Gaussian interval quadrature rule for exponential weights ⋆

Aleksandar S. Cvetković, [a,*] Gradimir V. Milovanović [b]

[a] *Department of Mathematics, Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia*

[b] *Mathematical Institute of the Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11001 Beograd, Serbia*

Abstract

Interval quadrature formulae of Gaussian type on $\mathbb{R}$ and $\mathbb{R}^+$ for exponential weight functions of the form $w(x) = \exp(-Q(x))$, where Q is a continuous function on its domain and such that all algebraic polynomials are integrable with respect to w, are considered. For a given set of nonoverlapping intervals and an arbitrary n, the uniqueness of the n-point interval Gaussian rule is proved. The results can be applied also to corresponding quadratures over $(-1, 1)$. An algorithm for the numerical construction of interval quadratures is presented. Finally, in order to illustrate the presented method, two numerical examples are done.

Key words: exponential weight; numerical integration; interval quadrature rule; weight coefficients; nodes.
AMS classification: 41A55, 65D30, 65D32.

1 Introduction

In this paper we consider Gaussian interval quadrature rule for the weight functions of the form

$$w_1(x) = \exp(-Q_1(x)), \quad x \in \mathbb{R}, \quad \text{and} \quad w_2(x) = \exp(-Q_2(x)), \quad x \in \mathbb{R}^+,$$

⋆ The authors were supported in part by the Serbian Ministry of Education and Science (Project: Approximation of integral and differential operators and applications, grant number #174015).
* Corresponding author.
E-mail addresses: aleksandarcvetkovic@pmf.ni.ac.rs (A.S. Cvetković), gvm@mi.sanu.ac.rs (G.V. Milovanović)

where Q_1 and Q_2 are supposed to be continuous functions on their domains, and given such that all algebraic polynomials are integrable with respect to w_1 and w_2. In this paper, when we want to refer to any of these two weights, we refer to the general weight w defined on I ($\mathbb{R}$ or $\mathbb{R}^+$). The index is added only when results are given specifically for a given weight.

The set of all algebraic polynomials is denoted by $\mathcal{P}$ and its subset of all algebraic polynomials of degree at most n by $\mathcal{P}_n$, where n is a natural number.

We also adopt the following definitions

$$H_n = \Big\{ \mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n \;\Big|\; h_k \geq 0,\ k = 1, \ldots, n \Big\},$$
$$H_n^h = \Big\{ \mathbf{h} = (h_1, \ldots, h_n) \in H_n \;\Big|\; \|\mathbf{h}\|_1 = \sum_{k=1}^{n} h_k \leq h \Big\}, \quad n \in \mathbb{N},\ h \geq 0.$$

Given $\mathbf{h} = (h_1, \ldots, h_n) \in H_n$, we define a Gaussian interval quadrature rule for the weight w to be

$$\int_I p(x) w(x) \mathrm{d}x = \sum_{k=1}^{n} \frac{\sigma_k}{w(I_k)} \int_{I_k} p(x) w(x) \mathrm{d}x, \quad p \in \mathcal{P}_{2n-1}, \tag{1.1}$$

where I_k, $k = 1, \ldots, n$, are nonoverlapping intervals, of the length $2h_k$, $k = 1, \ldots, n$, respectively, whose union is the proper subset of $\mathbb{R}^+$ in the case of the weight w_2. Quantities $w(I_k)$, $k = 1, \ldots, n$, are defined as

$$w(I_k) = \int_{I_k} w(x) \mathrm{d}x, \quad k = 1, \ldots, n.$$

Midpoints of the intervals I_k, $k = 1, \ldots, n$, are denoted by x_k, $k = 1, \ldots, n$, and are called the *nodes of the quadrature rule* (1.1), so that we have $I_k = (x_k - h_k, x_k + h_k)$, $k = 1, \ldots, n$. Accordingly the quantities σ_k, $k = 1, \ldots, n$, are called the *weights*.

In order to simplify our notation we define

$$I_k^1 = (x_k - h_k, x_k + h_k), \quad I_k^2 = (x_k - h_k, x_k), \quad k = 1, \ldots, n,$$

and $O_k = (x_k + h_k, x_{k+1} - h_{k+1})$, $k = 1, \ldots, n-1$, $O_n = (x_n + h_n, +\infty)$, and the interval $O_0^1 = (-\infty, x_1 - h_1)$, in case we are discussing the weight w_1 and $O_0^2 = (0, x_1 - h_1)$ if the weight w_2 is concerned. We use the notation O_0 for both intervals O_0^1 and O_0^2 and which interval is really referred would be clear from the context. Finally, the union of O_k, $k = 1, \ldots, n$, we denote by O^u, i.e., $O^u = \cup_{k=0}^n O_k$.

Given $\mathbf{h} = (h_1, \ldots, h_n) \in H_n$, we define the set of admissible nodes of the quadrature rule (1.1), to be the set

$$X_n^1(\mathbf{h}) = \Big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_k + h_k < x_{k+1} - h_{k+1}\Big\},$$
$$X_n^2(\mathbf{h}) = \Big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_1 - h_1,\ x_k + h_k < x_{k+1} - h_{k+1}\Big\},$$

for the weights w_1 and w_2, respectively. Note that the closure of X_n^1, for example, is given by

$$\overline{X_n^1(\mathbf{h})} = \Big\{\mathbf{x} = (x_1, \ldots, x_n) \mid\ x_k + h_k \le x_{k+1} - h_{k+1}\Big\}.$$

When dependence on the weight is not important, we denote by $X_n(\mathbf{h})$ the both sets.

Note that, according to continuity, we have

$$\lim_{h_k \to 0^+} \frac{1}{w(I_k)} \int_{I_k} p(x)w(x)\mathrm{d}x = p(x_k).$$

Accordingly, the Gaussian interval quadrature rule, for $\mathbf{h} = \mathbf{0}$, reduces to the standard Gaussian quadrature rule.

We are now able to formulate our main result.

Theorem 1.1 *Given $\mathbf{h} \in H_n$ there exists the unique Gaussian interval quadrature rule*

$$\int_I p(x)w(x)\mathrm{d}x = \sum_{k=1}^n \frac{\sigma_k}{w(I_k)} \int_{I_k} p(x)w(x)\mathrm{d}x, \quad p \in \mathcal{P}_{2n-1},$$

where intervals I_k, $k = 1, \ldots, n$, are nonoverlapping and I_k has the length $2h_k$. In the case of the weight w supported on $I = \mathbb{R}^+$, in addition, we have $x_1 - h_1 > 0$.

