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Bounds of the error of Gauss–Turán-type quadratures[☆]

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Abstract

We consider the remainder term of the Gauss–Turán quadrature formulae

$$R_{n,s}(f) = \int_{-1}^{1} w(t) f(t)\, \mathrm{d}t - \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v)$$

for analytic functions in some region of the complex plane containing the interval $[-1, 1]$ in its interior. The remainder term is presented in the form of a contour integral over confocal ellipses or circles. A strong error analysis is given for the case with a generalized class of weight functions, introduced recently by Gori and Micchelli. Also, we discuss a general case with an even weight function defined on $[-1, 1]$. Numerical results are included.

MSC: primary 41A55; secondary 65D30; 65D32

Keywords: Gauss–Turán quadrature; *s*-orthogonal polynomial; Multiple nodes; Remainder term for analytic functions; Contour integral representation; Error expansion; Error bounds

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1. Introduction

We consider the error term $R_{n,s}(f)$ in the Gauss–Turán quadrature formula with multiple nodes (see [26])

$$\int_{-1}^{1} w(t)f(t)\,\mathrm{d}t = \sum_{v=1}^{n}\sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v) + R_{n,s}(f). \tag{1.1}$$

Here, w is an integrable weight function on the interval $(-1, 1)$. It is well-known that the quadrature formula (1.1) is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, and that its nodes are the zeros of the corresponding s-orthogonal polynomial $\pi_{n,s}(t)$ of degree n. For more details on Gauss–Turán quadratures and s-orthogonal polynomials see the book [9] and survey paper [14], as well as very recent papers [15,19].

An analysis for analytic functions of the reminder term $R_{n,s}(f)$ of Gauss–Turán quadratures (1.1), with generalized Chebyshev weight functions, was given in [16–18]. The aim in this paper is to extend this analysis to certain wide classes of weight functions. After some general considerations in Sections 2, we study in Section 3 the case with a generalized class of weight functions, introduced recently by Gori and Micchelli [10], and then in sequel we develop an error analysis for even weight functions defined on $[-1, 1]$.

2. Remainder term for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and D be its interior. If the integrand f is an analytic function in D and continuous on $\overline{D}$, then we take as our starting point the well-known expression of the remainder term $R_{n,s}(f)$ in the form of the contour integral (cf. [20])

$$R_{n,s}(f) = \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma} K_{n,s}(z)f(z)\,\mathrm{d}z. \tag{2.1}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}, \quad z \notin [-1, 1], \tag{2.2}$$

where

$$\varrho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\, w(t)\,\mathrm{d}t, \quad n \in \mathbb{N} \tag{2.3}$$

and $\pi_{n,s}(t)$ is the corresponding s-orthogonal polynomial with respect to the measure $w(t)\,\mathrm{d}t$ on $(-1, 1)$.

Suppose that $\pi_{n,s}(t) = \kappa_n t^n + \kappa_{n-1}t^{n-1} + \cdots$, with $\kappa_n = \kappa_n^{(s)} > 0$, and also that the weight $w(t)$ is an even function on $[-1, 1]$.

Lemma 2.1. *Let $w(-t) = w(t)$. In the expansion*

$$\frac{1}{\pi_{n,s}(z)} = \sum_{j=0}^{+\infty} b_{n,j}^{(s)} z^{-n-j}, \quad |z| \geqslant 1,$$

for $n > 1$ there holds

$$b_{n,2j+1}^{(s)} = 0, \quad b_{n,2j}^{(s)} > 0 \quad (j = 0, 1, \ldots),$$

and for $n = 1$ there holds $1/\pi_{1,s}(z) = b_{1,0}^{(s)} z^{-1}$, where $b_{1,0}^{(s)} = 1/\kappa_1 > 0$.

Proof. It can be done in an analogous way as one of Lemma 1 in Stenger's paper [25], or by taking a new weight function $\overline{w}_{n,s}$, defined by (see [2, pp. 214–226]) $\overline{w}_{n,s}(t) = [\pi_{n,s}(t)]^{2s} w(t)$. Since this weight is also even on $[-1, 1]$, we have $\pi_{n,s}(-t) = (-1)^n \pi_{n,s}(t)$ (cf. [9]). Now, applying Lemma 1 from [25] the proof is finished. □

Thus, in our case, for $|z| \geqslant 1$, we have $[\pi_{n,s}(z)]^{-1} = \sum_{j=0}^{+\infty} b_{n,2j}^{(s)} z^{-n-2j}$ and

$$\frac{1}{[\pi_{n,s}(z)]^{2s+1}} = z^{-n(2s+1)} \left(\sum_{j=0}^{+\infty} b_{n,2j}^{(s)} z^{-2j} \right)^{2s+1},$$

i.e.,

$$\frac{1}{[\pi_{n,s}(z)]^{2s+1}} = \sum_{j=0}^{+\infty} \overline{b}_{n,2j}^{(s)} z^{-n(2s+1)-2j}. \tag{2.4}$$

It is clear that $\overline{b}_{n,2j+1}^{(s)} = 0$ and $\overline{b}_{n,2j}^{(s)} > 0$.

