Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
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doi:10.3906/mat-2203-82

On sharpening and generalization of Rivlin's inequality

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Received: 13.03.2022 • **Accepted/Published Online:** 30.03.2022 • **Final Version:** ..2022

Abstract: An inequality due to T. J. Rivlin from 1960 states that if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then

$$\max_{|z|=r}|P(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1}|P(z)|$$

for $0 \leq r \leq 1$. In this paper, we prove some generalizations of the above Rivlin's inequality which sharpens Rivlin's inequality as a special case. Some important consequences of these results are also discussed and some related inequalities are obtained.

Key words: Polynomials, zeros, inequalities

1. Introduction

If $P(z)$ is a polynomial of degree n, then the well-known Bernstein inequalities [2] on polynomials are given by

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \tag{1.1}$$

and

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)|, \tag{1.2}$$

whenever $R \geq 1$.

The inequality (1.1) is a direct consequence of Bernstein's theorem on the derivative of a trigonometric polynomial [9], and the inequality (1.2) follows from the maximum modulus theorem (see [7, Problem 269]). The reverse analogue of the inequality (1.2) whenever $R \leq 1$ is given by Varga [10], and he proved that, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=r}|P(z)| \geq r^n \max_{|z|=1}|P(z)|, \tag{1.3}$$

whenever $0 \leq r \leq 1$. The equality in (1.3) holds whenever $P(z) = az^n$. For the class of polynomials having no zeros inside the unit circle, it was Rivlin [8] who proved that, if $P(z)$ is a polynomial of degree n having no

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2010 *AMS Mathematics Subject Classification:* 30A10

zeros in $|z|<1$, then for $0\leq r\leq 1$

$$\max_{|z|=r}|P(z)|\geq\left(\frac{1+r}{2}\right)^n\max_{|z|=1}|P(z)|. \tag{1.4}$$

Equality holds in (1.4) if $P(z)=(z+a)^n$ whenever $|a|=1$.

For more information about this kind of inequalities, we refer to the monographs [3, 6].

Aziz [1] generalized the Rivlin's inequality (1.4) by proving that, if $P(z)$ has no zeros in $|z|<K, K\geq 1$, then

$$\max_{|z|=r}|P(z)|\geq\left(\frac{K+r}{K+1}\right)^n\max_{|z|=1}|P(z)|, \tag{1.5}$$

for $0\leq r\leq 1$. Although, the above inequality (1.5) is best possible with equality holding for polynomials $P(z)=(z+a)^n$ satisfying $|a|=K$, definitely the bound given in inequality (1.5) does not address the issue of how far the zeros lie outside the circle $|z|=K$. Now naturally a question arises; is there any way to refine the inequality (1.5) for the class of polynomials satisfying the hypotheses of the inequality (1.5), by capturing some information on the moduli of zeros? Can we obtain a bound via two extreme coefficients of $P(z)$ which are informative about the distance of zeros from the origin? In view of the example for the equality case in (1.5) which holds with the property $|a_0|/|a_n|=K^n$, it should be possible to improve upon the bound for polynomials $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ having no zeros in $|z|<K$, $K\geq 1$, satisfying $|a_0|/|a_n|\neq K^n$.

In this paper, we approach this side of the inequality and obtain a bound which sharpens the inequalities (1.5) and (1.4) significantly.

2. Main results

Theorem 2.1 *If $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ has no zeros in $|z|<K$, $K\geq 1$, then*

$$\max_{|z|=r}|P(z)|\geq\left\{\left(\frac{K+r}{K+1}\right)^n+\frac{1}{K^{n-1}}\left[\frac{|a_0|-|a_n|K^n}{|a_0|+|a_n|}\right]\left(\frac{1-r}{K+1}\right)^n\right\}\max_{|z|=1}|P(z)|, \tag{2.1}$$

for $0\leq r\leq 1$. The result is sharp and equality holds in (2.1) if $P(z)=(z+K)^n$ and also for $P(z)=z+a$ for any a with $|a|\geq K$.

When $K=1$, Theorem 2.1 reduces to the following sharpened form of Rivlin's inequality (1.4), which was independently proved recently by Kumar [5].