2 Preliminary results

We need the following interpolatory lemma.

Lemma 2.1 (i) *Given* $\mathbf{h} \in H_n$*, the set of nodes* $\mathbf{x} \in \overline{X_n(\mathbf{h})}$*, and the sequence of numbers* $j_k \in \{1, 2\}$*,* $k = 1, \dots, n$*, the interpolation problem*

$$\frac{1}{w(I_k^m)} \int_{I_k^m} p(x)w(x)\mathrm{d}x = f_{k,m}, \quad m = 1, j_k;\ k = 1, \dots, n,$$

has the unique solution $p \in \mathcal{P}_N$*, where* $N + 1 = \sum_{k=1}^n j_k$*.*

(ii) *Given* $\mathbf{h} \in H_n$*, the set of nodes* $\mathbf{x} \in \overline{X_n(\mathbf{h})}$ *and the sequence of numbers* $j_k \in \{1, 2\}$*,* $k = 1, \dots, n$*, for every* $c \in \mathbb{R}$*, there exists the unique polynomial* $p_c(x) = cx^{N+1} + q_c(x)$*,* $q_c \in \mathcal{P}_N$*, where* $N + 1 = \sum_{k=1}^n j_k$*, such that*

$$\frac{1}{w(I_k^m)} \int_{I_k^m} p(x)w(x)\mathrm{d}x = 0, \quad m = 1, j_k;\ k = 1, \dots, n,$$

which has exactly j_k *zeros in every interval* I_k*, and* $p_c = cp_1$*.*

Proof. For the proof of part (i) we consider the appropriate homogenous interpolation problem, i.e., the one with $f_{k,m} = 0$, $m = 1, j_k$, $k = 1, \dots, n$. We note that for $j_k = 2$ there must be a zero of the polynomial p inside every interval I_k^2 and also at least one inside every interval $I_k \backslash I_k^2$. If $j_k = 1$ it must be at least one zero inside I_k in total at least j_k zeros inside every interval I_k, $k = 1, \dots, n$, i.e., p must have at least $N + 1$ zeros. Since $p \in \mathcal{P}_N$, it must be $p \equiv 0$.

For the proof of part (ii) we just note that we can rewrite the system of equations as

$$\frac{1}{w(I_k^m)} \int_{I_k^m} q_c(x)w(x)\mathrm{d}x = -\frac{c}{w(I_k^m)} \int_{I_k^m} x^{N+1}w(x)\mathrm{d}x, \quad m = 1, j_k;\ k = 1, \dots, n,$$

which according to part (i) has the unique solution. Also, as we can see p_c is linear in c, and has exactly j_k, zeros in each interval I_k, $k = 1, \dots, n$. $\square$

In the case $f_{k,m} = 0$, $m = 1, 2$, for some $k \in \{1, \dots, n\}$, and $h_k = 0$, we note that $p(x_k) = 0$, by continuity. Since p has two zeros in I_k, for $h_k > 0$, there must be some zero of p' inside I_k, and by continuity we have $p'(x_k) = 0$, for $h_k = 0$. This means that for $f_{k,m} = 0$, $m = 1, 2$, and $h_k = 0$, our interpolation conditions become $p(x_k) = p'(x_k) = 0$.

The following lemma is rather trivial and we do not give its proof. The proof is based on the fact that, according to the assumptions we made, polynomials are integrable with respect to w.

Lemma 2.2 (i) *Given $\varepsilon > 0$ and $n \in \mathbb{N}_0$, there exists $\delta > 0$ such that*

$$0 \leq \int_M^{+\infty} x^n w(x)dx < \varepsilon,$$

provided $M > \delta$.

(ii) *Given any $\varepsilon > 0$ and $n \in \mathbb{N}_0$, there exists $\delta > 0$ such that*

$$0 \leq \int_{-\infty}^{-M} (-x)^n w_1(x)dx < \varepsilon,$$

provided $M > \delta$.

We need the following lemma which is similar to the one given in [8].

Lemma 2.3 *Given $h \geq 0$, there exists an $M > 0$ such that for every $\mathbf{h} \in H_n^h$ the nodes of the quadrature rule* (1.1) *are uniformly bounded, i.e.,*

$$|x_k| < M, \quad k = 1, \ldots, n.$$

Proof. Suppose that the statement of this lemma is wrong. Then for every $M > 0$ there exist some $\mathbf{h}_M \in H_n^h$, such that $|x_k| \geq M$, for some $k = 1, \ldots, n$. Since nodes are ordered, suppose that we have $\nu \in \{0, 1, \ldots, n-1\}$ nodes which are bounded and $n - \nu$ nodes which are not. Since, we have ν nodes which are bounded, we can always create a sequence such that all bounded nodes as well as the respective lengths are convergent. Denote those sequences by $\mathbf{x}^j = (x_1^j, \ldots, x_n^j)$ and $\mathbf{h}^j = (h_1^j, \ldots, h_n^j)$, $j \in \mathbb{N}$.

We distinguish two cases. First consider the weight $w_2 = \exp(-Q_2(x))$, where Q_2 is continuous on $\mathbb{R}^+$. Consider a sequence of polynomials p_j of degree $2n-1$ with leading coefficient -1, satisfying the following interpolation problems

$$\frac{1}{w_2(I_{k,j}^m)} \int_{I_{k,j}^m} p_j(x) w_2(x) \mathrm{d}x = 0, \quad m = 1, 2;\ k = 1, \ldots, n-1,$$

$$\frac{1}{w_2(I_{n,j})} \int_{I_{n,j}} p_j(x) w_2(x) \mathrm{d}x = 0,$$

where $I_{k,j} \equiv I^1_{k,j} = (x^j_k - h^j_k, x^j_k + h^j_k)$ and $I^2_{k,j} = (x^j_k - h^j_k, x^j_k)$ for $j \in \mathbb{N}$. According to Lemma 2.1 part (ii), the polynomial p_j exists for every $j \in \mathbb{N}$ and it has exactly two zeros in each $I_{k,j}$, $k = 1, \ldots, n-1$, and has one additional zero in $I_{n,j}$. Note that this polynomial annihilates the quadrature sum in (1.1), and is positive on the set $(0, x^j_n - h^j_n) \backslash \cup_{k=1}^{n-1} I_{k,j}$ and negative on $(x^j_n + h^j_n, +\infty)$. We are going to show that integral of such a polynomial cannot be zero, and hence, we show that the quadrature rule (1.1) whose nodes are not bounded for $\mathbf{h} \in H^h_n$ cannot exist.