If $b_{n,2k}^{(s)}$ are known, by using [11, Eq. 0.314], it is possible to determine $\overline{b}_{n,2j}^{(s)}$ in the following way

$$\overline{b}_{n,0}^{(s)} = b_{n,0}^{2s+1}, \quad \overline{b}_{n,2j}^{(s)} = \frac{1}{j b_{n,0}^{(s)}} \sum_{k=1}^{j} (2k(s+1) - j)\, b_{n,2k}^{(s)}\, \overline{b}_{n,2j-2k}^{(s)}.$$

Lemma 2.2. *For $\varrho_{n,s}(z)$ in* (2.3) *there holds*

$$\varrho_{n,s}(z) = \sum_{j=0}^{+\infty} c_{n,2j}^{(s)} z^{-n-2j-1}, \quad |z| > 1, \tag{2.5}$$

where $c_{n,2j}^{(s)} > 0$ $(n, s, j \in \mathbb{N}_0)$.

Proof. This result is an immediate consequence of Lemma 3 from [25] and the representation (2.3), i.e.,

$$\varrho_{n,s}(z) = \int_{-1}^{1} \frac{\pi_{n,s}(z)}{z - t} \overline{w}_{n,s}(t)\, \mathrm{d}t,$$

where $\overline{w}_{n,s}(t)$ is the new weight function introduced above in the proof of Lemma 2.1. □

An alternative expression for the kernel (2.2) can be given as $K_{n,s}(z)=R_{n,s}(g_z)$, where $g_z(t)=1/(z-t)$ (cf. [20,16]).

If (1.1) is a symmetric quadrature formula, i.e., if $w(t)$ is an even weight function on $[-1,1]$ and Γ is a symmetric curve (e.g., a circle with center at origin and radius ϱ (>1), or a confocal ellipse), then the function $\hat{g}_z(t)=z/(z^2-t^2)$ is even on $[-1,1]$. In this case for the nodes in (1.1) satisfy (cf. [9]) $\tau_\nu=-\tau_{n-\nu+1}$ ($\nu=1,2,\ldots,n$) (if n is odd then $\tau_{[n/2]+1}=0$). For the coefficients we have $A_{i,\nu}=(-1)^i A_{i,n-\nu+1}$ (if n is odd and i is odd then $A_{i,[n/2]+1}=0$).

In [21] it was proved that $A_{i,\nu}>0$ for even i. If we consider the functions $\overline{g}_z(t)=g_z(t)-\hat{g}_z(t)=t/(z^2-t^2)$ and $\overline{f}(t)=f(t)-f_{\overline{s}}(t)=\frac{1}{2}(f(t)-f(-t))$, where $f_{\overline{s}}(t)=\frac{1}{2}(f(t)+f(-t))$, which are continuous and odd, and whose derivatives of an arbitrary order are continuous functions on $[-1,1]$ which are even if i is odd, and odd if i is even.

Now, it is not difficult to conclude that $R_{n,s}(\overline{f})=0$ and $R_{n,s}(\overline{g})=0$, $K_{n,s}(z)=R_{n,s}(g_z)=R_{n,s}(\hat{g}_z)$ and $R_{n,s}(f)=R_{n,s}(f_{\overline{s}})$.

Therefore, some analogous bounds of the modulus of the remainder term, which for Gauss-type quadrature formula have been derived by Sherer and Schira (see [22, Section 2]), hold for Gauss–Turán type quadrature formula.

The integral representation (2.1) leads directly to the error estimate

$$|R_{n,s}(f)|\leqslant\frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.6}$$

where $\ell(\Gamma)$ is the length of the contour Γ.

A general estimate can be obtained by Hölder inequality. Thus,

$$|R_{n,s}(f)|=\frac{1}{2\pi}\left|\oint_\Gamma K_{n,s}(z)f(z)\,\mathrm{d}z\right|\leqslant\frac{1}{2\pi}\left(\oint_\Gamma|K_{n,s}(z)|^r|\,\mathrm{d}z|\right)^{1/r}\left(\oint_\Gamma|f(z)|^{r'}|\,\mathrm{d}z|\right)^{1/r'},$$

i.e.,

$$|R_{n,s}(f)|\leqslant\frac{1}{2\pi}\,\|K_{n,s}\|_r\,\|f\|_{r'}, \tag{2.7}$$

where $1\leqslant r\leqslant+\infty$, $1/r+1/r'=1$, and

$$\|f\|_r:=\begin{cases}\left(\oint_\Gamma|f(z)|^r|\,\mathrm{d}z|\right)^{1/r} & 1\leqslant r<+\infty,\\ \max\limits_{z\in\Gamma}|f(z)| & r=+\infty.\end{cases}$$

