Milovanović and Mir *Journal of Inequalities and Applications* (2020) 2020:136
https://doi.org/10.1186/s13660-020-02404-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalizations of Zygmund-type integral inequalities for the polar derivative of a complex polynomial

Gradimir V. Milovanović[1,2*] and Abdullah Mir[3]

*Correspondence:
gvm@mi.sanu.ac.rs
[1]Serbian Academy of Sciences and Arts, Belgrade, Serbia
[2]Faculty of Science and Mathematics, University of Niš, Niš, Serbia
Full list of author information is available at the end of the article

Abstract

Some Zygmund-type integral inequalities for the polar derivatives of complex polynomials, inspired by the classical Bernstein-type inequalities that relate the uniform norms of polynomials and their derivatives on the unit circle, are investigated. The obtained results sharpen as well as generalize some already known L^{δ}-estimates between polynomials and their polar derivatives.

MSC: 30A10; 30C10; 30D15

Keywords: Polar derivative; Zygmund-type inequality; Integral norm

1 Introduction and preliminaries

Let $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a complex polynomial of degree n and $P'(z)$ be its derivative. There is a well-known classical Bernstein result for two polynomials $P(z)$ and $Q(z)$ with degree of $P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$. Namely, the inequality $|P(z)| \leq |Q(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leq |Q'(z)|$ on $|z| = 1$. In particular, this result allows one to establish the famous Bernstein inequality [3] in the uniform norm on the unit circle, i.e., if $P(z)$ is a polynomial of degree n, it is true that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

This inequality (1.1) was proved by Bernstein in 1912, and is best possible with equality holding for monomials $P(z) = cz^n$, where c is an arbitrary complex number. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) can be sharpened. In fact, Erdős conjectured and later Lax [8] proved that, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

In 1969 Malik [9] proved an extension of (1.2) under the condition that $P(z) \neq 0$ in $|z| < k$, where $k \geq 1$, i.e.,

$$\max_{|z|=1} \left| P'(z) \right| \leq \frac{n}{1+k} \max_{|z|=1} \left| P(z) \right|. \tag{1.3}$$

This inequality was generalized in different directions. Chan and Malik [4] proved that, if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $\mu \geq 1$, is a lacunary polynomial of degree n and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} \left| P'(z) \right| \leq \frac{n}{1+k^{\mu}} \max_{|z|=1} \left| P(z) \right|. \tag{1.4}$$

Inequality (1.4) was independently proved by Qazi (see [17], Lemma 1), who also under the same conditions proved that

$$\max_{|z|=1} \left| P'(z) \right| \leq \frac{n}{1+S_0(\mu)} \max_{|z|=1} \left| P(z) \right|, \tag{1.5}$$

where

$$S_0(\mu) = k^{\mu+1} \left(\frac{(\frac{\mu}{n})|\frac{a_\mu}{a_0}| k^{\mu-1} + 1}{(\frac{\mu}{n})|\frac{a_\mu}{a_0}| k^{\mu+1} + 1} \right) \geq k^{\mu}. \tag{1.6}$$

For a polynomial $P(z)$ of degree n, now we define the so-called the polar derivative of $P(z)$ with respect to the point α as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

This polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ in the following sense:

$$\lim_{\alpha \to \infty} \left\{ \frac{D_\alpha P(z)}{\alpha} \right\} = P'(z),$$

uniformly with respect to z for $|z| \leq R$ and $R > 0$.

Aziz [1] was among the first who expanded some of the above inequalities by replacing the ordinary derivative with polar polynomial derivatives. In fact, In 1988, Aziz proved that, if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k$, with $k \geq 1$, then for each complex number α such that $|\alpha| \geq 1$,

$$\max_{|z|=1} \left| D_\alpha P(z) \right| \leq n \left(\frac{|\alpha| + k}{1+k} \right) \max_{|z|=1} \left| P(z) \right|. \tag{1.7}$$

Dividing both sides of inequality (1.7) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain (1.3).

