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Some inequalities for symmetric functions and an application to orthogonal polynomials ☆

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Abstract

We present some sharp inequalities for symmetric functions and give an application to orthogonal polynomials.

Keywords: Symmetric functions; Polynomials; Zeros; Linear functional; Moments; Orthogonal polynomial; Recurrence relation

1. Introduction

Symmetric functions are important in several branches of mathematics, especially in approximation theory, probability theory, combinatorics and algebra, and they have many applications in different areas (see [5, Chapter 1] for details about symmetric functions).

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Let $Q(x)$ be a polynomial of degree $n(\in \mathbb{N})$ with zeros λ_ν, $\nu = 1, \dots, n$, i.e.,

$$Q(x) = C \prod_{k=1}^{n} (x - \lambda_\nu), \quad C \neq 0. \tag{1.1}$$

It is well known that the coefficients of the polynomial (1.1) can be represented, using symmetric functions, in the following form:

$$Q(x) = C\left(x^n - \sigma_{n,1}x^{n-1} + \sigma_{n,2}x^{n-2} - \cdots + (-1)^n \sigma_{n,n}\right),$$

where $\sigma_{n,k}$, $k = 1, \dots, n$, are the so-called *elementary symmetric functions*,

$$\sigma_{n,k} = \sum_{(i_1,\dots,i_k)} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \dots, n,$$

and where the summation is performed over all combinations $(i_1, \dots, i_k)$ of the basic set $\{1, \dots, n\}$. Thus,

$$\sigma_{n,1} = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \qquad \sigma_{n,2} = \lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n, \quad \dots,$$
$$\sigma_{n,n} = \lambda_1\lambda_2 \cdots \lambda_n.$$

For the convenience we put $\sigma_{n,0} = 1$ and $\sigma_{n,k} = 0$, $k > n$ or $k < 0$. When we want to refer to the all elementary symmetric functions, we use notation $\boldsymbol{\sigma}_n = (\sigma_{n,0}, \dots, \sigma_{n,n})$, where $\boldsymbol{\sigma}_n$ represents a vector with $n + 1$ components.

There are several classical inequalities with symmetric functions (cf. [3,6,8,12,13,15–17]). For some recent results see [1,4,7,11,14]. For example, some general results on the positivity of symmetric functions have been recently obtained by Timofte [14].

In this paper we present the positivity for a special family of symmetric polynomials $p_k^n(\boldsymbol{\sigma}_n)$ and give some applications to orthogonal polynomials. The paper is organized as follows. The main inequality $p_k^n(\boldsymbol{\sigma}_n) > 0$ (Theorem 2.3) is stated in Section 2 and its proof is given in Section 3. A determinant representation of $p_k^n(\boldsymbol{\sigma}_n)$ is presented in Section 4 and some special cases are analyzed in Section 5. Finally, Section 6 is devoted to some applications to linear functionals and orthogonal polynomials.

2. Inequalities

In this paper we assume that all zeros of the polynomial (1.1) are positive, i.e.,

$$\lambda_\nu > 0, \quad \nu = 1, \dots, n.$$

Let the derivatives of $Q(x)$ at the point zero be denoted by $Q^{(k)}(0)$, i.e.,

$$Q^{(k)}(0) = \left.\frac{d^k Q(x)}{dx^k}\right|_{x=0}, \quad k \in \mathbb{N}_0.$$

Obviously, we have

$$Q^{(k)}(0) = (-1)^{n-k} k! C \sigma_{n,n-k}, \quad k = 0, 1, \dots, n. \tag{2.1}$$

We also define the sequence

$$Q_k = \frac{1}{k!}\frac{d^k}{dx^k}\frac{1}{Q(x)}\bigg|_{x=0}, \quad k \in \mathbb{N}_0.$$

Lemma 2.1. *The sequence* Q_k, $k \in \mathbb{N}_0$, *satisfies the following recurrence relation*:

$$\sigma_{n,n} Q_k = \frac{(-1)^n \delta_{k,0}}{k!\, C} + (-1)^{k-1} \sum_{\nu=\max\{0,k-n\}}^{k-1} (-1)^\nu \sigma_{n,n-k+\nu} Q_\nu, \quad k \in \mathbb{N}_0. \tag{2.2}$$

If the sum is empty, we consider it to be zero.

Proof. Put $f(x) = Q(x)$ and $g(x) = 1/Q(x)$, obviously we have $fg = 1$. If we apply the Leibnitz rule for the derivative of a product, we get

$$\frac{d^k}{dx^k}(fg)\bigg|_{x=0} = \sum_{\nu=0}^{k}\binom{k}{\nu} f^{(k-\nu)}(0) g^{(\nu)}(0) = \delta_{k,0}.$$

Substituting $f^{(\nu)}(0) = Q^{(\nu)}(0)$, $g^{(\nu)}(0) = Q_\nu$, and using (2.1), the previous equation reduces to

$$\sigma_{n,n} Q_k = \frac{(-1)^n \delta_{k,0}}{k!\, C} + (-1)^{k-1} \sum_{\nu=0}^{k-1} (-1)^\nu \sigma_{n,n-k+\nu} Q_\nu, \quad k \in \mathbb{N}_0.$$

According to the fact that $\sigma_{n,k} = 0$ for $k < 0$, we can truncate the summation in the previous form and so we get (2.2).

Equality in (2.2) holds even for the choice $k = 0$, in which case it reduces to $Q_0 = 1/Q(0) = (-1)^n/(C\sigma_{n,n})$. □

For $k > 0$, in (2.2) we have the homogenous difference equation ($\delta_{k,0} = 0$), which generates the solution $\sigma_{n,n}^k Q_k = p_k^n(\boldsymbol{\sigma}_n) Q_0$, where $p_k^n(\boldsymbol{\sigma}_n)$, $k \in \mathbb{N}$, is a polynomial in $\sigma_{n,0}, \ldots, \sigma_{n,n}$.

Now, we can state the following result:

Lemma 2.2. *The solution of the difference equation* (2.2) *admits a representation of the following form*:

$$\sigma_{n,n}^k Q_k = p_k^n(\boldsymbol{\sigma}_n) Q_0, \quad k \in \mathbb{N},$$

where p_k^n *is a polynomial of degree* k *of the elementary symmetric functions* $\sigma_{n,\nu}$, $\nu = 0, 1, \ldots, n$.