The case $r=+\infty$ ($r'=1$) gives

$$|R_{n,s}(f)|\leqslant\frac{1}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\oint_\Gamma|f(z)||\,\mathrm{d}z|\right)\leqslant\frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right),$$

i.e., (2.6). On the other side, for $r=1$ ($r'=+\infty$), the estimate (2.7) reduces to

$$|R_{n,s}(f)|\leqslant\frac{1}{2\pi}\left(\oint_\Gamma|K_{n,s}(z)||\,\mathrm{d}z|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.8}$$

which is evidently stronger than the previous, because of inequality

$$\oint_{\Gamma} |K_{n,s}(z)|\,|\mathrm{d}z| \leqslant \ell(\Gamma)\left(\max_{z\in\Gamma} |K_{n,s}(z)|\right).$$

Also, the case $r = r' = 2$ could be of some interest.

For getting the estimate (2.6) or (2.8) it is necessary to study the magnitude of $|K_{n,s}(z)|$ on Γ or the quantity $L_{n,s}(\Gamma) := (1/2\pi)\oint_{\Gamma}|K_{n,s}(z)|\,|\mathrm{d}z|$, respectively.

Taking the contour Γ as a confocal ellipse with foci at the points ∓ 1 and sum of semi-axes $\varrho > 1$, and w as one of the four generalized weight functions:

(a) $w_1(t) = (1-t^2)^{-1/2}$, (b) $w_2(t) = (1-t^2)^{1/2+s}$,
(c) $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, (d) $w_4(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$,

we studied in detail the estimates (2.6) and (2.8) in [16–18]. In that cases certain analytical results can be done. Namely, it is well-known that the Chebyshev polynomials of first kind T_n are s-orthogonal subject to $w_1(t)$ on $[-1, 1]$ for each $s \geqslant 0$ (see [1]), and that for three other weights $w_i(t)$, $i = 2, 3, 4$, the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, and W_n, which are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}, \quad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin\theta/2},$$

respectively (see [6,20]). Such weights, however, depend on s. Notice that the weight function in (d) can be omitted from an investigation because of $W_n(-t) = (-1)^n V_n(t)$.

The error term in Gaussian quadratures ($s = 0$) for these weights was studied in the case $r = +\infty$ and $r = 1$ by Gautschi and Varga [8] and Hunter [12], respectively (see also [3–5,7,23,24,13]).

3. Error estimates for Gauss–Turán quadratures with Gori–Micchelli weight functions

Recently, Gori and Micchelli [10] have introduced for each n a class of weight functions defined on $[-1, 1]$ for which explicit Gauss–Turán quadrature formulae can be found for all s. In the other words, these classes of weight functions have the peculiarity that the corresponding s-orthogonal polynomials, of the same degree, are independent of s. This class includes certain generalized Jacobi weight functions $w_{n,\mu}(t) = |U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^{\mu}$, where $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$ (Chebyshev polynomial of the second kind) and $\mu > -1$. In this case, the Chebyshev polynomials $T_n(t)$ appear to be s-orthogonal polynomials.

For simplicity, in this section we consider the case $s = 1$ and $\mu = 1/2$, i.e.,

$$w(t) = w_{n,1/2}(t) = |U_{n-1}(t)/n|^2\sqrt{1-t^2}. \tag{3.1}$$

We take the contour Γ as an ellipse with foci at the points ± 1 and sum of semi-axes $\varrho > 1$,

$$E_{\varrho} = \{z \in \mathbb{C}:\ z = \tfrac{1}{2}(\varrho \mathrm{e}^{\mathrm{i}\theta} + \varrho^{-1}\mathrm{e}^{-\mathrm{i}\theta}),\ 0 \leqslant \theta < 2\pi\}. \tag{3.2}$$

Now, we consider the estimate (2.8) for $s=1$, when $\Gamma = E_\varrho$. Since $z=\frac{1}{2}(\xi+\xi^{-1})$, $\xi=\varrho e^{i\theta}$, and $|\mathrm{d}z| = 2^{-1/2}\sqrt{a_2-\cos 2\theta}\,\mathrm{d}\theta$, where we put

$$a_j = a_j(\varrho) = \frac{1}{2}(\varrho^j+\varrho^{-j}), \quad j\in\mathbb{N},\ \varrho>1, \tag{3.3}$$

we have

$$L_{n,1}(E_\varrho) = \frac{1}{2\pi\sqrt{2}}\int_0^{2\pi}\frac{|\varrho_{n,1}(z)|(a_2-\cos 2\theta)^{1/2}}{|\pi_{n,1}(z)|^3}\,\mathrm{d}\theta. \tag{3.4}$$

