Available online at www.sciencedirect.com
SCIENCE DIRECT®
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 173 (2005) 383–388
www.elsevier.com/locate/cam

Letter to the Editor

Remarks on "Orthogonality of some sequences of the rational functions and Müntz polynomials"☆

Gradimir V. Milovanović*, Aleksandar S. Cvetković

Department of Mathematics, Faculty of Electronic Engineering, University of Niš, P.O. Box 73, 18000 Niš, Serbia and Montenegro

Received 29 January 2004; received in revised form 22 March 2004

Abstract

We give some remarks to results presented in Marinković et al. (J. Comput. Appl. Math. 163 (2004) 119). Namely, these results are direct consequences from Milovanović et al. (J. Comput. Appl. Math. 99 (1998) 299) and some of them are equivalent up to bilinear mappings.

Keywords: Rational functions; Müntz polynomials; Orthogonality; Inner product

1. Introduction

Let $M_{\gamma,\delta}^{\alpha,\beta}: \mathbb{C} \to \mathbb{C} (\alpha, \beta, \gamma, \delta \in \mathbb{R})$ be a nonsingular bilinear transformation

$$M_{\gamma,\delta}^{\alpha,\beta}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \Delta = \alpha\delta - \gamma\beta \neq 0, \quad z \in \mathbb{C}.$$

It is known that a composition of bilinear transformations is also a bilinear transformation, as well as that nonsingular bilinear transformations, with the composition of functions as an operation, form a group which is isomorphic with the multiplicative group of nonsingular matrices of type 2×2 (cf. [4, p. 135]).

☆ The authors were supported in part by the Serbian Ministry of Science, Technology and Development (Project #2002: Applied Orthogonal Systems, Constructive Approximation and Numerical Methods).
* Corresponding author.
E-mail addresses: grade@elfak.ni.ac.yu (G.V. Milovanović), aca@elfak.ni.ac.yu (A.S. Cvetković).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.05.007

As in [3], for an arbitrary sequence of complex numbers $\{a_\nu\}_{\nu\in\mathbb{N}_0}$, with the property $|a_\nu|<1$, we consider the sequence of rational functions (Malmquist–Takenaka basis)

$$w_n(s)=\frac{\prod_{\nu=0}^{n-1}(s-a_\nu)}{\prod_{\nu=0}^{n}(s-1/\overline{a}_\nu)},\quad n\in\mathbb{N}_0, \tag{1.1}$$

which is orthogonal with respect to the inner product defined by

$$(u,v)=\frac{1}{2\pi\mathrm{i}}\oint_{|s|=1}u(s)\overline{v(s)}\frac{\mathrm{d}s}{s}=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(\mathrm{e}^{\mathrm{i}\theta})\overline{v(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\theta \tag{1.2}$$

(cf. [5, sections 9.1 and 10.7], [3], or the survey paper [2]). Note that on the circle $|s|=1$ we have $s=1/\overline{s}$ as well as $(w_n,w_m)=\|w_n\|^2\delta_{n,m}$, where $\|w_n\|^2=|a_0a_1\cdots a_n|^2/(1-|a_n|^2)$.

For different complex numbers α_ν, $\nu\in\mathbb{N}_0$, such that $c\alpha_\nu\overline{\alpha}_\mu-a(\alpha_\nu+\overline{\alpha}_\mu)-b>0$, $\nu,\mu\in\mathbb{N}_0$, in [1] the authors considered the system of rational functions

$$W_n(z)=\frac{\prod_{\nu=0}^{n-1}(z-(a\overline{\alpha}_\nu+b)/(c\overline{\alpha}_\nu-a))}{\prod_{\nu=0}^{n}(z-\alpha_\nu)}=\frac{\prod_{\nu=0}^{n-1}(z-M_{c,-a}^{a,b}(\overline{\alpha}_\nu))}{\prod_{\nu=0}^{n-1}(z-\alpha_\nu)} \tag{1.3}$$

and proved its orthogonality with respect to

$$(W_n,W_m)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma}W_{\max\{m,n\}}(z)\overline{W_{\min\{m,n\}}\left(\frac{a\overline{z}+b}{c\overline{z}-a}\right)}\frac{\mathrm{d}z}{cz-a}, \tag{1.4}$$

where the contour Γ is given by $c|z|^2-a(z+\overline{z})-b=0$, $a,b,c\in\mathbb{R}$, $a^2+bc>0$. This inner product is defined only on the set $\mathscr{W}=\{W_\nu|\nu\in\mathbb{N}_0\}$ and not on the linear span of that set. Note that it cannot be extended to a linear span of $\mathscr{W}$ over $\mathbb{C}$ because that would require for example $(\lambda W_0,\mu W_0)=\lambda\overline{\mu}(W_0,W_0)=\overline{\lambda}\mu(W_0,W_0)$, which can only be true if $\lambda,\mu\in\mathbb{R}$.

