

Multiple Orthogonal Polynomials on the Semicircle and Corresponding Quadratures of Gaussian Type ¹

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In this paper we consider multiple orthogonal polynomials defined using orthogonality conditions spread out over r different measures are considered. We study multiple orthogonal polynomials on the real line, as well as on the semicircle (complex polynomials orthogonal with respect to the complex-valued inner products $(f, g)_k = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w_k(e^{i\theta})d\theta$, for $k = 1, 2, \dots, r$). For $r = 1$, in the real case we have the ordinary orthogonal polynomials, and in complex case orthogonal polynomials on the semicircle, introduced by Gautschi and Milovanović [7]. Multiple orthogonal polynomials satisfy a linear recurrence relation of the order $r + 1$. This is a generalization of the second order linear recurrence relation for ordinary monic orthogonal polynomials ($r = 1$). Using the discretized Stieltjes-Gautschi procedure, we compute recurrence coefficients and also zeros of multiple orthogonal polynomials, as well as the weight coefficients for the corresponding quadrature formulas of Gaussian type. Some numerical examples are also included.

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1. Introduction

Multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite-Padé approximation of a system of r (Markov) functions. There are several papers by Nikishin, Sorokin, de Bruin, Piñeiro, Aptekarev etc.

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Historically, Hermite-Padé approximation was introduced by Hermite to prove the transcendence of e . Multiple orthogonal polynomials can be used to give a constructive proof of irrationality and transcendence of certain real numbers (see [15]).

Starting with a problem that arise in the evaluation of computer graphics illumination models, Borges [4] has examined the problem of numerically evaluating a set of r definite integrals taken with respect to distinct weight functions but related by a common integrand and interval of integration. It is interesting that the nodes of an optimal set of such quadratures are the zeros of type II multiple orthogonal polynomials. However, Borges has not used multiple orthogonality. In order to construct Gaussian quadratures we consider the multiple orthogonality.

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy $r \in \mathbb{N}$ orthogonality conditions.

Let $r \geq 1$ be an integer and let w_1, w_2, \dots, w_r be r weight functions on the real line so that the support of each w_i is a subset of an interval E_i . Let $\vec{n} = (n_1, n_2, \dots, n_r)$ be a vector of r nonnegative integers, which is called a *multi-index* with length $|\vec{n}| = n_1 + n_2 + \dots + n_r$.

There are two types of multiple orthogonal polynomials (see [17]):

1° *Type I multiple orthogonal polynomials.*

Here we want to find a vector of polynomials $(A_{\vec{n},1}, A_{\vec{n},2}, \dots, A_{\vec{n},r})$ such that each $A_{\vec{n},i}$ is polynomial of degree $n_i - 1$ and the following orthogonality conditions hold:

$$(1.1) \quad \sum_{j=1}^r \int_{E_j} A_{\vec{n},j} x^k w_j(x) dx = 0, \quad k = 0, 1, 2, \dots, |\vec{n}| - 2.$$

2° *Type II multiple orthogonal polynomials.*

Type II multiple orthogonal polynomial is monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ such that satisfies the following orthogonality conditions:

$$(1.2) \quad \int_{E_1} P_{\vec{n}}(x) x^k w_1(x) dx = 0, \quad k = 0, 1, \dots, n_1 - 1,$$

$$(1.3) \quad \int_{E_2} P_{\vec{n}}(x) x^k w_2(x) dx = 0, \quad k = 0, 1, \dots, n_2 - 1,$$

⋮

$$(1.4) \quad \int_{E_r} P_{\vec{n}}(x) x^k w_r(x) dx = 0, \quad k = 0, 1, \dots, n_r - 1.$$

The conditions (1.2)–(1.4) give $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients $a_{k,\vec{n}}$ of the polynomial $P_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k,\vec{n}} x^k$, where $a_{|\vec{n}|,\vec{n}} = 1$. But the matrix of coefficients of this system can be singular and we need some additional conditions on the r weight functions to provide the uniqueness of the multiple orthogonal polynomial.

If the polynomial $P_{\vec{n}}(x)$ is unique, then we say that \vec{n} is *normal index* and if all indices are normal then we have a *complete system*.

For $r = 1$ in both cases we have the ordinary orthogonal polynomials. We will consider only the type II multiple orthogonal polynomials.