Theorem 3.1. *Let* E_ϱ $(\varrho>1)$ *be given by* (3.2), $b_{2n}=\varrho^{2n}+\varrho^{-2n}/4$ *and* a_{2n} *be defined by* (3.3). *Then, for the weight function* (3.1), *the quantity* $L_{n,1}(\mathscr{E}_\varrho)$ *can be expressed in the form*

$$L_{n,1}(\mathscr{E}_\varrho) = \frac{1}{2n^2\varrho^{3n}}\int_0^{\pi}\sqrt{\frac{(a_{2n}-\cos\theta)(b_{2n}+\cos\theta)}{(a_{2n}+\cos\theta)^3}}\,\mathrm{d}\theta. \tag{3.5}$$

Furthermore, the following estimate

$$L_{n,1}(E_\varrho) \leqslant \frac{\pi}{2n^2}\sqrt{\frac{1-x-9x^2+29x^3+4x^4}{x(x-1)^5}} \tag{3.6}$$

holds, where $x=\varrho^{4n}$.

Proof. For $z\in\mathscr{E}_\varrho$, i.e., $z=\frac{1}{2}(\xi+\xi^{-1})$, $\xi=\varrho e^{i\theta}$, we have

$$\pi_{n,s}(z) = T_n(z) = \frac{1}{2}\left(\xi^n+\xi^{-n}\right)$$

and, according to (2.3) and (3.1),

$$\varrho_{n,s}(z) = \frac{1}{n^2}\int_{-1}^{1}\frac{T_n(t)^{2s+1}U_{n-1}(t)^2}{z-t}\sqrt{1-t^2}\,\mathrm{d}t.$$

Using the equalities $2T_nU_{2n-1}=U_{3n-1}+U_{n-1}$ and $2(1-t^2)U_{(k+1)n-1}U_{kn-1}=T_n-T_{(2k+1)n}$ (for $k=1$ and $k=2$), the previous integral becomes

$$\varrho_{n,s}(z) = \frac{1}{16n^2}\int_{-1}^{1}\frac{T_n(t)^{2s-2}(2T_n(t)-T_{3n}(t)-T_{5n}(t))}{(z-t)\sqrt{1-t^2}}\,\mathrm{d}t. \tag{3.7}$$

In the simplest case $s=1$, (3.7) reduces to

$$\varrho_{n,1}(z) = \frac{\pi}{8n^2}\frac{(2\xi^n+\xi^{-n})(\xi^n-\xi^{-n})}{\xi^{3n}(\xi-\xi^{-1})}$$

i.e.,

$$|\varrho_{n,1}(z)| = \frac{\pi}{4n^2\varrho^{3n}}\frac{(a_{2n}-\cos 2n\theta)^{1/2}(b_{2n}+\cos 2n\theta)^{1/2}}{(a_2-\cos 2\theta)^{1/2}},$$

where $b_{2n}=\varrho^{2n}+\varrho^{-2n}/4$ and a_j defined by (3.3).

Now, by substitution this expression in (3.4), we obtain

$$L_{n,1}(E_\varrho) = \frac{1}{2n^2\varrho^{3n}} \int_0^\pi \sqrt{\frac{(a_{2n} - \cos 2n\theta)(b_{2n} + \cos 2n\theta)}{(a_{2n} + \cos 2n\theta)^3}}\, \mathrm{d}\theta,$$

or (3.5), because of periodicity of the integrand.

A upper bound of $L_{n,1}(E_\varrho)$ can be derived, applying the Cauchy's inequality, in the form

$$L_{n,1}(E_\varrho) \leqslant \frac{\sqrt{\pi}}{2n^2\varrho^{3n}} \left(\int_0^\pi \frac{(a_{2n} - \cos\theta)(b_{2n} + \cos\theta)}{(a_{2n} + \cos\theta)^3}\, \mathrm{d}\theta \right)^{1/2},$$

i.e., (3.6). □

In a similar way, as in [18] for generalized Chebyshev weights $w_i(t)$, $i = 1, 2, 3, 4$, after some calculations and using Lemma 2.2, we can obtain a simple estimate of the remainder term in the form

$$|R_{n,1}(f)| \leqslant \frac{\pi}{2n^2} \left(\max_{z \in E_\varrho} |f(z)| \right) \frac{2\varrho^{4n} + \varrho^{2n} - 1}{\varrho^{2n}(\varrho^{2n} - 1)^3}. \tag{3.8}$$

It is easy to show that the bound obtained in (3.6) is smaller than one in (3.8), because of the inequality

$$\frac{1 - x - 9x^2 + 29x^3 + 4x^4}{x(x-1)^5} < \left(\frac{2x + \sqrt{x} - 1}{\sqrt{x}(\sqrt{x} - 1)^3} \right)^2 \quad (x > 1).$$

The function $\varrho \mapsto \log_{10}(L_{n,1}(E_\varrho))$, as well as its bounds which appear on the right sides in (3.6) and (3.8) are given in Fig. 1. As we can see, the bound (3.6) is very precise especially for larger values of n and ϱ.