Over the last four decades many different authors produced a large number of results pertaining to the polar derivative of polynomials. More information on this topic can be found in the books of Milovanović et al. [11], Rahman and Schmeisser [18], and Marden [10]. One can also see in the literature (for example, refer [6, 7, 12, 13, 15, 16, 19]) the latest research and development in this direction. Recently, Mir [12] extended (1.7) to the class of lacunary type polynomials $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $\mu \ge 1$, not vanishing in $|z| < k$, $k \ge 1$, by first proving an L^δ-norm estimate of the polar derivative $D_\alpha P(z)$ with $|\alpha| \ge 1$ and then from the result so obtained produced the desired generalization of (1.7). More precisely, Mir proved the following result.

Theorem 1.1 (cf. [12]) *If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a lacunary polynomial of degree n having no zero in $|z| < k$, where $k \ge 1$, then for every $\delta > 0$ and for every complex number α such that $|\alpha| \ge 1$,*

$$\left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \le n\left(|\alpha| + k^\mu\right) C_\delta\left(k^\mu\right) \left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}}, \tag{1.8}$$

where

$$C_\delta\left(k^\mu\right) = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|k^\mu + e^{i\beta}\right|^\delta d\beta\right\}^{-\frac{1}{\delta}}. \tag{1.9}$$

If we let $\delta \to \infty$ in (1.8), noting that $C_\delta(k^\mu) \to 1/(1 + k^\mu)$, we get

$$\max_{|z|=1}\left|D_\alpha P(z)\right| \le n\left(\frac{|\alpha| + k^\mu}{1 + k^\mu}\right) \max_{|z|=1}\left|P(z)\right|, \tag{1.10}$$

which clearly represents a generalization of (1.7). In the same paper, Mir [12] further generalized inequalities (1.8) and (1.10) by proving the following result.

Theorem 1.2 (cf. [12]) *If $P(z) = z^s(a_0 + \sum_{j=\mu}^{n-s} a_j z^j)$, $1 \le \mu \le n - s$, $0 \le s \le n - 1$, is a polynomial of degree n having s-fold zeros at the origin and the remaining $n - s$ zeros in $|z| \ge k$, $k \ge 1$, then for every complex number α such that $|\alpha| \ge 1$,*

$$\left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \le \left\{(n-s)\left(|\alpha| + k^\mu\right) C_\delta\left(k^\mu\right) + s|\alpha|\right\} \left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}}, \tag{1.11}$$

where $C_\delta(k^\mu)$ is as defined in (1.9).

Making $\delta \to \infty$ in (1.11), we get

$$\max_{|z|=1}\left|D_\alpha P(z)\right| \le \left(\frac{n(|\alpha| + k^\mu) + s(|\alpha| - 1)k^\mu}{1 + k^\mu}\right) \max_{|z|=1}\left|P(z)\right|, \tag{1.12}$$

which clearly is a generalization of (1.10).

The main aim of this paper is to further generalize and strengthen all the previous mentioned inequalities by establishing some general Zygmund-type integral inequalities for polynomials involving the polar derivative. The obtained results represent polar derivative analogues of some classical Bernstein-type inequalities in the uniform norm on the unit circle and include also several interesting generalizations and refinements of some L^δ-norm inequalities for polynomials.

2 Main results

Here, we first prove the following more general inequality which includes not only (1.8) as a special case, but also derived polar derivative analogues of some classical Bernstein-type inequalities that relate the uniform norm of a polynomial to that of its derivative on the unit disk.

Theorem 2.1 *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, is a lacunary polynomial of degree n such that $P(z) \ne 0$ in $|z| < k$, where $k \ge 1$, then for all complex numbers α and β, with $|\alpha| \ge 1$ and $\delta \ge 1$,*

$$\left\{\int_0^{2\pi} \left|e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{1+S_0(\mu)}\right)P(e^{i\theta})\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \le n\big[\big(|\alpha| + S_0(\mu)\big) + |\beta|\big(|\alpha|-1\big)\big]C_\delta\big(S_0(\mu)\big)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^\delta d\theta\right\}^{\frac{1}{\delta}}, \tag{2.1}$$

where

$$C_\delta\big(S_0(\mu)\big) = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|S_0(\mu) + e^{it}\right|^\delta dt\right\}^{-\frac{1}{\delta}} \tag{2.2}$$

and $S_0(\mu)$ is defined by formula (1.6).