Corollary 2.2 *If $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n, having no zeros in $|z|<1$, then*

$$\max_{|z|=r}|P(z)|\geq\left\{\left(\frac{1+r}{2}\right)^n+\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|}\right]\left(\frac{1-r}{2}\right)^n\right\}\max_{|z|=1}|P(z)|, \tag{2.2}$$

whenever $0\leq r\leq 1$. Equality holds in (2.2) if $P(z)=(z+a)^n$ whenever $|a|=1$ and also for $P(z)=z+a$ for any a with $|a|\geq 1$.

Since $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ has all its zeros in $|z|\geq K$, we always have the situation

$$\frac{|a_0|-|a_n|K^n}{|a_0|+|a_n|}\geq 0.$$

Therefore, for all polynomials satisfying the hypotheses of Theorem 2.1 excepting those satisfying $|a_0| = |a_n|K^n$, our above inequality (2.1) sharpens the inequality (1.5).

As a consequence of Theorem 2.1, we obtain a result on the location of zeros of a polynomial in a disc as discussed below. We have

$$|P(0)| = |a_0| \geq \left\{ \left(\frac{K}{K+1}\right)^n + \frac{1}{K^{n-1}} \left[\frac{|a_0| - |a_n|K^n}{|a_0| + |a_n|}\right] \left(\frac{1}{K+1}\right)^n \right\} \max_{|z|=1} |P(z)|,$$

or equivalently

$$\frac{|a_0|(|a_0| + |a_n|)K^{n-1}}{K^{2n-1}(|a_0| + |a_n|) + |a_0| - |a_n|K^n} \geq \frac{1}{(K+1)^n} \max_{|z|=1} |P(z)|, \tag{2.3}$$

whenever $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ has no zeros in $|z| < K$, $K \geq 1$.

Now let m with $m \leq 1$ be the absolute value of a zero of smallest absolute value of $P(z)$. Also let $Q(z) = P(m^2 z)$. Then $Q(z)$ has no zeros in $|z| < 1/m$ such that $1/m \geq 1$, and hence from (2.3) it follows that

$$\frac{|a_0|(|a_0| + |a_n|m^{2n})}{|a_0| + |a_n|m^{2n} + (|a_0|m^n - |a_n|)m^{n-1}} \geq \frac{1}{(m+1)^n} \max_{|z|=1} |Q(z)|,$$

or equivalently

$$\frac{|a_0|(|a_0| + |a_n|m^{2n})}{|a_0| + |a_n|m^{2n} + (|a_0|m^n - |a_n|)m^{n-1}} \geq \frac{1}{(m+1)^n} \max_{|z|=m^2} |P(z)|.$$

Therefore, we arrived at the following result.

Corollary 2.3 *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n. *If*

$$\max_{|z|=R^2} |P(z)| > \frac{|a_0|(|a_0| + |a_n|R^{2n})(R+1)^n}{|a_0| + |a_n|R^{2n} + (|a_0|R^n - |a_n|)R^{n-1}},$$

for some $1 \geq R > 0$, *then* $P(z)$ *has at least one zero in* $|z| < R$.

By applying Theorem 2.1 to the reciprocal polynomial $z^n P(1/z)$ we get an inequality for the class of polynomials having all its zeros in $|z| \leq K$, $K \leq 1$. To elaborate it, if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ has all its zeros in $|z| \leq K$, $K \leq 1$, then $Q(z) = z^n P(1/z)$ has no zeros in $|z| < 1/K$, $1/K \geq 1$. Applying Theorem 2.1 to $Q(z)$ with $r = 1/R$, $R \geq 1$, we get

$$\max_{|z|=1/R} |Q(z)| \geq \left(\frac{1/K + 1/R}{1/K + 1}\right)^n + K^{n-1} \left[\frac{|a_n| - |a_0|/K^n}{|a_0| + |a_n|}\right] \left(\frac{1 - 1/R}{1/K + 1}\right)^n,$$

or equivalently

$$\frac{1}{R^n} \max_{|z|=R} |P(z)| \geq \left(\frac{1/K + 1/R}{1/K + 1}\right)^n + K^{n-1} \left[\frac{|a_n| - |a_0|/K^n}{|a_0| + |a_n|}\right] \left(\frac{1 - 1/R}{1/K + 1}\right)^n.$$

With the simplification of the above, we arrive at the following result.