There are two distinct cases for which the type II multiple orthogonal polynomials are given (see [17]).

1. *Angelesco systems*—For this systems the intervals E_i , on which the weight functions are supported, are disjoint, i.e., $E_i \cap E_j = \emptyset$ for $1 \leq i \neq j \leq r$.

2. *AT systems*—AT systems are such that all weight functions are supported on the same interval E and we also require that the $|\vec{n}|$ functions

$$w_1(x), xw_1(x), \dots, x^{n_1-1}w_1(x), w_2(x), xw_2(x), \dots, x^{n_2-1}w_2(x), \dots, w_r(x), xw_r(x), \dots, x^{n_r-1}w_r(x)$$

form a Chebyshev system on E for each multi-index \vec{n} .

The following two theorems have been proved in [17]:

Theorem 1.1. *In an Angelesco system the type II multiple orthogonal polynomial $P_{\vec{n}}(x)$ factors into r polynomials $\prod_{j=1}^r q_{n_j}(x)$, where each q_{n_j} has exactly n_j zeros on E_j .*

Theorem 1.2. *In an AT system the type II multiple orthogonal polynomial $P_{\vec{n}}(x)$ has exactly $|\vec{n}|$ zeros on E . For the type I vector of multiple orthogonal polynomials, the linear combination $\sum_{j=1}^r A_{\vec{n},j}(x)w_j(x)$ has exactly $|\vec{n}| - 1$ zeros on E .*

For each of the weight functions $w_k, k = 1, 2, \dots, r$,

$$(1.5) \quad (f, g)_k = \int_{E_k} f(x)g(x)w_k(x)dx$$

denotes the corresponding inner product of f and g .

In this paper in Section 2 we consider recurrence relations for some cases of type II multiple orthogonal polynomials. Then in Section 3 we present a

numerical procedure for construction of type II multiple orthogonal polynomials based on the Stieltjes-Gautschi procedure [5]. An optimal set of quadrature formulas with method for calculating the nodes and weight coefficients of such quadratures are considered in Section 4. Finally, in Section 5 we transfer the concept of multiple orthogonality on the unit semicircle in complex plane. We introduce multiple orthogonal polynomials on the semicircle and corresponding quadratures of Gaussian type. Also, some numerical examples are included.

2. Recurrence relations

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation. There is an interesting recurrence relation of order $r + 1$ for the type II multiple orthogonal polynomials with nearly diagonal multi-index.

Let $n \in \mathbb{N}$ and write it as $n = kr + j$, with $0 \leq j < r$. The *nearly diagonal multi-index* $\vec{s}(n)$ corresponding to n is given by

$$\vec{s}(n) = (\underbrace{k + 1, k + 1, \dots, k + 1}_j, \underbrace{k, k, \dots, k}_{r-j}).$$

Denote the corresponding type II multiple orthogonal polynomials by $P_n(x) = P_{\vec{s}(n)}(x)$.

The following recurrence relation

$$(2.1) \quad xP_m(x) = P_{m+1}(x) + \sum_{i=0}^r a_{m,r-i}P_{m-i}(x), \quad m \geq 1,$$

holds with initial conditions $P_0(x) = 1$ and $P_i(x) = 0$ for $i = -1, -2, \dots, -r$ (see [16]).

Setting $m = 0, 1, \dots, n - 1$ in (2.1), we get

$$H_n \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{bmatrix} - P_n(x) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

i.e.,

$$(2.2) \quad H_n \mathbf{P}_n(x) = x\mathbf{P}_n(x) - P_n(x)\mathbf{e}_n,$$

where

$$\mathbf{P}_n(x) = [P_0(x) \ P_1(x) \ \dots \ P_{n-1}(x)]^T, \quad \mathbf{e}_n = [0 \ 0 \ \dots \ 0 \ 1]^T,$$

3. Numerical construction of multiple orthogonal polynomials

In [13] we have obtained an effective numerical method for constructing the Hessenberg matrix H_n . We use some kind of the Stieltjes procedure (cf. [5]) and call it as the *discretized Stieltjes-Gautschi procedure*. At first, we express the elements of H_n in terms of the inner products (1.5) and then we use the corresponding Gaussian formulas to discretize these inner products. Of course, we suppose that the type II multiple orthogonal polynomials exist with respect to the inner products $(\cdot, \cdot)_k$, $k = 1, 2, \dots, r$, given by (1.5).

