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Generalized Factorial Functions, Numbers and Polynomials [1]

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This paper is dedicated to Prof. Bl. Sendov on the occasion of his 70th anniversary

The generalized factorial functions and numbers and some classes of polynomials associated with them are considered. The recurrence relations, several representations, asymptotic and other properties of such numbers and polynomials, as well as the corresponding generating functions are investigated.

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1. Introduction and preliminaries

In 1971 Kurepa (see [10, 11]) defined so-called the left factorial $!n$ by:

$$!0 = 0, \quad !n = \sum_{k=0}^{n-1} k! \quad (n \in \mathbb{N})$$

and extended it to the complex half-plane $\operatorname{Re} z > 0$ as

$$K(z) = !z = \int_0^{+\infty} \frac{t^z - 1}{t - 1}\, e^{-t}\, dt.$$

Such function can be also extended analytically to the whole complex plane by $K(z) = K(z+1) - \Gamma(z+1)$, where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt \quad (\operatorname{Re} z > 0) \qquad \text{and} \qquad z\Gamma(z) = \Gamma(z+1).$$

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Kurepa [11] proved that $K(z)$ is a meromorphic function with simple poles at the points $z_k = -k$ $(k \in \mathbb{N} \setminus \{2\})$. Slavić [23] found the representation

$$K(z) = -\frac{\pi}{e}\cot \pi z + \frac{1}{e}\left(\sum_{n=1}^{+\infty} \frac{1}{n!n} + \gamma\right) + \sum_{n=0}^{+\infty} \Gamma(z-n),$$

where γ is Euler's constant. These formulas were mentioned also in the book [14].

A number of problems and hypotheses, especially in number theory, were posed by Kurepa and then considered by several mathematicians. For example, Kurepa [10] asked if

$$\gcd(!n, n!) = 2 \quad (n = 2, 3, \ldots),$$

where $\gcd(a, b)$ denotes the greatest common divisor of integers a and b. This conjecture, known as the *left factorial hypothesis* (KH), is still an open problem in number theory. There are several statements equivalent to KH. An equivalent formulation of KH appears in the book [8, Problem B44],

$$!n \not\equiv 0 \pmod{n} \quad \text{for all} \;\; n > 2.$$

Kurepa [12] also showed that KH can be reduced to primes so that KH is equivalent to the following assertion

$$!p \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2.$$

For details see [18, 19, 20], as well as a recent survey written by Ivić and Mijajlović, [9].

Recently, Milovanović [16] defined and studied a sequence of the *factorial functions* $\{M_m(z)\}_{m=-1}^{+\infty}$, where $M_{-1}(z) = \Gamma(z)$ and $M_0(z) = K(z)$. Namely,

$$M_m(z) = \int_0^{+\infty} \frac{t^{z+m} - Q_m(t,z)}{(t-1)^{m+1}}\, e^{-t}\, dt \quad (\operatorname{Re} z > -(m+1)), \tag{1.1}$$

where the polynomials $Q_m(t; z)$, $m = -1, 0, 1, 2, \ldots$, are given by

$$Q_{-1}(t,z) = 0, \quad Q_m(t,z) = \sum_{k=0}^{m} \binom{m+z}{k} (t-1)^k. \tag{1.2}$$

Since

$$M_m(z) = M_m(z+1) - M_{m-1}(z+1) \quad (m \in \mathbb{N}_0), \tag{1.3}$$

similar to the gamma function, the functions $z \mapsto M_m(z)$, for each $m \in \mathbb{N}_0$, can be extended analytically to the hole complex plane, starting from the corresponding analytic extension of the gamma function.

Suppose that we have analytic extensions for all functions $z \mapsto M_\nu(z)$, $\nu < m$. Let the function $z \mapsto M_m(z)$ be defined by (1.1) for z in the half-plane $\operatorname{Re} z > -(m+1)$. Using successively (1.3), we define at first $M_m(z)$ for z in the strip $-(m+2) < \operatorname{Re} z < -(m+1)$, then for z such that $-(m+3) < \operatorname{Re} z < -(m+2)$, etc. In this way, we obtain the function $M_m(z)$ in the hole complex plane.

