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Orthogonal polynomials and Gaussian quadrature rules related to oscillatory weight functions☆

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This paper is dedicated to Professor Olav Njåstad on the occasion of his 70th birthday

Abstract

In this paper we consider polynomials orthogonal with respect to an oscillatory weight function $w(x) = x\mathrm{e}^{\mathrm{i}m\pi x}$ on $[-1, 1]$, where m is an integer. The existence of such polynomials as well as several of their properties (three-term recurrence relation, differential equation, etc.) are proved. We also consider related quadrature rules and give applications of such quadrature rules to some classes of integrals involving highly oscillatory integrands.

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1. Introduction

Polynomials orthogonal on the semicircle $\Gamma = \{z \in \mathbb{C} \mid z = \mathrm{e}^{\mathrm{i}\theta},\ 0 \leqslant \theta \leqslant \pi\}$ have been introduced and investigated by Gautschi and Milovanović [7]. The inner product was given by $(f, g) = \int_\Gamma f(z)g(z)(iz)^{-1}\,\mathrm{d}z$, i.e., $(f, g) = \int_0^\pi f(\mathrm{e}^{\mathrm{i}\theta})g(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$. This inner product is not Hermitian, but the corresponding (monic)

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orthogonal polynomials $\{\pi_k\}$ exist uniquely and, because of the property $(zf, g) = (f, zg)$, they satisfy the fundamental three-term recurrence relation. The general case of complex polynomials orthogonal with respect to a *complex weight function* was considered in [6]. A generalization of such polynomials on a circular arc was given by de Bruin [4], and further investigations were done by Milovanović and Rajković [19].

In this paper we use a complex oscillatory weight function $w(x)$ defined on $[-1, 1]$ by $w(x) = x\mathrm{e}^{\mathrm{i}m\pi x}$, where m is an integer different from zero. Introducing the measure

$$\mathrm{d}\mu_m(x) := \mathrm{d}\mu(x) = x\mathrm{e}^{\mathrm{i}m\pi x}\chi([-1, 1]; x)\mathrm{d}x, \quad m \in \mathbb{Z}\backslash\{0\},$$

where $\chi(A; \cdot)$ is the characteristic function of the set A, we consider polynomials orthogonal with respect to the moment functional

$$L(f) = \int_{-1}^{1} f(x)w(x)\mathrm{d}x = \int f(x)\mathrm{d}\mu, \tag{1.1}$$

i.e., with respect to the following non-Hermitian inner product

$$(f, g) = \int f(x)g(x)\mathrm{d}\mu. \tag{1.2}$$

Since this weight function $w(x)$ alternates in sign in the interval of orthogonality $[-1, 1]$, the existence of orthogonal polynomials is not assured. A proof of the existence is given in Section 2. The three-term recurrence relation for orthogonal polynomials is considered in Section 3. Numerical values of the recursion coefficients for some values of m are given and two conjectures are stated. A differential equation and related problems are studied in Section 4. Finally, the numerical construction of Gaussian quadrature rules related to orthogonal polynomials with respect to the previous moment functional as well as applications of such quadratures to some classes of integrals involving highly oscillatory integrands are discussed in Section 5.

2. Existence of orthogonal polynomials

Let a linear functional L be given on the linear space of all algebraic polynomials. The values of the linear functional L at the set of monomials are called moments and they are denoted by μ_k. Thus, $L(x^k) = \mu_k$, $k \in \mathbb{N}_0$. In [2, p. 7], the following definition can be found.

Definition 2.1. A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial sequence with respect to a moment functional L provided for all nonnegative integers m and n,

- $P_n(x)$ is a polynomial of degree n,
- $L(P_n(x)P_m(x)) = 0$ for $m \neq n$,
- $L(P_n^2(x)) \neq 0$.

If a sequence of orthogonal polynomial exists for a given linear functional L, then L is called a quasi-definite linear functional. Under the condition $L(P_n^2(x)) > 0$, the functional L is called positive definite (see [2]).

Using only linear algebraic tools, the following theorem can be stated (see [2, p. 11]).

Theorem 2.2. *The necessary and sufficient conditions for the existence of a sequence of orthogonal polynomials with respect to the linear functional L are that for each $n \in \mathbb{N}$ the Hankel determinants*

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-2} \end{vmatrix} \neq 0. \tag{2.1}$$

To prove the existence of orthogonal polynomials with respect to the linear functional L given by (1.1), the corresponding moments are needed. Because of shortness, we set $m\pi = \zeta$, where m ($\neq 0$) is an integer. Using an integration by parts we can get the following recurrence relation for the moments

$$\begin{aligned} \mu_{k+1} &= \int_{-1}^{1} x^{k+2} \mathrm{e}^{\mathrm{i}\zeta x} \mathrm{d}x = \frac{1}{\mathrm{i}\zeta} x^{k+2} \mathrm{e}^{\mathrm{i}\zeta x} \Big|_{-1}^{1} - \frac{k+2}{\mathrm{i}\zeta} \int_{-1}^{1} x^{k+1} \mathrm{e}^{\mathrm{i}\zeta x}\, \mathrm{d}x \\ &= \frac{1}{\mathrm{i}\zeta} (\mathrm{e}^{\mathrm{i}\zeta} - \mathrm{e}^{-\mathrm{i}\zeta} (-1)^{k+2}) - \frac{k+2}{\mathrm{i}\zeta}\, \mu_k, \end{aligned} \tag{2.2}$$

with the initial condition

$$\mu_0 = \int_{-1}^{1} x \mathrm{e}^{\mathrm{i}\zeta x}\, \mathrm{d}x = \frac{1}{\mathrm{i}\zeta} (\mathrm{e}^{\mathrm{i}\zeta} + \mathrm{e}^{-\mathrm{i}\zeta}) - \frac{1}{(\mathrm{i}\zeta)^2} (\mathrm{e}^{\mathrm{i}\zeta} - \mathrm{e}^{-\mathrm{i}\zeta}). \tag{2.3}$$

Since $\mathrm{e}^{\mathrm{i}\zeta} = (-1)^m$, the equalities (2.2) and (2.3) become

$$\mu_{k+1} = \frac{(-1)^m}{\mathrm{i}\zeta} (1 - (-1)^{k+2}) - \frac{k+2}{\mathrm{i}\zeta} \mu_k, \quad \mu_0 = 2 \frac{(-1)^m}{\mathrm{i}\zeta}. \tag{2.4}$$

The moments can be expressed in the following form

$$\mu_k = \frac{(-1)^{m+k} (k+1)!}{(\mathrm{i}\zeta)^{k+1}} \sum_{v=0}^{k} \frac{(1 + (-1)^v)(-\mathrm{i}\zeta)^v}{(v+1)!}. \tag{2.5}$$

Conjugating (2.5) an important equality for these moments can be given

$$\overline{\mu}_k = \frac{(-1)^{m+k} (k+1)!}{(-\mathrm{i}\zeta)^{k+1}} \sum_{v=0}^{k} \frac{(1 + (-1)^v)(\mathrm{i}\zeta)^v}{(v+1)!} = (-1)^{k+1} \mu_k, \tag{2.6}$$

since in the sum representing moments only terms with even v are not zero.

Theorem 2.3. *For every integer $m (\neq 0)$, the sequence of orthogonal polynomials with respect to the weight function $w(x) = x\mathrm{e}^{\mathrm{i}m\pi x}$, supported on the interval $[-1, 1]$, exists uniquely.*

Proof. According to Theorem 2.2, to prove this result we need only to prove that Hankel determinants for the sequence of moments (2.5) are not equal to zero, for any given nonzero integer m.

It can be observed that the moments are rational functions in $\zeta = m\pi$, with simple powers in the denominator. If we take from the ith row of the Hankel determinant Δ_n the factor $\frac{(-1)^m}{\mathrm{i}\zeta}\left(\frac{-1}{\mathrm{i}\zeta}\right)^{i-1}$, and from the vth column the factor $\left(\frac{-1}{\mathrm{i}\zeta}\right)^{v-1}$, our determinant becomes the Hankel determinant $\widetilde{\Delta}_n$ for the following sequence of moments

$$\widetilde{\mu}_k = (k+1)! \sum_{v=0}^{k} \frac{(1+(-1)^v)(-\mathrm{i}\zeta)^v}{(v+1)!}.$$

Notice that $\widetilde{\mu}_k = \widetilde{\mu}_k(-\mathrm{i}\zeta)$ is a polynomial (with rational coefficients) in $\mathrm{i}\zeta$ of degree k. A relation between the determinants Δ_n and $\widetilde{\Delta}_n$ is the following

$$\Delta_n = \frac{(-1)^{n(n+m-1)}}{(\mathrm{i}\zeta)^{n^2}}\widetilde{\Delta}_n.$$

It means that $\Delta_n \neq 0$ if and only if the determinant $\widetilde{\Delta}_n$ is not equal to zero. The value of $\widetilde{\Delta}_n$ is a polynomial in $\mathrm{i}\zeta$ (=$\mathrm{i}m\pi$) with rational coefficients. Since m is an integer, then $\mathrm{i}m\pi$ cannot be a zero of such a polynomial, since $\mathrm{i}m\pi$ is not an algebraic number. The number $\mathrm{i}m\pi$ can be a zero of a polynomial with rational coefficients if and only if that polynomial is identically zero. It is just left to prove that the determinant $\widetilde{\Delta}_n$ is not a polynomial which vanishes identically.

To prove this fact it is enough to note that the free factor of the polynomial $\widetilde{\Delta}_n$ ($\equiv \widetilde{\Delta}_n(\mathrm{i}\zeta)$) in $\mathrm{i}\zeta$, i.e., $\widetilde{\Delta}_n(0)$, is different from zero. If we take only free coefficients in the polynomials $\widetilde{\mu}_k$, $k = 0, 1, \ldots$ (i.e., $\widetilde{\mu}_k(0)$, $k = 0, 1, \ldots$) and make the corresponding Hankel determinant Δ_n^*, its value will be the value of the free factor in the polynomial represented by the determinant $\widetilde{\Delta}_n$. The Hankel determinant Δ_n^* is made with the sequence of moments $\mu_k^* = \widetilde{\mu}_k(0) = 2(k+1)!$. But, this is exactly the sequence of moments for the generalized Laguerre polynomials with $\alpha = 1$, multiplied by factor 2, and it cannot be equal to zero, because the sequence of the generalized Laguerre polynomials exists. □

We also need the values of the determinants Δ_n^*, which can easily be evaluated since they are connected with the generalized Laguerre polynomials with $\alpha = 1$. It is not so difficult to obtain

$$\Delta_n^* = 2^n \prod_{k=1}^{n-1} k!(k+1)! \tag{2.7}$$

We are also interested in a modified Hankel determinant, which can be expressed in terms of the moments $\widetilde{\mu}_k$. Namely,

$$\Delta_n' = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_n & \mu_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix} = \frac{(-1)^{n(n+m-1)+1}}{(\mathrm{i}\zeta)^{n^2+1}}\widetilde{\Delta}_n',$$

where $\widetilde{\Delta}_n'$ is the corresponding modified Hankel determinant for the moments $\widetilde{\mu}_k$. As before, the determinant $\widetilde{\Delta}_n'$ ($\equiv \widetilde{\Delta}_n'(\mathrm{i}\zeta)$) is an algebraic polynomial in $\mathrm{i}\zeta$ with rational coefficients. Its free term can easily be calculated using the corresponding determinant for the generalized Laguerre polynomials with the parameter $\alpha = 1$, i.e.,

$$\widetilde{\Delta}'_n(0) = n(n+1)\Delta_n^* = 2^n n(n+1) \prod_{k=1}^{n-1} k!(k+1)! \tag{2.8}$$

In the sequel we will consider only the case when $m > 0$, since for $m < 0$ we have $\mu_k(m) = \int x^k \mathrm{d}\mu_m = \overline{\int x^k \mathrm{d}\mu_{-m}} = \overline{\mu_k(-m)}$. This means that all results generated for $m > 0$ can be applied to the case $m < 0$ by a simple conjugation. Hence, we assume $m \in \mathbb{N}$.