Proof. For $k = 1$ the statement is obvious, since

$$\sigma_{n,n} Q_1 = \sigma_{n,n-1} Q_0 \quad \text{and} \quad p_1^n(\boldsymbol{\sigma}_n) = \sigma_{n,n-1}.$$

Assuming it is true for Q_1, we are able to prove the statement for Q_2, since

$$\sigma_{n,n}^2 Q_2 = -\sum_{\nu=0}^{1}(-1)^\nu \sigma_{n,n-k+\nu}\sigma_{n,n} Q_\nu$$
$$= -\big(\sigma_{n,n}\sigma_{n,n-2} - \sigma_{n,n-1} p_1^n(\boldsymbol{\sigma}_n)\big) Q_0 = p_2^n(\boldsymbol{\sigma}_n) Q_0,$$

where $p_2^n(\boldsymbol{\sigma}_n) = \sigma_{n,n-1}^2 - \sigma_{n,n}\sigma_{n,n-2}$. Repeating the same arguments, we can prove that our statement holds for $k \leqslant n$. Starting from that point, we can apply the induction.

Assuming that statement holds for $Q_{k-n}, \ldots, Q_{k-1}$, we prove that it is true for Q_k, since

$$\sigma_{n,n}^k Q_k = (-1)^{k-1} \sum_{\nu=k-n}^{k-1} (-1)^\nu \sigma_{n,n}^{k-\nu-1} \sigma_{n,n-k+\nu} \sigma_{n,n}^\nu Q_\nu$$
$$= (-1)^{k-1} Q_0 \sum_{\nu=k-n}^{k-1} (-1)^\nu \sigma_{n,n}^{k-\nu-1} \sigma_{n,n-k+\nu} p_\nu^n(\boldsymbol{\sigma}_n) = p_k^n(\boldsymbol{\sigma}_n) Q_0,$$

where obviously we have

$$p_k^n(\boldsymbol{\sigma}_n) = (-1)^{k-1} \sum_{\nu=k-n}^{k-1} (-1)^\nu \sigma_{n,n}^{k-\nu-1} \sigma_{n,n-k+\nu} p_\nu^n(\boldsymbol{\sigma}_n), \quad k > n. \qquad \square$$

Adopting $p_0^n(\boldsymbol{\sigma}_n) = 1$ and $p_k^n(\boldsymbol{\sigma}_n) = 0$, $k < 0$, we rewrite the recurrence (2.2) for the sequence Q_k, $k \in \mathbb{N}_0$, into the recurrence for the sequence $p_k^n(\boldsymbol{\sigma}_n)$, $k \in \mathbb{N}_0$. Thus, we have

$$p_k^n(\boldsymbol{\sigma}_n) = \delta_{k,0} + (-1)^{k-1} \sum_{\nu=\max\{0,k-n\}}^{k-1} (-1)^\nu \sigma_{n,n}^{k-\nu-1} \sigma_{n,n-k+\nu} p_\nu^n(\boldsymbol{\sigma}_n), \quad k \in \mathbb{N}_0. \tag{2.3}$$

Using the previously defined quantities, we can state our main result.

Theorem 2.3. *Provided all zeros λ_ν, $\nu = 1, \ldots, n$, counting multiplicities, of the polynomial Q are positive, we have*

$$p_k^n(\boldsymbol{\sigma}_n) > 0, \quad k \in \mathbb{N}_0. \tag{2.4}$$

As an illustration, we give values of the polynomials p_k^n, $k \in \mathbb{N}$, for the case when the polynomial Q has only two zeros. Thus, we have the following statement:

Theorem 2.4. *Suppose that the polynomial Q is of the second degree, then*

$$p_k^2(\boldsymbol{\sigma}_2) = \sum_{\nu=0}^{[k/2]} a_{2,\nu}^k \sigma_{2,1}^{k-2\nu} (\sigma_{2,0}\sigma_{2,2})^\nu, \quad k \in \mathbb{N}_0, \tag{2.5}$$

where the coefficients $a_{2,\nu}^k$, $\nu, k \in \mathbb{N}_0$, satisfy the following recurrences:

$$a_{2,\nu}^k = a_{2,\nu}^{k-1} - a_{2,\nu-1}^{k-2}, \quad \nu = 1, \ldots, [k/2]-1, \qquad a_{2,0}^k = 1, \quad k \in \mathbb{N}_0,$$
$$a_{2,\nu}^{2\nu} = (-1)^\nu, \qquad a_{2,\nu}^{2\nu+1} = 0, \quad \nu \in \mathbb{N}_0, \qquad a_{2,\nu}^k = 0, \quad \nu \notin \{0, 1, \ldots, [k/2]\}. \tag{2.6}$$

Moreover,

$$a_{2,\nu}^k = (-1)^\nu \binom{k-\nu}{\nu}, \quad k \geqslant 2\nu, \qquad a_{2,\nu}^k = 0, \quad k < 2\nu. \tag{2.7}$$

3. Proof of Theorem 2.3

We assume that the polynomial Q has M distinct zeros, denoted by μ_ν, $\nu = 1, \ldots, M$. Their multiplicities are denoted by M_ν, respectively, where

$$\sum_{\nu=1}^{M} M_\nu = n.$$

Proof of Theorem 2.3. Obviously, in the case $n = 1$, the polynomial Q has only one simple zero. There is nothing to prove, since using (2.3), we can calculate

$$p_k^1(\boldsymbol{\sigma}_1) = 1 > 0$$

and (2.4) holds.

In the sequel, we assume $n > 1$. First, we assume that for the zero μ_1 we have multiplicity $M_1 \geqslant 2$. Consider now the following polynomials:

$$P_k(x) = 1 + \int_0^x \frac{Q(t)}{\prod_{\nu=1}^M (t-\mu_\nu)} \frac{q(t)}{t-\mu_1} t^k \, dt, \quad q \in \mathcal{P}_{M-1}, \ k \in \mathbb{N}. \tag{3.1}$$

For different polynomials $q \in \mathcal{P}_{M-1}$ we have different polynomials P_k. For example, taking the special case $q \equiv 0$, we have $P_k \equiv 1$.

For a polynomial $q \in \mathcal{P}_{M-1}$ which is not identically zero, since

$$P_k'(x) = \frac{Q(x)}{\prod_{\nu=1}^M (x-\mu_\nu)} \frac{q(x)}{x-\mu_1} x^k,$$

we conclude that P_k' has zeros at μ_ν, $\nu = 1, \ldots, M$, of the multiplicities $M_\nu - 1 - \delta_{\nu,1}$, $\nu = 1, \ldots, M$, respectively, and a zero at the point zero of the multiplicity k. It is also easy to verify that the degree of P_k is $n + k - 1$.