Thus, inner product (1.4) can be extended on the linear span of the set $\mathscr{W}$ only if the linear span is taken over the reals. Denote that span by $\mathscr{L}(\mathscr{W})$. In that case we can compute (p,q), $p,q\in\mathscr{L}(\mathscr{W})$, only by knowing the expansions of $p=\sum_k p_kW_k$ and $q=\sum_k q_kW_k$, in which case we can compute $(p,q)=\sum_{n,m}p_nq_m(W_n,W_m)$.

If we drop max and min in definition (1.4), the inner product becomes well defined and equivalent to the inner product defined in (1.2) up to a bilinear transformation which is determined by a,b,c. Namely, then for such an inner product

$$(U,V)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma}U(z)\overline{V(z)}\,\frac{\mathrm{d}z}{cz-a}, \tag{1.5}$$

where $\Gamma=\{z\in\mathbb{C}:cz\overline{z}-a(z+\overline{z})-b=0,\ a,b,c\in\mathbb{R},\ a^2+bc>0\}$, its value remains the same on $\mathscr{L}(\mathscr{W})$ as for (1.4), but it can be extended (uniquely) on the linear span of $\mathscr{W}$ over $\mathbb{C}$, without having the unnatural property that expansions of elements should be known in order to compute inner product of two elements from the span. Note that $z=(a\overline{z}+b)/(c\overline{z}-a)$ when $z\in\Gamma$.

2. Orthogonality of rational functions (1.3)

Putting $s = M_{\gamma,\delta}^{\alpha,\beta}(z)$, where α and γ are chosen such that $\gamma a_\nu \neq \alpha$ and $\gamma/\overline{a}_\nu \neq \alpha$, in (1.1), we get

$$V_n(z) = w_n(M_{\gamma,\delta}^{\alpha,\beta}(z)) = A_n \frac{\gamma z + \delta}{z - M_{-\gamma,\alpha}^{\delta,-\beta}(1/\overline{a}_n)} \prod_{\nu=0}^{n-1} \frac{z - M_{-\gamma,\alpha}^{\delta,-\beta}(a_\nu)}{z - M_{-\gamma,\alpha}^{\delta,-\beta}(1/\overline{a}_\nu)}, \tag{2.1}$$

where

$$A_n = \frac{1}{\alpha - \gamma/\overline{a}_n} \prod_{\nu=0}^{n-1} \frac{\gamma a_\nu - \alpha}{\gamma/\overline{a}_\nu - \alpha} = \frac{1}{\gamma} w_n(\alpha/\gamma).$$

Taking $\alpha_\nu = M_{-\gamma,\alpha}^{\delta,-\beta}(1/\overline{a}_\nu)$, which gives $a_\nu = M_{\alpha,\beta}^{\gamma,\delta}(\overline{\alpha}_\nu)$, the function $V_n(z)$ can be rewritten in the following form:

$$V_n(z) = A_n \frac{\gamma z + \delta}{z - \alpha_n} \prod_{\nu=0}^{n-1} \frac{z - M_{\alpha^2-\gamma^2,\alpha\beta-\gamma\delta}^{\gamma\delta-\alpha\beta,\delta^2-\beta^2}(\overline{\alpha}_\nu)}{z - \alpha_\nu} = A_n(\gamma z + \delta) W_n(z), \tag{2.2}$$

where we put

$$a = \gamma\delta - \alpha\beta, \quad b = \delta^2 - \beta^2, \quad c = \alpha^2 - \gamma^2, \tag{2.3}$$

so that $W_n(z)$ is given by (1.3). Note that by this bilinear transformation $s = M_{\gamma,\delta}^{\alpha,\beta}(z)$, the unit circle is mapped to the curve $\{z \,:\, (\alpha^2 - \gamma^2) z\overline{z} + (\alpha\beta - \gamma\delta)(z + \overline{z}) + \beta^2 - \delta^2 = 0\}$, i.e., Γ, according to (2.3).

