Explicit formulas for five-term recurrence coefficients of orthogonal trigonometric polynomials of semi-integer degree ⋆

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Abstract

Orthogonal systems of trigonometric polynomials of semi-integer degree with respect to a weight function $w(x)$ on $[0, 2\pi)$ have been considered firstly by Turetzkii in [Uchenye Zapiski, Vypusk 1(149) (1959), 31–55, (translation in English in East J. Approx. 11 (2005) 337–359)]. It is proved that such orthogonal trigonometric polynomials of semi-integer degree satisfy five-term recurrence relation. In this paper we present explicit formulas for five-term recurrence coefficients for some weight functions.

Key words: Trigonometric polynomials of semi-integer degree; Orthogonality; Recurrence relation.
AMS classification: 42A05.

⋆ The authors were supported in part by the Serbian Ministry of Science and Environmental Protection (Project: Orthogonal Systems and Applications, grant number #144004G) and the Swiss National Science Foundation (SCOPES Joint Research Project No. IB7320–111079 "New Methods for Quadrature")
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1 Introduction

Let the weight function $w(x)$ be integrable and nonnegative on the interval $[0, 2\pi)$, vanishing there only on a set of a measure zero. For given weight function $w(x)$,

$$(f, g) = \int_0^{2\pi} f(x)g(x)w(x)\,\mathrm{d}x, \tag{1}$$

denotes the corresponding inner product of the functions f and g.

The trigonometric functions of the following form

$$A_{n+1/2}(x) = \sum_{\nu=0}^{n} \left(c_\nu \cos\left(\nu + \frac{1}{2}\right) x + d_\nu \sin\left(\nu + \frac{1}{2}\right) x \right), \tag{2}$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_n| + |d_n| \neq 0$, are called *trigonometric polynomials of semi-integer degree* $n + 1/2$. Coefficients c_n and d_n are *leading coefficients*.

For any positive integer n, with $\mathcal{T}_n^{1/2}$ we denote the set of all trigonometric polynomials of semi-integer degree at most $n + 1/2$, i.e., linear span of the set $\Big\{\cos(\nu + 1/2)x,\ \sin(\nu + 1/2)x,\ \nu = 0, 1, \ldots, n\Big\}$.

The trigonometric polynomial of semi-integer degree $n+1/2$, of the form (2), is called *orthogonal trigonometric polynomial of semi-integer degree* $n+1/2$ with respect to the weight function $w(x)$ if it is orthogonal on $[0, 2\pi)$ with respect to the inner product (1) to every trigonometric polynomial of semi-integer degree from $\mathcal{T}_{n-1}^{1/2}$, i.e., to every trigonometric polynomial of semi-integer degree less than or equal to $n - 1/2$. These trigonometric systems have applications in construction of quadrature formulas with maximal trigonometric degree of exactness. It is known that orthogonal trigonometric polynomial of semi-integer degree $A_{n+1/2}$ with given leading coefficients c_n and d_n, is uniquely determined (see [1, §3]).

We consider the following two choices of leading coefficients. For the choice $c_n = 1$, $d_n = 0$, we denote orthogonal trigonometric polynomial of semi-integer degree by $A^C_{n+1/2}$, and for the choice $c_n = 0$ and $d_n = 1$ by $A^S_{n+1/2}$. For the expanded forms of $A^C_{n+1/2}$ and $A^S_{n+1/2}$ we use the following notation

$$A^C_{n+1/2}(x) = \cos\Big(n + \frac{1}{2}\Big)x + \sum_{\nu=0}^{n-1}\Big(c_\nu^{(n)} \cos\Big(\nu + \frac{1}{2}\Big)x + d_\nu^{(n)} \sin\Big(\nu + \frac{1}{2}\Big)x\Big), \tag{3}$$

$$A^S_{n+1/2}(x) = \sin\Big(n + \frac{1}{2}\Big)x + \sum_{\nu=0}^{n-1}\Big(f_\nu^{(n)} \cos\Big(\nu + \frac{1}{2}\Big)x + g_\nu^{(n)} \sin\Big(\nu + \frac{1}{2}\Big)x\Big). \tag{4}$$