We show that the system of equations

$$P_k(\mu_\nu) = 1 + \int_0^{\mu_\nu} \frac{Q(t)}{\prod_{\nu=1}^M (t-\mu_\nu)} \frac{q(t)}{t-\mu_1} t^k \, dt = 0, \quad \nu = 1, \ldots, n, \tag{3.2}$$

has a solution $q \in \mathcal{P}_{M-1}$.

First, note that we can rewrite this system of equations in the form

$$\int_{\mu_\nu}^{\mu_{\nu+1}} \frac{Q(t)}{\prod_{\nu=1}^{M}(t-\mu_\nu)} \frac{q(t)}{t-\mu_1} t^k \, dt = -\delta_{\nu,0}, \quad \nu = 0, 1, \ldots, M-1, \tag{3.3}$$

where we use the convention $\mu_0 = 0$. To prove that the system (3.3) has a unique solution, it is enough to prove that the corresponding homogeneous system

$$\int_{\mu_\nu}^{\mu_{\nu+1}} \frac{Q(t)}{\prod_{\nu=1}^{M}(t-\mu_\nu)} \frac{q(t)}{t-\mu_1} t^k \, dt = 0, \quad \nu = 0, 1, \ldots, M-1, \tag{3.4}$$

has only the trivial solution $q \equiv 0$ in $\mathcal{P}_{M-1}$. Note that the polynomial

$$\frac{Q(t)}{\prod_{\nu=1}^{M}(t-\mu_\nu)} \frac{t^k}{t-\mu_1},$$

has a constant sign on the intervals $(\mu_\nu, \mu_{\nu+1})$, $\nu = 0, 1, \ldots, M-1$, since it has no zeros in these intervals. Therefore, the homogenous equations (3.4) imply that polynomial q must have at least one zero in each of the intervals $(\mu_\nu, \mu_{\nu+1})$, $\nu = 0, 1, \ldots, M-1$. This means that the polynomial q must have at least M zeros, the only polynomial from $\mathcal{P}_{M-1}$ satisfying this condition is, of course a polynomial which is identically zero.

This means that the system of equations (3.2) has a unique solution $q \in \mathcal{P}_{M-1}$. We denote that solution q^*. So that there exists (uniquely) polynomial P_k of the form (3.1), which has zeros at μ_ν, $\nu = 1, \ldots, M$, of the order $M_\nu - \delta_{\nu,1}$, $\nu = 1, \ldots, M$, denoted here P_k^*.

For the polynomial $(P_k^*)'$, we know that it has zeros of the order $M_\nu - 1 - \delta_{\nu,1}$ at the points μ_ν, $\nu = 1, \ldots, M$, and a zero of degree k at the point zero. Since it is of degree $k + n - 2$, there are $M - 1$ more zeros those are zeros of q^*. Using Role's theorem, we know that $(P_k^*)'$ must have at least one zero in each interval $(\mu_\nu, \mu_{\nu+1})$, $\nu = 1, \ldots, M-1$, since the polynomial P_k^* has zeros at the points μ_ν, $\nu = 1, \ldots, M$. There are $M - 1$ such zeros, so that the zeros of q^* ζ_ν, $\nu = 1, \ldots, M-1$, are simple and belong to the intervals $\zeta_\nu \in (\mu_\nu, \mu_{\nu+1})$, $\nu = 1, \ldots, M-1$. Since the polynomial

$$\frac{Q(x)}{\prod_{\nu=1}^{M}(t-\mu_\nu)} \frac{x^k}{x-\mu_1} q(x),$$

does not have any zeros in the interval $(0, \mu_1)$, it is of a constant sign there, P_k^* is also of the positive sign on the interval $(0, \mu_1)$. If it is not the case, then since $P_k^*(0) = 1$, there is at least one zero of the polynomial P_k^* in the interval $(0, \mu_1)$ suppose it is the point ζ. Then, according to the Role's theorem, there must be at least one zero of the polynomial $(P_k^*)'$ in the interval (ζ, μ_1), but this is a contradiction. Thus, the polynomial P_k^* is of the positive sign on the interval $(0, \mu_1)$.

Obviously, P_k^* cannot have zero in the interval (μ_1, μ_2), if it does then $(P_k^*)'$ must have two zeros in the interval (μ_1, μ_2) and those must be zeros of q^*, which is a contradiction. This leads to an observation that the polynomial $P_k^*(x)/(x - \mu_1)^{M_1 - 1}$ has a constant sign on the interval $(0, \mu_2)$ and that sign is $(-1)^{M_1 - 1}$.

Consider now the rational function $P_k/(x^{k+1}Q)$. It has poles of order $k + 1$ at the point zero and of order M_ν at points μ_ν, $\nu = 1, \ldots, M$. When x approaches the complex infinity, we have $P_k/(x^{k+1}Q) = O(x^{-2})$. Applying the Cauchy residue theorem to the function $P_k/(x^{k+1}Q)$, over the contour which has in its interior $\mathrm{co}\{0, \mu_1, \ldots, \mu_M\}$, we have

$$\frac{1}{k!}\left(\frac{P_k}{Q}\right)^{(k)}\Big|_{x=0} = -\sum_{\nu=1}^{M} \operatorname*{Res}_{x=\mu_\nu} \frac{P_k(x)}{x^{k+1}Q(x)}. \tag{3.5}$$

But since polynomial P_k is of the form (3.1), we know that $P_k^{(\nu)}(0) = \delta_{\nu,0}$, $\nu = 0, 1, \ldots, k$. Using the Leibniz rule, we get

$$\frac{1}{k!}\left(\frac{P_k}{Q}\right)^{(k)}\Big|_{x=0} = \frac{1}{k!}\sum_{\nu=0}^{k}\binom{k}{\nu} P^{(\nu)}(1/Q)^{(k-\nu)}\Big|_{x=0} = \frac{1}{k!}\left(\frac{1}{Q}\right)^{(k)}\Big|_{x=0} = Q_k.$$

Thus, for every polynomial P_k of the form (3.1), we have

$$\frac{1}{k!}\left(\frac{P_k}{Q}\right)^{(k)}\Big|_{x=0} = Q_k.$$

Now, choose $P_k = P_k^*$, using (3.5), we have

$$Q_k = -\frac{1}{\mu_1^{k+1}} \frac{(x-\mu_1)P_k^*(x)}{Q(x)}\Big|_{x=\mu_1}. \tag{3.6}$$