In general, for an arbitrary real $\zeta = m\pi$ ($m \notin \mathbb{Z}$), when the moments are given by (2.2) and (2.3), the existence of orthogonal polynomials is not assured. For example, equation $\Delta_3 = 0$ has as the smallest positive solution $\zeta \approx 7.13414399636896061399\ldots$.

3. Recurrence relation

Since the sequence of orthogonal polynomials with respect to the weight function $w(x) = x\mathrm{e}^{\mathrm{i}m\pi x}$ on $[-1, 1]$ exists, these (monic) polynomials satisfy the three-term recurrence relation

$$p_{n+1}(x) = (x - \mathrm{i}\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad n = 0, 1, \ldots, \tag{3.1}$$

with $p_0(x) = 1$ and $p_{-1}(x) = 0$. This kind of relation is provided by the property $(xf, g) = (f, xg)$ of the inner product (1.2). The recursion coefficients α_n and β_n can be expressed in terms of Hankel determinants as (cf. [7])

$$\mathrm{i}\alpha_n = \frac{\Delta'_{n+1}}{\Delta_{n+1}} - \frac{\Delta'_n}{\Delta_n} \quad (n \in \mathbb{N}_0) \quad \text{and} \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} \quad (n \in \mathbb{N}). \tag{3.2}$$

In this case, however, the values of Hankel determinants cannot be found easily, but, it is clear that the recursion coefficients are rational functions in $\zeta = m\pi$. Using our software package [3] we can generate coefficients even in symbolic form for some reasonable values of n (e.g., $n \leqslant 20$) and state the following conjecture:

Conjecture 3.1. *Let $a_n(z)$ and $c_n(z)$ be algebraic polynomials with integer coefficients of degree r_n and s_n, respectively, i.e., $a_n(z) = A_n z^{r_n} + \cdots$ and $c_n(z) = z^{s_n} + \cdots$. If $\zeta = m\pi$ and $n \geqslant 2$, then*

$$\alpha_n = \frac{a_n(\zeta^2)}{\zeta\, c_{n-1}(\zeta^2)c_n(\zeta^2)}, \quad \beta_n = B_n \frac{c_{n-2}(\zeta^2)c_n(\zeta^2)}{\zeta^2\, c_{n-1}(\zeta^2)^2},$$

where

$$A_n = \begin{cases} -\dfrac{n^2-1}{4} & (n\ odd), \\[2mm] \dfrac{n^2+10n+8}{4} & (n\ even), \end{cases} \qquad B_n = \begin{cases} 1 & (n\ odd), \\ -n^2 & (n\ even), \end{cases}$$

Table 1
Recursion coefficients α_n and β_n in (3.1) for $n \leqslant 3$

n	α_n	β_n
0	$\frac{2}{\zeta}$	$\frac{2(-1)^m}{\mathrm{i}\zeta}$
1	$\frac{-8}{\zeta(-2+\zeta^2)}$	$\frac{-2+\zeta^2}{\zeta^2}$
2	$\frac{2(-36+60\zeta^2-45\zeta^4+4\zeta^6)}{\zeta(-2+\zeta^2)(6-13\zeta^2+\zeta^4)}$	$-\frac{4(6-13\zeta^2+\zeta^4)}{\zeta^2(-2+\zeta^2)^2}$
3	$-\frac{2(-5184+32832\zeta^2-69552\zeta^4+15648\zeta^6-1259\zeta^8+26\zeta^{10}+\zeta^{12})}{\zeta(6-13\zeta^2+\zeta^4)(216-1224\zeta^2+165\zeta^4-14\zeta^6+\zeta^8)}$	$\frac{(-2+\zeta^2)(216-1224\zeta^2+165\zeta^4-14\zeta^6+\zeta^8)}{\zeta^2(6-13\zeta^2+\zeta^4)^2}$

and

$$r_n = \frac{n(n+1)}{2}, \quad s_n = \begin{cases} \dfrac{(n+1)^2}{4} & (n \ \mathit{odd}), \\ \dfrac{n(n+2)}{4} & (n \ \mathit{even}). \end{cases}$$

In Table 1, the first four recursion coefficients are given, where $\zeta = m\pi$. So, we have $c_1(z) = -2 + z$, $c_2(z) = 6 - 13z + z^2$, $c_3(z) = 216 - 1224z + 165z^2 - 14z^3 + z^4$. Increasing n, the complexity of expressions for three-term recurrence coefficients dramatically increases. For example, for $n = 4$ we have

$$\begin{aligned} a_4(z) = 2(&10497600 - 160963200z + 731274480z^2 - 231822000z^3 + 46761705z^4 \\ &- 6692544z^5 + 689193z^6 - 86568z^7 + 9764z^8 - 518z^9 + 8z^{10}) \end{aligned}$$

and

$$c_4(z) = 9720 - 113400z + 17361z^2 - 4932z^3 + 1101z^4 - 77z^5 + z^6.$$

so that, the corresponding three-term recurrence coefficients become quite complicated.

According to Table 1 we can conjecture that coefficients α_n and β_n are real. Furthermore, we can prove a more general result:

Lemma 3.2. *If a sequence of moments* $\{\mu_k\}_{k\in\mathbb{N}_0}$ *satisfies the condition* $\overline{\mu}_k = (-1)^{k+1}\mu_k$ $(k \in \mathbb{N}_0)$ *and the sequence of* (*monic*) *orthogonal polynomials* $\{p_n\}_{n\in\mathbb{N}_0}$ *exists, then*

$$(-1)^n\overline{p_n(-\overline{z})} = p_n(z) \tag{3.3}$$

and the coefficients in the three-term recurrence relation (3.1) *are real, i.e., they satisfy* $\mathrm{i}\alpha_n = -\overline{\mathrm{i}\alpha_n}$ $(k \geqslant 0)$ *and* $\beta_k = \overline{\beta_k}$ $(k \geqslant 1)$.

Proof. Following [2, p. 17] we have the following representation for the monic orthogonal polynomial $p_n(z)$ of degree n

$$p_n(z) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix},$$

where Δ_n is the Hankel determinant defined in (2.1). Putting $-\bar{z}$ instead of z, conjugating this equality and using $\bar{\mu}_k = (-1)^{k+1}\mu_k$ $(k \in \mathbb{N}_0)$, we get

$$\overline{\Delta_n p_n(-\bar{z})} = \begin{vmatrix} -\mu_0 & \mu_1 & -\mu_2 & \cdots & (-1)^{n+1}\mu_n \\ \mu_1 & -\mu_2 & \mu_3 & \cdots & (-1)^{n+2}\mu_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^n\mu_{n-1} & (-1)^{n+1}\mu_n & (-1)^{n+2}\mu_{n+1} & \cdots & (-1)^{2n}\mu_{2n-1} \\ 1 & -z & z^2 & \cdots & (-1)^n z^n \end{vmatrix}.$$

Notice that the moments for our weight function satisfy (2.6). If we take -1 from every row (except the last one) with odd index in the previous determinant and then take -1 from every even column, we obtain

$$\overline{\Delta_n p_n(-\bar{z})} = (-1)^{\lfloor (n+1)/2 \rfloor}(-1)^{\lceil n/2 \rceil} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix} = (-1)^n \Delta_n p_n(z).$$

Applying the same argument to the Hankel determinant, we conclude that $\overline{\Delta}_n = \Delta_n$. Since $\Delta_n \neq 0$, we get property (3.3).

Now, putting $-\bar{z}$ instead of z in (3.1) and conjugating it, we get

$$\overline{p_{n+1}(-\bar{z})} = (-z - \overline{\mathrm{i}\alpha_n})\overline{p_n(-\bar{z})} - \overline{\beta_n}\,\overline{p_{n-1}(-\bar{z})},$$

i.e., $p_{n+1}(z) = (z + \overline{\mathrm{i}\alpha_n})p_n(z) - \overline{\beta_n}p_{n-1}(z)$, where we used (3.3). Comparing this recurrence with (3.1) and using the uniqueness of $p_n(z)$ we obtain the desired statement. $\square$

In a section on differential equations (Section 4), the following statement is proved about a Laguerre-Freud type of nonlinear recurrence relations satisfied by the three-term recurrence coefficients (see [11]).

Theorem 3.3. *For $n \geqslant 4$, the three-term recurrence coefficients satisfy the following nonlinear recurrence equations*:

$$\begin{aligned}
\beta_{n+1} = \frac{1}{\zeta\beta_n} \Big\{ & \alpha_{n-2} + \alpha_{n-2}^3 + \alpha_{n-1} + \alpha_{n-1}^3 - \beta_{n-2}(5\alpha_{n-3} + 8\alpha_{n-2} + 2\alpha_{n-1}) \\
& - 3\beta_{n-1}(\alpha_{n-2} + \alpha_{n-1}) + \beta_n(2\alpha_{n-2} - 2\alpha_{n-1} - 3\alpha_n) \\
& + 2n[(\alpha_{n-3} - \alpha_{n-1})\beta_{n-2} + (\alpha_{n-2} - \alpha_n)\beta_n] \\
& + \zeta[\beta_{n-2}(\alpha_{n-1} - \alpha_{n-3})(\alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3}) + (\beta_{n-2} - \beta_n)^2 \\
& + \beta_n(\alpha_n - \alpha_{n-2})(\alpha_n + \alpha_{n-1} + \alpha_{n-2}) + \beta_{n-3}\beta_{n-2}] \\
& + \frac{\beta_n - \beta_{n-2}}{\alpha_{n-1} - \alpha_{n-2}}[1 + 3\alpha_{n-2}^2 - 2\beta_{n-2} - \beta_{n-1} - \beta_n - n(\beta_n - \beta_{n-2}) \\
& + \zeta(\alpha_{n-3}\beta_{n-2} + 2\alpha_{n-2}(\beta_{n-2} - \beta_n) - \alpha_n\beta_n)] \Big\},
\end{aligned}$$

$$\begin{aligned}
\alpha_{n+1} = \frac{2}{\zeta\beta_{n+1}} & \{-1 + (\alpha_{n-1} - \alpha_n)(\alpha_{n-2} - \alpha_n) + \beta_{n-1} + \beta_n + 2\beta_{n+1} \\
& + n[(\alpha_n - \alpha_{n-1})(\alpha_{n-3} + \alpha_{n-2} - \alpha_{n-1} - \alpha_n) - \beta_{n-1} + \beta_{n+1}]\} \\
& + \frac{1}{\beta_{n+1}}\{2\alpha_n(\beta_{n-1} - \beta_{n+1}) + 2\alpha_{n-2}\beta_{n-1} + (\alpha_n - \alpha_{n-1}) \\
& \times (\alpha_{n-3}^2 + \alpha_{n-3}\alpha_{n-2} + \alpha_{n-2}^2 - \alpha_{n-1}^2 - \alpha_{n-1}\alpha_n - \alpha_n^2 - \beta_{n-2} + \beta_n)\} \\
& + \frac{\alpha_n - \alpha_{n-1}}{\alpha_{n-2} - \alpha_{n-3}} \Big\{ [1 + 3\alpha_{n-3}^2 - \beta_{n-3} - \beta_{n-2} - n(\beta_{n-3} - \beta_{n-1})] \\
& - \frac{1}{\beta_n}[\alpha_{n-4}\beta_{n-3} + 2\alpha_{n-3}(\beta_{n-3} - \beta_{n-1}) - \beta_{n-1}\alpha_{n-1}] \Big\},
\end{aligned}$$

with initial conditions given in Table 1.