Taking that for inner products $(\cdot, \cdot)_{j+\ell r} = (\cdot, \cdot)_j$ ($\ell \in \mathbb{Z}$), the following result holds (see [13]):

Theorem 3.1. *The type II multiple monic orthogonal polynomials $\{P_n\}$, with nearly diagonal multi-index, satisfy the recurrence relation*

$$(3.1) \quad P_{n+1}(x) = (x - a_{n,r})P_n(x) - \sum_{k=0}^{r-1} a_{n,k}P_{n-r+k}(x), \quad n \geq 1,$$

where

$$a_{n,0} = \frac{(xP_n, P_{[(n-r)/r]})_{\nu+1}}{(P_{n-r}, P_{[(n-r)/r]})_{\nu+1}}$$

and

$$a_{n,k} = \frac{\left(xP_n - \sum_{i=0}^{k-1} a_{n,i}P_{n-r+i}, P_{[(n-r+k)/r]} \right)_{\nu+k+1}}{(P_{n-r+k}, P_{[(n-r+k)/r]})_{\nu+k+1}}, \quad k = 1, 2, \dots, r.$$

Here, we put $n = \ell r + \nu$, where $\ell = [n/r]$ and $\nu \in \{0, 1, \dots, r-1\}$ ($[t]$ is integer part of t).

We use alternatively recurrence relation and given formulas for coefficients. Knowing P_0 we compute $a_{0,r}$, then knowing $a_{0,r}$ we compute P_1 , and then again $a_{1,r}$ and $a_{1,r-1}$, etc. Continuing in this manner, we can generate as many polynomials, and therefore as many of the recurrence coefficients as are desired.

All of the necessary inner products can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulas with respect to the corresponding weight function w_i , $i = 1, 2, \dots, r$.

4. Quadrature formulae of Gaussian type

For the problem which has been examined by Borges [4], it is shown that it is not efficient to use a set of r Gauss-Christoffel quadrature formulas because valuable information is wasted.

Borges has introduced a performance ratio, defined as:

$$R = \frac{\text{Overall degree of precision} + 1}{\text{Number of integrand evaluation}}.$$

If we use the set of r Gauss-Christoffel quadrature formulas we have $R = 2/r$ and hence $R < 1$ for all $r > 2$. If we select a set of n distinct nodes, common for all quadrature formulas, weight coefficients for each of r quadrature formulas can be chosen in that way that a performance ratio is $R = 1$. Because the selection of nodes is arbitrary, the quadrature formulas may not be the best possible.

The aim is to find an optimal set of nodes, by mimicking the development of the Gauss-Christoffel quadrature formulas.

Denote with $W = \{w_1, w_2, \dots, w_r\}$ an AT system.

We introduce the following definition:

Definition 4.1. Let W be an AT system (the weight functions w_i , $i = 1, \dots, r$ are supported on the interval E), $\vec{n} = (n_1, n_2, \dots, n_r)$ be a multi-index, and $n = |\vec{n}|$. Set of quadrature formulas of the form:

$$(4.1) \quad \int_E f(x) w_m(x) dx \approx \sum_{i=1}^n A_{m,i} f(x_i), \quad m = 1, 2, \dots, r$$

will be called an optimal set with respect to (W, \vec{n}) if and only if the weight coefficients, $A_{m,i}$, and the nodes, x_i , satisfy the following equations:

$$(4.2) \quad \begin{aligned} \sum_{i=1}^n A_{m,i} &= \int_E w_m(x) dx \\ \sum_{i=1}^n A_{m,i} x_i &= \int_E x w_m(x) dx \\ &\vdots \\ \sum_{i=1}^n A_{m,i} x_i^{n+n_m-1} &= \int_E x^{n+n_m-1} w_m(x) dx \end{aligned}$$

for $m = 1, 2, \dots, r$.

The next generalization of fundamental theorem of Gauss-Christoffel quadrature formulas holds (for proof see [13]).