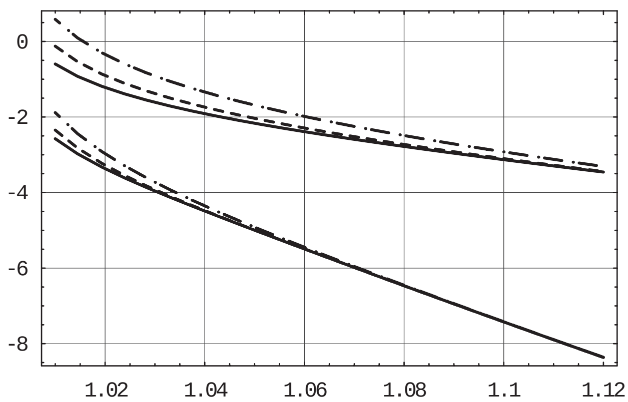

Fig. 1. Log_{10} of the values $L_{n,1}(E_\varrho)$ (solid line) and its bounds given by (3.6) (dashed line) and (3.8) (dot-dashed line) for $n = 10$ (top) and $n = 30$ (bottom).

4. Error estimates for even weight functions

In this section we consider the Gauss–Turán quadrature rule with respect to an even weight functions ($w(-t)=w(t)$ on $[-1,1]$). We suppose that the contour Γ is a central circle with a radius $1+\varepsilon$, where $\varepsilon>0$.

Theorem 4.1. *Let f be an analytic function in* $\{z: \ |z|<1+2\varepsilon\}$ $(\varepsilon>0)$, *i.e.,*

$$f(z)=\sum_{k=0}^{+\infty} a_k z^k, \quad |z|<1+2\varepsilon, \tag{4.1}$$

and the coefficients $\overline{b}_{n,2j}^{(s)}$ *and* $c_{n,2k-2j}^{(s)}$ *be given as in* (2.4) *and* (2.5), *respectively. Then, the remainder term* $R_{n,s}(f)$ *in the Gauss–Turán quadrature formula* (1.1) *can be expressed in the form*

$$R_{n,s}(f)=\sum_{k=0}^{+\infty} a_{2n(s+1)+2k}\, e_{n,k}^{(s)}, \tag{4.2}$$

where $e_{n,k}^{(s)}=\sum_{j=0}^{k}\overline{b}_{n,2j}^{(s)}\, c_{n,2k-2j}^{(s)}$.

Proof. By using the representation (2.2), from (2.4) and Lemma 2.2, we have

$$K_{n,s}(z)=\frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}=\left(\sum_{j=0}^{+\infty}\overline{b}_{n,2j}^{(s)}\, z^{-n(2s+1)-2j}\right)\left(\sum_{j=0}^{+\infty} c_{n,2j}^{(s)} z^{-n-2j-1}\right),$$

i.e.,

$$K_{n,s}(z)=\sum_{k=0}^{+\infty} e_{n,k}^{(s)} z^{-2n(s+1)-2k-1}, \quad |z|>1. \tag{4.3}$$

Now, we obtain

$$R_{n,s}(f)=\frac{1}{2\pi i}\oint_{\Gamma}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} a_j\, e_{n,k}^{(s)}\, z^{j-2n(s+1)-2k-1}\,\mathrm{d}z. \tag{4.4}$$

Both series, (4.1) and the last in (4.3), converge uniformly and absolutely in the annulus $\{z: 1+\frac{1}{2}\varepsilon<|z|<1+\frac{3}{2}\varepsilon\}$ and, therefore, the double sum in (4.4) also converges uniformly and absolutely in this annulus. So we may interchange integration and summation in (4.4). All terms integrate to zero except those for which $j=2n(s+1)+2k$. Summing the residues for $j-2n(s+1)-2k-1=-1$ we obtain (4.2). □

Let μ_j be moments of the weight function $w(t)$, i.e., $\mu_j=\int_{-1}^{1} w(t)t^j\,\mathrm{d}t$ $(j=0,1,\ldots)$. In the same time introduce the quantities

$$\overline{\mu}_j=\sum_{v=1}^{n}\sum_{i=0}^{2s} A_{i,v}\,\frac{j!}{(j-i)!}\,\tau_v^{j-i} \quad (j=0,1,\ldots). \tag{4.5}$$

According to Theorem 4.1, by putting $f(t) = g_k(t) = t^{2n(s+1)+2k}$, we obtain the following consequence:

Corollary 4.2. $e_{n,k}^{(s)} = \mu_{2n(s+1)+2k} - \overline{\mu}_{2n(s+1)+2k}$, $k = 0, 1, \ldots$.