In the same paper [16] the numbers $M_m(n)$ were introduced. For nonnegative integers $n, m \in \mathbb{N}_0$ we have

$$M_m(0) = 0, \quad M_m(n) = \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \sum_{k=i}^{n-1} k! \binom{m+n}{k+m+1}. \tag{1.4}$$

The numbers $M_m(n)$ can be expressed in terms of the *derangement numbers* (cf. [22, p. 65], [5, p. 182], [16])

$$S_k = k! \sum_{\nu=0}^{k} \frac{(-1)^\nu}{\nu!} \qquad (k \geq 0) \tag{1.5}$$

in the form

$$M_m(n) = \sum_{k=0}^{n-1} \binom{m+n}{k+m+1} S_k. \tag{1.6}$$

The numbers (1.5) satisfy the recurrence relation $S_k = kS_{k-1} + (-1)^k$ with $S_0 = 1$. Also, it is easy to prove that

$$S_{k+2} = (k+1)(S_{k+1} + S_k) \qquad (k \geq 0).$$

Notice that $S_0 = 1$, $S_1 = 0$, $S_2 = 1$, $S_3 = 2$, $S_4 = 9$, $S_5 = 44$, ... and $0 \leq S_k < S_{k+1}$ for $k \in \mathbb{N}$. Their generating function is given by (see [24, p. 147] and [17, Example 3])

$$\sum_{k=0}^{+\infty} S_k \frac{x^k}{k!} = \frac{e^{-x}}{1-x}\,.$$

In this paper we consider the factorial functions and numbers $M_m(n)$, some classes of polynomials associated with them, as well as some other related problems.

The paper is organized as follows. In Section 2 we investigate the numbers $M_m(n)$. Generating functions for such numbers are given in Section 3. Factorial polynomials are defined and investigated in Section 4. Finally, some integral representations of factorial functions $M_m(z)$ are derived in Section 5.

2. The numbers $M_m(n)$

For a fixed $m \in \mathbb{N}$, using (1.6) we obtain ([16])

$$\begin{aligned} M_m(1) &= 1, \quad M_m(2) = m+2, \quad M_m(3) = \frac{1}{2}(m^2+5m+8), \\ M_m(4) &= \frac{1}{6}(m^3+9m^2+32m+60), \quad \text{etc.} \end{aligned}$$

In general,

$$n!M_m(n+1) = \sum_{\nu=0}^{n} A_\nu^{(m,n)} m^\nu \qquad (A_n^{(m,n)} = 1).$$

Thus, for a fixed n, we have $M_m(n+1) \sim m^n/n!$ as $m \to +\infty$.

Some values of the numbers $M_m(n)$ are given in Table 1. The first row ($m = -1$) represents factorials $M_{-1}(n) = \Gamma(n) = (n-1)!$, and the second one ($m = 0$) gives the Kurepa numbers (left facorials) $M_0(n) = K(n) = !n$.

Taking (1.3), i.e.,

$$M_\nu(n+1) - M_{\nu-1}(n+1) = M_\nu(n) \tag{2.1}$$

for $\nu = 0, 1, \ldots, m$, we obtain

$$M_m(n+1) - M_{-1}(n+1) = \sum_{\nu=0}^{m} M_\nu(n).$$

So, we get the following representation:

Lemma 2.1. *For each $n \in \mathbb{N}$,*

$$M_m(n+1) = n! + \sum_{\nu=0}^{m} M_\nu(n).$$

Lemma 2.2. *For each fixed $\nu \in \mathbb{N}_0$ we have*

$$\lim_{n\to+\infty} \frac{M_\nu(n)}{M_{\nu-1}(n)} = 1. \tag{2.2}$$

$m \setminus n$	1	2	3	4	5	6	7	8
-1	1	1	2	6	24	120	720	5040
0	1	2	4	10	34	154	874	5914
1	1	3	7	17	51	205	1079	6993
2	1	4	11	28	79	284	1363	8356
3	1	5	16	44	123	407	1770	10126
4	1	6	22	66	189	596	2366	12492
5	1	7	29	95	284	880	3246	15738

Table 1: The numbers $M_m(n)$ for $m = -1, 0, 1, \ldots, 5$ and $n = 1, 2, \ldots, 8$

P r o o f. First, we note that all sequences $\{M_m(n)\}_{n=1}^{+\infty}$ $(m \geq -1)$ are increasing, as well as

$$\begin{aligned}\lim_{n\to+\infty} \frac{M_0(n)}{M_{-1}(n)} &= \lim_{n\to+\infty} \frac{K(n)}{\Gamma(n)} = \lim_{n\to+\infty} \frac{K(n)-K(n-1)}{\Gamma(n)-\Gamma(n-1)} \\ &= \lim_{n\to+\infty} \frac{\Gamma(n)}{\Gamma(n)-\Gamma(n-1)} = 1.\end{aligned}$$