Now, by $s = M_{\gamma,\delta}^{\alpha,\beta}(z)$ and starting from (1.1) and (1.2), we have

$$\begin{aligned}
\|w_n\|^2 \delta_{n,m} &= \frac{1}{2\pi \mathrm{i}} \oint_{|s|=1} w_n(s) \overline{w_m(s)} \frac{\mathrm{d}s}{s} \\
&= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} V_n(z) \overline{V_m(z)} \frac{(\alpha\delta - \beta\gamma)\, \mathrm{d}z}{(\alpha z + \beta)(\gamma z + \delta)} \\
&= \frac{A_n \overline{A}_m (\alpha\delta - \beta\gamma)}{2\pi \mathrm{i}} \oint_{\Gamma} W_n(z) \overline{W_m(z)} \frac{\gamma\overline{z} + z}{\alpha z + \beta}\, \mathrm{d}z,
\end{aligned}$$

where we used (2.1) and (2.2). Since $(\gamma\overline{z} + \delta)/(\alpha z + \beta) = \Delta/(cz - a)$, when $z \in \Gamma (\Delta = \alpha\delta - \gamma\beta)$, we get

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} W_n(z) \overline{W_m(z)} \frac{\mathrm{d}z}{cz - a} = \frac{\|w_n\|^2}{|A_n|^2 \Delta^2} \delta_{n,m},$$

i.e., functions (1.3) are orthogonal with respect to the inner product (1.5).

Note that we choose only nonsingular bilinear transformations, hence $\Delta \neq 0$. Also, $A_n \neq 0$ can be assured since our bilinear transformation is chosen such that $\gamma a_\nu \neq \alpha$. Note that the condition $|a_\nu| < 1$ is equivalent to the condition that points α_ν and $M_{c,-a}^{a,b}(\overline{\alpha}_\nu)$ lie on different sides of Γ and have the property

$c|\alpha_\nu|^2 - 2a\,\mathrm{Re}(\alpha_\nu) - b > 0$. Thus, it is clear that we do not require $c\alpha_\nu\overline{\alpha}_\nu - a(\alpha_\nu + \overline{\alpha}_\nu) - b > 0$, $\nu, \mu \in \mathbb{N}_0$ as in [1].[1]

For arbitrary $a, b, c \in \mathbb{R}$, $a^2 + bc > 0$, we can find nonsingular real bilinear transformation $M_{\gamma,\delta}^{\alpha,\beta}$, which coefficients satisfy (2.3). For $c \neq 0$ and $b \neq 0$, the solution can be found in the following parameterized form:

$$
\begin{aligned}
&\alpha = \sqrt{c}\,\cosh\,\theta, \quad \beta = \sqrt{b}\,\sinh\,\phi, \quad \gamma = \sqrt{c}\,\sinh\,\theta, \quad \delta = \sqrt{b}\,\cosh\,\phi,\\
&a = \sqrt{bc}\,\sinh(\theta - \phi), \quad \text{provided}\ \ b, c > 0,\\
&\alpha = \sqrt{-c}\,\sinh\,\theta, \quad \beta = \sqrt{b}\,\sinh\,\phi, \quad \gamma = \sqrt{-c}\,\cosh\,\theta, \quad \delta = \sqrt{b}\,\cosh\,\phi,\\
&a = \sqrt{-cb}\,\cosh\,(\theta - \phi), \quad \text{provided}\ \ c < 0, \quad b > 0.
\end{aligned}
$$

Similar solutions can be given also in the cases $c > 0$, $b < 0$ and $c < 0$, $b < 0$. For $c > 0$, $b < 0$, the solutions can be obtained from the case $c < 0$, $b > 0$, by simple change of the names c with b, α with δ and γ with β. For $c < 0$, $b < 0$, in solution for $c > 0$, $b > 0$, simply change b with $-b$, c with $-c$, α with γ and β with δ. Note that conditions $\gamma a_\nu \neq \alpha$ and $\gamma/\overline{a}_\nu \neq \alpha$ can be satisfied since one free parameter exists.