Theorem 4.1. *Let W be an AT system, $\vec{n} = (n_1, n_2, \dots, n_r)$, $n = |\vec{n}|$. Consider the quadrature formulas:*

$$(4.3) \quad \int_E f(x) w_m(x) dx \approx \sum_{i=1}^n A_{m,i} f(x_i)$$

where $m = 1, 2, \dots, r$.

These formulas form an optimal set with respect to (W, \vec{n}) if and only if:

1° They are exact for all polynomials of degree $\leq n - 1$.

2° The polynomial $q(x) = \prod_{i=1}^n (x - x_i)$ is the type II multiple orthogonal polynomial $P_{\vec{n}}$ with respect to W .

Notice that all zeros of the type II multiple orthogonal polynomial $P_{\vec{n}}$ are distinct and located in the interval E (Theorem 1.2).

For $r = 1$ in Definition 4.1 we have the Gauss-Christoffel quadrature formulas.

For the case of the nearly diagonal multi-indices $\vec{s}(n)$ we can compute the nodes $x_i, i = 1, 2, \dots, n$, of the Gaussian type quadrature formulas as eigenvalues of the corresponding banded Hessenberg matrix H_n . Then from corresponding recurrence relation it follows that the eigenvector associated with x_i is given by $\mathbf{P}_n(x_i)$. We can use this fact to compute the weight coefficients $A_{m,i}$ by requiring that each rule correctly generate the first n modified moments.

Denote by

$$V_n = [\mathbf{P}_n(x_1) \ \mathbf{P}_n(x_2) \ \dots \ \mathbf{P}_n(x_n)]$$

the matrix of the eigenvectors of H_n , each normalized so that the first component is equal to 1. Then, the weight coefficients $A_{m,i}$ can be found by solving systems of linear equations

$$V_n \cdot \begin{bmatrix} A_{m,1} \\ A_{m,2} \\ \vdots \\ A_{m,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{*(m)} \\ \mu_1^{*(m)} \\ \vdots \\ \mu_{n-1}^{*(m)} \end{bmatrix}, \quad m = 1, 2, \dots, r,$$

where

$$\mu_i^{*(m)} = \int_E P_i(x) w_m(x) dx, \quad m = 1, 2, \dots, r, \quad i = 0, 1, \dots, n - 1,$$

are modified moments, $P_i = P_{s(i)}$.

All of modified moments can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulas with respect to the corresponding weight function w_m , $m = 1, 2, \dots, r$.

5. Multiple orthogonal polynomials on the semicircle

Polynomials orthogonal on the semicircle were introduced by Gautschi and Milovanović [7].

Let w be a weight function which is positive and integrable on the open interval $(-1, 1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc

$$D_+ = \{z \in \mathbb{C} : |z| < 1, \text{Im } z > 0\}.$$

Consider the following inner product

$$(5.1) \quad [f, g] = \int_{\Gamma} f(z)g(z)w(z)(iz)^{-1} dz = \int_0^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta}) d\theta,$$

where Γ is the circular part of ∂D_+ and all integrals are assumed to exist (possibly) as appropriately defined improper integrals.

This inner product (5.1) is not Hermitian and the existence of the corresponding orthogonal polynomials, therefore, is not guaranteed.

We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$(5.2) \quad [\pi_k, \pi_l] \begin{cases} = 0 & \text{if } k \neq l, \\ \neq 0 & \text{if } k = l, \end{cases} \quad k, l = 0, 1, 2, \dots;$$

we assume π_k monic of degree k .

Gautschi, Landau and Milovanović [6] have established the existence of orthogonal polynomials $\{\pi_k\}$ assuming only that

$$(5.3) \quad \text{Re} [1, 1] = \text{Re} \int_0^{\pi} w(e^{i\theta}) d\theta \neq 0.$$

They have represented π_n as a linear (complex) combination of p_n and p_{n-1} ($\{p_k\}$ is corresponding ordinary orthogonal polynomials sequence (real) with respect to the same weight function w):

$$\pi_n(z) = p_n(z) - i\theta_{n-1}p_{n-1}(z), \quad n = 0, 1, 2, \dots$$

Polynomials orthogonal on the semicircle also satisfy the tree-term recurrence relation:

$$\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k\pi_{k-1}(z), \quad k = 0, 1, 2, \dots,$$

with initial conditions $\pi_{-1}(z) = 0, \pi_0(z) = 1$.