Using Stolz' theorem and relation (2.1) we get

$$\begin{aligned}\lim_{n\to+\infty} \frac{M_\nu(n)}{M_{\nu-1}(n)} &= \lim_{n\to+\infty} \frac{M_\nu(n)-M_\nu(n-1)}{M_{\nu-1}(n)-M_{\nu-1}(n-1)} \\ &= \lim_{n\to+\infty} \frac{M_{\nu-1}(n)}{M_{\nu-2}(n)} = \cdots = \lim_{n\to+\infty} \frac{M_0(n)}{M_{-1}(n)} = 1.\end{aligned}$$

■

Since

$$\frac{M_m(n)}{M_{-1}(n)} = \frac{M_0(n)}{M_{-1}(n)} \cdot \frac{M_1(n)}{M_0(n)} \cdots \frac{M_m(n)}{M_{m-1}(n)},$$

using (2.2) we get the following result:

Theorem 2.3. *For each $m \in \mathbb{N}_0$ we have*

$$\lim_{n\to+\infty} \frac{M_m(n)}{(n-1)!} = 1 \quad \textit{and} \quad \lim_{n\to+\infty} \frac{M_m(n)}{n!} = 0. \tag{2.3}$$

As we can see the asymptotic relation $M_m(n+1) \sim nM_m(n)$ $(n \to +\infty)$ holds. Furthermore, for a sufficiently large n, we can prove an inequality:

Theorem 2.4. *For each $m \in \mathbb{N}_0$ there exists an integer $n_m \in \mathbb{N}$ such that for every $n \geq n_m$ the following inequality*

$$M_m(n+1) \leq nM_m(n) \tag{2.4}$$

holds.

Proof. First, we note that for $m = -1$, the inequality (2.4) reduces to the well-known functional equation for the gamma function, $\Gamma(n+1) = n\Gamma(n)$, $n \geq 1$.

For $m = 0$, inequality (2.4) reduces to $!(n+1) \leq n(!n)$, which is not true for $n = 1$. But, it can be proved for each $n \geq n_0 = 2$. Indeed, for $n = 2$, (2.4) becomes an equality $!3 = 2(!2)$, i.e., $0! + 1! + 2! = 2(0! + 1!) = 4$. Suppose now that (2.4) holds for some $n = k \geq 2$, i.e.,

$$M_0(k+1) \leq kM_0(k).$$

Adding $M_{-1}(k+2) = \Gamma(k+2) = (k+1)\Gamma(k+1)$ to both sides in the previous inequality, we get

$$\begin{aligned} M_0(k+1) + M_{-1}(k+2) &\leq kM_0(k) + (k+1)M_{-1}(k+1) \\ &= (k+1)(M_0(k) + M_{-1}(k+1)) - M_0(k). \end{aligned}$$

Using recurrence relation (2.1) for $\nu = 0$ and a fact that $M_0(k) > 0$, we obtain

$$M_0(k+2) \leq (k+1)M_0(k+1).$$

Notice that $M_m(4) \leq 3M_m(3)$ for $m = 1, 2, 3, 4$ (see Table 1), so that $n_m = 3$ for such values of m.

For $m \geq 5$ we can prove inequality (2.4) for $n = m$, i.e., $M_m(m+1) \leq mM_m(m)$. This means that we can take $n_m = m$ for $m \geq 5$. According to (1.3) and (1.6) this inequality can be represented in the following equivalent forms:

$$M_{m-1}(m+1) \leq (m-1)M_m(m)$$

and

$$\sum_{k=0}^{m} \binom{2m}{k+m} S_k \leq (m-1) \sum_{k=0}^{m-1} \binom{2m}{k+m+1} S_k.$$

Because of $S_k = kS_{k-1} + (-1)^k$, the last inequality reduces to

$$\sum_{k=0}^{m} \binom{2m}{k+m} (-1)^k \leq \sum_{k=1}^{m} \binom{2m}{k+m} (m-1-k) S_{k-1}.$$

Since the sum on the left-hand side of the previous inequality is equal to $\binom{2m-1}{m-1}$ (cf. [21, p. 607]), we get

$$\binom{2m-1}{m} \leq \sum_{k=1}^{m} \binom{2m}{k+m}(m-1-k)S_{k-1},$$

i.e.,

$$\sum_{k=2}^{m-3} \binom{2m}{k+m}(m-1-k)S_{k-1} + A_m + B_m \geq 0, \tag{2.5}$$

where

$$A_m = (m-2)\binom{2m}{m+1} - \binom{2m-1}{m} = \frac{2m^2-5m-1}{m+1}\binom{2m-1}{m}$$

and

$$B_m = \binom{2m}{2m-2}S_{m-3} - \binom{2m}{2m}S_{m-1} = S_{m-3}\left[m(2m-1) - \frac{S_{m-1}}{S_{m-3}}\right].$$

Since $S_{m-1}/S_{m-3} = (m-2)(m-1+(-1)^m/S_{m-3})$ we have that

$$B_m = (m-2)S_{m-3}\left[m+4+\frac{6}{m-2} - \frac{(-1)^m}{S_{m-3}}\right] > 0$$

for $m \geq 5$. Notice also that $A_m > 0$ for $m \geq 3$.