Suppose that the first ν nodes and lengths are convergent and that other $n-\nu$ nodes are not, i.e., suppose that $x^j_k > M_j$, $k = \nu+1, \ldots, n$, where the sequence M_j tends to infinity. Then we have

$$\begin{aligned}\int\limits_{\mathbb{R}^+} p_j(x) w_2(x) \mathrm{d}x &= \int\limits_{O^u_j \backslash O_{n,j}} p_j w_2(x) \mathrm{d}x + \int\limits_{O_{n,j}} p_j(x) w_2(x) \mathrm{d}x \\ &= \int\limits_{O^u_j \backslash O_{n,j}} p_j w_2(x) \mathrm{d}x - \int\limits_{O_{n,j}} (-p_j)(x) w_2(x) \mathrm{d}x,\end{aligned}$$

where $O^u_j = \cup_{k=0}^n O_{k,j}$ and $O_{k,j} = (x^j_k + h^j_k, x^j_{k+1} - h^j_{k+1})$, $k = 1, \ldots, n-1$, $O_{n,j} = (x^j_n + h^j_n, +\infty)$.

Since the nodes x^j_k and the lengths h^j_k, $k = 1, \ldots, n-\nu$, are convergent, there exist some interval (α, β), $\alpha, \beta \in \mathbb{R}^+$, independent of j, with the property $(\alpha, \beta) \subset O^u_j \backslash O_{n,j}$ for $j > j_0$. Then we have

$$\begin{aligned}\int\limits_{O^u_j \backslash O_{n,j}} p_j(x) w_2(x) \mathrm{d}x &\geq \int\limits_{\alpha}^{\beta} p_j(x) w_2(x) \mathrm{d}x \\ &\geq \int\limits_{\alpha}^{\beta} (x-\alpha)^{2n-1-2n_1} (\beta - x)^{2n_1 - 1} w_2(x) \mathrm{d}x = J > 0,\end{aligned}$$

where n_1 denotes the number of intervals $I_{k,j}$ with the property $x^j_k + h^j_k < \alpha$ and where we used the simple fact that $x - x^j_k \pm h^j_k > x - \alpha$, $k = 1, \ldots, n_1$ and $x^j_k \pm h^j_k - x > \beta - x$, $k = n_1 + 1, \ldots, n$, for $x \in (\alpha, \beta)$. Note that the lower bound J does not depend of j.

Denote $M^j_+ = x^j_n + h^j_n$. For the integral over $O_{n,j}$ we have the following inequality

$$\int\limits_{O_{n,j}} (-p_j)(x) w_2(x) \mathrm{d}x \leq \int\limits_{M^j_+}^{+\infty} x^{2n-1} w_2(x) \mathrm{d}x = J_j,$$

where we used the fact that $x > x - x_k^j \pm h_k^j$, $k = 1, \ldots, n$, for $x > M_+^j$. Note that the integral J_j can be made arbitrarily small according to the fact that polynomials are integrable, i.e., we have $J_j \to 0$ as j tends to $+\infty$.

We obviously have

$$0 = \int_{\mathbb{R}^+} p_j(x) w_2(x) \mathrm{d}x \geq J - J_j \to J > 0,$$

as j tends to infinity, which is a contradiction.

For the weight w_1 we use the same construction. We choose some sequence such that n_1 nodes are diverging to $-\infty$ and n_2 nodes are diverging to $+\infty$ and $n - n_1 - n_2$ nodes which are bounded. We can always assume that the sequences of the nodes x_k^j and the lengths h_k^j, $k = n_1 + 1, \ldots, n - n_2$ are convergent. Then we can also extract some interval $(\alpha, \beta) \subset O_j^u$, $\alpha, \beta \in \mathbb{R}$, for $j > j_0$.

First consider a sequence of polynomials p_j^1 of degree $2n - 1$, with the leading coefficient -1, which exists according to Lemma 2.1, part (ii), with the property

$$\frac{1}{w_1(I_{k,j}^m)} \int_{I_{k,j}^m} p_j^1(x) w_1(x) \mathrm{d}x = 0, \quad k = 1, \ldots, n-1,$$

$$\frac{1}{w_1(I_{n,j})} \int_{I_{n,j}} p_j^1(x) w_1(x) \mathrm{d}x = 0.$$

For this sequence of polynomials we have that the quadrature sums in (1.1) are zero. For the integral we have

$$\int_{\mathbb{R}} p_j^1(x) w_1(x) \mathrm{d}x = \int_{O_j^u \setminus O_{n,j}} p_j^1(x) w_1(x) \mathrm{d}x - \int_{O_{n,j}} (-p_j^1)(x) w_1(x) \mathrm{d}x.$$

Using the same arguments as for the weight w_2, we can give the following bound

$$\int_{O_j^u \setminus O_{n,j}} p_j^1(x) w_1(x) \mathrm{d}x \geq \int_{\alpha}^{\beta} (x - \alpha)^{2n - 2n_1} (\beta - x)^{2n_1 - 1} w_1(x) \mathrm{d}x = J^1 > 0.$$

