\sqrt{714}-\sqrt{170\left(10626+53\sqrt{714}\right)}\right)\pi}{14280}$	$\frac{\left(77\left(2567+4\sqrt{1122}\right)-\sqrt{2618\left(14659854+36115\sqrt{1122}\right)}\right)\pi}{2419032}$
A_2	$\frac{\left(35-\sqrt{35}+2\sqrt{5\left(49-\sqrt{35}\right)}\right)\pi}{280}$	$\frac{\left(1309-2\sqrt{714}+\sqrt{170\left(10626-53\sqrt{714}\right)}\right)\pi}{14280}$	$\frac{\left(77\left(2567-4\sqrt{1122}\right)+\sqrt{2618\left(14659854-36115\sqrt{1122}\right)}\right)\pi}{2419032}$
B_2	$\frac{\left(35-\sqrt{35}-2\sqrt{5\left(49-\sqrt{35}\right)}\right)\pi}{280}$	$\frac{\left(1309-2\sqrt{714}-\sqrt{170\left(10626-53\sqrt{714}\right)}\right)\pi}{14280}$	$\frac{\left(77\left(2567-4\sqrt{1122}\right)-\sqrt{2618\left(14659854-36115\sqrt{1122}\right)}\right)\pi}{2419032}$
C_0		$\frac{4\pi}{15}$	$\frac{80\pi}{231}$
C_2			$\frac{\pi}{77}$
d	11	13	15

References

[1] G. Birkhoff, D.M. Young, Numerical quadrature of analytic and harmonic functions, J. Math. Phys. 29 (1950) 217–221.
[2] P.J. Davis, P. Rabinowitz, Methods of Numerical Integration, Academic Press, New York, 1975.
[3] F. Lether, On Birkhoff–Young quadrature of analytic functions, J. Comput. Appl. Math. 2 (1976) 81–84.
[4] J.N. Lyness, Quadrature methods based on complex function values, Math. Comput. 23 (1969) 601–619.
[5] J.N. Lyness, L.M. Delves, On numerical contour integration round a closed contour, Math. Comput. 21 (1967) 561–577.
[6] J.N. Lyness, C.B. Moler, Numerical differentiation of analytic functions, SIAM J. Numer. Anal. 4 (1967) 202–210.
[7] G.V. Milovanović, Numerical Analysis, Part II, Naučna Knjiga, Belgrade, 1985 (Third Edition 1991) (Serbian).
[8] G.V. Milovanović, R.Ž. Đorđević, On a generalization of modified Birkhoff–Young quadrature formula, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fis. No. 735–No. 762 (1982) 130–134.
[9] D.Đ. Tošić, A modification of the Birkhoff–Young quadrature formula for analytic functions, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fis. No. 602–No. 633 (1978) 73–77.
[10] D.M. Young, An error bound for the numerical quadrature of analytic functions, J. Math. Phys. 31 (1952) 42–44.