Corollary 2.4 *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ has all its zeros in $|z|\le K$, $K\le 1$, then*

$$\max_{|z|=R}|P(z)|\ge\left\{\left(\frac{K+R}{K+1}\right)^n+K^{n-1}\left[\frac{|a_n|K^n-|a_0|}{|a_n|+|a_0|}\right]\left(\frac{R-1}{K+1}\right)^n\right\}\max_{|z|=1}|P(z)|,\tag{2.4}$$

whenever $R\ge 1$. The result is best possible and equality holds in (2.4) if $P(z)=(z+K)^n$ and also for $P(z)=z+a$ for any a with $|a|\le K$.

Aziz [1] also derived an inequality that, if $P(z)$ is a polynomial of degree n having no zeros in $|z|<K$, K≤ 1, then

$$\max_{|z|=r}|P(z)|\ge\left(\frac{K+r}{K+1}\right)^n\max_{|z|=1}|P(z)|,\tag{2.5}$$

whenever $0<r\le K^2$. We shall sharpen the inequality (2.5) in the following result.

Theorem 2.5 *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z|<K, K\le 1$, then*

$$\max_{|z|=r}|P(z)|\ge\left\{\left(\frac{K+r}{K+1}\right)^n+\left[\frac{|a_0|-|a_n|K^n}{|a_0|+|a_n|}\right]\left(\frac{\lambda}{K+1}\right)^n\right\}\max_{|z|=1}|P(z)|,\tag{2.6}$$

whenever $0<r\le K^2$, and $\lambda=\min\{1-r,K+r\}$. Equality holds in (2.6) if $P(z)=(z+K)^n$ and also for $P(z)=z+a$ for any a with $|a|\ge K$.

We can apply Theorem 2.5 to the reciprocal polynomial $z^nP(1/z)$ and proceeding in the lines of the proof of Corollary 2.4 as explained above to get the following immediate consequence of Theorem 2.5.

Corollary 2.6 *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disc $|z|\le K$, $K\ge 1$, then*

$$\max_{|z|=R}|P(z)|\ge\left\{\left(\frac{K+R}{K+1}\right)^n+\left[\frac{|a_n|K^n-|a_0|}{|a_0|+|a_n|}\right]\left(\frac{\mu}{K+1}\right)^n\right\}\max_{|z|=1}|P(z)|,\tag{2.7}$$

whenever $R\ge K^2$ and $\mu=\min\{R-1,(R+K)/K\}$. The result is best possible and equality holds in (2.7) if $P(z)=(z+K)^n$ and also for $P(z)=z+a$ for any a with $|a|\le K$.

It was Govil [4] who generalized the inequality (1.4) by studying the relative growth of a polynomial $P(z)$ having no zeros in the open unit disc, with respect to two circles $|z|=r$ and $|z|=R$ whenever $0\le r<R\le 1$. He proved that, if $P(z)$ has no zeros in $|z|<1$ then for $0\le r<R\le 1$

$$\max_{|z|=r}|P(z)|\ge\left(\frac{1+r}{1+R}\right)^n\max_{|z|=R}|P(z)|.\tag{2.8}$$

Our next result sharpens (2.8) and is stated below.

Theorem 2.7 *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ has no zeros in $|z|<1$ then for $0\le r<R\le 1$, we have*

$$\max_{|z|=r}|P(z)|\ge\left\{\left(\frac{1+r}{1+R}\right)^n+R^{n-1}\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right]\left(\frac{R-r}{1+R}\right)^n\right\}\max_{|z|=R}|P(z)|,\tag{2.9}$$

The result is best possible and equality holds in (2.9) for $P(z)=(z+a)^n$, where $|a|=1$, and $P(z)=z+a$, where $|a|\ge 1$.

It may be remarked that when $R = 1$, Theorem 2.7 gives the sharpened version of Rivlin's inequality stated in Corollary 2.2.