In [2] we proved that orthogonal trigonometric polynomials of semi-integer degree $A^C_{k+1/2}(x)$ and $A^S_{k+1/2}(x)$, $k \in \mathbb{N}$, satisfy the following five-term recurrence relations:

$$\begin{aligned} A^C_{k+1/2}(x) &= (2\cos x - \alpha_k^{(1)})A^C_{k-1/2}(x) - \beta_k^{(1)} A^S_{k-1/2}(x) \\ &\qquad -\alpha_k^{(2)} A^C_{k-3/2}(x) - \beta_k^{(2)} A^S_{k-3/2}(x), \end{aligned} \tag{5}$$

and

$$\begin{aligned} A^S_{k+1/2}(x) &= (2\cos x - \delta_k^{(1)})A^S_{k-1/2}(x) - \gamma_k^{(1)} A^C_{k-1/2}(x) \\ &\qquad -\gamma_k^{(2)} A^C_{k-3/2}(x) - \delta_k^{(2)} A^S_{k-3/2}(x), \end{aligned} \tag{6}$$

where recurrence coefficients are given by $\alpha_1^{(2)} = \beta_1^{(2)} = \gamma_1^{(2)} = \delta_1^{(2)} = 0$, and

$$\begin{aligned} \alpha_k^{(1)} &= \frac{I^S_{k-1}J^C_{k-1} - I_{k-1}J_{k-1}}{D_{k-1}}, \quad & \alpha_k^{(2)} &= \frac{I^C_{k-1}I^S_{k-2} - I_{k-1}I_{k-2}}{D_{k-2}}, \\ \beta_k^{(1)} &= \frac{I^C_{k-1}J_{k-1} - I_{k-1}J^C_{k-1}}{D_{k-1}}, \quad & \beta_k^{(2)} &= \frac{I_{k-1}I^C_{k-2} - I^C_{k-1}I_{k-2}}{D_{k-2}}, \\ \gamma_k^{(1)} &= \frac{I^S_{k-1}J_{k-1} - I_{k-1}J^S_{k-1}}{D_{k-1}}, \quad & \gamma_k^{(2)} &= \frac{I_{k-1}I^S_{k-2} - I^S_{k-1}I_{k-2}}{D_{k-2}}, \\ \delta_k^{(1)} &= \frac{I^C_{k-1}J^S_{k-1} - I_{k-1}J_{k-1}}{D_{k-1}}, \quad & \delta_k^{(2)} &= \frac{I^S_{k-1}I^C_{k-2} - I_{k-1}I_{k-2}}{D_{k-2}}, \end{aligned} \tag{7}$$

where $D_{k-j} = I^C_{k-j}I^S_{k-j} - I^2_{k-j}$, $j = 1, 2$, and

$$\begin{aligned} I^C_\nu &= (A^C_{\nu+1/2}, A^C_{\nu+1/2}), \quad & J^C_\nu &= (2\cos x A^C_{\nu+1/2}, A^C_{\nu+1/2}), \\ I^S_\nu &= (A^S_{\nu+1/2}, A^S_{\nu+1/2}), \quad & J^S_\nu &= (2\cos x A^S_{\nu+1/2}, A^S_{\nu+1/2}), \\ I_\nu &= (A^C_{\nu+1/2}, A^S_{\nu+1/2}), \quad & J_\nu &= (2\cos x A^C_{\nu+1/2}, A^S_{\nu+1/2}). \end{aligned} \tag{8}$$

Knowing recurrence coefficients in five-term recurrence relations (5) and (6) we can obtain coefficients of expanded forms (3) and (4) for $A^C_{n+1/2}$ and $A^S_{n+1/2}$.

Theorem 1 *Coefficients* $c_\nu^{(k)}, d_\nu^{(k)}, f_\nu^{(k)}, g_\nu^{(k)}$, $k \in \mathbb{N}$, $\nu = 0, 1, \ldots, k-1$, *of the representations* (3) *and* (4) *can be computed by the following formulas:*

$$\begin{aligned} & c_0^{(1)} = 1 - \alpha_1^{(1)}, \quad d_0^{(1)} = -\beta_1^{(1)}, \quad f_0^{(1)} = -\gamma_1^{(1)}, \quad g_0^{(1)} = -1 - \delta_1^{(1)}; \\ & c_0^{(2)} = 1 + c_0^{(1)} - \alpha_2^{(1)}c_0^{(1)} - \beta_2^{(1)}f_0^{(1)} - \alpha_2^{(2)}, \\ & d_0^{(2)} = -d_0^{(1)} - \alpha_2^{(1)}d_0^{(1)} - \beta_2^{(1)}g_0^{(1)} - \beta_2^{(2)}, \end{aligned}$$