Complexity in three-term recurrence coefficients should not prevent numerical calculation. Using the recurrence relation for the moments (2.2), μ_k can be calculated and then, using the Chebyshev algorithm, the recursion coefficients can be constructed. Since the Chebyshev algorithm is ill-conditioned (see [5]), an arithmetic with higher precision is needed. Another way to calculate the three-term recurrence coefficients is the Stieltjes–Gautschi procedure (see [5,14]). Using this procedure the recursion coefficients can be constructed for larger values of m.

First, we discuss the stability in the computation of the moments using (2.4). Numerical examples show that if m is small, e.g., 1, 2, 3 and so, then (2.4) is not numerically stable for calculation of the moments. For example, for $m = 1$ with 16 decimal digits mantissa (double precision or D-arithmetic), the relative error in the moment μ_{50} is 10^{17}. If we increase m enough, for example $m = 7$, the relation (2.4) has better stability and the relative error in μ_{50}, in double precision arithmetics is about 10^{-10}. If we increase m further, (2.4) becomes quite stable.

It should be emphasized that in the cases where $m = 1, 2, 3$, (2.5) can be used for calculations of the moments. With increasing m, (2.5) becomes unstable for calculation since it represents an ill-conditioned numerical series.

Using relations (2.5) and (2.4), the moments can be calculated numerically stable. However, a construction of three-term recurrence coefficients is more complicated. There is one algorithm connecting

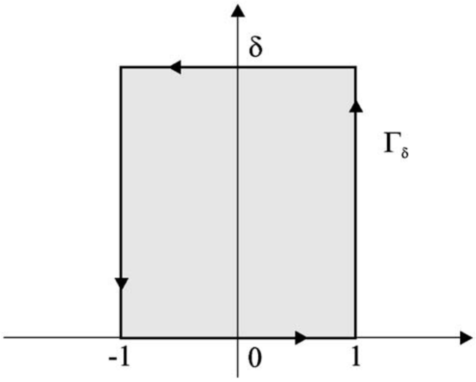

Fig. 1. The contour of integration.

moments and recursive coefficients known as the Chebyshev algorithm. It is also known that this algorithm is ill-conditioned (see [5,14]).

We have conducted some numerical experiments and get the following results. In the case m is small $1, 2, 3, \ldots$, the Chebyshev algorithm is quite stable if the Q-arithmetic (with 34 decimal digits mantissa) is used. The relative errors in coefficients α_{30}, β_{30} are 10^{-27} and 10^{-8} in Q-arithmetic and D-arithmetic, respectively. Increasing m, for example $m=50$, the relative errors in coefficients increase and they become 1 and 10^{-6} in D-arithmetic and Q-arithmetic, respectively. For $m=10^3$, the corresponding relative errors in α_{30}, β_{30} are about 1 even in Q-arithmetic. Thus, this means that the Chebyshev algorithm cannot be used in constructions (even in Q-arithmetic), when m is sufficiently large, e.g., $m>20$.

Another more stable way for constructing the recursion coefficients is the Stieltjes–Gautschi procedure. In order to apply this procedure we need the following auxiliary result:

Lemma 3.4. *For each algebraic polynomial g of degree at most $2N$ the formula*

$$\int_{-1}^{1} g(x)\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x = \frac{\mathrm{i}(-1)^m}{m\pi}\sum_{k=1}^{N}\lambda_k^L\left[g\left(-1+\mathrm{i}\,\frac{\tau_k^L}{m\pi}\right)-g\left(1+\mathrm{i}\,\frac{\tau_k^L}{m\pi}\right)\right] \tag{3.4}$$

holds, where τ_k^L and λ_k^L are the parameters of the N-point Gauss–Laguerre quadrature rule.

Proof. Applying a complex integration method [15] to $\int_{\Gamma_\delta} g(z)\mathrm{e}^{\mathrm{i}m\pi z}\,\mathrm{d}z$ over the rectangular contour Γ_δ (see Fig. 1), letting $\delta \to +\infty$ and using the N-point Gauss–Laguerre quadrature rule, we get (3.4). □

The integrals which appear in Darboux formulas for the recursion coefficients,

$$\mathrm{i}\alpha_k = \frac{(xp_k, p_k)}{(p_k, p_k)} = \frac{\int_{-1}^{1} p_k^2(x)x^2\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x}{\int_{-1}^{1} p_k^2(x)x\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x} \quad (k\geqslant 0), \qquad \beta_0=\mu_0=\frac{2(-1)^m}{\mathrm{i}m\pi},$$

$$\beta_k = \frac{(p_k, p_k)}{(p_{k-1}, p_{k-1})} = \frac{\int_{-1}^{1} p_k^2(x)x\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x}{\int_{-1}^{1} p_{k-1}^2(x)x\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x} \quad (k\geqslant 1),$$

can be computed exactly, except for rounding errors, taking N sufficiently large in the previous lemma. In order to obtain the first n coefficients α_k, β_k $(k=0,1,\ldots,n-1)$, we need $N \geqslant n$.

In our numerical experiments, the Stieltjes–Gautschi procedure proves itself worthy. For small values of m, for example smaller than 25, it is ill-conditioned. For example, the relative errors in three-term recurrence coefficients α_{50} and β_{50} are 10^{-7} for $m=20$ in D-arithmetic. Increasing m, the corresponding relative errors become of the machine precision magnitude for all three-term recurrence coefficients, except the coefficient α_1. But it is not a problem, because for this coefficient we have an explicit expression (see Table 1).

Using Mathematica we calculate the first 30 recursion coefficients for $m = 1, 10^2, 10^4, 10^6, 10^9$. All computations are done using a combination of the Chebyshev method and the Stieltjes–Gautschi procedure. It should be noted that only for $m=1$ the Chebyshev algorithm is used in Q-arithmetic. All other coefficients are calculated using the Stieltjes–Gautschi procedure. The first 30 recursion coefficients for $m=1$ and 100 are presented in Table 2. Numbers in parentheses indicate decimal exponents.

According to a very extensive numerical calculations, using a package of routines written in Mathematica (see [3]), we can state the following conjecture: (It cannot be seen from a small number of coefficients as in Table 2.)

Conjecture 3.5. *For the recursion coefficients, the following asymptotic relations are true*

$$\alpha_k \to 0, \quad \beta_k \to \frac{1}{4}, \quad \text{as} \quad k \to +\infty. \tag{3.5}$$

Note that Magnus' theorem (see [10]) cannot be applied to prove this conjecture, since the weight function has zero value inside the interval $[-1,1]$. Our weight function does not fit to the classes of Nuttal and Whery (see [20]), either.

Finally, the nonlinear recurrence relations for the three-term recurrence coefficients, given in Theorem 3.3, are numerically unstable. However, they can be used for a symbolic construction, since they have smaller complexity than the Chebyshev algorithm. However, if we are able to do computations in some higher arithmetics, these relations can be used also for numerical construction, again with a lower complexity than the Chebyshev algorithm.

4. Differential equation and related problems

First we prove a result for the first derivative of our orthogonal polynomials.

Theorem 4.1. *For* $(monic)$ *polynomials* p_n *orthogonal with respect to the weight function* $w(x)=x\mathrm{e}^{\mathrm{i}m\pi x}$ $(m \in \mathbb{N})$ *on* $[-1,1]$, *the following equation*

$$-x\phi p_n' = p_2^n p_n + q_2^n p_{n-1}, \quad n \in \mathbb{N}, \tag{4.1}$$

holds, where p_2^n *and* q_2^n *are polynomials of second degree and* $\phi = 1 - x^2$.

Proof. Let $w(x) = x\mathrm{e}^{\mathrm{i}\zeta x}$, where $\zeta = m\pi$. Then we have the following differential equation

$$(x\phi w)' + \psi w = 0, \quad \psi = -[(2+\mathrm{i}\zeta x)\phi + x\phi'].$$

Table 2
The recursion coefficients α_k, β_k, $k = 0, 1, \ldots, 29$, with respect to the weight $w(x) = x e^{\mathrm{i} m\pi x}$ on $[-1, 1]$, for $m = 1$ (left) and $m = 100$ (right)