If the polynomial $P(z)$ has all its zeros on $|z| \leq 1$, then the reciprocal polynomial $Q(z) = z^n P(1/z)$ also has all its zeros in $|z| \geq 1$. Now if $1 \leq R < r$, then $1/r < 1/R \leq 1$. Therefore applying Theorem 2.7 to the polynomial $Q(z)$ we get

$$\max_{|z|=1/r} |Q(z)| \geq \left\{ \left(\frac{1 + 1/r}{1 + 1/R} \right)^n + 1/R^{n-1} \left[\frac{|a_n| - |a_0|}{|a_n| + |a_0|(1/R^n)} \right] \left(\frac{1/R - 1/r}{1 + 1/R} \right)^n \right\} \max_{|z|=1/R} |Q(z)|,$$

which with simplification yields

$$\frac{1}{r^n} \max_{|z|=r} |P(z)| \geq \left\{ \frac{R^n}{r^n} \left(\frac{r+1}{R+1} \right)^n + \frac{R}{r^n} \left[\frac{|a_n| - |a_0|}{R^n|a_n| + |a_0|} \right] \left(\frac{r-R}{R+1} \right)^n \right\} \frac{1}{R^n} \max_{|z|=R} |P(z)|,$$

Thus we arrived at the following result.

Corollary 2.8 *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ has all its zeros on $|z| \leq 1$ then for $1 \leq R < r$, we have*

$$\max_{|z|=r} |P(z)| \geq \left\{ \left(\frac{1+r}{1+R} \right)^n + \frac{1}{R^{n-1}} \left[\frac{|a_n| - |a_0|}{|a_0| + |a_n|R^n} \right] \left(\frac{r-R}{1+R} \right)^n \right\} \max_{|z|=R} |P(z)|. \tag{2.10}$$

The result is best possible and equality holds in (2.10) for $P(z) = (z+a)^n$, where $|a| = 1$, and $P(z) = z + a$, where $|a| \geq 1$.

Remark 2.9 The above Corollary 2.8 sharpens the Corollary to Theorem 1 given in the paper due to Govil [4].

3. Auxiliary results

In order to establish our results stated in the previous section, we need to prove some fundamental inequalities involving nonnegative real numbers.

Lemma 3.1 *If $a \geq K^m$, $b \geq K$, $K \geq 1$, and m is any positive integer, then*

$$\frac{1}{K^{m-1}} \frac{a - K^m}{a+1} \cdot \frac{b-K}{b+1} + \frac{1}{K^{m-1}} \frac{a - K^m}{a+1} + \frac{b-K}{b+1} \geq \frac{1}{K^m} \frac{ab - K^{m+1}}{(ab+1)}. \tag{3.1}$$

Proof We need to show

$$\frac{1}{K^{m-1}} \frac{a - K^m}{a+1} \cdot \frac{b-K}{b+1} + \frac{1}{K^{m-1}} \frac{a - K^m}{a+1} + \frac{b-K}{b+1} - \frac{1}{K^m} \frac{ab - K^{m+1}}{(ab+1)} \geq 0.$$

Equivalently, it suffices to show that

$$\begin{aligned} K(ab+1)(a-K^m)(b-K) \quad &+ \quad K(ab+1)(a-K^m)(b+1) \\ &+ \quad K^m(b-K)(ab+1)(a+1) \\ &- \quad (ab-K^{m+1})(a+1)(b+1) \geq 0. \end{aligned}$$

In view of the fact that $K \geq 1$, and $(a+1)(b+1) = ab+1+a+b$, to establish the above claim, we need to prove

$$\begin{aligned}(ab+1)(a-K^m)(b-K) + K(ab+1)(a-K^m)(b+1)& \\ + K^m(b-K)(ab+1)(a+1) - (ab-K^{m+1})(ab+1)& \\ - (ab-K^{m+1})(a+b) \geq 0.& \qquad (3.2)\end{aligned}$$