In the limiting case, when $\delta \to \infty$, the result is best possible for $\beta = 0$ and equality in (2.1) holds for $P(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of μ and $\alpha \ge 1$. Setting $\mu = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a lacunary polynomial of degree n such that $P(z) \ne 0$ in $|z| < k$, where $k \ge 1$, then for all complex numbers α and β, with $|\alpha| \ge 1$ and $\delta \ge 1$,*

$$\left\{\int_0^{2\pi} \left|e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{1+S_1}\right)P(e^{i\theta})\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \le n\big[\big(|\alpha| + S_1\big) + |\beta|\big(|\alpha|-1\big)\big]B_\delta\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^\delta d\theta\right\}^{\frac{1}{\delta}}, \tag{2.3}$$

where

$$S_1 = k^2\left(\frac{|\frac{a_1}{na_0}| + 1}{|\frac{a_1}{na_0}|k^2 + 1}\right)$$

and

$$B_\delta = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|S_1 + e^{it}\right|^\delta dt\right\}^{-\frac{1}{\delta}}.$$

Remark 2.3 For $k = 1$, we have $S_1 = 1$ and Corollary 2.2 reduces to a result of Mir and Wani [16] for $\delta \ge 1$. Dividing the two sides of (2.3) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 2.4 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a lacunary polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, where $k \geq 1$, then for each complex number β and $\delta \geq 1$,*

$$\left\{\int_0^{2\pi} \left|\mathrm{e}^{\mathrm{i}\theta} P'\left(\mathrm{e}^{\mathrm{i}\theta}\right) + \frac{n\beta}{1+S_1} P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^{\delta} \mathrm{d}\theta\right\}^{\frac{1}{\delta}} \leq n\left(|\beta|+1\right) B_\delta \left\{\int_0^{2\pi} \left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^{\delta} \mathrm{d}\theta\right\}^{\frac{1}{\delta}}, \tag{2.4}$$

where S_1 and B_δ are as defined in Corollary 2.2.

Taking $\beta = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.5 *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, is a lacunary polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for each complex number α such that $|\alpha| \geq 1$ and $\delta \geq 1$, we have*

$$\left\{\int_0^{2\pi} \left|D_\alpha P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^{\delta} \mathrm{d}\theta\right\}^{\frac{1}{\delta}} \leq n\left(|\alpha| + S_0(\mu)\right) C_\delta\left(S_0(\mu)\right) \left\{\int_0^{2\pi} \left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^{\delta} \mathrm{d}\theta\right\}^{\frac{1}{\delta}}, \tag{2.5}$$

where $S_0(\mu)$ is defined by formula (1.6) and $C_\delta(S_0(\mu))$ is as defined by (2.2).

In the limiting case, when $\delta \to \infty$, the result is best possible and equality in (2.5) holds for $P(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of μ and $\alpha \geq 1$.

Remark 2.6 If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \neq 0$ in $|z| < k$, $k \geq 1$, then by Lemma 3.5 for $s = 0$, we have $\psi_0(\mu) = S_0(\mu) \geq k^\mu$. Taking $a = |\alpha| \geq 1$, $b = S_0(\mu)$, $c = k^\mu$ in Lemma 3.3, we obtain the following inequality:

$$\frac{|\alpha| + S_0(\mu)}{\{\int_0^{2\pi} |S_0(\mu) + \mathrm{e}^{\mathrm{i}\beta}|^{\delta} \,\mathrm{d}\beta\}^{1/\delta}} \leq \frac{|\alpha| + k^\mu}{\{\int_0^{2\pi} |\mathrm{e}^{\mathrm{i}\beta} + k^\mu|^{\delta} \,\mathrm{d}\beta\}^{1/\delta}},$$

which holds for each $\delta \geq 1$. This shows that Corollary 2.5 sharpens the bound in (1.8).

Remark 2.7 Inequality (2.5) has also been recently established by Kumar [7].

Dividing both the sides of (2.5) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get Theorem 2.1 of Gardner and Weems [5] (see also Mir, Dewan, and Singh [14]).

It is important to mention that inequalities involving polynomials in the uniform norm on the unit circle in the complex plane are special cases of the polynomial inequalities involving the integral norm. For example, if we let $\delta \to \infty$ in (2.5), noting that $C_\delta(S_0(\mu)) \to 1/(1 + S_0(\mu))$, we get the following refinement of (1.10).