k	α_k	β_k	α_k	β_k
0	0.6366197723675813	i0.6366197723675813	0.636619772367581343(−2)	−i0.636619772367581343(−2)
1	−0.3235841294789642	0.7973576327153245	−0.258017503993376924(−6)	0.999979735763271532
2	−0.5954161768781585(−1)	0.1629220776952933	0.254657585433565251(−1)	−0.405247775618572461(−4)
3	0.7910334466320025(−1)	0.3051351241301063	−0.636961626800292598(−2)	1.000101349111133839
4	−0.8764276869073403(−1)	0.2128993040344742	0.509431340532140238(−1)	−0.162012052723457109(−3)
5	0.8473746274827025(−1)	0.2987622570884405	−0.191210397675282768(−1)	1.000872216947472743
6	−0.8688561816172646(−1)	0.2117636703524410	0.828351751096263442(−1)	−0.363939709018379531(−3)
7	0.8570745958769821(−1)	0.2980998244194351	−0.382936614600099358(−1)	1.003027613759062012
8	−0.8666844850324216(−1)	0.2111804398870007	0.121228565536061484	−0.644714605572716494(−3)
9	0.8608925862853529(−1)	0.2977791164282434	−0.639763468973776280(−1)	1.007564442098398045
10	−0.8659857109140611(−1)	0.2108828566593321	0.166292349800760667	−0.100075337165944050(−2)
11	0.8627247767865183(−1)	0.2976026890742332	−0.963400995285357488(−1)	1.015772811174387681
12	−0.8657405623922258(−1)	0.2107122791921952	0.218321255798082199	−0.142531313953930180(−2)
13	0.8637268365256313(−1)	0.2974961702027002	−0.135681191786866237	1.029298977634625351
14	−0.8656574858964479(−1)	0.2106058739335615	0.277793639272715674	−0.190714016219807481(−2)
15	0.8643279816867875(−1)	0.2974271916096470	−0.182478840649006230	−1.050259840871622525
16	−0.8656375162024725(−1)	0.2105351818496800	0.345451197686132546	−0.242906045156949549(−2)
17	0.8647142245760357(−1)	0.2973800585167439	−0.237474578129759936	1.081441619533492662
18	−0.8656429264180822(−1)	0.2104858842700285	0.422413386788202912	−0.296671384695609290(−2)
19	0.8649758482146797(−1)	0.2973464643640837	−0.301786269554184218	1.126639104687113826
20	−0.8656581806481852(−1)	0.2104501631104796	0.510349931697269430	−0.348772831212214243(−2)
21	0.8651606334559308(−1)	0.2973216937975279	−0.377080215465214900	1.191239358421633902
22	−0.8656765827803345(−1)	0.2104234637586969	0.611754722209078155	−0.395173484446661438(−2)
23	0.8652956531748510(−1)	0.2973029138909545	−0.465844702939731706	1.283252896277182427
24	−0.8656952073777001(−1)	0.2104029899018435	0.730404564816816487	−0.431171525927172178(−2)
25	0.8653971170304653(−1)	0.2972883414189559	−0.571848592026866151	1.415214321987903186
26	−0.8657128152632596(−1)	0.2103869488493026	0.872173116479886438	−0.451722285521488598(−2)
27	0.8654751824392116(−1)	0.2972768093450591	−0.700955412047560028	1.607892405307178560
28	−0.8657289425490870(−1)	0.2103741489011420	1.046573605525170660	−0.451997413886477718(−2)
29	0.8655364621426911(−1)	0.2972675283097719	−0.862666748047142974	1.898085932425570343

Using this equation we can, also, derive the following differential equation

$$(x\phi p_n w)' - (x\phi p_n' - p_n\psi)w = 0.$$

If we multiply the previous equation with $x^k,\ k \in \mathbb{N}_0$, and integrate over $(-1, 1)$, we get

$$\int_{-1}^{1} x^k (x\phi p_n w)' \mathrm{d}x = \int x^k (x\phi p_n' - p_n\psi)\mathrm{d}\mu = x^{k+1}\phi p_n w\Big|_{-1}^{1} - k\int x^k \phi p_n \,\mathrm{d}\mu,$$

where the right-hand side in the last equality was obtained using integration by parts. Its first term is equal to zero, as well as the second one provided $k+2<n$, since p_n is an orthogonal polynomial with respect to $\mathrm{d}\mu = w(x)\mathrm{d}x$. Thus, for $k+2<n$, we have $\int x^k(x\phi p_n' - p_n\psi)\mathrm{d}\mu = 0$. Using this fact, we can write

the following expansion

$$-x\phi p_n' + p_n\psi = \sum_{k=n-2}^{n+3} \theta_n^k p_k, \tag{4.2}$$

where θ_n^k are some constants. This can be reduced, by using three-term recurrence relation (3.1), to

$$-x\phi p_n' + p_n\psi = p_3^n p_n + q_2^n p_{n-1}, \tag{4.3}$$

where we can express explicitly the polynomials p_3^n and q_2^n in the forms

$$\begin{aligned} p_3^n = {} & \theta_n^{n+3}x^3 + [\theta_n^{n+2} - \mathrm{i}\theta_n^{n+3}(\alpha_n + \alpha_{n+1} + \alpha_{n+2})]x^2 \\ & + [\theta_n^{n+1} - \mathrm{i}\theta_n^{n+2}(\alpha_n + \alpha_{n+1}) - \theta_n^{n+3}[\beta_{n+1} + \beta_{n+2} + \alpha_n(\alpha_{n+1} + \alpha_{n+2}) + \alpha_{n+1}\alpha_{n+2}]]x \\ & + \theta_n^n - \mathrm{i}\alpha_n\theta_n^{n+1} - (\alpha_n\alpha_{n+1} + \beta_{n+1})\theta_n^{n+2} \\ & + \mathrm{i}[\alpha_n(\beta_{n+2} + \alpha_{n+1}\alpha_{n+2}) + \alpha_{n+2}\beta_{n+1}]\theta_n^{n+3} - \frac{\theta_n^{n-2}}{\beta_{n-1}}, \\ q_2^n = {} & -\beta_n\theta_n^{n+3}x^2 + \left[\frac{\theta_n^{n-2}}{\beta_{n-1}} - \beta_n\theta_n^{n+2} + \mathrm{i}\beta_n\theta_n^{n+3}(\alpha_{n+1} + \alpha_{n+2})\right]x \\ & + \theta_n^{n-1} - \beta_n\theta_n^{n+1} + \mathrm{i}\alpha_{n+1}\beta_n\theta_n^{n+2} + \beta_n(\alpha_{n+1}\alpha_{n+2} + \beta_{n+2})\theta_n^{n+3} - \mathrm{i}\alpha_{n-1}\frac{\theta_n^{n-2}}{\beta_{n-1}}. \end{aligned}$$

When we subtract the term with p_n on the left-hand side of (4.3), we get (4.1), where it is adopted $p_2^n = p_3^n - \psi$. It is obvious from (4.1) that p_2^n has to be of second degree, since the polynomial on the left-hand of the mentioned equation is of degree $n+2$. $\square$

Actually, a little more can be stated. Using the previous theorem for our polynomials, we have

$$\theta_n^{n+3} = \mathrm{i}\zeta, \quad \theta_n^{n+2} = n + 4 - \zeta(\alpha_{n+2} + \alpha_{n+1} + \alpha_n). \tag{4.4}$$

In the sequel we also need to define

$$K_n(x, y) = \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\|p_k\|^2} \quad \text{and} \quad k_n(x) = K_n(x, x).$$

Lemma 4.2. *We have the following equations*

$$\theta_n^k\|p_k\|^2 + \theta_k^n\|p_n\|^2 = \int \psi p_n p_k \,\mathrm{d}\mu, \quad \beta_{n+1}\theta_n^{n+1} - \theta_{n+1}^n = \int x\phi k_n \mathrm{d}\mu,$$

i.e.,

$$\begin{aligned} 2\beta_{n+1}\theta_n^{n+1} &= \frac{1}{\|p_n\|^2}\int \psi p_{n+1}p_n \,\mathrm{d}\mu + \int x\phi k_n \,\mathrm{d}\mu, \\ 2\theta_{n+1}^n &= \frac{1}{\|p_n\|^2}\int \psi p_{n+1}p_n \,\mathrm{d}\mu - \int x\phi k_n \,\mathrm{d}\mu, \quad 2\theta_n^n = \frac{1}{\|p_n\|^2}\int \psi p_n^2 \,\mathrm{d}\mu. \end{aligned}$$

Proof. According to (4.2) we have

$$
\begin{aligned}
\theta_n^k \|p_k\|^2 &= \int_{-1}^{1} (-x\phi p_n' + p_n\psi) w p_k \,\mathrm{d}x = -\int_{-1}^{1} (x\phi p_n w)' p_k \,\mathrm{d}x \\
&= -x\phi p_n w p_k\big|_{-1}^{1} + \int_{-1}^{1} x\phi p_n w p_k' \mathrm{d}x = -\theta_k^n \|p_n\|^2 + \int \psi p_n p_k \mathrm{d}\mu,
\end{aligned} \tag{4.5}
$$

which is the first equality. The second one can be proved using the Christoffel–Darboux identity (cf. [14]). Thus, we have

$$
\beta_{n+1}\theta_n^{n+1} - \theta_{n+1}^n = \frac{1}{\|p_n\|^2} \int x\phi(p_n p_{n+1}' - p_{n+1} p_n') \,\mathrm{d}\mu = \int x\phi k_n \,\mathrm{d}\mu.
$$

Last two equalities can be obtained by substituting $k = n+1$ in the first equality and performing some calculations. □

In the sequel, we adopt the following short notation

$$
\gamma_n = \frac{1}{\beta_{n+1}} \int x\phi k_n \,\mathrm{d}\mu. \tag{4.6}
$$

Lemma 4.3. *Let $\zeta = m\pi$. The following equations*

$$
\begin{aligned}
\frac{1}{\|p_{n+1}\|^2} \int \psi p_{n+1} p_n \,\mathrm{d}\mu &= 4\mathrm{i}(\alpha_{n+1} + \alpha_n) + \mathrm{i}\zeta(\beta_{n+2} + \beta_{n+1} + \beta_n) \\
&\quad - \mathrm{i}\zeta[\alpha_{n+1}(\alpha_{n+1} + \alpha_n) + \alpha_n^2 + 1], \\
\frac{1}{\|p_n\|^2} \int \psi p_n^2 \,\mathrm{d}\mu &= -2 + 4(\beta_{n+1} - \alpha_n^2 + \beta_n) \\
&\quad - \zeta(\alpha_{n+1}\beta_{n+1} + 2\alpha_n\beta_{n+1} - \alpha_n - \alpha_n^3 + 2\alpha_n\beta_n + \alpha_{n-1}\beta_n),
\end{aligned}
$$

hold, with conventions $\beta_{-n} = 0$ $(n \geqslant 0)$ and $\alpha_{-n} = 0$ $(n \geqslant 1)$.