Since the first term in (2.5) is equal to zero and others are positive, we conclude that the inequality (2.5) is true for each $m \geq 5$. Thus, this proves the existence of the numbers n_m for each m.

The proof of (2.4) for $m \in \mathbb{N}$ and $n \geq n_m$ can be given by induction in m. Namely, supposing that (2.4) holds for each $\nu < m \in \mathbb{N}$ and $n \geq n_\nu$ ($n_1 = \cdots = n_4 = 3$, $n_\nu = \nu$ for $\nu > 4$) we prove the inequality $M_m(n+1) \leq nM_m(n)$ for $n \geq n_m = m$. In order to do this we apply induction in n, in the same way as for $m = 0$.

Since (2.4) holds for $m = 0$ and $n \geq n_0 = 2$, it means that (2.4) holds for each $m \in \mathbb{N}_0$ and $n \geq n_m$. ∎

Remark 2.5. Theorem 2.4 establishes only the existence of the numbers n_m. The minimal values of m_n can be expressed in the following way, if we define

$$a_\nu = \nu^2 + \nu - 1, \quad b_\nu = \nu^2 + 3\nu, \quad I_\nu = \{m \in \mathbb{Z} : m \in [a_\nu, b_\nu]\},$$

for each $\nu \in \mathbb{N}_0$. Notice that $a_{\nu+1} = b_\nu + 1$, $I_0 = \{-1, 0\}$, as well as $I_1 \cup I_2 \cup \cdots = \mathbb{N}$. *Then for each* $\nu \in \mathbb{N}$ *and* $m \in I_\nu$ *we have* $n_m = \nu + 2$.

For example, for $\nu = 1$, i.e., $m \in \{1, 2, 3, 4\}$, we have $n_m = 3$. It was noted in the proof of the previous theorem. For $m \in \{5, 6, 7, 8, 9, 10\}$ $(\nu = 2)$ we have $n_m = 4$, etc. Notice that $n_m < m$.

3. Generating functions for the numbers $M_m(n)$

Definition 3.1. Let $m \in \mathbb{N}_0$. The exponential generating function of the sequence $\{M_m(n)\}_{n=0}^{+\infty}$ is given by

$$g_m(x) = \sum_{n=0}^{+\infty} M_m(n) \frac{x^n}{n!}. \tag{3.1}$$

According to Theorem 2.3, the expansion (3.1) converges in the unit circle $|x| < 1$. Notice also that $g_m(0) = 0$.

Remark 3.2. Because of $M_m(0) = 0$ we have

$$g_m(x) = \sum_{n=1}^{+\infty} M_m(n) \frac{x^n}{n!}. \tag{3.2}$$

This modification in previous definition enables us to define the exponential generating function of the sequence $\{\Gamma(n)\}_{n=1}^{+\infty}$, i.e., $\{(n-1)!\}_{n=1}^{+\infty}$. Thus, for $m = -1$, (3.2) reduces to

$$g_{-1}(x) = \sum_{n=1}^{+\infty} \Gamma(n) \frac{x^n}{n!} = \sum_{n=1}^{+\infty} \frac{x^n}{n} = \log \frac{1}{1-x} \qquad (|x| < 1). \tag{3.3}$$

Theorem 3.3. *The generating functions* g_m $(m = 0, 1, \ldots)$ *satisfy the following relation*

$$g_{m+1}(x) = g_m(x) + e^x \int_0^x e^{-t} g_m(t)\, dt, \qquad m \geq 0, \tag{3.4}$$

where

$$g_0(x) = e^{x-1}(\mathrm{Ei}\,(1) - \mathrm{Ei}\,(1-x)), \tag{3.5}$$

and $\mathrm{Ei}\,(x)$ *is the exponential integral defined by*

$$\mathrm{Ei}\,(x) = \mathrm{v.p.} \int_{-\infty}^x \frac{e^t}{t}\, dt \qquad (x > 0). \tag{3.6}$$