Corollary 2.8 *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, is a lacunary polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for each complex number α such that $|\alpha| \geq 1$, we have*

$$\max_{|z|=1} \left|D_\alpha P(z)\right| \leq n\left(\frac{|\alpha| + S_0(\mu)}{1 + S_0(\mu)}\right) \max_{|z|=1} \left|P(z)\right|, \tag{2.6}$$

where $S_0(\mu)$ is as defined in formula (1.6).

As an application of Corollary 2.5, we shall also prove the following result which provides a generalization of (1.11) and many other related results.

Theorem 2.9 *If $P(z) = z^s(a_0 + \sum_{j=\mu}^{n-s} a_j z^j)$, with $0 \le s \le n-1$ and $1 \le \mu \le n-s$, is a lacunary polynomial of degree n having s-fold zeros at the origin and the remaining $n-s$ zeros in $|z| \ge k$, where $k \ge 1$, then for every complex number α such that $|\alpha| \ge 1$ and $\delta \ge 1$, the following inequality*

$$\begin{aligned}\left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \\ \le \left[(n-s)\left(|\alpha| + \psi_s(\mu)\right)C_\delta\left(\psi_s(\mu)\right) + s|\alpha|\right]\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}}\end{aligned} \tag{2.7}$$

holds, where

$$\psi_s(\mu) = k^{\mu+1}\left\{\frac{(\frac{\mu}{n-s})|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{(\frac{\mu}{n-s})|\frac{a_\mu}{a_0}|k^{\mu+1} + 1}\right\}$$

and

$$C_\delta\left(\psi_s(\mu)\right) = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\psi_s(\mu) + e^{i\beta}\right|^\delta d\beta\right\}^{-\frac{1}{\delta}}.$$

In the limiting case, when $\delta \to \infty$, the result is best possible for $s = 0$ and equality in (2.7) holds for $P(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of μ and $\alpha \ge 1$.

Remark 2.10 For $s = 0$, Theorem 2.9 reduces to Corollary 2.5.

Remark 2.11 By Lemma 3.5, we have $\psi_s(\mu) \ge k^\mu$. Taking $a = |\alpha| \ge 1$, $b = \psi_s(\mu)$, $c = k^\mu$ in Lemma 3.3, we get, for each $\delta \ge 1$,

$$\frac{|\alpha| + \psi_s(\mu)}{\{\int_0^{2\pi} |\psi_s(\mu) + e^{i\beta}|^\delta d\beta\}^{1/\delta}} \le \frac{|\alpha| + k^\mu}{\{\int_0^{2\pi} |k^\mu + e^{i\beta}|^\delta d\beta\}^{1/\delta}}.$$

This shows that Theorem 2.9 sharpens the bound in (1.11).

Remark 2.12 Making $\delta \to \infty$ in (2.7), we get

$$\max_{|z|=1}\left|D_\alpha P(z)\right| \le \left(\frac{n(|\alpha| + \psi_s(\mu)) + s(|\alpha| - 1)\psi_s(\mu)}{1 + \psi_s(\mu)}\right)\max_{|z|=1}\left|P(z)\right|, \tag{2.8}$$

where $\psi_s(\mu)$ is as defined in Theorem 2.9. It is easy to verify that, for every complex number α with $|\alpha| \ge 1$ and $k \ge 1$, the function

$$x \mapsto \frac{n(|\alpha| + x) + s(|\alpha| - 1)x}{1 + x}, \quad x \ge 0,$$

is a nonincreasing function in x. If we combine this fact with Lemma 3.5, according to which $\psi_s(\mu) \ge k^\mu$ for $\mu \ge 1$, we find that the right-hand side of (2.8) does not exceed the right-hand side of (1.12). Thus (2.8) represents a refinement of (1.12).

3 Auxiliary results

In order to prove our main results (Theorems 2.1 and 2.9), we need the following lemmas.

Lemma 3.1 *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $\mu \geq 1$, be a lacunary polynomial of degree n without zeros in the disc $|z| < k$, where $k \geq 1$. Then on $|z| = 1$,*

$$S_0(\mu)\left|P'(z)\right| \leq \left|Q'(z)\right| \tag{3.1}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu \leq 1, \tag{3.2}$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$ and $S_0(\mu)$ is as defined by (1.6).