Proof. In order to prove these equalities we need to calculate integrals of the forms $\int x^\nu p_{n+1} p_n \,\mathrm{d}\mu$ and $\int x^\nu p_n^2 \,\mathrm{d}\mu$ $(\nu = 0, 1, 2, 3)$. We evaluate these integrals using the three-term recurrence relation, since we have

$$
\begin{aligned}
&x p_k = p_{k+1} + \mathrm{i}\alpha_k p_k + \beta_k p_{k-1}, \quad x^2 p_k = x(x p_k) = (x p_{k+1}) + \mathrm{i}\alpha_k (x p_k) + \beta_k (x p_{k-1}), \\
&x^3 p_k = x(x^2 p_k) = (x^2 p_{k+1}) + \mathrm{i}\alpha_k (x^2 p_k) + \beta_k (x^2 p_{k-1}).
\end{aligned}
$$

According to these formulas, we evaluate integrals substituting $k := n+1$ and $k := n$ in the previous equalities and taking scalar products with p_n. □

Theorem 4.4. *Let $\zeta = m\pi$. The polynomials p_2^n and q_2^n, which appear in* (4.1), *have the forms*

$$
p_2^n = n x^2 - \mathrm{i} u_n x + v_n \quad \textit{and} \quad q_2^n = -\beta_n[\mathrm{i}\zeta x^2 + (2n + 2 - \zeta\alpha_n)x - \mathrm{i}\delta_n],
$$

where

$$
\begin{aligned}
u_n &= (n+2)(\alpha_n+\alpha_{n+1}) - \frac{\zeta}{2}[1+\alpha_{n+1}^2+\alpha_n(\alpha_n+\alpha_{n+1})+\beta_n-\beta_{n+1}-\beta_{n+2}] + \frac{\mathrm{i}\gamma_n}{2},\\
v_n &= 1+n\beta_n-(n+2)(\alpha_n\alpha_{n+1}+\beta_{n+1}) - \frac{\mathrm{i}\alpha_n\gamma_n}{2}\\
&\quad + \frac{\zeta}{2}[(\alpha_{n+1}+\alpha_n)(\alpha_n\alpha_{n+1}+\beta_{n+1})-\alpha_n\beta_{n+2}-(\alpha_n+\alpha_{n-1})\beta_n],\\
\delta_n &= (n+2)\alpha_{n+1}+n\alpha_{n-1}+\mathrm{i}\,\frac{\gamma_n+\gamma_{n-1}}{2}\\
&\quad + \frac{\zeta}{2}[\beta_{n-1}+\beta_{n+2}-\alpha_{n+1}(\alpha_{n+1}+\alpha_n)-\alpha_{n-1}(\alpha_n+\alpha_{n-1})],
\end{aligned}
$$

with conventions $\beta_{-n}=0$ $(n\geqslant 0)$ *and* $\alpha_{-n}=0$ $(n\geqslant 1)$.

Proof. Taking $k=n-2$ in (4.5) and recalling that x^3p_{n-2}' is a polynomial of degree n with leading coefficient $n-2$, we have

$$
\begin{aligned}
\theta_n^{n-2}\|p_{n-2}\|^2 &= \int x\phi p_{n-2}'p_n\,\mathrm{d}\mu\\
&= \int xp_{n-2}'p_n\,\mathrm{d}\mu - \int x^3p_{n-2}'p_n\,\mathrm{d}\mu = -(n-2)\|p_n\|^2,
\end{aligned} \tag{4.7}
$$

where the orthogonality of polynomials is used. Using (4.4), we get

$$
\beta_n\left[\frac{\theta_n^{n-2}}{\beta_{n-1}\beta_n} - \theta_n^{n+2} + \mathrm{i}\theta_n^{n+3}(\alpha_{n+1}+\alpha_{n+2})\right] = \beta_n(-2n-2+\zeta\alpha_n).
$$

According to Lemmas 4.2 and 4.3, the free term in q_2^n can be expressed as it is given. Similarly, using (4.7), (4.4) and Lemma 4.2, an expression for p_2^n can be derived. $\square$

Theorem 4.5. *The polynomials* p_2^n *and* q_2^n *satisfy the following recurrence relations*

$$
\begin{aligned}
p_2^{n+1} &= -\,q_2^n\frac{x-\mathrm{i}\alpha_n}{\beta_n} + p_2^{n-1} + q_2^{n-1}\frac{x-\mathrm{i}\alpha_{n-1}}{\beta_{n-1}},\\
q_2^{n+1} &= -\,x\phi + (x-\mathrm{i}\alpha_n)p_2^n + q_2^{n-1}\frac{\beta_n}{\beta_{n-1}}\\
&\quad +\,(x-\mathrm{i}\alpha_n)\left[q_2^n\,\frac{x-\mathrm{i}\alpha_n}{\beta_n} - p_2^{n-1} - q_2^{n-1}\frac{x-\mathrm{i}\alpha_{n-1}}{\beta_{n-1}}\right].
\end{aligned}
$$

Proof. Starting from two equations

$$
-x\phi p_n' = p_2^np_n + q_2^np_{n-1}, \quad -x\phi p_{n-1}' = p_2^{n-1}p_{n-1} + q_2^{n-1}p_{n-2}, \tag{4.8}
$$

multiplying the first equation with $(x-\mathrm{i}\alpha_n)$ and the second one with β_n and then subtracting, we obtain

$$\begin{aligned}
&-x\phi[(x-\mathrm{i}\alpha_n)p_n'-\beta_n p_{n-1}'+p_n]+x\phi p_n=-x\phi[(x-\mathrm{i}\alpha_n)p_n-\beta_n p_{n-1}]'+x\phi p_n\\
&\quad=-x\phi p_{n+1}'+x\phi p_n=(x-\mathrm{i}\alpha_n)p_2^n p_n+[q_2^n(x-\mathrm{i}\alpha_n)-\beta_n p_2^{n-1}]p_{n-1}-\beta_n q_2^{n-1}p_{n-2}\\
&\quad=\left\{(x-\mathrm{i}\alpha_n)p_2^n+q_2^{n-1}\frac{\beta_n}{\beta_{n-1}}\right\}p_n\\
&\qquad+\left\{q_2^n(x-\mathrm{i}\alpha_n)-\beta_n p_2^{n-1}-q_2^{n-1}(x-\mathrm{i}\alpha_{n-1})\frac{\beta_n}{\beta_{n-1}}\right\}p_{n-1}\\
&\quad=-\left\{q_2^n\frac{x-\mathrm{i}\alpha_n}{\beta_n}-p_2^{n-1}-q_2^{n-1}\frac{x-\mathrm{i}\alpha_{n-1}}{\beta_{n-1}}\right\}p_{n+1}\\
&\qquad+\left\{(x-\mathrm{i}\alpha_n)p_2^n+q_2^{n-1}\frac{\beta_n}{\beta_{n-1}}+(x-\mathrm{i}\alpha_n)\right.\\
&\qquad\times\left.\left[q_2^n\frac{x-\mathrm{i}\alpha_n}{\beta_n}-p_2^{n-1}-q_2^{n-1}\frac{x-\mathrm{i}\alpha_{n-1}}{\beta_{n-1}}\right]\right\}p_n.
\end{aligned}$$

Now, we can just read terms with p_{n+1} and p_n on the right-hand side and recognize them as p_2^{n+1} and q_2^{n+1}. □

Theorem 4.6. *We have*

$$p_2^{n+1}+p_2^n+q_2^n\frac{x-\mathrm{i}\alpha_n}{\beta_n}=(1+\mathrm{i}m\pi x)\phi,\quad n\in\mathbb{N}. \tag{4.9}$$

Proof. Using the first equation from Theorem 4.5 we find

$$p_2^{n+1}+\frac{x-\mathrm{i}\alpha_n}{\beta_n}q_2^n=p_2^{n-1}+\frac{x-\mathrm{i}\alpha_{n-1}}{\beta_{n-1}}q_2^{n-1}.$$

If we add to both sides p_2^n, we conclude that the expression stated in the theorem is independent of n, i.e.,

$$p_2^{n+1}+p_2^n+\frac{x-\mathrm{i}\alpha_n}{\beta_n}q_2^n=p_2^n+p_2^{n-1}+\frac{x-\mathrm{i}\alpha_{n-1}}{\beta_{n-1}}q_2^{n-1}.$$

The proof can be completed by a direct calculation of p_2^2, p_2^1 and q_2^1. Using Table 1, the following values can be calculated

$$p_2^1=1+\mathrm{i}\zeta x+x^2,\quad q_2^1=\frac{\mathrm{i}}{\zeta}[2+4\mathrm{i}\zeta x+(2-\zeta^2)x^2],$$

$$p_2^2=\frac{2}{(2-\zeta^2)^2}[8+(18-\zeta^2)\mathrm{i}\zeta x+(2-\zeta^2)^2x^2].$$

A direct calculation finishes the proof. □

Beside p_2^1, p_2^2 and q_2^1 (given before), we need also as a starting value

$$q_2^2 = 4\frac{2\mathrm{i}(-12-4\zeta^2+\zeta^4)+2\zeta(36-2\zeta^4)x+\mathrm{i}(-12+32\zeta^2-15\zeta^4+\zeta^6)x^2}{\zeta(2-\zeta^2)^2}$$

in order to use the recurrence relations from Theorem 4.5.

Lemma 4.7. *Let $\zeta = m\pi$. Then*

$$\begin{aligned}
\mathrm{i}\,\frac{\alpha_{n+1}\gamma_{n+1}-\alpha_n\gamma_{n-1}}{2} &= 1-\beta_{n+1}+nc_n-(n+3)c_{n+2}\\
&\quad -\frac{\zeta}{2}[a_nc_n-a_{n+2}c_{n+2}+\alpha_{n+1}\beta_{n+3}-\alpha_n\beta_{n-1}], \qquad (4.10)\\
\beta_{n+2}\gamma_{n+1}-\beta_n\gamma_{n-1} &= -\mathrm{i}[(\alpha_{n+2}+2\alpha_{n+1})\beta_{n+2}+(2\alpha_n+\alpha_{n-1})\beta_n\\
&\quad + a_{n+1}(\beta_{n+1}+2c_{n+1}-A_{n+1}-1)], \qquad (4.11)
\end{aligned}$$

$$-\mathrm{i}\,\frac{\gamma_{n+1}-\gamma_{n-1}}{2} = (n+3)a_{n+2}-na_n-\frac{\zeta}{2}[A_{n+2}-A_n+b_n-b_{n+3}], \qquad (4.12)$$

where $a_n=\alpha_n+\alpha_{n-1}$, $b_n=\beta_n+\beta_{n-1}$, $c_n=\alpha_n\alpha_{n-1}+\beta_n$ and $A_n=\alpha_n^2+\alpha_{n-1}^2+\alpha_n\alpha_{n-1}$.

Proof. In order to obtain the first Eq. (4.10) we use (4.9) at $x=0$, i.e., $p_2^{n+1}(0)+p_2^n(0)-\mathrm{i}(\alpha_n/\beta_n)q_2^n(0)=1$, and for (4.11) we use the definition of γ_n, given by (4.6), from which we conclude that

$$\beta_{n+2}\gamma_{n+1} = \int x\phi\left(\frac{p_{n+1}^2}{\|p_{n+1}\|^2}+\frac{p_n^2}{\|p_n\|^2}\right)\mathrm{d}\mu+\beta_n\gamma_{n-1}.$$

These integrals can be calculated in the same fashion as is it presented in the proof of Lemma 4.3. For the second integral, we have

$$\int x\phi\,\frac{p_n^2}{\|p_n\|^2}\,\mathrm{d}\mu = -\mathrm{i}\left[(\alpha_{n+1}+2\alpha_n)\beta_{n+1}+(2\alpha_n+\alpha_{n-1})\beta_n-\alpha_n(1+\alpha_n^2)\right].$$

Finally, to prove (4.12) we consider terms with x in both sides in (4.9)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p_2^{n+1}+p_2^n+\frac{x-\mathrm{i}\alpha_n}{\beta_n}q_2^n\right)\Bigg|_{x=0} = \mathrm{i}m\pi.$$

In terms of Theorem 4.5, we have $-u_{n+1}-u_n+\delta_n+\alpha_n(2n+1-\zeta\alpha_n)=m\pi$. A direct calculation finishes the proof. □

Now we are ready to prove Theorem 3.3 from Section 3.