In the case $c = 0$, note that a cannot be zero, because of the condition $a^2 + bc > 0$, and we can choose

$$
\alpha = \gamma = t \in \mathbb{R}\backslash\{0\}, \quad \beta = \frac{1}{2}\left(\frac{bt}{a} - \frac{a}{t}\right), \quad \delta = \frac{1}{2}\left(\frac{bt}{a} + \frac{a}{t}\right).
$$

In this case $\alpha/\gamma = 1$ which means that our circle $|s| = 1$ is mapped by this transformation into the line $z = -b/a$.

3. Orthogonality of Müntz polynomials

We take a sequence of complex numbers $\{\lambda_\nu | \nu \in \mathbb{N}_0\}$, with the following properties $\mathrm{Re}(\lambda_\nu\overline{\lambda}_\mu) > 1$, ν, $\mu \in \mathbb{N}_0$, and $|\lambda_\nu| > 1$, $\nu \in \mathbb{N}_0$. As Müntz polynomials in $\mathscr{X} = \{x^{\lambda_\nu} : \nu \in \mathbb{N}_0\}$, we define the linear span of $\mathscr{X}$ over $\mathbb{C}$ and denote it by $\mathscr{L}(\mathscr{X})$. Also, we define the product $\odot$ of two monomials from $\mathscr{X}$ as $x^{\lambda_\nu} \odot x^{\lambda_\mu} = x^{\lambda_\nu\lambda_\mu}$. It can be extended naturally to $\mathscr{L}(\mathscr{X})$ assuming it is linear. Next, we define a bilinear functional on $\mathscr{L}(\mathscr{X})^2$ in the following way:

$$
[p, q] = \int_0^1 (p \odot \overline{q})(x)\,\frac{\mathrm{d}x}{x^2}, \quad p, q \in \mathscr{L}(\mathscr{X}). \tag{3.1}
$$

The linearity and symmetry $[p, q] = \overline{[q, p]}$ of this functional are evident. In [3] it was proved that it is positive-definite, using a representation of it as a bilinear form over the $\mathbb{C}^n$. For any two polynomials $p = \sum_{\nu=0}^n p_\nu x^{\lambda_\nu}$ and $q = \sum_{\nu=0}^n q_\nu x^{\lambda_\nu}$ in $\mathscr{L}(\mathscr{X})$, we have

$$
[p, q] = \sum_{\nu,\mu=0}^n \frac{p_\nu q_\mu}{\lambda_\nu\overline{\lambda}_\mu - 1}. \tag{3.2}
$$

Note that conditions $\mathrm{Re}(\lambda_\nu\overline{\lambda}_\mu) > 1$, $\nu, \mu \in \mathbb{N}_0$, are needed in order to have an integrability of $x^{\lambda_\nu\overline{\lambda}_\mu - 2}$ over $(0,\ 1)$. However, in proving positive definiteness of the bilinear form it is enough to prove that diagonal

[1] There is a miss-print in this condition in [1, Eq. (2.2)].

elements in the corresponding determinant are positive and that all main minors are positive. Thus, we are concerned with the following determinants:

$$D_k = \det\left[\frac{1}{\lambda_\nu \overline{\lambda}_\mu - 1}\right]_{\nu,\mu=0}^{k} = \frac{D_{k-1}}{|\lambda_k|^2 - 1} \prod_{\nu=0}^{k-1} \frac{|\lambda_k - \lambda_\nu|^2}{|\lambda_\nu \overline{\lambda}_k - 1|^2}, \quad k = 1, \ldots, n,$$

where $D_0 = 1/(|\lambda_0|^2 - 1) > 0$ (see [3]). According to conditions $|\lambda_\nu| > 1$, $\nu \in \mathbb{N}_0$, the diagonal elements are positive, as well as $D_k > 0$ for all k.

There is also a beautiful connection between rational functions (1.1) and Müntz polynomials P_n, $n \in \mathbb{N}_0$, orthogonal with respect to $[.,.]$. Namely, it was proved that

$$P_n(x) = \frac{1}{2\pi \mathrm{i}} \oint_G w_n(s) x^s \, \mathrm{d}s, \quad n \in \mathbb{N}_0,$$

where the single contour G surrounds all the points $\lambda_\nu = 1/\overline{a}_\nu$, $\nu = 0, 1, \ldots, n$. Also, $[P_n, P_m] = (w_n, w_m)$.