The above lemma is implicit in Qazi [17].

The next two lemmas have been recently proved by Govil and Kumar [6].

Lemma 3.2 *If p and q are arbitrary positive real numbers such that $q \geq px$, where $x \geq 1$, and if β is any real number such that $0 \leq \beta < 2\pi$, then*

$$\frac{q + py}{x + y} \leq \left|\frac{q + pe^{i\beta}}{x + e^{i\beta}}\right|$$

for each $y \geq 1$.

Lemma 3.3 *For real numbers $a \geq 1$, $b \geq c \geq 1$, and $\delta > 0$, we have the following inequality*:

$$\frac{a + b}{\{\int_0^{2\pi} |e^{i\theta} + b|^\delta \,d\theta\}^{1/\delta}} \leq \frac{a + c}{\{\int_0^{2\pi} |e^{i\theta} + c|^\delta \,d\theta\}^{1/\delta}}.$$

We need also the following lemma proved by Aziz and Shah [2].

Lemma 3.4 *Let $P(z)$ be a polynomial of degree n and $Q(z)$ be defined by $Q(z) = z^n\overline{P(1/\bar{z})}$. Then, for every $\delta > 0$, the following inequality*

$$\int_0^{2\pi}\int_0^{2\pi} \left|Q'\left(e^{i\theta}\right) + e^{i\varphi}P'\left(e^{i\theta}\right)\right|^\delta \,d\varphi \,d\theta \leq 2\pi n^\delta \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\delta \,d\theta$$

holds.

Lemma 3.5 *Let $P(z)$ be a lacunary polynomial of degree n having s-fold zeros at the origin and the remaining $n - s$ zeros in $|z| \geq k$, where $k \geq 1$, i.e.,*

$$P(z) = z^s\left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right), \quad 0 \leq s \leq n - 1, 1 \leq \mu \leq n - s.$$

Then

$$\psi_s(\mu) = k^{\mu+1}\left\{\frac{(\frac{\mu}{n-s})|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{(\frac{\mu}{n-s})|\frac{a_\mu}{a_0}|k^{\mu+1} + 1}\right\} \geq k^\mu.$$

Proof We put $P(z) = z^s\phi(z)$, with

$$\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu,$$

and suppose that $\phi(z) \neq 0$ in the disc $|z| < k$, where $k \geq 1$.

Then on applying inequality (3.2) of Lemma 3.1 to the polynomial $\phi(z)$ of degree $n-s$, we get

$$\left(\frac{\mu}{n-s}\right)\left|\frac{a_\mu}{a_0}\right| k^\mu \leq 1,$$

which is equivalent to

$$\left(\frac{\mu}{n-s}\right)\left|\frac{a_\mu}{a_0}\right| k^\mu (k-1) \leq k-1,$$

which implies

$$\left(\frac{\mu}{n-s}\right)\left|\frac{a_\mu}{a_0}\right| k^{\mu+1} + 1 \leq \left(\frac{\mu}{n-s}\right)\left|\frac{a_\mu}{a_0}\right| k^\mu + k,$$

from which Lemma 3.5 follows. □

4 Proofs of main results

Proof of Theorem 2.1 By hypothesis we have $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \neq 0$ in $|z| < k$, where $k \geq 1$. Supposing $Q(z) = z^n\overline{P(1/\bar{z})}$, we conclude that $P(z) = z^n\overline{Q(1/\bar{z})}$ and it can be easily checked that, for each θ $(0 \leq \theta < 2\pi)$,

$$nP(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}\theta}P'(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}(n-1)\theta}\overline{Q'(\mathrm{e}^{\mathrm{i}\theta})},$$

as well as for polar derivative

$$D_\alpha P(\mathrm{e}^{\mathrm{i}\theta}) = nP(\mathrm{e}^{\mathrm{i}\theta}) + (\alpha - \mathrm{e}^{\mathrm{i}\theta})P'(\mathrm{e}^{\mathrm{i}\theta}),$$

where $\alpha \in \mathbb{C}$. Using these equalities, we obtain

$$\begin{aligned} \left|D_\alpha P(\mathrm{e}^{\mathrm{i}\theta})\right| &\leq \left|nP(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{\mathrm{i}\theta}P'(\mathrm{e}^{\mathrm{i}\theta})\right| + |\alpha|\left|P'(\mathrm{e}^{\mathrm{i}\theta})\right| \\ &= \left|Q'(\mathrm{e}^{\mathrm{i}\theta})\right| + |\alpha|\left|P'(\mathrm{e}^{\mathrm{i}\theta})\right|. \end{aligned} \tag{4.1}$$