Proof of Theorem 3.3. In order to be able to solve the system of linear equations given in Lemma 4.7 we first need to prove that this system has a solution. We can examine two pairs of linear equations, the first pair of equations (4.10) and (4.12), and second pair (4.11) and (4.12).

A sufficient condition that the first system of equations has a solution is that $\alpha_{n+1} \neq \alpha_n$ for every $n \in \mathbb{N}_0$. Using the expressions given in (3.2) for α-coefficients, we find

$$\mathrm{i}(\alpha_{n+1} - \alpha_n) = \frac{\Delta'_{n+2}\Delta_{n+1}\Delta_n - 2\Delta'_{n+1}\Delta_{n+2}\Delta_n + \Delta'_n\Delta_{n+2}\Delta_{n+1}}{\Delta_{n+2}\Delta_{n+1}\Delta_n}.$$

In order to have $\alpha_{n+1} \neq \alpha_n$, it is enough to prove that the free term in the numerator on the right side in the previous expression is not equal to zero. According to (2.7) and (2.8), this free term can be expressed as follows:

$$\begin{aligned} &2^{3(n+1)} \prod_{j=0}^{n+1} j!(j+1)! \prod_{j=0}^{n} j!(j+1)! \prod_{j=0}^{n-1} j!(j+1)! \\ &\quad \times [(n+2)(n+3) - 2(n+1)(n+2) + n(n+1)] \\ &\quad = 2^{3n+4} \prod_{j=0}^{n+1} j!(j+1)! \prod_{j=0}^{n} j!(j+1)! \prod_{j=0}^{n-1} j!(j+1)! \neq 0. \end{aligned}$$

To be able to solve the second pair of equations we need $\beta_{n+2} \neq \beta_n$, $n \in \mathbb{N}$. Using (2.7) we can also evaluate the free term in the numerator of the difference $\beta_{n+2}-\beta_n=\Delta_{n+1}(\Delta_{n+3}\Delta_n^2-\Delta_{n+2}^2\Delta_{n-1})/(\Delta_n^2\Delta_{n+2}^2)$ in the form

$$\begin{aligned} &2^{n+3} \prod_{i=0}^{n+2} i!(i+1)! 2^{2n} \prod_{i=0}^{n-1} i!^2(i+1)!^2 - 2^{2n+4} \prod_{i=0}^{n+1} i!^2(i+1)!^2 2^{n-1} \prod_{i=0}^{n-2} i!(i+1)! \\ &\quad = 2^{3(n+1)}(n+1)!^2(n+2)!^2 \prod_{i=0}^{n} i!(i+1)! \prod_{i=0}^{n-1} i!^2(i+1)!^2(4n+6) \neq 0. \end{aligned}$$

Since polynomials in ζ $(=m\pi)$ in the numerators for the differences $\alpha_{n+1} - \alpha_n$ and $\beta_{n+2} - \beta_n$ are not identically zero, they cannot be equal to zero for any nonzero integer m, since again $m\pi$, $m \in \mathbb{Z}\backslash\{0\}$, is a transcendental number. This means that the systems of equations (4.10)–(4.12) and (4.11)–(4.12) have unique solutions for any given n.

The following solutions for γ-coefficients can be obtained by solving systems

$$\begin{aligned} -\mathrm{i}\gamma_n = \frac{-2}{\alpha_n - \alpha_{n-1}} &\left\{1 + (n+2)(a_{n+1}\alpha_{n-1} - c_{n+1}) + (n-1)(\beta_{n-1} - \alpha_{n-1}^2) - \beta_n\right\} \\ &+ \zeta \left\{a_n\alpha_{n-1} - a_{n+1}\alpha_{n+1} + \beta_{n+2} + \frac{(a_{n-1} + \alpha_{n-1})\beta_{n-1} - (a_n + \alpha_{n+1})\beta_{n+1}}{\alpha_n - \alpha_{n-1}}\right\}, \end{aligned}$$

$$\begin{aligned} -\mathrm{i}\gamma_n = \frac{-2}{\alpha_{n+2} - \alpha_{n+1}} &\{1 + (n+1)(c_{n+1} - a_{n+1}\alpha_{n+2}) - (n+4)(\beta_{n+3} - \alpha_{n+2}^2) - \beta_{n+2}\} \\ &+ \zeta \left\{a_{n+2}\alpha_{n+2} - a_{n+1}\alpha_n + \beta_n - \frac{(a_{n+3} + \alpha_{n+2})\beta_{n+3} + (a_{n+1} + \alpha_{n+2})\beta_{n+1}}{\alpha_{n+2} - \alpha_{n+1}}\right\}, \end{aligned}$$

$$
\begin{aligned}
-\mathrm{i}\gamma_n = & \frac{-1}{\beta_{n+1}-\beta_{n-1}}\left\{(\alpha_{n+1}+2\alpha_n)\beta_{n+1}+(2\alpha_{n-1}+\alpha_{n-2})\beta_{n-1}+a_n(\beta_n+2c_n-A_n-1)\right\} \\
& - \frac{\beta_{n-1}}{\beta_{n+1}-\beta_{n-1}}\{2[(n+2)a_{n+1}-(n-2)a_{n-1}]+\zeta[A_{n-1}-A_{n+1}-b_{n-1}+b_{n+2}]\},
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{i}\gamma_n = & \frac{-1}{\beta_{n+3}-\beta_{n+1}}\{(\alpha_{n+3}+2\alpha_{n+2})\beta_{n+3}+(2\alpha_{n+1}+\alpha_n)\beta_{n+1} \\
& + a_{n+2}(\beta_{n+2}+2c_{n+2}-A_{n+2}-1)\} \\
& - \frac{\beta_{n+3}}{\beta_{n+3}-\beta_{n+1}}\{2[(n+5)a_{n+3}-(n+1)a_{n+1}]+\zeta[A_{n+1}-A_{n+3}-b_{n+1}+b_{n+4}]\},
\end{aligned}
$$

where a_n, b_n, c_n and A_n are as in Lemma 4.7.

Using these equations we can get nonlinear recurrence relations for the three-term recurrence coefficients. The two equations presented in this theorem are just two out of five possible. It is important to note that the nonlinear recurrence relations, which can be obtained by equating the first and third and the second and fourth solutions for γ_n, are the same. □

The previous consideration leads to the conclusion that the coefficients γ_n are purely imaginary numbers. This simple fact has a great impact on the monic polynomial Q_2^n $(=\mathrm{i}q_2^n/(\zeta\beta_n))$. Namely,

$$
\begin{aligned}
Q_2^n = & x^2+\mathrm{i}\left[\alpha_n - 2\frac{n+1}{\zeta}\right]x - \frac{1}{\zeta}\left[(n+2)\alpha_{n+1}+n\alpha_{n-1}+\mathrm{i}\frac{\gamma_n+\gamma_{n-1}}{2}\right] \\
& - \frac{1}{2}\left[\beta_{n-1}+\beta_{n+2}-\alpha_{n+1}(\alpha_{n+1}+\alpha_n)-\alpha_{n-1}(\alpha_n+\alpha_{n-1})\right].
\end{aligned}
$$

It can be seen that the roots of q_2^n have to be of the form $\mathrm{i}y_1$ and $\mathrm{i}y_2$ or $\pm x_1+\mathrm{i}y_1$, since their sum is a purely imaginary number and their product is real. All numerical experiments have shown that roots of q_2^n are purely imaginary numbers. It was seen also that with increasing n, one zero tends to infinity over the positive part of the imaginary axis and that the second one tends to zero again over the positive imaginary axis. An idea of such behavior of the zeros is also supported in the expression for Q_2^n. As we can see, the sum of the zeros increases with n. The location of the zeros of the polynomial q_2^n is connected with the zeros of the orthogonal polynomial p_n.

Theorem 4.8. *If the polynomial q_2^n does not have common zeros with p_n, then the zeros of p_n are simple. If τ is a multiple zero of p_n with multiplicity k, then it is also a zero of q_2^n with multiplicity $k-1$. The polynomial p_n can have a zero with multiplicity at most three.*

Proof. It is easy to check that p_n and p_{n-1} do not have zeros in common. Namely, if they do have some common zero $x=\tau$, then using the three-term recurrence relation, we can conclude that τ is a zero of all other polynomials p_k, $k=n-2,\dots,1$. Then, since τ is a zero of p_1, its value is $\mathrm{i}\alpha_0$. It is easy to check that p_2 does not have $\mathrm{i}\alpha_0$ as its zero.

Now suppose that q_2^n and p_n do not have zeros in common and p_n has a multiple zero τ. Then, using (4.1), we see that

$$0 = -\tau\phi(\tau)p_n'(\tau) = p_2^n(\tau)p_n(\tau) + q_2^n(\tau)p_{n-1}(\tau) = q_2^n(\tau)p_{n-1}(\tau) \neq 0,$$

which is a contradiction. Thus, all zeros of p_n are simple.

To conclude the rest, it is enough to see that $-x\phi p_n' - p_2^n p_n$ has the factor $(x-\tau)^{k-1}$. Hence, q_2^n has a zero τ of multiplicity $k-1$. If the zero of q_2^n is double, then p_n may have a zero with multiplicity three. □

Theorem 4.9. *The orthogonal polynomial p_n with respect to the weight $w(x) = x\exp(\mathrm{i}m\pi x)$ on $[-1, 1]$ is a solution of the following differential equation*

$$x\phi q_2^n p_n'' - [\psi q_2^n + x\phi(q_2^n)']p_n' + \left[(p_2^n)'q_2^n - p_2^n(q_2^n)' + \frac{q_2^n(\hat{p}_2^n q_2^n - p_2^n\hat{q}_2^n)}{x\phi}\right]p_n = 0,$$

where $\hat{p}_2^n = -q_2^{n-1}/\beta_{n-1}$ and $\hat{q}_2^n = p_2^{n-1} + (x - \mathrm{i}\alpha_{n-1})q_2^{n-1}/\beta_{n-1}$. The term $q_2^n(\hat{p}_2^n q_2^n - p_2^n\hat{q}_2^n)$ has also the factor $x\phi$.

Proof. Starting from two equations (4.8) and using the three-term recurrence relation, we can calculate

$$-x\phi p_{n-1}' = -\frac{q_2^{n-1}}{\beta_{n-1}}p_n + \left(p_2^{n-1} + q_2^{n-1}\frac{x - \mathrm{i}\alpha_{n-1}}{\beta_{n-1}}\right)p_{n-1} = \hat{p}_2^n p_n + \hat{q}_2^n p_{n-1}.$$

Substituting in this equation the value for p_{n-1}, calculated from the first equation and using Theorem 4.6, the proof of the statement is completed.

Since all terms, except $q_2^n(\hat{p}_2^n q_2^n - p_2^n\hat{q}_2^n)$, are divisible by $x\phi$, the mentioned term must be divisible by $x\phi$ as well. □

The following theorem can easily be proved. It can be used efficiently in the cases when zeros of the polynomial p_n of very high degree are calculated.