This equation can be rewritten in the form

$$\begin{aligned} p_k^n(\boldsymbol{\sigma}_n) = \frac{\sigma_{n,n}^k Q_k}{Q_0} &= -(-1)^n \frac{C\sigma_{n,n}^{k+1}}{\mu_1^{k+1}} \frac{P_k^*(x)}{(x-\mu_1)^{M_1-1}} \frac{(x-\mu_1)^{M_1}}{Q(x)}\Big|_{x=\mu_1} \\ &= -(-1)^n \frac{\sigma_{n,n}^{k+1}}{\mu_1^{k+1}} \frac{P_k^*(x)}{(x-\mu_1)^{M_1-1}}\Big|_{x=\mu_1} \frac{1}{\prod_{\nu=2}^{M}(\mu_1-\mu_\nu)^{M_\nu}}. \end{aligned} \tag{3.7}$$

Taking only the sign of the terms, from this equation we get

$$\begin{aligned} \operatorname{sgn}\bigl(p_k^n(\boldsymbol{\sigma}_n)\bigr) &= -(-1)^n \operatorname{sgn}\left(\frac{P_k^*(x)}{(x-\mu_1)^{M_1-1}}\Big|_{x=\mu_1}\right) \frac{1}{\prod_{\nu=2}^{M}\operatorname{sgn}(\mu_1-\mu_\nu)^{M_\nu}} \\ &= -(-1)^n(-1)^{M_1-1}\prod_{\nu=2}^{M}(-1)^{M_\nu} = (-1)^{n+\sum_{\nu=1}^{M} M_\nu} = 1. \end{aligned}$$

This proves inequality (2.4) in the case $M_1 \geqslant 2$.

In the case $M_1 = 1$, we consider the polynomials P_k of the following form:

$$P_k(x) = 1 + \int_0^x t^k \frac{Q(t)}{\prod_{\nu=1}^{M}(x-\mu_\nu)} q(t)\,dt, \quad q \in \mathcal{P}_{M-2}.$$

Using the same arguments as in the case $M_1 \geqslant 2$, we can prove that there exist (uniquely) q^* and respective P_k^*, such that

$$P_k^*(\mu_\nu) = 1 + \int_0^{\mu_\nu} t^k \frac{Q(t)}{\prod_{\nu=1}^{M}(x-\mu_\nu)} q^*(t)\,dt = 0, \quad \nu = 2, \ldots, M.$$

Also, it is easy to show that all $M-2$ zeros of q^* are contained in the intervals $(\mu_\nu, \mu_{\nu+1})$, $\nu = 2, \ldots, M-1$, and that the polynomial P_k^* has the positive sign on the interval $(0, \mu_2)$. Using (3.5), we find

$$Q_k = -\frac{1}{\mu_1^{k+1}} \frac{P_k^*(\mu_1)}{Q'(\mu_1)},$$

which gives

$$\operatorname{sgn}\bigl(p_k^n(\boldsymbol{\sigma}_n)\bigr) = -(-1)^n \frac{1}{\prod_{\nu=2}^M \operatorname{sgn}(\mu_1 - \mu_\nu)^{M_\nu}} = (-1)^{n+\sum_{\nu=1}^M M_\nu} = 1,$$

where we used the fact that $M_1 = 1$. This proves (2.4), also for the case $M_1 = 1$. □

There is one simple generalization of Theorem 2.3. Suppose all zeros of the polynomial Q are bigger than ζ, i.e., $\zeta < \lambda_\nu$, $\nu = 1, \ldots, n$, then we have

$$Q(x) = C \prod_{\nu=1}^n (x - \lambda_\nu) = C \prod_{\nu=1}^n \bigl(x - \zeta - (\lambda_\nu - \zeta)\bigr) = C \prod_{\nu=1}^n \bigl(y - \lambda_\nu^*\bigr) = Q^*(y),$$

where we put $\lambda_\nu^* = \lambda_\nu - \zeta$, $\nu = 1, \ldots, n$, and $y = x - \zeta$. We can express elementary symmetric functions of the polynomial Q^* using elementary symmetric functions of the polynomial Q; we have

$$\sigma_{n,k}^* = \sum_{(i_1,\ldots,i_k)} \lambda_{i_1}^* \cdots \lambda_{i_k}^* = \sum_{(i_1,\ldots,i_k)} (\lambda_{i_1} - \zeta) \cdots (\lambda_{i_k} - \zeta),$$

i.e.,

$$\sigma_{n,k}^* = \sum_{j=0}^k (-1)^{k-j} \binom{n-j}{k-j} \zeta^{k-j} \sigma_{n,j}. \tag{3.8}$$

We can apply Theorem 2.3 to the polynomial Q^*, since $\lambda_\nu^* > 0$, $\nu = 1, \ldots, n$. Therefore,

$$p_k^n\bigl(\boldsymbol{\sigma}_n^*\bigr) > 0, \quad k \in \mathbb{N}_0,$$

and

$$p_k^n\bigl(\boldsymbol{\sigma}_n^*\bigr) = \delta_{k,0} + (-1)^{k-1} \sum_{\nu=\max\{0,k-n\}}^{k-1} (-1)^\nu \bigl(\sigma_{n,n}^*\bigr)^{k-n-1} \sigma_{n,n-k}^* p_\nu^n\bigl(\boldsymbol{\sigma}_n^*\bigr),$$

where $\boldsymbol{\sigma}_n^* = (\sigma_{n,0}^*, \ldots, \sigma_{n,n}^*)$ and $\boldsymbol{\sigma}_n = (\sigma_{n,0}, \ldots, \sigma_{n,n})$. Using (3.8), we get

$$p_k^{n,\zeta}(\boldsymbol{\sigma}_n) = p_k^n\bigl(\boldsymbol{\sigma}_n^*\bigr) > 0$$

and the recurrence relation for the polynomials $p_k^{n,\zeta}(\boldsymbol{\sigma}_n)$ is given by

$$p_k^{n,\zeta}(\boldsymbol{\sigma}_n) = \delta_{k,0} + (-1)^{k-1} \sum_{\nu=\max\{0,k-n\}}^{k-1} (-1)^\nu \left(\sum_{j=0}^n (-1)^{n-j} \zeta^{n-j} \sigma_{n,j} \right)^{k-\nu-1}$$

$$\times\left(\sum_{j=0}^{n-k+\nu}(-1)^{n-k+\nu-j}\binom{n-j}{n-k+\nu-j}\zeta^{n-k+\nu-j}\sigma_{n,j}\right)p_\nu^{n,\zeta}(\boldsymbol{\sigma}_n). \tag{3.9}$$

We have proved the following result:

Theorem 3.1. *If the zeros of the polynomial Q are bigger than ζ, then*

$$p_k^{n,\zeta}(\boldsymbol{\sigma}_n) > 0, \quad k \in \mathbb{N}_0,$$

where the polynomials $p_k^{n,\zeta}(\boldsymbol{\sigma}_n)$, $k \in \mathbb{N}_0$, are generated using the recurrence (3.9).