It is easy to see that $e_{n,k}^{(s)} > 0$, i.e., $\mu_{2n(s+1)+2k} > \overline{\mu}_{2n(s+1)+2k}$. It follows, e.g., from the representation $e_{n,k}^{(s)} = R_{n,s}(g_k) = \int_{-1}^{1} \Phi_n(t) g_k^{(2n(s+1))}\, dt$, where $\Phi_n(t)$ (> 0) is the influence function (see [9]).

If we prove that the quantities $\overline{\mu}_{2n(s+1)+2k}$ are positive for a sufficiently large k, then the previous inequality shows that $\overline{\mu}_{2n(s+1)+2k} \to 0$, when $k \to +\infty$.

In order to prove $\overline{\mu}_{2n(s+1)+2k} > 0$ we take that n, s are fixed, and consider the part of the double sum in (4.5),

$$\frac{(2n(s+1)+2k)!}{(2n(s+1)+2k-1)!}\,(A_{1,1}\tau_1^{2n(s+1)+2k-1} + A_{1,n}\tau_n^{2n(s+1)+2k-1}) \tag{a}$$

$$+\ \frac{(2n(s+1)+2k)!}{(2n(s+1)+2k-2)!}\,(A_{2,1}\tau_1^{2n(s+1)+2k-2} + A_{2,n}\tau_n^{2n(s+1)+2k-2}). \tag{b}$$

If $A_{1,1} < 0$, then by summing in (a) and (b) we obtain the positive expressions.

If $A_{1,1} > 0$, we sum separately, firstly the quantities on the left sides in (a) and (b), and then the quantities on the right sides. For instance the sum of the quantities on the left sides of (a) and (b) gives

$$\frac{(2n(s+1)+2k)!}{(2n(s+1)+2k-2)!}\,\tau_1^{2n(s+1)+2k-2}\left[\frac{A_{1,1}\tau_1}{2n(s+1)+2k-1} + A_{2,1}\right] > 0,$$

since $A_{2,1} > 0$, and $A_{1,1}\,\tau_1/(2n(s+1)+2k-1) \to 0$, when $k \to +\infty$.

For the other parts of the double sum in (4.5) the same conclusion holds. Therefore, there exists k_0 such that for all $k \geqslant k_0$ we have $\overline{\mu}_{2n(s+1)+2k} > 0$. Also, $\lim_{k\to+\infty} e_{n,k}^{(s)} = 0$, and the sequence $(e_{n,k}^{(s)})_{k=0,1,\ldots}$ is bounded. The same conclusion can be derived from the following corollary of Theorem 4.1.

Corollary 4.3. $e_{n,k}^{(s)} = \mu_{2n(s+1)+2k}[1 + \mathrm{o}(1)]$.

Proof. Let

$$x_0 = \max_{\nu=1,2,\ldots,n} |\tau_\nu| \quad (\tau_\nu = \tau_\nu^{(n,s)}), \quad \text{where} \quad \pi_{n,s}(\tau_\nu) = 0.$$

Let us define a positive constant $L \equiv L_{n,s}$ such that

$$L x_0^{2n(s+1)} = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} |A_{i,\nu}|\, \frac{(2n(s+1))!}{(2n(s+1)-i)!}\, |\tau_\nu|^{2n(s+1)-i}.$$

For $k=0,1,\ldots,$ we have

$$\begin{aligned}
&\sum_{v=1}^{n}\sum_{i=0}^{2s}|A_{i,v}|\,\frac{(2n(s+1)+2k)!}{(2n(s+1)-i+2k)!}\,|\tau_v|^{2n(s+1)+2k-i}\\
&\quad\leqslant x_0^{2k}\,\frac{(2n(s+1)+2k)\cdots(2n(s+1)+1)}{(2n(s+1)+2k-2s)\cdots(2n(s+1)+1-2s)}\\
&\qquad\times\sum_{v=1}^{n}\sum_{i=0}^{2s}|A_{i,v}|\,\frac{(2n(s+1))!}{(2n(s+1)-i)!}\,|\tau_v|^{2n(s+1)-i}\\
&\quad= Lx_0^{2n(s+1)+2k}\,\frac{(2n(s+1)+2k)\cdots(2n(s+1)+1)}{(2n(s+1)+2k-2s)\cdots(2n(s+1)+1-2s)}.
\end{aligned}$$

Since $x_0<1$, we have

$$\mu_{2n(s+1)+2k}>2\int_{(1+x_0)/2}^{1}w(t)t^{2n(s+1)+2k}\,\mathrm{d}t\geqslant A\left[\frac{1}{2}(x_0+1)\right]^{2n(s+1)+2k},$$

where $A=2\int_{(1+x_0)/2}^{1}w(t)\,\mathrm{d}t>0$. Further,

$$x_0^{2n(s+1)+2k}\leqslant\left[\frac{2x_0}{1+x_0}\right]^{2n(s+1)+2k}\frac{1}{A}\,\mu_{2n(s+1)+2k}.$$