$$
\begin{aligned}
f_0^{(2)} &= f_0^{(1)} - \delta_2^{(1)} f_0^{(1)} - \gamma_2^{(1)} c_0^{(1)} - \gamma_2^{(2)}, \\
g_0^{(2)} &= 1 - g_0^{(1)} - \delta_2^{(1)} g_0^{(1)} - \gamma_2^{(1)} d_0^{(1)} - \delta_2^{(2)};
\end{aligned}
$$

for $k \geq 2$*:*

$$
\begin{aligned}
c_{k-1}^{(k)} &= c_{k-2}^{(k-1)} - \alpha_k^{(1)}, \quad d_{k-1}^{(k)} = d_{k-2}^{(k-1)} - \beta_k^{(1)}, \\
f_{k-1}^{(k)} &= f_{k-2}^{(k-1)} - \gamma_k^{(1)}, \quad g_{k-1}^{(k)} = g_{k-2}^{(k-1)} - \delta_k^{(1)};
\end{aligned}
$$

for $k \geq 3$*:*

$$
\begin{aligned}
c_0^{(k)} &= c_0^{(k-1)} + c_1^{(k-1)} - \alpha_k^{(1)} c_0^{(k-1)} - \beta_k^{(1)} f_0^{(k-1)} - \alpha_k^{(2)} c_0^{(k-2)} - \beta_k^{(2)} f_0^{(k-2)}, \\
c_{k-2}^{(k)} &= 1 + c_{k-3}^{(k-1)} - \alpha_k^{(1)} c_{k-2}^{(k-1)} - \beta_k^{(1)} f_{k-2}^{(k-1)} - \alpha_k^{(2)}, \\
d_0^{(k)} &= -d_0^{(k-1)} + d_1^{(k-1)} - \alpha_k^{(1)} d_0^{(k-1)} - \beta_k^{(1)} g_0^{(k-1)} - \alpha_k^{(2)} d_0^{(k-2)} - \beta_k^{(2)} g_0^{(k-2)}, \\
d_{k-2}^{(k)} &= d_{k-3}^{(k-1)} - \alpha_k^{(1)} d_{k-2}^{(k-1)} - \beta_k^{(1)} g_{k-2}^{(k-1)} - \beta_k^{(2)}, \\
f_0^{(k)} &= f_0^{(k-1)} + f_1^{(k-1)} - \gamma_k^{(1)} c_0^{(k-1)} - \delta_k^{(1)} f_0^{(k-1)} - \gamma_k^{(2)} c_0^{(k-2)} - \delta_k^{(2)} f_0^{(k-2)}, \\
f_{k-2}^{(k)} &= f_{k-3}^{(k-1)} - \gamma_k^{(1)} c_{k-2}^{(k-1)} - \delta_k^{(1)} f_{k-2}^{(k-1)} - \gamma_k^{(2)}, \\
g_0^{(k)} &= -g_0^{(k-1)} + g_1^{(k-1)} - \gamma_k^{(1)} d_0^{(k-1)} - \delta_k^{(1)} g_0^{(k-1)} - \gamma_k^{(2)} d_0^{(k-2)} - \delta_k^{(2)} g_0^{(k-2)}, \\
g_{k-2}^{(k)} &= 1 + g_{k-3}^{(k-1)} - \gamma_k^{(1)} d_{k-2}^{(k-1)} - \delta_k^{(1)} g_{k-2}^{(k-1)} - \delta_k^{(2)};
\end{aligned}
$$

and for $k \geq 4$*, for* $\nu = 1, \ldots, k-3$*:*

$$
\begin{aligned}
c_\nu^{(k)} &= c_{\nu-1}^{(k-1)} + c_{\nu+1}^{(k-1)} - \alpha_k^{(1)} c_\nu^{(k-1)} - \beta_k^{(1)} f_\nu^{(k-1)} - \alpha_k^{(2)} c_\nu^{(k-2)} - \beta_k^{(2)} f_\nu^{(k-2)}, \\
d_\nu^{(k)} &= d_{\nu-1}^{(k-1)} + d_{\nu+1}^{(k-1)} - \alpha_k^{(1)} d_\nu^{(k-1)} - \beta_k^{(1)} g_\nu^{(k-1)} - \alpha_k^{(2)} d_\nu^{(k-2)} - \beta_k^{(2)} g_\nu^{(k-2)}, \\
f_\nu^{(k)} &= f_{\nu-1}^{(k-1)} + f_{\nu+1}^{(k-1)} - \gamma_k^{(1)} c_\nu^{(k-1)} - \delta_k^{(1)} f_\nu^{(k-1)} - \gamma_k^{(2)} c_\nu^{(k-2)} - \delta_k^{(2)} f_\nu^{(k-2)}, \\
g_\nu^{(k)} &= g_{\nu-1}^{(k-1)} + g_{\nu+1}^{(k-1)} - \gamma_k^{(1)} d_\nu^{(k-1)} - \delta_k^{(1)} g_\nu^{(k-1)} - \gamma_k^{(2)} d_\nu^{(k-2)} - \delta_k^{(2)} g_\nu^{(k-2)}.
\end{aligned}
$$