Substituting the following

$$\begin{aligned}(ab+1)(a-K^m)(b-K) - (ab-K^{m+1})(ab+1)& \\ = -(ab+1)(K(a-K^m) + K^m(b-K)),&\end{aligned}$$

$$K(ab+1)(a-K^m)(b+1) = Kb(ab+1)(a-K^m) + K(ab+1)(a-K^m),$$

and

$$K^m(b-K)(ab+1)(a+1) = K^m a(b-K)(ab+1) + K^m(b-K)(ab+1)$$

in (3.2) and simplifying, we will be finally left to show that

$$Kb(ab+1)(a-K^m) + K^m a(b-K)(ab+1) - (ab-K^{m+1})(a+b) \geq 0.$$

Since for $a \geq 1$, $b \geq 1$, the inequality $(ab+1) \geq (a+b)$ holds, so it is sufficient to prove that

$$Kb(a-K^m) + K^m a(b-K) - (ab-K^{m+1}) \geq 0.$$

But

$$\begin{aligned}Kb(a-K^m) + K^m a(b-K) - (ab-K^{m+1})& \\ = (K-1)(a-K^m)(b-1) + K^m(a-1)(b-1) + (K-1)a \geq 0,&\end{aligned}$$

and hence the proof is complete. □

When $K > 0$ we have the following result analogous to Lemma 3.1.

Lemma 3.2 *If* $a \geq K^m$, $b \geq K$, $K > 0$, *and* m *is any positive integer, then*

$$\frac{a-K^m}{a+K^m} \cdot \frac{b-K}{b+K} + \frac{a-K^m}{a+K^m} + \frac{b-K}{b+K} \geq \frac{ab-K^{m+1}}{ab+K^{m+1}}. \qquad (3.3)$$

Proof The proof is almost in line with that of Lemma 3.1, but needs some modifications and records slightly different observations. For the sake of completeness let us present the proof.

Our claim is

$$\frac{a-K^m}{a+K^m} \cdot \frac{b-K}{b+K} + \frac{a-K^m}{a+K^m} + \frac{b-K}{b+K} - \frac{ab-K^{m+1}}{ab+K^{m+1}} \geq 0.$$

Equivalently it suffices to show that

$$\begin{aligned}(ab+K^{m+1})(a-K^m)(b-K) + (ab+K^{m+1})(a-K^m)(b+K)& \\ + (b-K)(ab+K^{m+1})(a+K^m) - (ab-K^{m+1})(a+K^m)(b+K) \geq 0.&\end{aligned}$$

Since $(a+K^m)(b+K)=ab+K^{m+1}+aK^m+bK$, to establish our claim, we need to prove that

$$\begin{aligned}(ab+K^{m+1})(a-K^m)(b-K)+(ab+K^{m+1})(a-K^m)(b+K)\\ +(b-K)(ab+K^{m+1})(a+K^m)-(ab-K^{m+1})(ab+K^{m+1})\\ -(ab-K^{m+1})(aK^m+bK)\geq 0.\end{aligned} \tag{3.4}$$

Substituting the following:

$$\begin{aligned}(ab+K^{m+1})(a-K^m)(b-K)-(ab-K^{m+1})(ab+K^{m+1})\\ =-(ab+K^{m+1})(K(a-K^m)+K^m(b-K)),\end{aligned}$$

$$(ab+K^{m+1})(a-K^m)(b+K)=b(ab+K^{m+1})(a-K^m)+K(ab+K^{m+1})(a-K^m),$$

and

$$(b-K)(ab+K^{m+1})(a+K^m)=a(b-K)(ab+K^{m+1})+K^m(b-K)(ab+K^{m+1})$$

in (3.4), and simplifying, our claim reduces to

$$b(ab+K^{m+1})(a-K^m)+a(b-K)(ab+K^{m+1})-(ab-K^{m+1})(aK^m+bK)\geq 0.$$

Since $ab+K^{m+1}-aK^m-bK=(a-K^m)(b-K)\geq 0$, we must have $ab+K^{m+1}\geq aK^m+bK$ and therefore it is sufficient to prove that

$$b(a-K^m)+a(b-K)-(ab-K^{m+1})\geq 0.$$

But

$$b(a-K^m)+a(b-K)-(ab-K^{m+1})=(a-K^m)(b-K)\geq 0,$$

and hence the proof is complete. □

Lemma 3.3 *For any* $0\leq r\leq 1$ *and* $R_k\geq K\geq 1$, $1\leq k\leq n$, *we have*

$$\prod_{k=1}^{n}\frac{r+R_k}{1+R_k}\geq\left(\frac{K+r}{K+1}\right)^n+\frac{1}{K^{n-1}}\left[\frac{R_1R_2\cdots R_n-K^n}{R_1R_2\cdots R_n+1}\right]\left(\frac{1-r}{K+1}\right)^n. \tag{3.5}$$