Denote $M_j^- = x_1^j - h_1^j$ and $M_j^+ = x_n^j + h_n^j$, then using integrability of the polynomials with respect to w_1, we have

$$
\begin{aligned}
\int\limits_{O_{n,j}} (-p_j^1)(x)w_1(x)\mathrm{d}x &\le \int\limits_{M_j^+} (x+M_j^-)^{2n-1}w_1(x)\mathrm{d}x \\
&\le \sum_{k=0}^{n} \binom{n}{k} (M_j^-)^k \int\limits_{M_j^+}^{+\infty} x^{2n-1-k}w_1(x)\mathrm{d}x \\
&\le (1+M_j^-)^{2n-1} \int\limits_{M_j^+}^{+\infty} x^{2n-1}w_1(x)\mathrm{d}x \\
&\le \frac{(2M_j^-)^{2n-1}}{(M_j^+)^{2n}} \int\limits_{M_j^+}^{+\infty} x^{4n-1}w_1(x)\mathrm{d}x \le \frac{(2M_j^-)^{2n-1}\varepsilon}{(M_j^+)^{2n}} = J_j^1,
\end{aligned}
$$

for $M_j^+ > \delta^+$, according to Lemma 2.2 part (i). Again, note that J^1 is a constant which does not depend of j. This gives a natural bound

$$
0 = \int\limits_{\mathbb{R}} p_j(x)w_1(x)\mathrm{d}x \ge J^1 - J_j^1.
$$

Now, for the same sequence of nodes and lengths we introduce a sequence of polynomials p_j^2 of degree $2n-1$, with the leading coefficients 1, which are solutions of the following interpolation problem

$$
\begin{aligned}
&\frac{1}{w_1(I_{1,j})} \int\limits_{I_{1,j}} p_j^2(x)w_1(x)\mathrm{d}x = 0, \\
&\frac{1}{w_1(I_{k,j}^m)} \int\limits_{I_{k,j}^m} p_j^2(x)w_1(x)\mathrm{d}x = 0, \quad m = 1,2;\ k = 2,\dots,n,
\end{aligned}
$$

which exists according to Lemma 2.1, part (ii). For this sequence, again, the quadrature sum in (1.1) is annihilated, and for the integral we have

$$
\int\limits_{\mathbb{R}} p_j^2(x)w_1(x)\mathrm{d}x = -\int\limits_{O_{1,j}} (-p_j^2)(x)w_1(x)\mathrm{d}x + \int\limits_{O_j^u \setminus O_{1,j}} p_j^2(x)w_1(x)\mathrm{d}x,
$$

where we can estimate

$$
\int\limits_{O_j^u \setminus O_{1,j}} p_j^2(x)w_1(x)\mathrm{d}x \ge \int\limits_{\alpha}^{\beta} (x-\alpha)^{2n-2n_1-1}(\beta-x)^{2n_1}w_1(x)\mathrm{d}x = J^2 > 0.
$$

Note that J^2 does not depend of j and

$$\int_{O_{1,j}} (-p_j^2)(x)w_1(x)\mathrm{d}x \leq \int_{-\infty}^{-M_j^-} (M_j^+ - x)^{2n-1} w_1(x)\mathrm{d}x$$

$$\leq \frac{(2M_j^+)^{2n-1}}{(M_j^-)^{2n}} \int_{-\infty}^{-M_j^-} (-x)^{4n-1} w_1(x)\mathrm{d}x$$

$$\leq \frac{(2M_j^+)^{2n-1}}{(M_j^-)^{2n}}\varepsilon = J_j^2 \geq 0,$$

for $M_j^- > \delta^-$, according to Lemma 2.2 part (ii). This consideration gives a natural bound

$$0 = \int_{\mathbb{R}} p_j^2(x)w_1(x)\mathrm{d}x \geq J^2 - J_j^2.$$

We are going now to prove that at least one of the sequences J_j^1 and J_j^2 tends to zero. Suppose it is J_j^2, this produces a contradiction since

$$0 = \int_{\mathbb{R}} p_j^2 w_1(x)\mathrm{d}x \geq J^2 - J_j^2 \to J^2 > 0.$$

Suppose that $J_j^1 \to c \in \mathbb{R}^+ \cup \{+\infty\}$. Then obviously we have

$$J_j^2 = \frac{(2M_j^+)^{2n-1}\varepsilon}{(M_j^-)^{2n}} = \frac{2^{4n-2}\varepsilon^2}{\frac{(2M_j^-)^{2n-1}\varepsilon}{(M_j^+)^{2n}} M_j^+ M_j^-} \to 0,$$

for $M_j^+, M_j^- \to +\infty$, which is desired a contradiction. Using the same arguments we get a similar contradiction under hypotheses $J_j^2 \to c \in \mathbb{R}^+ \cup \{+\infty\}$. $\square$

Lemma 2.4 *Given $h \geq 0$, there exists some ε_0 such that for every $\mathbf{h} \in H_n^h$ and respective nodes $\mathbf{x} \in X_n(\mathbf{h})$, we have*

$$x_{k+1} - h_{k+1} - x_k - h_k > \varepsilon_0, \quad k = 1, \dots, n.$$

In addition, if we are studying the weight w_2 we have $\varepsilon_0 < x_1 - h_1$.

Proof. Assume the contrary. Then for every $\varepsilon > 0$ there would exist two sequences $\mathbf{x}^j$ and $\mathbf{h}^j$ such that

$$x_{\nu+1}^j - h_{\nu+1}^j - x_\nu^j - h_\nu^j \leq \varepsilon.$$

We may assume that the sequences $\mathbf{x}^j$ and $\mathbf{h}^j$ are convergent, since, those two sequences belong to the compact sets. Now, consider the sequence of polynomials p_j of degree $2n-2$, with the leading coefficient 1, which satisfies the following interpolation problem

$$\frac{1}{w(I^m_{k,j})}\int\limits_{I^m_{k,j}} p_j(x)w(x)\mathrm{d}x = 0, \quad m = 1,2;\ k = 1\ldots,\nu-1,$$

$$\frac{1}{w(I_{\nu,j})}\int\limits_{I^m_{\nu,j}} p_j(x)w(x)\mathrm{d}x = \frac{1}{w(I_{\nu+1,j})}\int\limits_{I^m_{\nu+1,j}} p_j(x)w(x)\mathrm{d}x = 0,$$

$$\frac{1}{w(I^m_{k,j})}\int\limits_{I^m_{k,j}} p_j(x)w(x)\mathrm{d}x = 0, \quad m = 1,2;\ k = \nu+2\ldots,n.$$

Every polynomial in this sequence annihilates the quadrature sum in (1.1) and also every polynomial is positive on $O^u_j \setminus O_{\nu,j}$ and is negative on $O_{\nu,j}$, so that we have

$$0 = \int\limits_I p_j(x)w(x)\mathrm{d}x = \int\limits_{O^u_j\setminus O_{\nu,j}} p_j(x)w(x)\mathrm{d}x - \int\limits_{O_{\nu,j}} (-p_j)(x)w(x)\mathrm{d}x.$$

Obviously first integral is bounded from below with for example

$$\int\limits_M^{+\infty} (x-M)^{2n-2}w(x)\mathrm{d}x > 0$$

and the second integral tends to zero as $j \to +\infty$, since

$$\begin{aligned} 0 &\le \int\limits_{O_{\nu,j}} (-p_j)(x)w(x)\mathrm{d}x \\ &\le (M + x^j_{\nu+1} + h^j_{\nu+1})^{2\nu-1}(M - x^j_\nu - h^j_\nu)^{2(n-\nu)-1} \max_{x\in O^j_\nu} w(x)\varepsilon, \end{aligned}$$

which produces a contradiction.