The condition $\mathrm{Re}(\lambda_\nu \overline{\lambda_\mu}) > 1$, $\nu, \mu \in \mathbb{N}_0$, is required only for the inner product $[.,.]$ to have a representation over integral (3.1). We can go opposite, i.e., we can define the inner product using the bilinear form (3.2). Then, in order to have a positive definiteness, as we mentioned, it is enough to have $|\lambda_\nu| > 1$, $\nu \in \mathbb{N}_0$. This means that, in this way, we can extend the collection of underlying sets $\mathscr{X}$. In the sequel, by $[.,.]$ we assume an inner product defined by the bilinear form (3.2) with $|\lambda_\nu| > 1$, $\nu \in \mathbb{N}_0$, which reduces to (3.1), provided $\mathrm{Re}[\lambda_\nu \overline{\lambda_\mu}] > 1$, $\nu, \mu \in \mathbb{N}_0$. All the properties for $[.,.]$, defined by (3.1), are valid for $[.,.]$, defined using (3.2), since in [3] they are proved using purely algebraic properties of D_n.

Introducing some complicated products of monomials on $\mathscr{X}_\alpha = \{x^{\alpha_\nu} : \nu \in \mathbb{N}_0\}$, for example as

$$x^{\alpha_\nu} * x^{\alpha_\mu} = x^{R_1(\alpha_\nu) R_1(\alpha_\mu) - R_2(\alpha_\nu) R_2(\alpha_\mu) + 1},$$

where R_1 and R_2 are any two polynomials and complex numbers α_ν, $\nu \in \mathbb{N}_0$, are chosen such that

$$\mathrm{Re}[R_1(\alpha_\nu)\overline{R_1(\alpha_\mu)} - R_2(\alpha_\nu)\overline{R_2(\alpha_\mu)}] > 0 \tag{3.3}$$

and that $R_2(\alpha_\nu) \neq 0$, $\nu \in \mathbb{N}_0$, we do not get anything new. An explanation of this fact follows.

Naturally, we can extend the domain of $*$ to the linear span of $\mathscr{X}_\alpha$ over $\mathbb{C}$, denote it by $\mathscr{L}(\mathscr{X}_\alpha)$, assuming it is linear and define a bilinear functional on $\mathscr{L}(\mathscr{X}_\alpha)$ as

$$\langle p, q \rangle = \int_0^1 (p * \overline{q})(x) \frac{\mathrm{d}x}{x^2}, \quad p, q \in \mathscr{L}(\mathscr{X}_\alpha).$$

As before, for two polynomials $p = \sum_{\nu=0}^{n} p_\nu x^{\alpha_\nu}$ and $q = \sum_{\nu=0}^{n} q_\nu x^{\alpha_\nu}$ in $\mathscr{L}(\mathscr{X}_\alpha)$, we have

$$\langle p, q \rangle = \sum_{\nu,\,\mu=0}^{n} \frac{p_\nu \overline{q}_\mu}{R_1(\alpha_\nu)\overline{R_1(\alpha_\mu)} - R_2(\alpha_\nu)\overline{R_2(\alpha_\mu)}}.$$

To prove definiteness it is enough to prove that the following determinants

$$D_k^\alpha = \det\left[\frac{1}{R_1(\alpha_\nu)\overline{R_1(\alpha_\mu)} - R_2(\alpha_\nu)\overline{R_2(\alpha_\mu)}}\right]_{\nu,\mu=0}^{k}, \quad k = 0, 1, \ldots, n$$

have positive main minors. But, taking $\lambda_\nu = R_1(\alpha_\nu)/R_2(\alpha_\nu)$, we have

$$D_k^\alpha = \prod_{\nu=0}^{k} \frac{1}{|R_2(\alpha_\nu)|^2} \det\left[\frac{1}{(R_1(\alpha_\nu)/R_2(\alpha_\nu))\overline{(R_1(\alpha_\mu)/\overline{R_2(\alpha_\mu)})} - 1}\right]_{\nu,\mu=0}^{k} = \prod_{\nu=0}^{k} \frac{1}{|R_2(\alpha_\nu)|^2} D_k.$$