P r o o f. Let $|x| < 1$ and g_m be defined by (3.1). Then

$$g'_m(x) = \sum_{n=1}^{+\infty} M_m(n) \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{+\infty} M_m(n+1) \frac{x^n}{n!}$$

and

$$g'_{m+1}(x) - g'_m(x) = \sum_{n=0}^{+\infty} \left(M_{m+1}(n+1) - M_m(n+1) \right) \frac{x^n}{n!} ,$$

i.e.,

$$g'_{m+1}(x) - g'_m(x) = \sum_{n=0}^{+\infty} M_{m+1}(n) \frac{x^n}{n!} = g_{m+1}(x).$$

Integrating this differential equation, we obtain

$$g_{m+1}(x) = e^x \int_0^x e^{-t} g'_m(t)\, dt. \tag{3.7}$$

Finally, an integration by parts gives

$$g_{m+1}(x) = g_m(x) + e^x \int_0^x e^{-t} g_m(t)\, dt.$$

According to (3.7) and (3.3), for $m = 0$ we have

$$g_0(x) = e^x \int_0^x \frac{e^{-t}}{1-t}\, dt = e^x \int_{1-x}^1 \frac{e^{u-1}}{u}\, du.$$

Using the exponential integral $\mathrm{Ei}\,(x)$ defined by (3.6) (see [1, p. 228]), the previous formula becomes

$$g_0(x) = e^{x-1} (\mathrm{Ei}\,(1) - \mathrm{Ei}\,(1-x)) \qquad (|x| < 1),$$

i.e., (3.5). ■

R e m a r k 3.4. The generating function for left factorial in the form (3.5) was obtained by Đ. Cvijović.

In order to find an explicit expression for $g_m(x)$ we denote its Laplace transform by $G_m(s)$. Then, from (3.4) it follows

$$G_{m+1}(s) = G_m(s) + \frac{1}{s-1}\, G_m(s) = \left(1 + \frac{1}{s-1}\right) G_m(s),$$

i.e.,

$$G_m(s) = \left(1 + \frac{1}{s-1}\right)^m G_0(s) = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{(s-1)^k} \cdot G_0(s).$$

The inverse transform gives

$$g_m(x) = g_0(x) + \sum_{k=1}^{m} \binom{m}{k} \frac{1}{(k-1)!} \int_0^x e^{x-t}(x-t)^{k-1} g_0(t)\, dt. \tag{3.8}$$

Using integration by parts in the integral convolutions on the right hand side in (3.8) yields

$$\begin{aligned} \int_0^x e^{x-t}(x-t)^{k-1} g_0(t)\, dt &= \frac{1}{k} \int_0^x e^{x-t}(x-t)^k \frac{dt}{1-t} \\ &= \frac{e^x}{k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x-1)^{k-\nu} \int_0^x e^{-t}(1-t)^{\nu-1}\, dt. \end{aligned}$$

Since

$$\int_0^x e^{-t}(1-t)^{\nu-1}\, dt = \begin{cases} e^{-x} g_0(x), & \text{if } \nu = 0, \\ P_{\nu-1}(0) - e^{-x} P_{\nu-1}(x), & \text{if } \nu \geq 1, \end{cases}$$

where

$$P_\nu(x) = (-1)^\nu \nu! \sum_{j=0}^{\nu} \frac{(x-1)^j}{j!}\,, \tag{3.9}$$

we get

$$\begin{aligned} \int_0^x e^{x-t}(x-t)^{k-1} g_0(t)\, dt &= \frac{(x-1)^k}{k} g_0(x) + \frac{e^x}{k} \sum_{\nu=1}^{k} \binom{k}{\nu} (x-1)^{k-\nu} P_{\nu-1}(0) \\ &\quad - \frac{1}{k} \sum_{\nu=1}^{k} \binom{k}{\nu} (x-1)^{k-\nu} P_{\nu-1}(x). \end{aligned}$$

According to this equality, (3.8) and (3.9) we have the following result:

Theorem 3.5. *For each $m \in \mathbb{N}_0$ the generating function $x \mapsto g_m(x)$ is given by*

$$g_m(x) = \frac{1}{m!}(A_m(x) g_0(x) + B_m(x) e^x - C_m(x)), \tag{3.10}$$

where $A_m(x)$, $B_m(x)$, and $C_m(x)$ are polynomials determined by

$$
\begin{aligned}
\frac{A_m(x)}{m!} &= \sum_{k=0}^{m} \binom{m}{k} \frac{(x-1)^k}{k!}, \\
\frac{B_m(x)}{m!} &= \sum_{\nu=0}^{m-1} \left(\sum_{k=1}^{m-\nu} \binom{m}{k+\nu} \frac{(-1)^{k-1}}{k} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \right) \frac{(x-1)^\nu}{\nu!}, \\
\frac{C_m(x)}{m!} &= \sum_{j=0}^{m-1} \left(\sum_{\nu=0}^{j} (-1)^\nu \binom{j}{\nu} \sum_{k=j+1}^{m} \frac{(-1)^{k-1}}{k-\nu} \binom{m}{k} \right) \frac{(x-1)^j}{j!},
\end{aligned}
$$

respectively.