Now, for $\delta \geq 1$, using Minkowski's inequality, we get

$$\begin{aligned} &\left\{\int_0^{2\pi}\left|S_0(\mu) + \mathrm{e}^{\mathrm{i}t}\right|^\delta \mathrm{d}t \int_0^{2\pi}\left|\mathrm{e}^{\mathrm{i}\theta}D_\alpha P(\mathrm{e}^{\mathrm{i}\theta}) + n\beta\left(\frac{|\alpha|-1}{1+S_0(\mu)}\right)P(\mathrm{e}^{\mathrm{i}\theta})\right|^\delta \mathrm{d}\theta\right\}^{\frac{1}{\delta}} \\ &= \left\{\int_0^{2\pi}\int_0^{2\pi}\left|S_0(\mu) + \mathrm{e}^{\mathrm{i}t}\right|^\delta \right. \\ &\quad \times \left.\left|\mathrm{e}^{\mathrm{i}\theta}D_\alpha P(\mathrm{e}^{\mathrm{i}\theta}) + n\beta\left(\frac{|\alpha|-1}{1+S_0(\mu)}\right)P(\mathrm{e}^{\mathrm{i}\theta})\right|^\delta \mathrm{d}\theta\,\mathrm{d}t\right\}^{\frac{1}{\delta}} \end{aligned}$$

$$\begin{aligned}
&\leq \left\{\int_0^{2\pi}\int_0^{2\pi}\left|S_0(\mu)+e^{it}\right|^\delta\left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\, dt\right\}^{\frac{1}{\delta}}\\
&\quad + n|\beta|\left(|\alpha|-1\right)\left\{\int_0^{2\pi}\int_0^{2\pi}\left|\frac{S_0(\mu)+e^{it}}{1+S_0(\mu)}\right|^\delta\left|P\left(e^{i\theta}\right)\right|^\delta d\theta\, dt\right\}^{\frac{1}{\delta}}\\
&\leq \left\{\int_0^{2\pi}\int_0^{2\pi}\left|S_0(\mu)+e^{it}\right|^\delta\left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\, dt\right\}^{\frac{1}{\delta}}\\
&\quad + n|\beta|\left(|\alpha|-1\right)(2\pi)^{1/\delta}\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}}.
\end{aligned} \tag{4.2}$$

Recall that $P(z)\neq 0$ in $|z|<k$, where $k\geq 1$, we have by inequality (3.1) of Lemma 3.1 for $0\leq\theta<2\pi$,

$$S_0(\mu)\left|P'\left(e^{i\theta}\right)\right|\leq\left|Q'\left(e^{i\theta}\right)\right|.$$

This gives, by taking $p=|P'(e^{i\theta})|$, $q=|Q'(e^{i\theta})|$, and $x=S_0(\mu)$ in Lemma 3.2, for each complex number α with $y=|\alpha|\geq 1$,

$$\begin{aligned}
&\left(\left|Q'\left(e^{i\theta}\right)\right|+|\alpha|\left|P'\left(e^{i\theta}\right)\right|\right)\left|S_0(\mu)+e^{it}\right|\\
&\quad\leq\left(S_0(\mu)+|\alpha|\right)\left|\left|Q'\left(e^{i\theta}\right)\right|+e^{it}\left|P'\left(e^{i\theta}\right)\right|\right|.
\end{aligned} \tag{4.3}$$