Finally, for the weight w_2, consider two sequences $\mathbf{h}^j$ and $\mathbf{x}^j$ for which for every $\varepsilon > 0$ we have $x^j_1 - h^j_1 < \varepsilon$. Then, consider a sequence of polynomials p_j, of degree $2n-1$ with leading coefficient 1, which is a solution of the following interpolation problem

$$\frac{1}{w(I_{1,j})}\int\limits_{I_{1,j}} p_j(x)w_2(x)\mathrm{d}x = 0,$$
$$\frac{1}{w(I_{k,j}^m)}\int\limits_{I_{k,j}^m} p_j(x)w_2(x)\mathrm{d}x = 0, \quad m = 1,2;\ k = 2,\ldots,n.$$

Obviously every polynomial p_j is positive on $O_j^u \backslash O_{1,j}$ and negative on $O_{1,j}$ and annihilates the quadrature sum in (1.1). Thus, we have

$$0 = \int\limits_{\mathbb{R}^+} p_j(x)w_2(x)\mathrm{d}x = \int\limits_{O_j^u\backslash O_{1,j}} p_j(x)w_2(x)\mathrm{d}x - \int\limits_{O_{1,j}} (-p_j)(x)w_2(x)\mathrm{d}x.$$

First integral is bounded from bellow by

$$\int\limits_M^{+\infty} (x-M)^{2n-1}w_2(x)\mathrm{d}x > 0,$$

while the second integral is bounded from above by

$$M^{2n-1}\int\limits_{O_{1,j}} w_2(x)\mathrm{d}x \to 0,$$

as $j \to +\infty$. $\square$

According to previous two lemmas we can define the set

$$X_n^{\varepsilon_0,M} = \Big\{\mathbf{x}\in\mathbb{R}^n \ \Big|\ |x_n| < M,\ x_{\nu+1} - h_{\nu+1} - x_\nu - h_\nu > \varepsilon_0,\ \nu = 1\ldots,n\Big\}$$

for the weight w_1 and

$$X_n^{\varepsilon_0,M} = \Big\{\mathbf{x}\in\mathbb{R}^n \ \Big|\ \varepsilon_0 < x_1 - h_1,\ x_n < M,\ x_{\nu+1} - h_{\nu+1} - x_\nu - h_\nu > \varepsilon_0,\ \nu = 1\ldots,n\Big\}$$

for the weight w_2, and also we know that nodes of the quadrature rules belong to these two sets for the respective weights.

Lemma 2.5 *For* $\mathbf{h} \in H_n^h$ *there exists some* $\delta_0 > 0$ *such that* $\sigma_k > \delta_0$*,* $k = 1,\ldots,n$*, where* σ_k*,* $k = 1,\ldots,n$*, are weights in the quadrature rule* (1.1)*.*

Proof. Assume the contrary. Then there would exist some sequences $\mathbf{h}^j$ and $\mathbf{x}^j$, such that we have $\sigma_\nu^j \to 0$, for some $\nu \in \{1,\ldots,n\}$ as j tends to infinity.

Since the sets H_n^h and $\overline{X_n^{\varepsilon_0,M}(\mathbf{h})}$ are compact we can extract convergent subsequences. Now, consider a sequence of polynomials p_j, of degree $2n-2$ with leading coefficient 1, which satisfy the following interpolation problem

$$\frac{1}{w(I_{k,j}^m)} \int_{I_{k,j}^m} p_j(x)w(x)\mathrm{d}x = 0, \quad m = 1, 2;\ k = 1, \dots, \nu-1, \nu+1, \dots, n.$$

For this sequence of polynomials we have

$$\begin{aligned}\int_I p_j(x)w(x)\mathrm{d}x &= \int_{O_j \cup I_{\nu,j}} p_j(x)w(x)\mathrm{d}x = \frac{\sigma_\nu^j}{w(I_{\nu,j})} \int_{I_{\nu,j}} p_j(x)w(x)\mathrm{d}x \\ &\leq \sigma_\nu^j (M + x_\nu^j + h_\nu^j)^{2(\nu-1)} (M - x_\nu^j + h_\nu^j)^{2(n-\nu)},\end{aligned}$$

according to the convergence of the sequences $\mathbf{h}^j$ and $\mathbf{x}^j$,

$$\lim_{j\to+\infty} \sigma_\nu^j \geq \frac{\int_M^{+\infty} (x-M)^{2n-2} w(x)dx}{(M - x_\nu - h_\nu)^{2(\nu-1)} (M - x_\nu + h_\nu)^{2(n-\nu)}} > 0,$$

which produces a contradiction. □

Trivially, the weights are bounded from above, since, for the polynomial $p \equiv 1$, we have

$$\int_I w(x)\mathrm{d}x = \sum_{k=1}^n \sigma_k > \sigma_\nu, \quad \nu = 1, \dots, n.$$

3 Proof of the main result

Finally, we are able to prove our main result. The proof is based on an idea given in [3]. The difference is that we are going to apply the technique to the system (1.1) directly, which in turn gives us opportunity to develop a numerical algorithm for the construction of the quadrature rule.