The polynomials $A_m(x)$, $B_m(x)$, and $C_m(x)$ for $1 \le m \le 6$ are presented in Table 2.

m	$A_m(x)$	$B_m(x)$	$C_m(x)$
1	x	1	1
2	x^2+2x-1	$2x+2$	$x+2$
3	x^3+6x^2+3x-4	$3x^2+12x+4$	x^2+6x+4
4	$x^4+12x^3+30x^2-4x-15$	$4x^3+36x^2+64x+6$	$x^3+12x^2+31x+6$
5	$x^5+20x^4+110x^3+140x^2$	$5x^4+80x^3+340x^2$	$x^4+20x^3+111x^2$
	$-95x-56$	$+350x-16$	$+158x-16$
6	$x^6+30x^5+285x^4+940x^3$	$6x^5+150x^4+1160x^3$	$x^5+30x^4+286x^3$
	$+555x^2-906x-185$	$+3090x^2+2004x-310$	$+968x^2+789x-310$

Table 2: The polynomials $A_m(x)$, $B_m(x)$, and $C_m(x)$ in (3.10) for $m = 1,2,3,4,5,6$

Remark 3.6. Starting from $A_0(x) = 1$, $B_0(x) = C_0(x) = 0$, the polynomials $A_m(x)$, $B_m(x)$, and $C_m(x)$ can be calculated recursively by

$$
\begin{aligned}
A_{m+1}(x) &= (m+1) \left(A_m(x) + \int_1^x A_m(t)\,dt \right), \\
B_{m+1}(x) &= (m+1) \left(B_m(x) + \int_0^x B_m(t)\,dt \right) + \beta_{m+1}, \\
C_{m+1}(x) &= e^x \int e^{-x} \varphi_m(x)\,dx,
\end{aligned}
$$

where $\varphi_m(x)$ is a polynomial of degree m defined by

$$\varphi_m(x) = (m+1)\left(C'_m(x) - \frac{1}{x-1}\int_1^x A_m(t)\,dt\right),$$

and $\beta_{m+1} = C_{m+1}(0) - (m+1)C_m(0)$.

Remark 3.7. It is clear that $A_m(x) > 0$ for $x \geq 1$ and $A_m(1) = m!$. Also, $B_m(0) = C_m(0)$. It can be proved that polynomials $A_m(x)$ have only real zeros distributed in $(-\infty, 1)$. Furthermore, the zeros of $A_m(x)$ and $A_{m+1}(x)$ mutually separate each other.

4. The factorial polynomials

Definition 4.1. Let $m \in \mathbb{N}_0$. The factorial polynomials $\{K_n^{(m)}(t)\}_{n\in\mathbb{N}}$ are defined by

$$G_m(t,x) = e^{xt} g_m(x) = \sum_{n=1}^{+\infty} K_n^{(m)}(t)\frac{x^n}{n!},$$

where g_m is defined by (3.1) and given by (3.10).

Using (3.1) and the numbers $M_m(k)$ it is easy to prove the following explicit representation of the factorial polynomials:

Theorem 4.2. *For each $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we have*

$$K_n^{(m)}(t) = \sum_{k=1}^{n} \binom{n}{k} M_m(k) t^{n-k}.$$

For example, for $m = 0$ and $n \leq 7$ we have

$$\begin{aligned}
K_1^{(0)}(t) &= 1,\\
K_2^{(0)}(t) &= 2t + 2,\\
K_3^{(0)}(t) &= 3t^2 + 6t + 4,\\
K_4^{(0)}(t) &= 4t^3 + 12t^2 + 16t + 10,\\
K_5^{(0)}(t) &= 5t^4 + 20t^3 + 40t^2 + 50t + 34,\\
K_6^{(0)}(t) &= 6t^5 + 30t^4 + 80t^3 + 150t^2 + 204t + 154,\\
K_7^{(0)}(t) &= 7t^6 + 42t^5 + 140t^4 + 350t^3 + 714t^2 + 1078t + 874.
\end{aligned}$$