On applying (4.1) and (4.3), we get, for each $\delta\geq 1$,

$$\begin{aligned}
&\int_0^{2\pi}\left|S_0(\mu)+e^{it}\right|^\delta dt\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\\
&\quad=\int_0^{2\pi}\int_0^{2\pi}\left|S_0(\mu)+e^{it}\right|^\delta\left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta dt\, d\theta\\
&\quad\leq\int_0^{2\pi}\int_0^{2\pi}\left|S_0(\mu)+e^{it}\right|^\delta\left[\left|Q'\left(e^{i\theta}\right)\right|+|\alpha|\left|P'\left(e^{i\theta}\right)\right|\right]^\delta dt\, d\theta\\
&\quad\leq\left(S_0(\mu)+|\alpha|\right)^\delta\int_0^{2\pi}\int_0^{2\pi}\left|\left|Q'\left(e^{i\theta}\right)\right|+e^{it}\left|P'\left(e^{i\theta}\right)\right|\right|^\delta dt\, d\theta.
\end{aligned} \tag{4.4}$$

Since for each $\delta\geq 1$ and arbitrary $a,b\in\mathbb{C}$ the equality

$$\int_0^{2\pi}\left|a+e^{it}b\right|^\delta dt=\int_0^{2\pi}\left||a|+e^{it}|b|\right|^\delta dt$$

holds, using Lemma 3.4 for each $\delta\geq 1$ and $|\alpha|\geq 1$, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\left|S_0(\mu)+e^{it}\right|^\delta dt\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\\
&\quad\leq\left(S_0(\mu)+|\alpha|\right)^\delta\int_0^{2\pi}\int_0^{2\pi}\left|Q'\left(e^{i\theta}\right)+e^{it}P'\left(e^{i\theta}\right)\right|^\delta dt\, d\theta\\
&\quad\leq\left(S_0(\mu)+|\alpha|\right)^\delta 2\pi n^\delta\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^\delta d\theta,
\end{aligned}$$

which on raising the power $1/\delta$ on both sides and then using in (4.2) gives (2.1), and this completes the proof of Theorem 2.1. □

Proof of Theorem 2.9 As in Lemma 3.5, we put $P(z) = z^s\phi(z)$, where

$$\phi(z) = a_0 + \sum_{j=\mu}^{n-s} a_j z^j, \quad 1 \le \mu \le n-s,$$

is a polynomial of degree $n-s$ having no zeros in $|z| < k$, where $k \ge 1$. Applying Corollary 2.5 to the polynomial $\phi(z)$ of degree $n-s$, we obtain, for $|\alpha| \ge 1$ and $\delta \ge 1$,

$$\left\{\int_0^{2\pi} \left|D_\alpha\phi\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \le \frac{(n-s)(|\alpha| + \psi_s(\mu))}{\{\frac{1}{2\pi}\int_0^{2\pi} |\psi_s(\mu) + e^{i\beta}|^\delta d\beta\}^{\frac{1}{\delta}}}\left\{\int_0^{2\pi} \left|\phi\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}}. \tag{4.5}$$

Now

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\ &= nz^s\phi(z) + (\alpha - z)\left(sz^{s-1}\phi(z) + z^s\phi'(z)\right) \\ &= z^s\left((n-s)\phi(z) + (\alpha - z)\phi'(z)\right) + \alpha s z^{s-1}\phi(z) \\ &= z^s D_\alpha\phi(z) + s\alpha z^{s-1}\phi(z), \end{aligned}$$

which implies

$$zD_\alpha P(z) = z^{s+1}D_\alpha\phi(z) + s\alpha P(z). \tag{4.6}$$

Hence, for $0 \le \theta < 2\pi$, then from (4.6) we get

$$\left|D_\alpha P\left(e^{i\theta}\right)\right| = \left|e^{i(s+1)\theta}D_\alpha\phi\left(e^{i\theta}\right) + s\alpha P\left(e^{i\theta}\right)\right|,$$

which gives, by using Minkowski's inequality for $\delta \ge 1$,

$$\begin{aligned} \left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} &= \left\{\int_0^{2\pi} \left|e^{i(s+1)\theta}D_\alpha\phi\left(e^{i\theta}\right) + s\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \\ &\le \left\{\int_0^{2\pi} \left|D_\alpha\phi\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \\ &\quad + s|\alpha|\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}}. \end{aligned} \tag{4.7}$$