In the case $\zeta = 0$, we have $p_k^{n,0}(\boldsymbol{\sigma}_n) = p_k^n(\boldsymbol{\sigma}_n)$.

4. Determinant representation of p_k^n

It is not surprising that our polynomials $p_k^n(\boldsymbol{\sigma}_n)$ can be represented in a determinant form. Namely, we have the following result:

Theorem 4.1. *The polynomial $p_k^n(\boldsymbol{\sigma}_n)$, $k \in \mathbb{N}$, admits the following determinant representation*:

$$p_k^n(\boldsymbol{\sigma}_n) = \begin{vmatrix} \beta_0 & -1 & & & & & & \mathbf{O} \\ \beta_1 & \beta_0 & -1 & & & & & \\ \beta_2 & \beta_1 & \beta_0 & -1 & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \\ \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 & -1 & & \\ & \ddots & \ddots & & \ddots & \ddots & \ddots & \\ & & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 & -1 \\ \mathbf{O} & & & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 \end{vmatrix}, \tag{4.1}$$

where $\beta_\nu = (-1)^\nu \sigma_{n,n}^\nu \sigma_{n,n-\nu-1}$, $\nu = 0, 1, \ldots, n-1$.

Proof. Introducing $\beta_\nu = (-1)^\nu \sigma_{n,n}^\nu \sigma_{n,n-\nu-1}$, $\nu = 0, 1, \ldots, n-1$, and $\beta_{-1} = -1$, the recurrence relation (2.3) becomes

$$\sum_{\nu=\max\{0,i-n\}}^{i} \beta_{i-\nu-1} p_\nu^n(\boldsymbol{\sigma}_n) = -\delta_{i,0}, \quad i \in \mathbb{N}_0.$$

For $i = 0, 1, \ldots, k$, $k \geqslant n$, it gives the following system of linear equations:

$$\begin{aligned}
&\beta_{-1} p_0^n(\boldsymbol{\sigma}_n) && = -1,\\
&\beta_0 p_0^n(\boldsymbol{\sigma}_n) + \beta_{-1} p_1^n(\boldsymbol{\sigma}_n) && = 0,\\
&\beta_1 p_0^n(\boldsymbol{\sigma}_n) + \beta_0 p_1^n(\boldsymbol{\sigma}_n) + \beta_{-1} p_2^n(\boldsymbol{\sigma}_n) && = 0,\\
& && \vdots\\
&\beta_{n-1} p_0^n(\boldsymbol{\sigma}_n) + \beta_{n-2} p_1^n(\boldsymbol{\sigma}_n) + \cdots + \beta_{-1} p_n^n(\boldsymbol{\sigma}_n) && = 0,\\
& && \vdots\\
&\beta_{n-1} p_{k-n}^n(\boldsymbol{\sigma}_n) + \beta_{n-2} p_{k-n+1}^n(\boldsymbol{\sigma}_n) + \cdots + \beta_{-1} p_k^n(\boldsymbol{\sigma}_n) && = 0.
\end{aligned}$$

Since the determinant of this system is equal to $(\beta_{-1})^k = (-1)^k \neq 0$, we can solve it for $p_k^n(\boldsymbol{\sigma}_n)$ using Cramer's rule, which leads to the following determinant representation:

$$p_k^n(\boldsymbol{\sigma}_n) = \frac{1}{(-1)^k} \begin{vmatrix}
\beta_{-1} & & & & & & & -1\\
\beta_0 & \beta_{-1} & & & & & & 0\\
\beta_1 & \beta_0 & \beta_{-1} & & & & & 0\\
\vdots & & & \ddots & & & & \vdots\\
\beta_{n-1} & \cdots & \beta_1 & \beta_0 & \beta_{-1} & & & 0\\
& \ddots & & & & \ddots & & \vdots\\
& & \beta_{n-1} & \cdots & \beta_1 & \beta_0 & \beta_{-1} & 0\\
& & & \beta_{n-1} & \cdots & \beta_1 & \beta_0 & 0
\end{vmatrix}.$$

Expanding this determinant with respect to the last column, we get (4.1). □

5. Special cases

5.1. Case of a single zero of multiplicity n

Suppose that polynomial Q has a single zero λ_1 of multiplicity n, i.e., let $Q(x) = (x - \lambda_1)^n$. In this special case, the elementary symmetric functions $\sigma_{n,k}$ have the following values:

$$\sigma_{n,k} = \binom{n}{k} \lambda_1^k, \quad k \in \mathbb{N}_0,$$

which is verified easily recalling the definition of the elementary symmetric functions and recalling the number of combinations of n elements of kth class.

Theorem 5.1. *If the polynomial Q has a single zero λ_1 of multiplicity n, then*

$$p_k^n(\boldsymbol{\sigma}_n) = \binom{k+n-1}{n-1} \lambda_1^{k(n-1)}.$$

Proof. Obviously, for $k = 0$, there is nothing to prove since our statement becomes

$$p_0^n(\boldsymbol{\sigma}_n) = \binom{0+n-1}{n-1} \lambda_1^{0(n-1)} = 1.$$

Thus, we assume $k > 0$. If we replace values for $p_k^n(\boldsymbol{\sigma}_n)$ given in the statement into (2.3), we get

$$\binom{k+n-1}{n-1} = (-1)^{k-1} \sum_{\nu=\max\{0,k-n\}}^{k-1} (-1)^\nu \binom{n}{n-k+\nu}\binom{\nu+n-1}{n-1},$$

which can be reduced to

$$\sum_{\nu=\max\{0,k-n\}}^{k} (-1)^\nu \binom{n}{k-\nu}\binom{\nu+n-1}{n-1} = 0. \tag{5.1}$$

Using an expansion of the geometric series, it is easy to verify the following expansion:

$$\frac{(1+x)^n}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x^{n-1}}{1+x}\right) = \sum_{\ell=0}^{+\infty} x^\ell \sum_{\nu=\max\{0,\ell-n\}}^{\ell} (-1)^\nu \binom{n}{\ell-\nu}\binom{\nu+n-1}{n-1}. \tag{5.2}$$

However, using the Bézout's theorem, we get

$$\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x^{n-1}}{1+x}\right) = \frac{d^{n-1}}{dx^{n-1}}\left(r_{n-2} + \frac{(-1)^{n-1}}{1+x}\right) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{(-1)^{n-1}}{1+x}\right) = \frac{(n-1)!}{(1+x)^n},$$

since r_{n-2} is a polynomial of degree $n-2$. Using this fact, (5.2) is transformed into

$$1 = \sum_{\ell=0}^{+\infty} x^\ell \sum_{\nu=\max\{0,\ell-n\}}^{\ell} (-1)^\nu \binom{n}{\ell-\nu}\binom{\nu+n-1}{n-1},$$

which means that all coefficients with x^ℓ, $\ell \in \mathbb{N}$, on the right-hand side must be zero, i.e., (5.1) holds for $k \in \mathbb{N}$. □

5.2. Case $n = 2$

In the case $n = 2$, we already stated Theorem 2.4 in Section 2. We give now a proof of this theorem.