Proof We prove the result by induction on n. The identity

$$\frac{r+R_1}{1+R_1}=\left(\frac{K+r}{K+1}\right)+\left[\frac{R_1-K}{R_1+1}\right]\left(\frac{1-r}{K+1}\right), \tag{3.6}$$

justifies the validity of (3.5) for $n=1$. Let us assume that (3.5) is true for $n=m$. Then using the result for

m and with the help of (3.6), we will have

$$\begin{aligned}\prod_{k=1}^{m+1}\frac{r+R_k}{1+R_k} &= \left(\prod_{k=1}^{m}\frac{r+R_k}{1+R_k}\right)\left(\frac{r+R_{m+1}}{1+R_{m+1}}\right)\\ &\geq \left[\left(\frac{K+r}{K+1}\right)^m+\frac{1}{K^{m-1}}\left[\frac{R_1R_2\cdots R_m-K^m}{R_1R_2\cdots R_m+1}\right]\left(\frac{1-r}{K+1}\right)^m\right]\\ &\quad\times\left[\left(\frac{K+r}{K+1}\right)+\left[\frac{R_{m+1}-K}{R_{m+1}+1}\right]\left(\frac{1-r}{K+1}\right)\right]\\ &= \left(\frac{K+r}{K+1}\right)^{m+1}+\frac{1}{K^{m-1}}\left[\frac{R_1R_2\cdots R_m-K^m}{R_1R_2\cdots R_m+1}\right]\left[\frac{R_{m+1}-K}{R_{m+1}+1}\right]\left(\frac{1-r}{K+1}\right)^{m+1}\\ &\quad+\left(\frac{K+r}{K+1}\right)^m\left(\frac{1-r}{K+1}\right)\left[\frac{R_{m+1}-K}{R_{m+1}+1}\right]\\ &\quad+\frac{1}{K^{m-1}}\left(\frac{1-r}{K+1}\right)^m\left(\frac{K+r}{K+1}\right)\left[\frac{R_1R_2\cdots R_m-K^m}{R_1R_2\cdots R_m+1}\right].\end{aligned}$$

Therefore we will have

$$\begin{aligned}\prod_{k=1}^{m+1}\frac{r+R_k}{1+R_k} &\geq \left(\frac{K+r}{K+1}\right)^{m+1}+\left(\frac{1-r}{K+1}\right)^{m+1}\\ &\quad\times\left[\frac{1}{K^{m-1}}\left[\frac{R_1R_2\cdots R_m-K^m}{R_1R_2\cdots R_m+1}\right]\left[\frac{R_{m+1}-K}{R_{m+1}+1}\right]\right.\\ &\quad\left.+\frac{R_{m+1}-K}{R_{m+1}+1}+\frac{1}{K^{m-1}}\frac{R_1R_2\cdots R_m-K^m}{R_1R_2\cdots R_m+1}\right].\end{aligned}$$

Applying Lemma 3.1 to the second term in the right hand side of the above inequality, we obtain

$$\prod_{k=1}^{m+1}\frac{r+R_k}{1+R_k}\geq\left(\frac{K+r}{K+1}\right)^{m+1}+\frac{1}{K^m}\left[\frac{R_1R_2\cdots R_{m+1}-K^{m+1}}{R_1R_2\cdots R_{m+1}+1}\right]\left(\frac{1-r}{K+1}\right)^{m+1},$$

by which the method of induction is complete. □

Applying again the principle of induction on n and proceeding similarly as in the proof of above Lemma 3.3, but using Lemma 3.2, instead of Lemma 3.1, we can obtain the following result.

Lemma 3.4 *For any* $0\leq r\leq 1$ *and* $R_k\geq K$, $1\leq k\leq n$, $K>0$, *we have*

$$\prod_{k=1}^{n}\frac{r+R_k}{1+R_k}\geq\left(\frac{K+r}{K+1}\right)^n+\left[\frac{R_1R_2\cdots R_n-K^n}{R_1R_2\cdots R_n+K^n}\right]\left(\frac{\lambda}{K+1}\right)^n, \tag{3.7}$$

where $\lambda=\min\{1-r,K+r\}$.