Under certain conditions zeros of polynomials orthogonal on the semicircle are in D_+ (see [7], [6], [11], [12]).

Multiple orthogonal polynomials on the semicircle are a generalization of orthogonal polynomials on the semicircle in the sense that they satisfy $r \in \mathbb{N}$ orthogonality conditions.

Let $r \geq 1$ be an integer and let w_1, w_2, \dots, w_r be r admissible weight functions. Let $\vec{n} = (n_1, n_2, \dots, n_r)$ be the multi-index with length $|\vec{n}| = n_1 + n_2 + \dots + n_r$. Multiple orthogonal polynomial on the semicircle is monic polynomial $\Pi_{\vec{n}}(z)$ of degree $|\vec{n}|$ such that satisfies the following orthogonality conditions:

$$(5.4) \quad \int_{\Gamma} \Pi_{\vec{n}}(z) z^k w_1(z)(iz)^{-1} dz = 0, \quad k = 0, 1, \dots, n_1 - 1,$$

$$(5.5) \quad \int_{\Gamma} \Pi_{\vec{n}}(z) z^k w_2(z)(iz)^{-1} dz = 0, \quad k = 0, 1, \dots, n_2 - 1,$$

⋮

$$(5.6) \quad \int_{\Gamma} \Pi_{\vec{n}}(z) z^k w_r(z)(iz)^{-1} dz = 0, \quad k = 0, 1, \dots, n_r - 1.$$

For $r = 1$ we have the ordinary orthogonal polynomials on the semicircle.

For any polynomial g the following equations hold

$$(5.7) \quad 0 = \int_{\Gamma} g(z)w_m(z) dz + \int_{-1}^1 g(x)w_m(x) dx$$

and

$$(5.8) \quad \int_{\Gamma} \frac{g(z)w_m(z)}{iz} dz = \pi g(0)w_m(0) + i \int_{-1}^1 \frac{g(x)w_m(x)}{x} dx$$

for $m = 1, 2, \dots, r$.

We consider only the nearly diagonal multi-indices.

Corresponding type II multiple orthogonal polynomials (real) $\{P_n\}$ satisfy recurrence relation (2.1). It is easy to see that for $m = 1, 2, \dots, r$ associated polynomials of second kind

$$(5.9) \quad Q_n^{(m)}(z) = \int_{-1}^1 \frac{P_n(z) - P_n(x)}{z - x} w_m(x) dx, \quad n = 0, 1, \dots,$$

satisfy the same recurrence relation (but with different initial conditions).

Denote zero moments with $\mu_0^{(m)}$, i.e.

$$(5.10) \quad \mu_0^{(m)} = \int_{\Gamma} \frac{w_m(z)}{iz} dz = \pi w_m(0) + i \int_{-1}^1 \frac{w_m(x)}{x} dx, \quad m = 1, 2, \dots, r.$$

Let

$$(5.11) \quad D_n = \begin{bmatrix} Q_{n-1}^{(1)}(0) - i\mu_0^{(1)}P_{n-1}(0) & \cdots & Q_{n-r}^{(1)}(0) - i\mu_0^{(1)}P_{n-r}(0) \\ Q_{n-1}^{(2)}(0) - i\mu_0^{(2)}P_{n-1}(0) & \cdots & Q_{n-r}^{(2)}(0) - i\mu_0^{(2)}P_{n-r}(0) \\ \vdots & & \vdots \\ Q_{n-1}^{(r)}(0) - i\mu_0^{(r)}P_{n-1}(0) & \cdots & Q_{n-r}^{(r)}(0) - i\mu_0^{(r)}P_{n-r}(0) \end{bmatrix}.$$

In similar way as in [6, Theorem 2.1], using equations (5.7), (5.8) for appropriately chosen polynomials g and orthogonality conditions (5.4)–(5.6), we can prove existence and uniqueness of multiple orthogonal polynomials on the semicircle with additional conditions that all matrices D_n are regular.