Using (4.5) in (4.7) and noting that

$$\left|\phi\left(e^{i\theta}\right)\right| = \left|e^{is\theta}\phi\left(e^{i\theta}\right)\right| = \left|P\left(e^{i\theta}\right)\right|,$$

it follows that, for every $|\alpha| \geq 1$ and $\delta \geq 1$,

$$\left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}} \leq \left(\frac{(n-s)(|\alpha| + \psi_s(\mu))}{\{\frac{1}{2\pi}\int_0^{2\pi} |\psi_s(\mu) + e^{i\beta}|^\delta d\beta\}^{1/\delta}} + s|\alpha|\right)\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\delta d\theta\right\}^{\frac{1}{\delta}},$$

which is inequality (2.7), and this completes the proof of Theorem 2.9. □

5 Conclusion

We establish some Zygmund-type integral inequalities for the polar derivative of polynomials that are inspired by the classical Bernstein-type inequalities that relate the sup-norm of a polynomial to that of its derivative on the unit circle. The obtained results sharpen and generalize some already known estimates that relate the L^δ-norms of polynomials and their polar derivatives.

Acknowledgements
The authors wish to thank the anonymous referees, the editor and Professor Narendra Govil for their valuable comments.

Funding
The work of the first author was partly supported by the Serbian Academy of Sciences and Arts (Project Φ-96).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. They read and approved the final manuscript.

Author details
[1]Serbian Academy of Sciences and Arts, Belgrade, Serbia. [2]Faculty of Science and Mathematics, University of Niš, Niš, Serbia. [3]Department of Mathematics, University of Kashmir, Srinagar, India.

Received: 20 March 2020 Accepted: 28 April 2020 Published online: 12 May 2020

References

1. Aziz, A.: Inequalities for the polar derivative of a polynomial. J. Approx. Theory **55**, 183–193 (1988)
2. Aziz, A., Shah, W.M.: L^p inequalities for polynomials with restricted zeros. Glas. Mat. **32**, 247–258 (1997)
3. Bernstein, S.: Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné. Mem. Acad. R. Belg. **4**, 1–103 (1912)
4. Chan, T.N., Malik, M.A.: On Erdős–Lax theorem. Proc. Indian Acad. Sci. **92**, 191–193 (1983)
5. Gardner, R.B., Weems, A.: A Bernstein type L^p inequality for a certain class of polynomials. J. Math. Anal. Appl. **219**, 472–478 (1998)
6. Govil, N.K., Kumar, P.: On L^p inequalities involving polar derivative of a polynomial. Acta Math. Hung. **152**, 130–139 (2017)
7. Kumar, P.: On Zygmund-type inequalities involving polar derivative of a lacunary-type polynomial. Bull. Math. Soc. Sci. Math. Roum. **62**, 163–172 (2019)
8. Lax, P.D.: Proof a a conjecture of P. Erdős on the derivative of a polynomial. Bull. Am. Math. Soc. **50**, 509–513 (1944)
9. Malik, M.A.: On the derivative of a polynomial. J. Lond. Math. Soc. **1**, 57–60 (1969)
10. Marden, M.: Geometry of Polynomials. Math. Surveys, vol. 3. Am. Math. Soc., Providence (1966)
11. Milovanović, G.V., Mitrinović, D.S., Rassias, T.M.: Topics in Polynomials: Extremal Problems, Inequalities, Zeros. World Scientific, Singapore (1994)
12. Mir, A.: Bernstein-type integral inequalities for a certain class of polynomials. Mediterr. J. Math. **16**, Art. 143 (2019)
13. Mir, A.: Generalizations of Bernstein and Turán-type inequalities for the polar derivative of a complex polynomial. Mediterr. J. Math. **17**, Art. 14 (2020)

14. Mir, A., Dewan, K.K., Singh, N.: Some L^p inequalities for polynomials. Funct. Approx. Comment. Math. **42**, 131–143 (2010)
15. Mir, A., Hussain, I.: On the Erdős–Lax inequality concerning polynomials. C. R. Acad. Sci. Paris, Ser. I **355**, 1055–1062 (2017)
16. Mir, A., Wani, A.: Polynomials with polar derivatives. Funct. Approx. Comment. Math. **55**, 139–144 (2016)
17. Qazi, M.A.: On the maximum modulus of polynomials. Proc. Am. Math. Soc. **115**, 337–343 (1992)
18. Rahman, Q.I., Schmeisser, G.: Analytic Theory of Polynomials. Oxford University Press, London (2002)
19. Zireh, A.: On the polar derivative of a polynomial. Bull. Iran. Math. Soc. **40**, 967–976 (2014)