In the sequel we need the following notation

$$\Delta_k^{h_k}(\pi_\ell w) = \partial_{x_k} \left(\frac{1}{w(I_k)} \int_{I_k} \pi_\ell w(x)\mathrm{d}x \right) = \frac{(\pi_\ell w)(x_k + h_k) - (\pi_\ell w)(x_k - h_k)}{w(I_k)}$$

$$-\frac{w(x_k+h_k)-w(x_k-h_k)}{w^2(I_k)}\int_{I_k}\pi_\ell(x)w(x)\mathrm{d}x.$$

Note that we have a continuity, i.e.,

$$\begin{aligned}\Delta_k^0(\pi_\ell w) &= \lim_{h_k\to 0^+}\Delta_k^{h_k}(\pi_\ell w)\\ &= \lim_{h_k\to 0^+}\frac{w(x_k+h_k)(\pi_\ell(x_k+h_k)-\pi_\ell(\zeta))-w(x_k-h_k)(\pi_\ell(x_k-h_k)-\pi_\ell(\zeta))}{w(\eta)2h_k}\\ &= \pi_\ell'(x_k),\end{aligned}$$

where, according to the mean value theorem, we used

$$\int_{I_k} w(x)\mathrm{d}x = w(\eta)2h_k, \quad \frac{1}{w(I_k)}\int_{I_k}\pi_\ell(x)w(x)\mathrm{d}x = \pi_\ell(\zeta).$$

Proof of Theorem 1.1. By π_ℓ, $\ell\in\mathbb{N}_0$, we denote polynomials orthonormal with respect to the weight w, and then we define a vector representing the weights $\sigma=(\sigma_1,\ldots,\sigma_n)$.

We are going to fix some $h\ge 0$, and we consider the following family of functions

$$F_\ell^{\mathbf{h}}(\sigma,\mathbf{x})=\sum_{k=1}^n\frac{\sigma_k}{w(I_k)}\int_{I_k}\pi_\ell(x)w(x)\mathrm{d}x-\delta_{\ell,0}, \quad \ell=0,1,\ldots,2n-1, \quad \mathbf{h}\in H_n^h,$$

as well as the family of mappings $\mathbf{F_h}(\sigma,\mathbf{x})=(F_0^{\mathbf{h}},F_1^{\mathbf{h}},\ldots,F_{2n-1}^{\mathbf{h}})$, $\mathbf{h}\in H_n^h$. As a domain of the functions $F_\ell^{\mathbf{h}}$, $\ell=0,1,\ldots,2n-1$, and the mapping $\mathbf{F^h}$, we consider the set $D=[\delta_0,\mu_0]^n\times\overline{X_n^{\varepsilon_0,M}(\mathbf{h})}$, where $\mu_0=\int_I w(x)\mathrm{d}x$. According to the property

$$\int_I \pi_\ell(x)w(x)\mathrm{d}x=\delta_{\ell,0}, \qquad \ell=0,1,\ldots,2n-1,$$

we conclude that any solution of the equation

$$\mathbf{F^h}(\sigma,\mathbf{x})=\mathbf{0},$$

i.e., of the system of equations

$$F_\ell^{\mathbf{h}}(\sigma,\mathbf{x})=\sum_{k=1}^n\frac{\sigma_k}{w(I_k)}\int_{I_k}\pi_\ell(x)w(x)\mathrm{d}x-\delta_{\ell,0}=0, \quad \ell=0,\ldots,2n-1, \tag{3.1}$$

gives a quadrature rule of the form (1.1).

First we establish easily that $F_\ell^{\mathbf{h}}$, $\ell = 0, 1, \ldots, 2n-1$, are continuous functions on their domain. Second we conclude that the partial derivatives of the functions $F_\ell^{\mathbf{h}}$ are given by

$$\partial_{\sigma_k} F_\ell^{\mathbf{h}} = \frac{1}{w(I_k)} \int\limits_{I_k} \pi_\ell(x) w(x) \mathrm{d}x, \quad \partial_{x_k} F_\ell^{\mathbf{h}} = \sigma_k \Delta_k^{h_k}(\pi_\ell w).$$

As it can be inspected we see that partial derivatives are continuous as functions on D, this implies that the mapping $\mathbf{F_h}$ is Fréchet differentiable (see [?], [11]).

Next we prove that the mapping $\mathbf{F_h}$ has a nonsingular Jacobian on D. First, we easily establish that the determinant of the Jacobian matrix can be expressed in the form

$$J = \left(\prod_{k=1}^{n} \sigma_k \right) \det[\, A \quad B \,], \tag{3.2}$$

where

$$A = \begin{bmatrix} \frac{1}{w(I_1)} \int\limits_{I_1} \pi_0(x) w(x) \mathrm{d}x & \cdots & \frac{1}{w(I_n)} \int\limits_{I_n} \pi_0(x) w(x) \mathrm{d}x \\ \vdots & \ddots & \vdots \\ \frac{1}{w(I_1)} \int\limits_{I_1} \pi_{2n-1}(x) w(x) \mathrm{d}x & \cdots & \frac{1}{w(I_n)} \int\limits_{I_n} \pi_{2n-1}(x) w(x) \mathrm{d}x \end{bmatrix}$$

and

$$B = \begin{bmatrix} \Delta_1^{h_1}(\pi_0 w) & \Delta_2^{h_2}(\pi_0 w) & \cdots & \Delta_n^{h_n}(\pi_0 w) \\ \Delta_1^{h_1}(\pi_1 w) & \Delta_2^{h_2}(\pi_1 w) & \cdots & \Delta_n^{h_n}(\pi_1 w) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_1^{h_1}(\pi_{2n-1} w) & \Delta_2^{h_2}(\pi_{2n-1} w) & \cdots & \Delta_n^{h_n}(\pi_{2n-1} w) \end{bmatrix}.$$

We know that the weights $\sigma_k \in [\delta_0, \mu_0]$, $k = 1, \ldots, n$, so if $J = 0$, then it must be that the rows of the determinant are linearly dependent. This leads to a conclusion that the following interpolation problem

$$\frac{1}{w(I_k)}\int\limits_{I_k} p(x)w(x)\mathrm{d}x = 0, \quad \Delta_k^{h_k}(pw) = 0, \quad k = 1,\ldots,n,$$

has a nontrivial solution for $p \in \mathcal{P}_{2n-1}$. Note that

$$\begin{aligned}\Delta_k^{h_k}(pw) &= \frac{(pw)(x_k+h_k)-(pw)(x_k-h_k)}{w(I_k)} \\ &\qquad -\frac{w(x_k+h_k)-w(x_k-h_k)}{w^2(I_k)}\int\limits_{I_k}\pi_\ell(x)w(x)\mathrm{d}x \\ &= \frac{(pw)(x_k+h_k)-(pw)(x_k-h_k)}{w(I_k)},\end{aligned}$$

since the second term is zero according to the first interpolation condition. Hence, a consequence of the assumption $J = 0$ is the existence of the solution of the following interpolation problem

$$\frac{1}{w(I_k)}\int\limits_{I_k} p(x)w(x)\mathrm{d}x = 0, \ \frac{(pw)(x_k+h_k)-(pw)(x_k-h_k)}{w(I_k)} = 0, \quad k = 1,\ldots,n.$$

According to the first interpolation condition, p must have a zero inside of each I_k. According to the second interpolation condition we have

$$(pw)(x_k+h_k) = (pw)(x_k-h_k) \quad \text{if } h_k \neq 0$$

or $p'(x_k) = 0$ if $h_k = 0$.