Proof of Theorem 2.4. In the case $n = 2$, the representation (4.1) reduces to a determinant of a tridiagonal matrix. Namely,

$$p_k^2(\boldsymbol{\sigma}_2) = \begin{vmatrix} \beta_0 & -1 & & & \mathrm{O} \\ \beta_1 & \beta_0 & -1 & & \\ & \beta_1 & \beta_0 & \ddots & \\ & & \ddots & \ddots & -1 \\ \mathrm{O} & & & \beta_1 & \beta_0 \end{vmatrix},$$

where $\beta_0 = \sigma_{2,1}$ and $\beta_1 = -\sigma_{2,2}\sigma_{2,0}$.

Using the well-known relation for determinants of the tridiagonal matrices (see [9]), we have the following recurrence $p_k^2(\boldsymbol{\sigma}_2) = \beta_0 p_{k-1}^2(\boldsymbol{\sigma}_2) + \beta_1 p_{k-2}^2(\boldsymbol{\sigma}_2)$, i.e.,

$$p_k^2(\boldsymbol{\sigma}_2) = \sigma_{2,1} p_{k-1}^2(\boldsymbol{\sigma}_2) - \sigma_{2,2}\sigma_{2,0} p_{k-2}^2(\boldsymbol{\sigma}_2), \quad k > 2. \tag{5.3}$$

The rest of the proof goes inductively. Namely, we suppose that

$$p_{k-1}^2(\boldsymbol{\sigma}_2) = \sum_{\nu=0}^{[(k-1)/2]} a_{2,\nu}^{k-1} \sigma_{2,1}^{k-1-2\nu} (\sigma_{2,0}\sigma_{2,2})^{\nu} \quad \text{and}$$

$$p_{k-2}^2(\boldsymbol{\sigma}_2) = \sum_{\nu=0}^{[(k-2)/2]} a_{2,\nu}^{k-2} \sigma_{2,1}^{k-2-2\nu} (\sigma_{2,0}\sigma_{2,2})^{\nu}.$$

Then, using (5.3), we find

$$\begin{aligned} p_k^2(\boldsymbol{\sigma}_2) &= \sum_{\nu=0}^{[(k-1)/2]} a_{2,\nu}^{k-1} \sigma_{2,1}^{k-2\nu} (\sigma_{2,0}\sigma_{2,2})^{\nu} - \sum_{\nu=0}^{[(k-2)/2]} a_{2,\nu}^{k-2} \sigma_{2,1}^{k-2-2\nu} (\sigma_{2,0}\sigma_{2,2})^{\nu+1} \\ &= a_{2,0}^{k-1} \sigma_{2,1}^{k} + \sum_{\nu=1}^{[(k-1)/2]} \left(a_{2,\nu}^{k-1} - a_{2,\nu-1}^{k-2}\right) \sigma_{2,1}^{k-2\nu} (\sigma_{2,2}\sigma_{2,0})^{\nu} \\ &\quad - \frac{1}{2}\left(1 + (-1)^k\right) a_{2,[(k-2)/2]}^{k-2} \sigma_{2,1}^{k-2[k/2]} (\sigma_{2,2}\sigma_{2,0})^{[k/2]}, \end{aligned}$$

and using this relation, we get the relations (2.6). It is easy to check that p_1^2 and p_2^2 have representations as it is stated in this theorem.

To prove the last relation, we can check directly

$$(-1)^{\nu} \binom{k-\nu}{\nu} = (-1)^{\nu} \binom{k-1-\nu}{\nu} - (-1)^{\nu-1} \binom{k-1-\nu}{\nu-1},$$

which, after dividing by $(-1)^{\nu}$, becomes the basic binomial identity (see [9, p. 53]). □

5.3. *Cases* $k = 3, 4, 5, 6$

In this subsection we present special cases for $k = 3, 4, 5, 6$ and n arbitrary. The case $k = 2$ is already known in the literature (see [10, p. 73]) and we present it for completeness, hence, for $k = 2$ we have

$$p_2^n(\boldsymbol{\sigma}_n) = \sigma_{n,n-1}^2 - \sigma_{n,n}\sigma_{n,n-2} > 0, \quad n \in \mathbb{N}.$$

For bigger values of k we can use some computer algebra, for example *Mathematica*, *Maple*, to construct the polynomials $p_k^n(\boldsymbol{\sigma}_n)$. We present our results in the following statement.

Theorem 5.2. *We have*

$$\begin{aligned} p_3^n(\boldsymbol{\sigma}_n) &= \sigma_{n,n-1}^3 - 2\sigma_{n,n-2}\sigma_{n,n-1}\sigma_{n,n} + \sigma_{n,n-3}\sigma_{n,n}^2, \\ p_4^n(\boldsymbol{\sigma}_n) &= \sigma_{n,n-1}^4 - 3\sigma_{n,n-2}\sigma_{n,n-1}^2\sigma_{n,n} + \sigma_{n,n-2}^2\sigma_{n,n}^2 + 2\sigma_{n,n-3}\sigma_{n,n-1}\sigma_{n,n}^2 \\ &\quad - \sigma_{n,n-4}\sigma_{n,n}^3, \\ p_5^n(\boldsymbol{\sigma}_n) &= \sigma_{n,n-1}^5 - 4\sigma_{n,n-2}\sigma_{n,n-1}^3\sigma_{n,n} + 3\sigma_{n,n-2}^2\sigma_{n,n-1}\sigma_{n,n}^2 + 3\sigma_{n,n-3}\sigma_{n,n-1}^2\sigma_{n,n}^2 \\ &\quad - 2\sigma_{n,n-3}\sigma_{n,n-2}\sigma_{n,n}^3 - 2\sigma_{n,n-4}\sigma_{n,n-1}\sigma_{n,n}^3 + \sigma_{n,n-5}\sigma_{n,n}^5, \end{aligned}$$