That means that

$$\begin{aligned}
&\sum_{v=1}^{n}\sum_{i=0}^{2s}|A_{i,v}|\,\frac{(2n(s+1)+2k)!}{(2n(s+1)-i+2k)!}\,|\tau_v|^{2n(s+1)+2k-i}\\
&\quad\leqslant\frac{L}{A}\,\mu_{2n(s+1)+2k}\left[\frac{2x_0}{1+x_0}\right]^{2n(s+1)+2k}\frac{(2n(s+1)+2k)\cdots(2n(s+1)+1)}{(2n(s+1)+2k-2s)\cdots(2n(s+1)+1-2s)}.
\end{aligned}$$

Consider now the expression

$$\frac{(2n(s+1)+2k)\cdots(2n(s+1)+1)}{(2n(s+1)+2k-2s)\cdots(2n(s+1)+1-2s)}=\frac{a(a+1)\cdots(a+2k-1)}{b(b+1)\cdots(b+2k-1)},$$

where we put $a=2n(s+1)+1\ (>0)$ and $b=2n(s+1)-2s+1\ (>0)$. We also have that $a\geqslant b$ (In fact, $a=b$ for $s=0$, otherwise $a>b$.).

By using the well-known fact that

$$\Gamma(\xi)=\lim_{k\to+\infty}\frac{k!k^{\xi}}{\xi(\xi+1)\cdots(\xi+k)}=\lim_{k\to+\infty}\frac{(2k-1)!(2k-1)^{\xi}}{\xi(\xi+1)\cdots(\xi+2k-1)},$$

we have

$$\frac{a(a+1)\cdots(a+2k-1)}{b(b+1)\cdots(b+2k-1)}\sim\frac{(2k-1)!(2k-1)^{a}/\Gamma(a)}{(2k-1)!(2k-1)^{b}/\Gamma(b)}=\frac{\Gamma(b)}{\Gamma(a)}\,(2k-1)^{a-b},$$

when $k \to +\infty$. Further, for $d = (1 + x_0)/(2x_0) > 1$ we have $[2x_0/(1 + x_0)]^{2k} = 1/d^{2k}$. Because of

$$\left[\frac{2x_0}{1+x_0}\right]^{2k} \frac{(2n(s+1)+2k)\cdots(2n(s+1)+1)}{(2n(s+1)+2k-2s)\cdots(2n(s+1)+1-2s)} \sim \frac{\Gamma(b)}{\Gamma(a)} \frac{(2k-1)^{a-b}}{d^{2k}},$$

when $k \to +\infty$, where $d > 1$, $a - b > 0$, and $\lim_{k\to+\infty} (2k-1)^{a-b}/d^{2k} = 0$, we conclude that

$$\begin{aligned}
&\left| \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} \frac{(2n(s+1)+2k)!}{(2n(s+1)-i+2k)!} \tau_v^{2n(s+1)+2k-i} \right| \\
&\quad \leqslant \sum_{v=1}^{n} \sum_{i=0}^{2s} |A_{i,v}| \frac{(2n(s+1)+2k)!}{(2n(s+1)-i+2k)!} |\tau_v|^{2n(s+1)+2k-i} \\
&\quad \leqslant \mu_{2n(s+1)} \frac{L}{A} \left[\frac{2x_0}{1+x_0}\right]^{2n(s+1)+2k} \frac{a(a+1)\cdots(a+2k-1)}{b(b+1)\cdots(b+2k-1)}
\end{aligned}$$

with

$$\frac{L}{A}\left[\frac{2x_0}{1+x_0}\right]^{2n(s+1)+2k} \frac{a(a+1)\cdots(a+2k-1)}{b(b+1)\cdots(b+2k-1)} \to 0, \quad k \to +\infty. \qquad \square$$

In some particular cases we can prove that the remainder $R_{n,s}(f)$ decreases monotonically to zero, when all a_{2k} in the expansion (4.1) are nonnegative. For similar investigations in the case of Gaussian quadratures ($s = 0$) see Stenger [25].

At the end we consider some error bounds. Namely, for the remainder term in Gauss–Turán quadrature formula (1.1) we can derive the following estimate

$$|R_{n,s}(f)| \leqslant \|R_{n,s}\| \|f\|. \tag{4.6}$$

We proved that the sequence $(e_{n,k}^{(s)})_{k=0,1,\ldots}$ (n, s-fixed) is bounded, and to converge to zero, when $k \to +\infty$. If a given sequence belongs to the space ℓ^p $(p \geqslant 1)$, we apply Hölder's inequality to (4.2) to obtain

$$|R_{n,s}(f)| \leqslant \left(\sum_{k=0}^{+\infty} (e_{n,k}^{(s)})^p\right)^{1/p} \left(\sum_{k=0}^{+\infty} |a_{2n(s+1)+2k}|^q\right)^{1/q}, \tag{4.7}$$

where $1/p + 1/q = 1$.