We represent polynomial Π_n as

$$(5.12) \quad \Pi_n(z) = P_n(z) + \theta_{n,1}P_{n-1}(z) + \theta_{n,2}P_{n-2}(z) + \cdots + \theta_{n,r}P_{n-r}(z).$$

Coefficients $\theta_{n,j}$, $j = 1, 2, \dots, r$ are solution of a system of linear equations

$$\sum_{j=1}^r \theta_{n,j} \left(Q_{n-j}^{(m)}(0) - i\mu_0^{(m)}P_{n-j}(0) \right) = i\mu_0^{(m)}P_n(0) - Q_n^{(m)}(0), \quad m = 1, 2, \dots, r.$$

Under condition that all matrices D_n in (5.11) are regular, the previous system has unique solution for all n .

Denote

$$(5.13) \quad [f, g]_j = \int_{\Gamma} f(z)g(z)w_j(z)(iz)^{-1} dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w_j(e^{i\theta}) d\theta.$$

We put also $[f, g]_{j+\ell r} = [f, g]_j$ for each $\ell \in \mathbb{Z}$.

In a similar way as in real case, we can prove that the multiple orthogonal polynomials on the semicircle satisfy the following recurrence relation of order $r + 1$:

$$(5.14) \quad z\Pi_m(z) = \Pi_{m+1}(z) + \sum_{i=0}^r \alpha_{m,r-i}\Pi_{m-i}(z), \quad m \geq 1,$$

with initial conditions $\Pi_0(z) = 1$, and $\Pi_{-1}(z) = \Pi_{-2}(z) = \cdots = \Pi_{-r}(z) = 0$.

Also, we can obtain the recursion coefficients and the multiple orthogonal polynomials on the semicircle using some kind of the discretized Stieltjes-Gautschi procedure.

Similarly as in the real case we can prove the following result:

Theorem 5.1. *Multiple orthogonal polynomials on the semicircle $\{\Pi_n\}$, with nearly diagonal multi-index, satisfy the recurrence relation*

$$(5.15) \quad \Pi_{n+1}(z) = (z - \alpha_{n,r})\Pi_n(z) - \sum_{k=0}^{r-1} \alpha_{n,k}\Pi_{n-r+k}(z), \quad n \geq 1,$$

where

$$(5.16) \quad \alpha_{n,0} = \frac{[z\Pi_n, \Pi_{[(n-r)/r}]_{\nu+1}}{[\Pi_{n-r}, \Pi_{[(n-r)/r}]_{\nu+1}}$$

and

$$(5.17) \quad \alpha_{n,k} = \frac{[z\Pi_n - \sum_{i=0}^{k-1} \alpha_{n,i}\Pi_{n-r+i}, \Pi_{[(n-r+k)/r}]_{\nu+k+1}}{[\Pi_{n-r+k}, \Pi_{[(n-r+k)/r}]_{\nu+k+1}}$$

for $k = 1, 2, \dots, r$.

Here, we put $n = \ell r + \nu$, where $\ell = [n/r]$ and $\nu \in \{0, 1, \dots, r - 1\}$ ($[t]$ is integer part of t).

We have to calculate all of the inner products (5.16)–(5.17), i.e., we have to calculate the integrals of the following type $\int_{\Gamma} \frac{z^j \Pi_l(z) w_k(z) dz}{iz}$.

For $j \geq 1$, because of (5.7), we can calculate these integrals exactly, except for rounding errors, by using the corresponding Gaussian quadratures.

For $j = 0$ we have

$$\int_{\Gamma} \frac{\Pi_l(z) w_k(z) dz}{iz} = \mu_0^{(k)} \Pi_l(0) + i \int_{-1}^1 \frac{\Pi_l(x) - \Pi_l(0)}{x} w_k(x) dx,$$

and we use the corresponding Gaussian quadratures and (5.10).

Knowing the recurrence coefficients we form a complex lower banded Hessenberg matrix H_n as in real case. The zeros of the multiple orthogonal polynomials on the semicircle are the eigenvalues of the complex Hessenberg matrix H_n .

Also, we can generate the corresponding quadrature formulae of Gaussian type:

$$\int_0^{\pi} f(e^{i\theta}) w_k(e^{i\theta}) d\theta \approx \sum_{\nu=1}^n \sigma_{k,\nu} f(\zeta_{\nu}), \quad k = 1, 2, \dots, r,$$

where for each $w_k, k = 1, 2, \dots, r$, the corresponding quadrature is exact for all polynomials of degree $\leq n + n_k - 1$.