Combining these facts, we conclude that p must have at least two zeros in each I_k, $k = 1,\ldots,n$. Hence, the polynomial p must have at least $2n$ zeros and only such a polynomial from $\mathcal{P}_{2n-1}$ is $p \equiv 0$. Hence, our interpolation problem has only trivial solutions. Accordingly, the corresponding determinant is not equal to zero.

Finally, we have that themapping $\mathbf{F}_\mathbf{h}$ has a nonsingular Jacobian on its domain. According to the implicit function theorem, the known solution for $\mathbf{h} = \mathbf{0}$ can be extended uniquely for all $\mathbf{h} \in H_n^h$, since the family $\mathbf{F}^\mathbf{h}$ depends continuously on $\mathbf{h}$. □

We emphasize here that the system of equations (3.1), with the same method of proof, can be used also for the weight functions of the form $w_3(x) = \exp(-Q_3(x))$, $x \in (-1,1)$, where Q_3 is continuous and such that polynomials are integrable with respect to w_3, to prove the existence and uniqueness of the quadrature rule (1.1). Here, the same argument applies that Jacobian J is non singular at the solution.

4 Numerical construction

We have the following simple result:

Theorem 4.1 *Nodes and weights in the quadrature rule* (1.1) *are continuous functions of* $\mathbf{h}$.

Proof. According to the implicit function Theorem, differentiability of $\mathbf{F}$ implies continuity with respect to the parameter. □

Theorem 4.2 *Let* $x_k^{\mathbf{h}_1}$, $\sigma_k^{\mathbf{h}_1}$, $k = 1, \dots, n$, *be nodes and weights of the Gaussian interval quadrature rule for the vector of the lengths* $\mathbf{h}_1$. *There exists some* $\varepsilon > 0$, *such that for all* $\mathbf{h}$, *with the property* $||\mathbf{h} - \mathbf{h}_1|| < \varepsilon$, *the Newton-Kantorovich process applied to the nonlinear system of equations*

$$F_\ell^{\mathbf{h}}(\sigma, \mathbf{x}) = 0, \quad \ell = 0, 1, \dots, 2n-1, \tag{4.1}$$

converges with the starting values $x_k^{\mathbf{h}_1}$, $\sigma_k^{\mathbf{h}_1}$, $k = 1, \dots, n$.

Proof. We note that Jacobian of the system of equations (4.1) is non-singular for the sequence of lengths $\mathbf{h}_1$ at the solution $\mathbf{x}^{\mathbf{h}_1}$ and $\sigma^{\mathbf{h}_1}$ and the convergence of the Newton-Kantorovich method can be established (cf. [6,11]). □

The previous two theorems can be used to design an algorithm for constructing quadrature rules (1.1). Namely, for constructing Gaussian interval quadrature rules for the weight w, with the given vector of the lengths h, we can state the following algorithm:

1° Choose $\mathbf{h}_0 = 0$, $\mathbf{x}_0 = \mathbf{x}^0$ and $\sigma_0 = \sigma^0$, where $\mathbf{x}^0$ and σ^0 are nodes and weights of the Gaussian quadrature rule for the weight w. Take $\mathbf{h}_1 = \mathbf{h}$.
2° Solve the system (3.1) using the Newton-Kantorovich method for the lengths $\mathbf{h}_1$ with the starting values $\mathbf{x}_1$ and σ_1.
3° If the method diverges take $\mathbf{h}_1 = \mathbf{h}_0 + (\mathbf{h}_1 - \mathbf{h}_0)/2$, $\mathbf{x}_1 = \mathbf{x}_0$ and $\sigma_1 = \sigma_0$ and go back to Step 2°.
4° Take $\mathbf{h}_0 = \mathbf{h}_1$, $\mathbf{x}_0 = \mathbf{x}_1$ and $\sigma_0 = \sigma_1$. If $\mathbf{h}_0 \neq \mathbf{h}$ go back to Step 2°.

5° Return $\mathbf{h}_0$, $\mathbf{x}_0$ and σ_0.

The Newton-Kantorovich method has the following form

$$\mathbf{y}_{k+1} = \mathbf{y}_k - J^{-1} F^{\mathbf{h}}(\mathbf{y}_k),$$

where J is given in (3.2) and where we are using $\mathbf{y} = (\sigma, \mathbf{x})$. As it can be seen we must be able to compute the integrals of the form

$$\int_{I_k} \pi_\ell(x) w(x)\, \mathrm{d}x, \quad \ell \in \mathbb{N}_0, \quad k = 1, \ldots, n. \tag{4.2}$$

Since the weight function w is continuous, we can apply the Gauss-Legendre quadrature for computing the integrals (4.2). Using the N-point Gauss-Legendre quadrature rule, with the nodes t_m and the weights A_m, $m = 1, \ldots, N$, we have the following approximation of the integral

$$\int_{I_k} \pi_m(x) w(x) \mathrm{d}x \approx h_k \sum_{\ell=1}^{N} A_\ell (\pi_m w)(h_k t_\ell + x_k).$$

It is trivial that the quadrature sum to the right converges to the integral on the left, since, $p_m(x)w(x)$ is a continuous function.

As an illustration we give two examples.

Example 4.1. We consider the construction of the Gaussian interval quadrature rule for the weight function $w_3(x) = (1 + \sin^2(\pi x))\chi_{[-1,1]}(x)$. The construction of polynomials π_n, $n \in \mathbb{N}_0$, orthonormal with respect to w_3 can be done using a software presented in [4]. The coefficients of the three-term recurrence relation

$$x\pi_n(x) = \beta_{n+1}\pi_{n+1}(x) + \alpha_n \pi_n(x) + \beta_n \pi_{n-1}(x),$$

are given in Table 4.1. Due to the symmetry argument we have $\alpha_n = 0$, $n \in \mathbb{N}_0$.