$$\begin{aligned} p_6^n(\boldsymbol{\sigma}_n) = {} & \sigma_{n,n-1}^8 - 5\sigma_{n,n-2}\sigma_{n,n-1}^4\sigma_{n,n} + 6\sigma_{n,n-2}^2\sigma_{n,n-1}^2\sigma_{n,n}^2 + 4\sigma_{n,n-3}\sigma_{n,n-1}^3\sigma_{n,n}^2 \\ & - \sigma_{n,n-2}^3\sigma_{n,n}^3 - 6\sigma_{n,n-3}\sigma_{n,n-2}\sigma_{n,n-1}\sigma_{n,n}^3 - 3\sigma_{n,n-4}\sigma_{n,n-1}^2\sigma_{n,n}^3 \\ & + \sigma_{n,n-3}^2\sigma_{n,n}^4 + 2\sigma_{n,n-4}\sigma_{n,n-2}\sigma_{n,n}^4 + 2\sigma_{n,n-5}\sigma_{n,n-1}\sigma_{n,n}^4 - \sigma_{n,n-6}\sigma_{n,n}^5, \end{aligned}$$

for any $n \in \mathbb{N}$.

Proof. Direct calculation. □

6. Application to the theory of orthogonal polynomials

Here we want to present an inspiration for our result. Suppose that we have a linear functional $\mathcal{L}$ acting on the space of all algebraic polynomials $\mathcal{P}$. The moments of the functional $\mathcal{L}$ are given by $m_k = \mathcal{L}(x^k)$, $k \in \mathbb{N}_0$. Suppose that the sequence of moments satisfies the following recurrence relation:

$$\sum_{\nu=0}^{k} \frac{m_\nu}{\nu!} \sum_{\ell=-j}^{j} a_\ell \frac{((\ell+\alpha)h)^{k-\nu}}{(k-\nu)!} = \delta_{k,0}, \quad k \in \mathbb{N}_0, \tag{6.1}$$

where h, α and a_ℓ, $\ell = -j, \dots, j$, are arbitrary real numbers.

Let a polynomial Q of degree $n = 2j$ be defined by

$$Q(x) = \sum_{\ell=-j}^{j} a_\ell x^{\ell+j}$$

and such that all its zeros be positive and different from 1. Hence, we can request that this polynomial Q is normalized such that $Q(1) = 1$.

We are interested in a representation of the linear functional $\mathcal{L}$. It is known (see [2]) that every linear functional $\mathcal{L}$ acting on the space of all polynomials can be represented with a function of the bounded variation ϕ using Stieltjes–Lebesgue integral

$$\mathcal{L}(p) = \int_{\mathbb{R}} p(x)\,d\phi(x), \quad p \in \mathcal{P}.$$

Under a condition that $\mathcal{L}$ is positive definite, the function ϕ is nondecreasing. For the special case when zeros of Q are bigger than 1, we can state that the respective linear functional $\mathcal{L}$ is positive definite. Moreover, we can reconstruct the measure which represents it. Thus, we have the following result:

Theorem 6.1. *If all zeros* λ_ν, $\nu = 1, \dots, n$, *counting multiplicities, of the polynomial* Q *are bigger than* 1, *then the linear functional* $\mathcal{L}$ *admits the representation*

$$\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i\, p\big((i+j-\alpha)h\big),$$

where

$$Q_i = \frac{1}{i!}\frac{d^i}{dx^i}\left(\frac{1}{Q(x)}\right)\bigg|_{x=0}, \quad i \in \mathbb{N}_0.$$

The functional $\mathcal{L}$ is positive definite.

Proof. Denote by f the generating function for the sequence of moments m_k, $k \in \mathbb{N}_0$, i.e., the function

$$f(u) = \sum_{k=0}^{+\infty} \frac{m_k}{k!} u^k.$$

The recurrence relation (6.1) for the moments has as a consequence the following relation:

$$f(u) Q\bigl(\exp(hu)\bigr) = \exp\bigl((j-\alpha)hu\bigr).$$

As it is well known, we can expand the rational function $1/Q(x)$ into the partial fraction decomposition

$$\frac{1}{Q(x)} = \sum_{\nu=1}^{M}\sum_{\ell=1}^{M_\nu} \frac{Q_{\nu,\ell}}{(x-\mu_\nu)^\ell}, \tag{6.2}$$

where we assume that Q has M distinct zeros μ_ν with the multiplicities M_ν, $\nu = 1, \dots, M$. Using this partial fraction decomposition, the expansion of the geometric series and the expansion of the function $\exp(x)$, we have for u sufficiently close to zero

$$\begin{aligned} f(u) &= \sum_{\nu=1}^{M}\sum_{\ell=1}^{M_\nu} \frac{Q_{\nu,\ell}\exp((j-\alpha)hu)}{(\exp(hu)-\mu_\nu)^\ell} \\ &= \sum_{\nu=1}^{M}\sum_{\ell=1}^{M_\nu} \frac{Q_{\nu,\ell}}{(-\mu_\nu)^\ell} \sum_{i=0}^{+\infty} \binom{\ell+i-1}{i} \frac{1}{\mu_\nu^i} \sum_{k=0}^{+\infty} \frac{((i+j-\alpha)h)^k}{k!} u^k \\ &= \sum_{k=0}^{+\infty} \frac{u^k}{k!} \sum_{i=0}^{+\infty} \bigl((i+j-\alpha)h\bigr)^k \sum_{\nu=1}^{M}\sum_{\ell=1}^{M_\nu} (-1)^\ell \binom{\ell+i-1}{i} \frac{Q_{\nu,\ell}}{\mu_\nu^{\ell+i}}. \end{aligned}$$

Now, from this equation we can identify the moments as

$$m_k = \sum_{i=0}^{+\infty} \bigl((i+j-\alpha)h\bigr)^k \sum_{\nu=1}^{M}\sum_{\ell=1}^{M_\nu} (-1)^\ell \binom{\ell+i-1}{i} \frac{Q_{\nu,\ell}}{\mu_\nu^{\ell+i}}, \quad k \in \mathbb{N}_0.$$

On the other side, from (6.2), we have

$$Q_i = \frac{1}{i!}\frac{d^i}{dx^i}\left(\frac{1}{Q(x)}\right)\bigg|_{x=0} = \sum_{\nu=1}^{M}\sum_{\ell=1}^{M_\nu} (-1)^\ell \binom{\ell+i-1}{i} \frac{Q_{\nu,\ell}}{\mu_\nu^{\ell+i}},$$

so that we can interpret our linear functional in the following form:

$$\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i\, p\bigl((i+j-\alpha)h\bigr).$$

Since we know that $\sigma_{n,n}^i Q_i = p_i^n(\boldsymbol{\sigma}_n) Q_0$, according to Lemma 2.2, and since

$$Q_0 = \prod_{\nu=1}^{n} \frac{-\lambda_\nu}{1-\lambda_\nu} > 0,$$

using Theorem 2.3, we conclude that $Q_i > 0$ and our functional $\mathcal{L}$ is positive definite. □

Of course, since our result cannot depend on the order of enumeration of the coefficients a_ℓ, $\ell = -j, \ldots, j$, of the polynomial Q, if all zeros of Q are positive and smaller than 1, then we can change the order of enumeration of the coefficients a_ℓ, $\ell = -j, \ldots, j$, and have all zeros of such Q bigger than 1. However, we need to change the sign of h and α also. Now, our representation Theorem 6.1 is applicable, so that we have the following corollary.