Note that (3.3) implies $|\lambda_\nu| > 1$; however, it does not imply that for every ν and μ we have $\mathrm{Re}(\lambda_\nu\overline{\lambda}_\mu) > 1$. But, there are sequences α_ν, $\nu \in \mathbb{N}_0$, for which λ_ν fulfills the requirement. The point is that we cannot claim directly a connection between $\langle .,.\rangle$ and $[.,.]$ if we stick to definition (3.1) for any sequence $\alpha_\nu \in \mathbb{N}_0$. But, if we use $[.,.]$, defined by (3.2), obviously there is a direct connection between $\langle .,.\rangle$ and $[.,.]$, i.e., $\lambda_\nu = R_1(\alpha_\nu)/R_2(\alpha_\nu)$, $\nu \in \mathbb{N}_0$. Thus, according to a result from [2] about positive definiteness for $[.,.]$, defined using (3.2), the positive definiteness of $\langle .,.\rangle$ follows directly.

It is clear from representations of the inner products $[.,.]$ and $\langle .,.\rangle$ over bilinear forms that if $P_n(x) = \sum_{k=0}^{n} p_k^n x^{\lambda_k}$ is the orthogonal polynomial with respect to $[.,.]$, defined using (3.2), then the corresponding orthogonal polynomial Q_n with respect to $\langle .,.\rangle$ is given by

$$Q_n(x) = \sum_{k=0}^{n} \frac{p_k^n x^{\alpha_k}}{R_2(\alpha_k)}.$$

It can be proved easily, since we have

$$\begin{aligned}[p,q] &= \sum_{\nu,\mu=0}^{n} \frac{p_\nu \overline{q}_\mu}{\lambda_\nu\overline{\lambda}_\mu - 1} = \sum_{\nu,\mu=0}^{n} \frac{p_\nu/R_2(\alpha_\nu)\overline{q_\mu/R_2(\alpha_\nu)}}{R_1(\alpha_\nu)\overline{R_1(\alpha_\mu)} - R_2(\alpha_\nu)\overline{R_2(\alpha_\mu)}} \\ &= \int_0^1 \left(\sum_{\nu=0}^{n} \frac{p_\nu x^{\alpha_\nu}}{R_2(\alpha_\nu)}\right) * \overline{\left(\sum_{\mu=0}^{n} \frac{q_\nu x^{\alpha_\nu}}{R_2(\alpha_\nu)}\right)} \frac{\mathrm{d}x}{x^2} = \langle \mathscr{I}p, \mathscr{I}q\rangle,\end{aligned}$$

where we define a linear bijective mapping $\mathscr{I} : \mathscr{L}(\mathscr{X}) \to \mathscr{L}(\mathscr{X}_\alpha)$ by $\mathscr{I}x^{\lambda_\nu} = x^{\alpha_\nu}/R_2(\alpha_\nu)$, where $\lambda_\nu = R_1(\alpha_\nu)/R_2(\alpha_\nu)$, $\nu \in \mathbb{N}_0$. Using $[P_n, P_m] = (w_n, w_m)$ (see [3]), we can also get $\langle Q_n, Q_m\rangle = \langle \mathscr{I}P_n, \mathscr{I}P_m\rangle = [P_n, P_m] = (w_n, w_m)$, $n, m \in \mathbb{N}_0$.

Therefore, special results presented in [1] can be recovered simply by taking $R_1(z) = \alpha z + \beta$ and $R_2(z) = \gamma z + \delta$, where α, β, γ and δ are determined according to a, b, c given by (2.3).

References

[1] S.D. Marinković, B. Danković, M.S. Stanković, P.M. Rajković, Orthogonality of some sequences of the rational functions and the Müntz polynomials, J. Comput. Appl. Math. 163 (2004) 419–427.

[2] G.V. Milovanović, Some generalized orthogonal systems and their connections, in: N. Bokan (Ed.), Proceedings of the Symposium “Contemporary Mathematics”, Belgrade, 1998, Faculty of Mathematics, University of Belgrade, 2000, pp. 181–200.

[3] G.V. Milovanović, B. Danković, S.Lj. Rančić, Some Müntz orthogonal systems, J. Comput. Appl. Math. 99 (1998) 299–310.

[4] D.S. Mitrinović, Complex Analysis, Gradjevinska Knjiga, Belgrade, 1997 (in Serbian).

[5] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, American Mathematical Society Colloquium Publication, Vol. 20, 5th Edition, American Mathematical Society, Providence, RI, 1969.