Theorem 4.10. *Under the assumption that the polynomials q_2^n and p_n do not have zeros in common, we have the following system of equations*

$$\frac{p_n''(x_\nu^n)}{p_n'(x_\nu^n)} + \frac{2}{x_\nu^n} + \frac{1}{x_\nu^n - 1} + \frac{1}{x_\nu^n + 1} + \mathrm{i}m\pi - \frac{(q_2^n)'(x_\nu^n)}{q_2^n(x_\nu^n)} = 0, \tag{4.13}$$

where x_ν^n, $\nu = 1, \ldots, n$, are distinct zeros of the polynomial p_n.

It is well-known that the QR-algorithm for finding zeros x_ν^n can be ill-conditioned, when the Jacobi matrix is not positive definite (see [13,21]). Using the previous theorem, under mild assumption and with appropriate starting values for the zeros x_ν^n, a numerical construction can be performed using a similar algorithm as given in [21].

5. Gaussian quadrature rule

In all numerical examples we have run, we were unable to find even one example of orthogonal polynomials with multiple zeros. This means that Theorem 4.8 is not sharp enough. It also seems that the zeros of the orthogonal polynomials are uniformly bounded, which is connected with the already mentioned conjecture about an asymptotic behavior given in (3.5). It is known that, if the three-term recurrence coefficients are uniformly bounded, the corresponding Jacobi matrix can be understood as a linear operator J acting on the Hilbert space ℓ^2 of all complex square-summable sequences. Furthermore, the uniform boundedness of the recursion coefficients implies the boundedness of this linear operator J (see [1]). It is also known that the zeros of all orthogonal polynomials are bounded by the norm of J.

Since we cannot claim that the zeros of orthogonal polynomials are simple, in applications we should be ready to apply Gaussian quadrature rules which deal with multiplicities. In the case of multiple zeros of orthogonal polynomials, the Gaussian quadrature rule has the following form (see [9,17])

$$G_n(f) = \sum_{\nu=1}^{n} \sum_{k=0}^{m_\nu - 1} w_{\nu,k}^{n} f^{(k)}(x_\nu^n). \tag{5.1}$$

According to considerations in the previous section, at most two nodes in (5.1) may have multiplicities. However, as we mentioned before, in all our examples we have encountered simple nodes and we have used the standard Gaussian quadrature rule

$$G_n(f) = \sum_{\nu=1}^{n} w_\nu^n f(x_\nu^n). \tag{5.2}$$

Distributions of the zeros of the orthogonal polynomials for $m = 2$ and 22 are presented in Fig. 2. Only the zeros of polynomials with degrees $n = 5(5)35$ are displayed. The corresponding distribution of the zeros for $m = 1000$ is given in Fig. 3 (left). For better visibility in the same figure (right) only the main group of zeros is presented.

We can see that the distribution of the zeros is such that all zeros are with positive imaginary part, except for one zero of polynomials of odd degree for which its real part is zero. From the figures we can also conclude that all zeros are in the half-strip $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1 \wedge \mathrm{Im}(z) > 0\}$, except maybe one zero for polynomials of odd degree with real part equal to zero. Also, it is obvious that if m is increasing, then the zeros of the polynomials are grouped around the points ± 1. This is the reason why the QR-algorithm in D-arithmetic cannot be used for a construction of zeros with a large m (e.g., $m = 10^9$). Zeros of polynomials for very large m are very close to each other, so that they cannot be distinguished in D-arithmetic.

Also, it can be seen that for a fixed m, if we increase degree of a polynomial, then the zeros tend to cover the interval $(-1, 1)$. This is, however, less obvious for larger m, because of the mentioned behavior of their grouping near the points ± 1. This behavior is also in good agreement with the conjectured asymptotic for the three-term recurrence coefficients (see [9,10,12]).

Since the zeros of the orthogonal polynomials are not contained in the supporting set of the measure μ, we cannot expect the quadrature rule (5.1) to converge, except for functions which are analytic in a certain complex domain $\mathscr{D} \supset [-1, 1]$. Spurious zeros (see [22]) for our sequence of orthogonal polynomials were not detected.

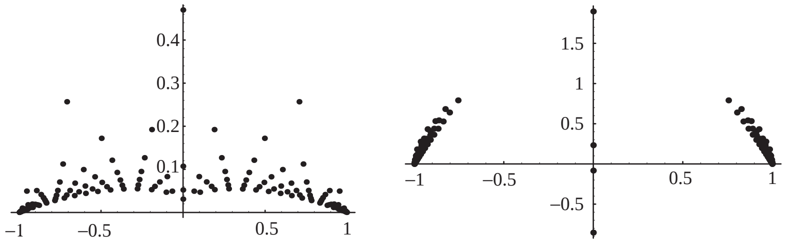

Fig. 2. Distribution of zeros for $m=2$ (left) and $m=22$ (right).

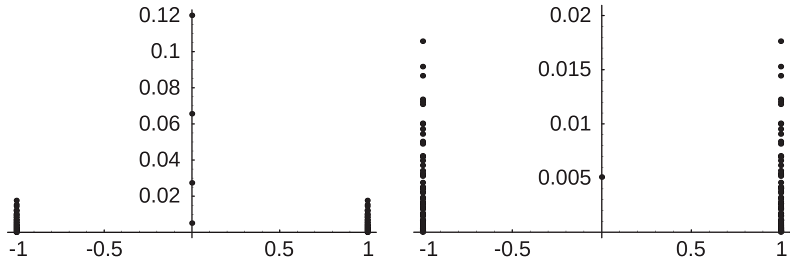

Fig. 3. Distribution of zeros for $m=1000$: all zeros (left) and selected zeros (right).

If we adopt the conjectures that the zeros are bounded and that there are no spurious zeros it can be claimed that the Gaussian quadrature rules are convergent (for analytic integrands) (see [18]).

For a construction of the Gaussian quadrature rule the QR-algorithm is used (see [5,8]), but in a modified form (see [16]). For example, in D-arithmetic for $m=10^3$, the maximal relative error in the constructed Gaussian weights with $n=40$ nodes is of magnitude 6×10^{-2} and 10^{-10} using the original and the modified version of the QR algorithm, respectively. Note also that the maximal relative error in the constructed nodes, in D-arithmetic using the QR-algorithm, is of magnitude 10^{-13} for the same values of m and n. In our experiments the QR-algorithm exhibits stability. For values of extremely large m (e.g., $m=10^9$), the QR-algorithm executed in D-arithmetic exhibits poor behavior. However, this is not due to ill-conditioning, but rather to the fact that in D-arithmetic the zeros of the orthogonal polynomials for $m=10^9$ cannot be distinguished for n sufficiently small (e.g., n of smaller order than 10^3). The phenomenon of bifurcation, encountered for generalized Bessel polynomials (see [21]), also appears for our orthogonal polynomials. In this case, a construction of the zeros of the orthogonal polynomials (the nodes in the Gaussian quadrature rules) should be performed using (4.13). Starting values for the zeros of the orthogonal polynomials in the Newton–Kantorovič method can be the zeros obtained by the QR-algorithm or some other approximation of the zeros based on the presented figures.

In Table 3 we give the nodes and the weights of the Gaussian formulas (5.2) (to 14 decimals only, to save space) for $n=10$ and 20 points, when the weight function is $w(x)=x\exp(\mathrm{i}10\pi x)$ $(m=10)$.

A possible application of these quadratures is in numerical calculation of integrals involving highly oscillatory integrands. We consider here the calculation of Fourier coefficients:

$$F_m(f)=C_m(f)+\mathrm{i}S_m(f)=\int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}m\pi x}\,\mathrm{d}x.$$

Table 3
Nodes x_ν^n and weights w_ν^n for $n=10$ and $n=20$, when $m=10$

x_ν^n	w_ν^n
$\pm0.99833323072985 + \mathrm{i}8.0272939519631(-3)$	$\pm2.6219203933839(-3) - \mathrm{i}1.6185441353473(-2)$
$\pm0.99100855292260 + \mathrm{i}4.3127353860182(-2)$	$\mp4.2536667302542(-4) - \mathrm{i}1.3094541072322(-2)$
$\pm0.97687553159991 + \mathrm{i}1.1013718224261(-1)$	$\mp1.0621304335557(-3) - \mathrm{i}2.4769593783534(-3)$
$\pm0.95373342919595 + \mathrm{i}2.1833923751031(-1)$	$\mp1.2031273072243(-4) - \mathrm{i}7.4485602824591(-5)$
$\pm0.91584375790176 + \mathrm{i}3.9367290146258(-1)$	$\mp8.9861059280033(-7) + \mathrm{i}4.3878859382919(-7)$
$\pm0.99846411789590 + \mathrm{i}3.8408078263538(-3)$	$\pm3.1046382565034(-3) - \mathrm{i}8.8805764375599(-3)$
$\pm0.99183153354877 + \mathrm{i}2.0347206579223(-2)$	$\pm1.9244416480364(-3) - \mathrm{i}1.2968640292577(-2)$
$\pm0.97957734645900 + \mathrm{i}5.0510187350466(-2)$	$\mp1.6177820964010(-3) - \mathrm{i}7.9111318842472(-3)$
$\pm0.96109698530124 + \mathrm{i}9.5210986930994(-2)$	$\mp1.8324015346507(-3) - \mathrm{i}2.0568347704191(-3)$
$\pm0.93538991039429 + \mathrm{i}1.5590937196538(-1)$	$\mp5.3091855800340(-4) - \mathrm{i}5.9554706994496(-5)$
$\pm0.90085641267460 + \mathrm{i}2.3495811423141(-1)$	$\mp3.6520885773394(-5) + \mathrm{i}4.3342540474941(-5)$
$\pm0.85489295111768 + \mathrm{i}3.3624578190485(-1)$	$\pm1.7102442502967(-6) + \mathrm{i}2.4540197057364(-6)$
$\pm0.79298840127057 + \mathrm{i}4.6668799320411(-1)$	$\pm4.5156899615527(-8) - \mathrm{i}4.7151995083521(-8)$
$\pm0.70627510516459 + \mathrm{i}6.4046420228669(-1)$	$\mp3.9409685347890(-10) + \mathrm{i}6.5422096655704(-11)$
$\pm0.57200700024404 + \mathrm{i}8.9689758832390(-1)$	$\pm1.3302611178019(-13) - \mathrm{i}1.8953246644963(-13)$

Since $\int_{-1}^{1} \exp(\mathrm{i}m\pi x)\mathrm{d}x = 0$, we have

$$F_m(f) = \int_{-1}^{1} \frac{f(x)-f(0)}{x}\, x\mathrm{e}^{\mathrm{i}m\pi x}\, \mathrm{d}x = \int g(x)\, \mathrm{d}\mu(x),$$

so that we can compute it using the Gaussian quadrature rules (5.2) of the function g defined by

$$g(x) = \frac{f(x)-f(0)}{x}, \quad g(0) = f'(0).$$

Under the assumption that f is analytic in some domain $\mathscr{D} \supset [-1,1]$, the numerical integration can be safely applied, since g is also analytic in $\mathscr{D}$.