The quantities $\sigma_{n,p}^{(s)} = (\sum_{k=0}^{+\infty} (e_{n,k}^{(s)})^p)^{1/p}$ are independent of f and can be computed. In particular, if $p \to +\infty$, we get

$$|R_{n,s}(f)| \leqslant v(n,s) \sum_{k=0}^{+\infty} |a_{2n(s+1)+2k}|, \tag{4.8}$$

where

$$v(n,s) = \sigma_{n,\infty}^{(s)} = \sup_{k=0,1,\ldots} e_{n,k}^{(s)} \tag{4.9}$$

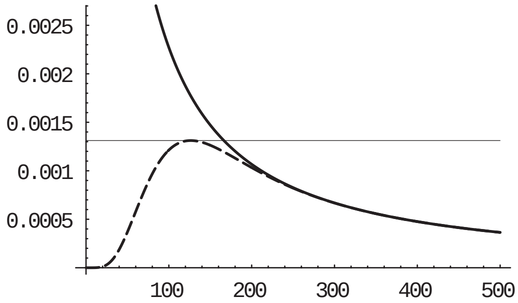

Fig. 2. Quantities $e_{n,k}^{(s)}$ (dashed line) and $\mu_{2n(s+1)+2k}$ (solid line) as functions in k (case $n = 10$, $s = 1$).

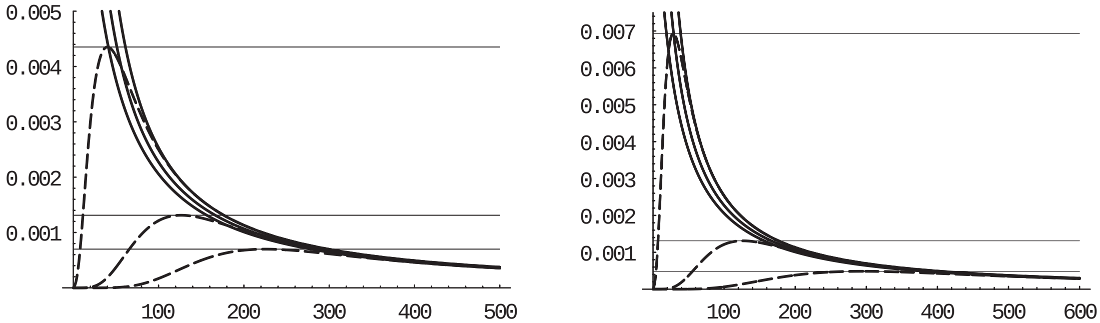

Fig. 3. Graphs for $n = 10$, $s = 0, 1, 2$ (left) and $s = 1$, $n = 5, 10, 15$ (right).

is a quantity which there exists always, what we proved above, and it can be calculated, e.g., by using Corollary 4.3.

In the case $p = q = 2$ the estimate (4.7) of type (4.6) gives

$$\|R_{n,s}\| = \left(\sum_{k=0}^{+\infty} (e_{n,k}^{(s)})^2\right)^{1/2} = W_{n,s}^{1/2}. \tag{4.10}$$

A calculation of $W_{n,s}$ in (4.10) for Gaussian rule $(s = 0)$ was given by Wilf [27] and Stenger [25].

Example 4.4. As an even weight function we take the generalized Gegenbauer weight function $w(x) = |x|^{-1/3}(1 - x^2)^{1/4}$ and consider the estimates given by (4.8) and (4.9).

A typical graph illustrating the relationship between $e_{n,k}^{(s)}$ and $\mu_{2n(s+1)+2k}$ is given in Fig. 2 (case $n=10$, $s = 1$). The supremum $v(10, 1) = \sup_{k\in\mathbb{N}_0} e_{10,k}^{(1)}$ is also displayed in the same figure.

Numerical values of $v(n,s)=\sup_{k\in\mathbb{N}_0} e_{n,k}^{(s)}$ for $s=0,1,2$ and $n=10$ are

$$v(10,0)=4.35\times 10^{-3},\quad v(10,1)=1.31\times 10^{-3},\quad v(10,2)=6.99\times 10^{-4}.$$

Similarly, for $n=5,10,15$ and $s=1$, these supremums are

$$v(5,1)=6.94\times 10^{-3},\quad v(10,1)=1.31\times 10^{-3},\quad v(15,1)=4.86\times 10^{-4}.$$

The corresponding graphs are displayed in Fig. 3.

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