In Table 4.1 we present the nodes x_k and the weights σ_k, $k = 1, \ldots, 20$, of the interval quadrature rule for the lengths h_k, $k = 1, \ldots, n$. Numbers in parentheses indicate decimal exponents.

In these computations we used the Gauss-Legendre quadrature rule with $N = 200$, which can be constructed easily using the software presented in [4]. We emphasize that only 6 iterations are needed to achieve the machine

n	β_n	h_n	x_n	σ_n
0	.1732050807568877(1)			
1	.5625357494354858	.5(−3)	−.9918786560396439	.1956804397339318(−1)
2	.4744097291915951	.5(−3)	−.9614381075560322	.4251776720722553(−1)
3	.5451748814372039	.5(−3)	−.9080446112010355	.6989649666054748(−1)
4	.4902516082531161	.5(−3)	−.8329679467843836	.1062769691409894
5	.5116615917925814	.5(−3)	−.7393850335196595	.1557774933444651
6	.4960434357623456	.5(−3)	−.6308744385377325	.2115648925047616
7	.5043712049750382	.5(−3)	−.5101786576975410	.2521640009724387
8	.4992622199415154	.5(−3)	−.3789105635382944	.2538678888971356
9	.5018594691915659	.5(−3)	−.2372713857587796	.2149883222579176
10	.5000692849492345	.5(−3)	−.8382147223781044(−1)	.1702023935242340
11	.5008805207452746	1.(−2)	.7773915805021710(−1)	.1695740403227302
12	.5002567008862471	1.(−2)	.2317904609560994	.2131028185764868
13	.5004970651019456	1.(−2)	.3737399160708616	.2526704845526047
14	.5002662750464688	1.(−2)	.5051421897004346	.2523796479631700
15	.5003247686077697	1.(−2)	.6258704045058415	.2128404700920239
16	.5002308842803479	1.(−2)	.7343498427170446	.1572976039307774
17	.5002353021643951	1.(−2)	.8278804340496836	.1074210611029361
18	.5001915456563984	1.(−2)	.9029506268697068	.7053778826824858(−1)
19	.5001820387838695	1.(−2)	.9565319074873180	.4317303040340147(−1)
20	.5001580813732057	1.(−2)	.9888837787522304	.2417878630451257(−1)

Table 4.1
Three-term recurrence coefficients β_n for the weight function $w_3(x) = (1 + \sin^2(\pi x))\chi_{[-1,1]}$. The lengths h_k, nodes x_k and weights σ_k in the Gaussian interval quadrature rule (1.1) for the weight function w_3

precision (m.p. $\approx 2.22 \times 10^{-16}$). The condition number of the Jacobian J is approximately equal to 10^2.

Example 4.2. As another example we consider a construction of the interval quadrature rule for the weight function $w_1(x) = (1+x^2)\exp(-x^2)$ on $\mathbb{R}$. Table 4.2 gives results and necessary data for the computation. We emphasize that in this case we needed only 5 iterations to achieve the machine precision. The condition number of the Jacobian was approximately 10^6. Hence, we used the higher precision arithmetics for computations. Finally, we emphasize that in

n	β_n	h_n	x_n	σ_n
0	.1630546158916783(1)			
1	.9128709291752769	.5(−2)	−.3612388438899722(1)	.3088689648398463(−4)
2	.1125462867742275(1)	.5(−2)	−.2709854370117718(1)	.4442533429127377(−2)
3	.1359179318946780(1)	.5(−2)	−.1926462245521316(1)	.8699337386563393(−1)
4	.1527966904418630(1)	.5(−2)	−.1180470276632278(1)	.4432275725131416
5	.1692733753984765(1)	.5(−2)	−.4110974306984462	.7945347145218178
6	.1834362605472096(1)	1.(−2)	.4109390457868029	.7946527364203003
7	.1969724283534903(1)	1.(−2)	.1180387980330989(1)	.4433091460680362
8	.2093412469129451(1)	1.(−2)	.1926426174233540(1)	.8701506670179548(−1)
9	.2211564341431752(1)	1.(−2)	.2709855088042445(1)	.4443848801467260(−2)
10	.2322528159157648(1)	1.(−2)	.3612424340490755(1)	.3089714047003200(−4)

Table 4.2
The three-term recurrence coefficients β_n for the weight $w_2(x) = (1 + x^2)\exp(-x^2)$ on $\mathbb{R}$. The lengths h_k, the nodes x_k and the weights σ_k in the Gaussian interval quadrature rule (1.1) for the weight function w_2.

the case of the classical weight functions, the corresponding algorithms for constructing Gaussian interval quadrature rules are much more efficient and they are presented in [7–10].

References

[1] B. Bojanov, P. Petrov, Gaussian interval quadrature formula, Numer. Math., **87** (2001), 625–643.

[2] B. Bojanov, P. Petrov, Uniqueness of the Gaussian interval quadrature formula, Numer. Math., **95** (2003), 53–62.

[3] B. Bojanov, P. Petrov, Gaussian interval quadrature formula for Tchebycheff systems, SIAM J. Numer. Anal., **43** (2005), 787–795.

[4] A.S. Cvetković, G.V. Milovanović, The Mathematica Package “OrthogonalPolynomials”, Facta Univ. Ser. Math. Inform., **19** (2004), 17–36.

[5] R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, Tensor Analysis and Applications, Springer-Verlag, 1998.

[6] C.T. Kelley, Iterative Methods for Linear and Nonlinear Equations, SIAM, Philadelphia, 1995

[7] G.V. Milovanović, A.S. Cvetković, Uniqueness and computation of Gaussian interval quadrature formula for Jacobi weight function, Numer. Math., **99** (2004), 141–162.

[8] G.V. Milovanović, A.S. Cvetković, Gauss-Laguerre interval quadrature rule, J. Comput. Appl. Math., **182** (2005), 433–446.

[9] G.V. Milovanović, A.S. Cvetković, Gauss-Radau and Gauss-Lobatto interval quadrature rules for Jacobi weight function, Numer. Math., **102** (2006), 523–542.

[10] G.V. Milovanović, A.S. Cvetković, *Gauss-Hermite interval quadrature rule*, Comput. Math. Appl., **54** (2007), 544–555.

[11] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.