In general, for some analytic function f, the approximation of the integral with respect to the measure $\exp(\mathrm{i}m\pi x)\mathrm{d}x$, can be given as

$$F_m(f) = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}m\pi x}\, \mathrm{d}x \approx \sum_{\nu=1}^{n} \frac{w_\nu^n}{x_\nu^n}\,(f(x_\nu^n) - f(0)). \tag{5.3}$$

Example 5.1. We can get an interesting result if we apply our quadrature rule for $m=10$ to the function $f(x) = \dfrac{x}{x^2+1/4}$. According to (5.3), we consider

$$S_{10}(f) = \int_{-1}^{1} \frac{x}{x^2+1/4} \sin(10\pi x)\, \mathrm{d}x \approx G_n(f) = \mathrm{Im}\left\{\sum_{\nu=1}^{n} \frac{w_\nu^n}{(x_\nu^n)^2+1/4}\right\}.$$

Table 4
Gaussian approximations $G_n(f)$ and $\widetilde{G}_n(f)$ for $S_{10}(f)$ and $f(x) = x/(x^2 + 1/4)$

n	$G_n(f)$	$\widetilde{G}_n(f)$
10	−0.0509124802888631	−0.0509120068454231
20	−0.0509124798498521	−0.0509120064064121
30	−0.0509124699339274	−0.0509119964904873
40	−0.0509120078597894	
50	−0.0509120064014030	
60	−0.0509120064013063	
70	−0.0509120064013063	

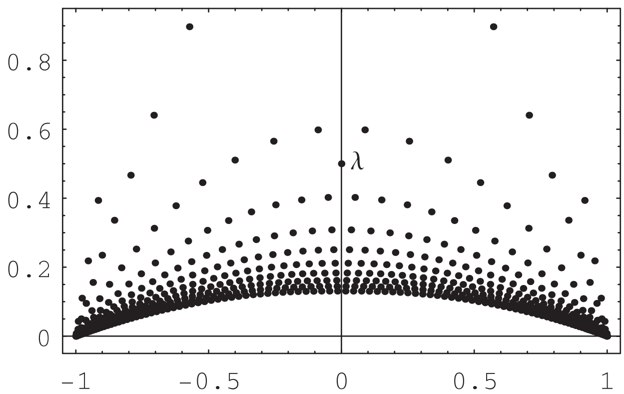

Fig. 4. Distribution of zeros for polynomials orthogonal with respect to the weight function $w(x) = x \exp(\mathrm{i}10\pi x)$ on $[-1, 1]$.

The Gaussian approximations for $n = 10(10)70$ are given in Table 4. The function f has simple poles at $z = \pm\mathrm{i}/2$. Finding the residuum at the point $\lambda = \mathrm{i}/2$, we obtain $R = \mathrm{Im}\{2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}/2}[f(z)\mathrm{e}^{\mathrm{i}10\pi z}]\} = 4.734434401198 \cdot 10^{-7}$.

If we simply add this value to $G_n(f)$, we can significantly improve the results for $n < 40$, as can be checked in Table 4 (the column $\widetilde{G}_n(f) = G_n(f) + R$). The reason for such behavior of the quadrature rules we can find in the zero distribution for polynomials orthogonal with respect to the weight $w(x) = x \exp(\mathrm{i}10\pi x)$ on $[-1, 1]$ (see Fig. 4). As we can see, while the convex hull of the zeros includes the point $\lambda = \mathrm{i}/2$ (cases for $n \leqslant 30$), this singularity has an influence on the Gaussian approximations $G_n(f)$. But, when zeros drop below λ (for $n \geqslant 40$), this influence ceases.

If we increase m, for example taking $m = 30, 100, 10^6$, the convergence is rather faster. Table 5 shows $G_n(f)$ for $m = 30$ and 100. For $m = 10^6$, the relative error in $G_{10}(f)$ is smaller than 10^{-60}.

This faster convergence can be understood easily, since values of the residuum at $z = \mathrm{i}/2$ are decreasing exponentially with m, so it cannot harm the convergence.

Example 5.2. In this example, we present poor convergence results for an entire function. Consider $f(x) = \cos(10^5 x^3 + x)$, i.e., the integral

$$I_m(f) = \int_{-1}^{1} \cos(10^5 x^3 + x) x \mathrm{e}^{\mathrm{i}m\pi x}\, \mathrm{d}x.$$

Table 5
Gaussian approximations $G_n(f)$ for $S_{30}(f)$ and $S_{100}(f)$

m	n	$G_n(f)$
30	10	−0.0169759131766787207976
	20	−0.0169759131766780460809
	30	−0.0169759131766780460809
100	10	−0.00050929580138121841037438653708
	20	−0.00050929580138121841037438653707

Table 6
Gaussian approximation $G_n(f)$ for the integral $I_m(f)$, $m = 10^3$

n	10	20	30	40	...	200
$G_n(f)$	1.2(516)i	−7.1(1224)i	−7.9(1967)i	5.2(2725)i	...	−2.2(15192)i

Table 7
Gaussian approximation $G_n(f)$ and corresponding relative error r_n for the integral $I_m(f)$, $m = 10^6$

n	$-\mathrm{i}G_n(f)$	r_n
6	3.662375550592334(−7)	4.0(−8)
8	3.662375697084937(−7)	1.1(−10)
10	3.662375697480965(−7)	2.9(−13)
12	3.662375697482014(−7)	m.p.

Taking $m = 10^3$ and applying the Gaussian quadrature rule (5.2), we get the results given in Table 6. These results are not obtained in the standard D-arithmetic, but by using an extended exponent arithmetic (in MATHEMATICA package). Since the absolute value of $I_m(f)$ has to be smaller than 1, it can be seen that we do not have convergence for $n \leqslant 200$.

The reason for such poor behavior of the quadrature rules is easy to understand. The quadrature sum has the form

$$G_n(f) = \sum_{\nu=1}^{n} w_\nu^n \cos(10^5 (x_\nu^n)^3 + x_\nu^n),$$

as it can be seen from Fig. 3, zeros of orthogonal polynomials for $m = 10^3$ have imaginary values about 0.02, while real parts are close to 1 for $n \leqslant 35$, so that values of the cos-function are to be evaluated in points with imaginary parts $\approx 6 \times 10^3$, i.e.,

$$\cos(10^5 (x_\nu^n)^3 + x_\nu^n) \approx \cos(10^5 (1 + 0.02\mathrm{i})^3 + 1 + 0.02\mathrm{i}) \approx \cos(10^5 + 6 \times 10^3 \mathrm{i}),$$

and these values are huge. It takes much more than $n = 200$ for the zeros of orthogonal polynomials to come over the interval $[-1, 1]$. The same behavior occurs when an integration of any such function is considered, for example $\cos(10^5 x)$.

However, if m is increased to 10^6, the convergence is evident (see Table 7). An application of quadrature rules to the function $f(x) = \cos(10^5 x^3 + x)$ gives results to machine precision with only twelve points in the quadrature sum. Table 7 shows the results $-\mathrm{i}G_n(f)$ and the corresponding relative errors $r_n = |(G_n(f) - I_m(f))/I_m(f)|$ (m.p. stands for machine precision in double precision arithmetic ($\approx 2.22 \cdot 10^{-16}$)). Here it is important to note that the imaginary part of the zeros is of magnitude 10^{-6} while real parts are still close to 1 for $n = 12$, so that the demonstrated effect for $m = 10^3$ cannot appear.

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References

[1] B. Beckerman, M. Castro, On the determinacy of the complex Jacobi operators, Publication ANO 446, Université Lille, 2002.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[3] A.S. Cvetković, G.V. Milovanović, The Mathematica Package "Orthogonal Polynomials", Facta Univ. Ser. Math. Inform 19 (2004).
[4] M.G. de Bruin, Polynomials orthogonal on a circular arc, J. Comput. Appl. Math. 31 (1990) 253–266.
[5] W. Gautschi, Algorithm 726: ORTHPOL—A package of routines for generating orthogonal polynomials and Gauss-type quadrature rules, ACM Trans. Math. Software 10 (1994) 21–62.
[6] W. Gautschi, H.J. Landau, G.V. Milovanović, Polynomials orthogonal on the semicircle, II, Constr. Approx. 3 (1987) 389–404.
[7] W. Gautschi, G.V. Milovanović, Polynomials orthogonal on the semicircle, J. Approx. Theory 46 (1986) 230–250.
[8] G.H. Golub, J.H. Welsch, Calculation of Gauss quadrature rule, Math. Comput. 23 (1986) 221–230.
[9] P. Gonzales-Vera, G. Lopez, R. Orive, J.C. Santos, On the convergence of quadrature formula for complex weight functions, J. Math. Anal. Appl. 189 (1995) 514–532.
[10] A.P. Magnus, Toeplitz matrix techniques and convergence of complex weight Pade approximation, J. Comput. Appl. Math. 19 (1987) 23–28.
[11] A.P. Magnus, Painlevé-type of differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, J. Comput. Appl. Math. 57 (1995) 215–237.
[12] A. Mate, P. Nevai, A generalization of Poincare theorem for recurrence equations, J. Approx. Theory 63 (1990) 92–97.
[13] G.V. Milovanović, Complex orthogonality on the semicircle with respect to Gegenbauer weight: theory and applications, in: Th.M. Rassias (Ed.), Topics in Mathematical Analysis, Ser. Pure Math., 11, World Scientific Publishing, Teaneck, NJ, 1989, pp. 695–722.
[14] G.V. Milovanović, Numerical Analysis, Part I, Naučna knjiga, Belgrade, 1991 (Serbian).
[15] G.V. Milovanović, Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures, Comput. Math. Appl. 36 (1998) 19–39.
[16] G.V. Milovanović, A.S. Cvetković, Note on construction of weights in Gauss-type quadrature rule, Facta Univer. Ser. Math. Inform. 15 (2000) 69–83.
[17] G.V. Milovanović, A.S. Cvetković, Complex Jacobi matrices and quadrature rules, FILOMAT 17 (2003) 117–134.
[18] G.V. Milovanović, A.S. Cvetković, An application of little $1/q$-Jacobi polynomials to summation of certain series, Facta Univ. Ser. Math. Inform. 18 (2003) 31–46.
[19] G.V. Milovanović, P.M. Rajković, On polynomials orthogonal on a circular arc, J. Comput. Appl. Math. 51 (1994) 1–13.
[20] J. Nuttal, C.J. Wherry, Gaussian integration for complex weight functions, J. Inst. Maths. Appl. 21 (1978) 165–170.
[21] L. Pasquini, Accurate computation of the zeroes of the generalized Bessel polynomials, Numer. Math. 86 (2000) 507–538.
[22] H. Stahl, Spurious poles in Pade approximation, J. Comput. Appl. Math. 99 (1998) 511–527.