Since

$$\begin{aligned}\frac{d}{dt}\,K_n^{(m)}(t) &= \sum_{k=1}^{n}\binom{n}{k}(n-k)M_m(k)t^{n-1-k}\\ &= \sum_{k=1}^{n-1}\binom{n-1}{k}M_m(k)t^{n-1-k},\end{aligned}$$

we have the following differentiation formula

$$\frac{d}{dt}\,K_n^{(m)}(t) = nK_{n-1}^{(m)}(t).$$

Also, we have

$$\frac{d^\nu}{dt^\nu}\,K_n^{(m)}(t) = n(n-1)\cdots(n-\nu+1)K_{n-\nu}^{(m)}(t) \qquad (0<\nu<n).$$

Expanding $K_n^{(m)}(t+s)$ in Taylor series and using the previous formula we obtain

$$K_n^{(m)}(t+s) = \sum_{\nu=0}^{+\infty}\frac{1}{\nu!}\frac{d^\nu}{dt^\nu}\,K_n^{(m)}(t)s^\nu = \sum_{\nu=0}^{n-1}\binom{n}{\nu}K_{n-\nu}^{(m)}(t)s^n,$$

i.e.,

$$K_n^{(m)}(t+s) = \sum_{\nu=1}^{n}\binom{n}{\nu}K_\nu^{(m)}(t)s^{n-\nu}.$$

It is clear that

$$\frac{d^\nu}{dt^\nu}\,K_n^{(m)}(t) > 0 \qquad (0\le\nu<n)$$

for each $t\ge 0$. Therefore, the polynomials $K_n^{(m)}(t)$ have no positive real zeros.

A simple representation of these polynomials in terms of zeros can be done:

Theorem 4.3. *Let τ_ν $(\nu=1,\ldots,n)$ be zeros of $K_{n+1}^{(m)}(t)$, i.e., $K_{n+1}^{(m)}(t) = (n+1)\prod_{\nu=1}^{n}(t-\tau_\nu)$. Then*

$$K_n^{(m)}(t) = \frac{1}{n+1}\sum_{k=1}^{n}\frac{K_{n+1}^{(m)}(t)}{t-\tau_k} = \sum_{k=1}^{n}\prod_{\substack{\nu=1\\ \nu\ne k}}^{n}(t-\tau_\nu).$$

5. The factorial functions $M_m(z)$

In this section we consider again the factorial functions $M_m(z)$ defined by (1.1) and (1.2). First, we put

$$Y_m(t,z)=\frac{t^{z+m}-Q_m(t,z)}{(t-1)^{m+1}} \qquad (\operatorname{Re} z>-(m+1)).$$

Using the binomial series

$$t^{z+m}=(1+t-1)^{m+z}=\sum_{k=0}^{+\infty}\binom{m+z}{k}(t-1)^k \qquad (|t-1|<1),$$

we see that, for $0<t<2$,

$$Y_m(t,z)=\sum_{k=0}^{+\infty}\binom{m+z}{k+m+1}(t-1)^k.$$

The function $Y_m(t,z)$ can be expressed in terms of the hypergeometric function ${}_2F_1$, defined by

$${}_2F_1(a,b,c;x)=\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\cdot\frac{x^k}{k!}$$

for $|x|<1$, and by continuation elsewhere. It is well-known that

$${}_2F_1(a,b,c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \xi^{b-1}(1-\xi)^{c-b-1}(1-x\xi)^{-a}\,d\xi$$

in the x plane cut along the real axis from 1 to ∞, if $\operatorname{Re} c>\operatorname{Re} b>0$ (cf. [2, p. 65]). Here it is understood that $\arg\xi=\arg(1-\xi)=0$ and $(1-x\xi)^{-a}$ has its principal value.

According to (1.1) we note that $M_m(z)$ can be interpreted as the Laplace transform of the function $t\mapsto Y_m(t,z)$ at the point $s=1$. Therefore, we put

$$F_m(s,z)=\mathcal{L}[Y_m(t,z)]=\int_0^{+\infty} G_m(t,z)e^{-st}\,dt, \tag{5.1}$$

where z is a complex parameter such that $\operatorname{Re} z>-(m+1)$, and $M_m(z)=F_m(1,z)$.