4. Proofs of main results

Proof of Theorem 2.1 Let $z_k = R_k \mathrm{e}^{\mathrm{i}\varphi_k}$, $1 \le k \le n$, be the zeros of $P(z)$. Observe that $R_k \ge K$, $K \ge 1$, $1 \le k \le n$, since $P(z) \ne 0$ in $|z| < K$, $K \ge 1$. Then for any $0 \le r \le 1$, and $0 \le \varphi \le 2\pi$, we have

$$\begin{aligned}\frac{|P(r\mathrm{e}^{\mathrm{i}\varphi})|}{|P(\mathrm{e}^{\mathrm{i}\varphi})|} &= \prod_{k=1}^{n} \frac{|r\mathrm{e}^{\mathrm{i}\varphi} - R_k \mathrm{e}^{\mathrm{i}\varphi_k}|}{|\mathrm{e}^{\mathrm{i}\varphi} - R_k \mathrm{e}^{\mathrm{i}\varphi_k}|} \\ &= \prod_{k=1}^{n} \frac{|r\mathrm{e}^{\mathrm{i}(\varphi-\varphi_k)} - R_k|}{|\mathrm{e}^{\mathrm{i}(\varphi-\varphi_k)} - R_k|} \\ &= \prod_{k=1}^{n} \left(\frac{r^2 + R_k^2 - 2rR_k \cos(\varphi - \varphi_k)}{1 + R_k^2 - 2rR_k \cos(\varphi - \varphi_k)} \right)^{1/2} \\ &\ge \prod_{k=1}^{n} \frac{r + R_k}{1 + R_k}.\end{aligned}$$

Therefore we have

$$|P(r\mathrm{e}^{\mathrm{i}\varphi})| \ge \prod_{k=1}^{n} \frac{r + R_k}{1 + R_k} |P(\mathrm{e}^{\mathrm{i}\varphi})|. \tag{4.1}$$

Now applying Lemma 3.3 to the right hand side of the inequality (4.1) and using the fact that

$$R_1 R_2 \cdots R_n = \frac{|a_0|}{|a_n|},$$

we get the required inequality. □

Proof of Theorem 2.5 Since the polynomial $P(z)$ has all its zeros in $|z| \ge K$, $K \le 1$, we can express $P(z) = a_n \prod_{k=1}^{n} (z - R_k \mathrm{e}^{\mathrm{i}\alpha_k})$ where $R_k \ge K$, $1 \le k \le n$. Then it is a simple exercise to check the validity of

$$\frac{|P(r\mathrm{e}^{\mathrm{i}\alpha})|}{|P(\mathrm{e}^{\mathrm{i}\alpha})|} = \prod_{k=1}^{n} \frac{|r\mathrm{e}^{\mathrm{i}\alpha} - R_k \mathrm{e}^{\mathrm{i}\alpha_k}|}{|\mathrm{e}^{\mathrm{i}\alpha} - R_k \mathrm{e}^{\mathrm{i}\alpha_k}|} \ge \prod_{k=1}^{n} \frac{r + R_k}{1 + R_k}, \tag{4.2}$$

whenever $r \le K^2$ and $o \le \alpha \le 2\pi$.

Now applying Lemma 3.4 to the right hand side of the above inequality (4.2), we get the required inequality. □

Proof of Theorem 2.7 By assumption, the polynomial $P(z)$ has no zeros in $|z| < 1$. But then the polynomial $P(Rz)$ has no zeros in $|z| < 1/R$, where $1/R \ge 1$. Observe that the polynomial $P(Rz)$ satisfies the hypotheses of Theorem 2.1, and therefore using Theorem 2.1 for $P(Rz)$ we have

$$\max_{|z|=r/R} |P(Rz)| \ge \left\{ \left(\frac{1/R + r/R}{1/R + 1} \right)^n + R^{n-1} \left[\frac{|a_0| - |a_n R^n|(1/R^n)}{|a_0| + |a_n R^n|} \right] \left(\frac{1 - r/R}{1/R + 1} \right)^n \right\} \max_{|z|=1} |P(Rz)|,$$

which is equivalent to (2.9). □

Acknowledgment

The work of the second author was supported in part by the Serbian Academy of Sciences and Arts, Belgrade (Φ-96).

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