The nodes of such optimal set of quadratures are zeros of multiple orthogonal polynomial on the semicircle, i.e. eigenvalues of the Hessenberg matrix H_n . Using the corresponding eigenvectors we obtain the weight coefficients in a similar way as in the real case.

6. Numerical examples

Numerical examples for real case can be found in [13]. Here we give an example of multiple Jacobi polynomials on the semicircle. We take the weight functions

$$w_m(z) = (1 - z)^\alpha(1 + z)^{\beta_m}, \quad m = 1, 2, \dots, r,$$

where $\alpha, \beta_m > -1, m = 1, 2, \dots, r$, and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$ in order to have an AT system.

Numerical results suggest that the zeros of these multiple Jacobi polynomials on the semicircle for $n \geq 2$ are simple and always contained in the upper unit half disc D_+ .

In the next table the nodes (ζ_ν) and weights ($\sigma_{j,\nu}, j = 1, \dots, r$) for quadrature formulas of Gaussian type are given. (We give parameters with only seven digits in order to save space.)

The case $n = 8, 10, r = 2, \alpha = 1, \beta_1 = 1/2, \beta_2 = 1/4$

ν	ζ_ν	$\sigma_{1,\nu}$	$\sigma_{2,\nu}$
1	-0.9788997+0.0026247 <i>i</i>	0.0580377 - 0.3542335 <i>i</i>	-0.0188189 - 0.3832973 <i>i</i>
2	-0.8760521+0.0159867 <i>i</i>	-0.3374499 + 0.9347142 <i>i</i>	-0.2794998 + 0.9968886 <i>i</i>
3	-0.6630103+0.0461827 <i>i</i>	0.0739024 - 1.5920321 <i>i</i>	-0.2597682 - 1.7364585 <i>i</i>
4	-0.3560514+0.0994357 <i>i</i>	-2.4651138 + 1.8318576 <i>i</i>	-2.2947968 + 2.1521091 <i>i</i>
5	-0.0384815+0.1615833 <i>i</i>	4.0225993 + 0.2344999 <i>i</i>	4.4344593 - 0.3611610 <i>i</i>
6	0.2730107+0.1200963 <i>i</i>	1.7070884 - 1.8265961 <i>i</i>	1.5093451 - 2.0000640 <i>i</i>
7	0.6153346+0.0567013 <i>i</i>	0.0142094 - 0.0333422 <i>i</i>	-0.0129471 + 0.0126075 <i>i</i>
8	0.8781496+0.0170913 <i>i</i>	0.0683192 - 0.0148059 <i>i</i>	0.0636190 - 0.0182020 <i>i</i>
1	-0.9880607+0.0012045 <i>i</i>	-0.2444479 + 0.3791317 <i>i</i>	-0.2090496 + 0.4295410 <i>i</i>
2	-0.9278047+0.0074739 <i>i</i>	0.5579302 - 0.9679792 <i>i</i>	0.3858939 - 1.1059521 <i>i</i>
3	0.9180556+0.0092898 <i>i</i>	0.0053745 - 0.0079086 <i>i</i>	0.0018335 - 0.0057780 <i>i</i>
4	-0.7967483+0.0218339 <i>i</i>	-1.0054363 + 1.2162234 <i>i</i>	-0.9308388 + 1.3867825 <i>i</i>
5	0.7356568+0.0308872 <i>i</i>	0.1766943 - 0.0092201 <i>i</i>	0.1691405 - 0.0162823 <i>i</i>
6	-0.5916927+0.0466799 <i>i</i>	0.7261829 - 1.5490023 <i>i</i>	0.4441955 - 1.7913086 <i>i</i>
7	0.4779421+0.0651869 <i>i</i>	-0.0093143 - 0.1648483 <i>i</i>	-0.0616815 - 0.1002179 <i>i</i>
8	-0.3288006+0.0861789 <i>i</i>	-3.0832225 + 1.3083385 <i>i</i>	-3.0114444 + 1.7142267 <i>i</i>
9	0.1891105+0.1151987 <i>i</i>	2.3710585 - 2.1396074 <i>i</i>	2.1832689 - 2.4192835 <i>i</i>
10	-0.0616346+0.1363395 <i>i</i>	3.6467732 + 1.1149342 <i>i</i>	4.1702746 + 0.5706947 <i>i</i>

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