Corollary 6.2. *If all zeros λ_ν, $\nu = 1, \ldots, n$, counting multiplicities, of the polynomial Q are positive and smaller than* 1, *then the linear functional $\mathcal{L}$ admits the representation*

$$\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i^* p\big(-(i+j+\alpha)h\big),$$

where

$$Q_i^* = \frac{1}{i!} \frac{d^i}{dx^i} \left(\frac{1}{x^n Q(1/x)} \right) \bigg|_{x=0}, \quad i \in \mathbb{N}_0.$$

The functional $\mathcal{L}$ is positive definite.

Using the same arguments, we can state the similar results provided a sequence of moments satisfies the following recurrence relation:

$$\sum_{\nu=0}^{k} \frac{m_\nu}{\nu!} \sum_{\ell=-j}^{j-1} a_\ell \frac{((\ell+\alpha+1/2)h)^{k-\nu}}{(k-\nu)!} = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{6.3}$$

Then, we define polynomial Q in the following way:

$$Q(x) = \sum_{\ell=-j}^{j-1} a_\ell \, x^{\ell+j}.$$

Obviously, the polynomial Q has degree $n = 2j - 1$. Using the same arguments as before, we can state the following result:

Theorem 6.3. *If all zeros λ_ν, $\nu = 1, \ldots, n$, counting multiplicities, of the polynomial Q are bigger than* 1, *then the linear functional $\mathcal{L}$ admits the representation*

$$\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i \, p\big((i+j-\alpha-1/2)h\big),$$

where

$$Q_i = \frac{1}{i!}\frac{d^i}{dx^i}\left(\frac{1}{Q(x)}\right)\bigg|_{x=0}, \quad i \in \mathbb{N}_0.$$

The functional $\mathcal{L}$ is positive definite.

For the case when all zeros are positive and smaller than 1, we have the following corollary:

Corollary 6.4. *If all zeros λ_ν, $\nu = 1, \dots, n$, counting multiplicities, of the polynomial Q are positive and smaller than* 1, *then the linear functional $\mathcal{L}$ admits the representation*

$$\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i^* p\big(-(i + j + \alpha - 1/2)h\big),$$

where

$$Q_i^* = \frac{1}{i!}\frac{d^i}{dx^i}\left(\frac{1}{x^n Q(1/x)}\right)\bigg|_{x=0}, \quad i \in \mathbb{N}_0.$$

The functional $\mathcal{L}$ is positive definite.

As an illustrative example, we consider the case when the polynomial Q has only one zero λ_1 with multiplicity M_1 (in this case, of course, $n = M_1$). Since the polynomial Q is normalized ($Q(1) = 1$), we have

$$Q(x) = \left(\frac{x - \lambda_1}{1 - \lambda_1}\right)^{M_1}.$$

Now, from this equation we can read

$$a_\ell = \frac{(-1)^{\ell+j}}{(1 - \lambda_1)^{M_1}}\binom{M_1}{\ell + j}\lambda_1^{\ell+j}, \quad \ell = -j, \dots, [M_1/2],$$

where $j = [(M_1 + 1)/2]$. So that moments satisfy the following recurrence relation

$$\sum_{\nu=0}^{k} \frac{m_\nu}{\nu!} \sum_{\ell=-j}^{[M_1/2]} a_\ell \frac{((\ell + \alpha + j - M_1/2)h)^{k-\nu}}{(k - \nu)!} = \delta_{k,0}, \quad k \in \mathbb{N}_0.$$

Assuming $\lambda_1 > 1$, we know, according to Theorems 6.1 and 6.3, that the sequence of moments can be represented as a sequence of moments of the linear functional $\mathcal{L}$ of the following form:

$$\mathcal{L}(p) = \sum_{i=0}^{+\infty} Q_i p\big((i - \alpha + M_1/2)h\big), \qquad Q_i = \frac{(M_1)_i}{i!}\frac{(\lambda_1 - 1)^{M_1}}{\lambda_1^{M_1+i}}.$$

It is easy to see that the functional $\mathcal{L}$ is positive definite as we expect according to the previous theorems.

It can be checked easily that this linear functional coincides with the Meixner linear functional of the first kind (see [2]). Actually, the Meixner polynomials of the first kind (see [2, p. 161]) are orthogonal with respect to the linear functional

$$\mathcal{L}^M(p) = \sum_{i=0}^{+\infty} p(i)\frac{c^i(\beta)_i}{i!}, \quad p \in \mathcal{P},\ c \in (0,1),\ \beta > 0.$$

Our linear functional $\mathcal{L}$ coincides with this one if we choose $c = 1/\lambda_1$, $\beta = M_1$, $\alpha = 0$, $h = 1$ and if we apply the shift for $M_1/2$.

The case $\lambda_1 \in (0,1)$, applying the Corollaries 6.2 and 6.4, leads again to the Meixner polynomials of the first kind which are orthogonal with respect to the linear functional

$$\mathcal{L}^M(p) = \sum_{i=0}^{+\infty} p\bigl(-(i+\beta)\bigr)\frac{c^{-i}(\beta)_i}{i!}, \quad p \in \mathcal{P},\ c > 1,\ \beta > 0,$$

where we need to choose $c = 1/\lambda_1$, $\beta = M_1$, $h = 1$ and again we need to shift for $M_1/2$.

This gives directly the following result:

Theorem 6.5. *Assume that a sequence of moments m_k, $k \in \mathbb{N}_0$, satisfies the recurrence relation* (6.1) *or* (6.3), *with zeros of Q, being all bigger than* 1 *or positive and smaller than* 1. *Then, there exists a sequence of polynomials orthogonal with respect to the corresponding linear functional $\mathcal{L}$.*

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