Theorem 5.1. *For* $\operatorname{Re} z > -(m+1)$, *the factorial functions* $z \mapsto M_m(z)$ *have the integral representation*

$$M_m(z) = \frac{z(z+1)\cdots(z+m)}{m!}\int_0^1 \xi^{z-1}(1-\xi)^m e^{(1-\xi)/\xi}\Gamma\Big(z, \frac{1-\xi}{\xi}\Big)\,d\xi,$$

where $\Gamma(z,x)$ *is the incomplete gamma function defined by*

$$\Gamma(z,x) = \int_x^{+\infty} t^{z-1}e^{-t}\,dt. \tag{5.2}$$

Proof. Since

$$(k+m+1)! = (m+1)!(m+2)_k \qquad \text{and} \qquad (1-z)_k = \frac{(-1)^k\Gamma(z)}{\Gamma(z-k)},$$

we have

$$\binom{m+z}{k+m+1} = \frac{\Gamma(m+z+1)}{\Gamma(z-k)(k+m+1)!} = \frac{\Gamma(m+z+1)}{\Gamma(z)(m+1)!}\cdot\frac{(1-z)_k(-1)^k}{(m+2)_k},$$

so that

$$\begin{aligned} Y_m(t,z) &= \frac{\Gamma(m+z+1)}{\Gamma(z)(m+1)!}\sum_{k=0}^{+\infty}\frac{(1-z)_k(1)_k}{(m+2)_k}\cdot\frac{(1-t)^k}{k!} \\ &= \frac{\Gamma(m+z+1)}{\Gamma(z)(m+1)!}\,{}_2F_1(1-z,1,m+2;1-t), \end{aligned}$$

or, by continuation,

$$Y_m(t,z) = \frac{\Gamma(m+z+1)}{\Gamma(z)m!}\int_0^1 (1-\xi)^m(1-(1-t)\xi)^{z-1}d\xi.$$

According to (5.1) we have

$$\begin{aligned} F_m(s,z) &= \frac{z(z+1)\cdots(z+m)}{m!}\int_0^{+\infty} e^{-st}\int_0^1 (1-\xi)^m\xi^{z-1}(t+\alpha)^{z-1}\,d\xi\,dt \\ &= \frac{z(z+1)\cdots(z+m)}{m!}\int_0^1 \xi^{z-1}(1-\xi)^m\int_0^{+\infty} e^{-st}(t+\alpha)^{z-1}\,dt\,d\xi \end{aligned}$$

where $\alpha = (1-\xi)/\xi$.

Since

$$\mathcal{L}[(t+\alpha)^{z-1}] = \frac{e^{\alpha s}\Gamma(z,\alpha s)}{s^z} \qquad (\operatorname{Re} s > 0),$$

where $\Gamma(z,x)$ is the incomplete gamma function defined by (5.2), we get

$$F_m(s,z) = \frac{z(z+1)\cdots(z+m)}{m!\,s^z}\int_0^1 \xi^{z-1}(1-\xi)^m e^{\alpha s}\Gamma(z,\alpha s)\,d\xi.$$

Finally, for $s = 1$ we obtain the result of the theorem. ∎

Changing variables $(1-\xi)/\xi = x$ we get an alternative form of the previous theorem:

Corollary 5.2. *For* $\operatorname{Re} z > -(m+1)$, *the factorial functions* $z \mapsto M_m(z)$ *have the integral representation*

$$M_m(z) = \frac{z(z+1)\cdots(z+m)}{m!}\int_0^{+\infty} \frac{x^m e^x\Gamma(z,x)}{(x+1)^{z+m+1}}\,dx.$$

In a special case when $m = 0$, we get the integral representation of the Kurepa's function

$$K(z) = z\int_0^1 \xi^{z-1}e^{(1-\xi)/\xi}\Gamma\Big(z,\frac{1-\xi}{\xi}\Big)\,d\xi = z\int_0^{+\infty}\frac{e^x\Gamma(z,x)}{(x+1)^{z+1}}\,dx,$$

which holds for $\operatorname{Re} z > -1$. In 1995 one of us [15] derived the Chebyshev expansion for $K(1+z)$ and $1/K(1+z)$, as well as the power series expansion of $K(a+z)$, $a \geq 0$, and determined numerical values of their coefficients $b_\nu(a)$ for $a = 0$ and $a = 1$. Using an asymptotic behaviour of $b_\nu(a)$, when $\nu \to \infty$, a transformation of series with much faster convergence was obtained. For similar expansions of the gamma function see e.g. Davis [6], Luke [13], Fransén and Wrigge [7], and Bohman and Fröberg [4].

Remark 5.3. The function $x \mapsto x^{-z}e^x\Gamma(z,x)$ can be expanded in continuous fractions (cf. [3, Chapter 9])

$$x^{-z}e^x\Gamma(z,x) = \cfrac{1}{x+\cfrac{1-z}{1+\cfrac{1}{x+\cfrac{2-z}{1+\cdots}}}}.$$

Remark 5.4. The function $z \mapsto M_m(z)$ has zeros at $z = -n$, $n = 0, 1, \ldots, m$.

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