Proof. Substituting $A_{\nu+1/2}^C(x)$ and $A_{\nu+1/2}^S(x)$, $\nu = k-2, k-1, k$, given by (3) and (4) in recurrence relations (5) and (6) and comparing coefficients multiplying $\cos(\nu+1/2)x$ and $\sin(\nu+1/2)x$, $\nu = 0, 1, \ldots, k$, on the left and on the right hand sides of obtained equalities, we get what is stated. □

Also, in [2], numerical method for construction of the corresponding quadratures with maximal trigonometric degree of exactness was presented. This method is based on the five-term recurrence relations for orthogonal trigonometric polynomials $A_{n+1/2}^C$ and $A_{n+1/2}^S$. In fact, the main problem in procedure presented in [2] is calculation of five-term recurrence coefficients.

In this paper, in Section 2, for some special weight functions explicit formulas for five-term recurrence coefficients as well as explicit formulas for coefficients of expanded forms (3) and (4) are presented.

In [2] is proved that the case of an symmetric weight function on $(0, 2\pi)$, i.e., case when $w(x) = w(2\pi - x)$, $x \in (0, 2\pi)$, reduces to algebraic polynomials, and five-term recurrence relations reduce to the three-term recurrence relations (see [2, Section 3]). So, in this paper we will not consider such weights.

2 Explicit formulas

For some special weight functions $w(x)$ we can find explicit formulas for recurrence coefficients $\alpha_k^{(j)}$, $\beta_k^{(j)}$, $\gamma_k^{(j)}$, $\delta_k^{(j)}$, $j = 1, 2$, $k \in \mathbb{N}$, as well as for integrals I_k^C, I_k^S, I_k, J_k^C, J_k^S, J_k, $k \in \mathbb{N}_0$, and for coefficients $c_\nu^{(k)}$, $d_\nu^{(k)}$, $f_\nu^{(k)}$, $g_\nu^{(k)}$, $\nu = 0, 1, \ldots, k$, $k \in \mathbb{N}$ ($c_k^{(k)} = g_k^{(k)} = 1$, $d_k^{(k)} = f_k^{(k)} = 0$).

First, we consider weight functions $w_m(x) = 1 + \sin mx$, $m \in \mathbb{N}$. In the sequel we need the following integrals ($\delta_{\nu,\mu}$ is Kronecker delta function, $k, \ell \in \mathbb{N}_0$):

$$
\begin{aligned}
&\int_0^{2\pi} \cos(k+1/2)x \cos(\ell+1/2)x w_m(x)\,\mathrm{d}x = \pi\delta_{k,\ell}, \qquad (9)\\
&\int_0^{2\pi} \sin(k+1/2)x \sin(\ell+1/2)x w_m(x)\,\mathrm{d}x = \pi\delta_{k,\ell},\\
&\int_0^{2\pi} \cos(k+1/2)x \sin(\ell+1/2)x w_m(x)\,\mathrm{d}x\\
&\qquad = \frac{\pi}{2}(\delta_{k,\ell-m} + \delta_{k,m-\ell-1} - \delta_{k,\ell+m}),\ k \ge 1,\\
&\int_0^{2\pi} \cos(x/2) \sin(\ell+1/2)x w_m(x)\,\mathrm{d}x = \frac{\pi}{2}(\delta_{1,m-\ell} + \delta_{0,\ell-m}),\\
&\int_0^{2\pi} \cos x \cos(k+1/2)x \cos(\ell+1/2)x w_m(x)\,\mathrm{d}x = \frac{\pi}{2}\delta_{k,\ell\pm1},\ k \ge 1,\\
&\int_0^{2\pi} \cos x \cos(x/2) \cos(\ell+1/2)x w_m(x)\,\mathrm{d}x = \frac{\pi}{2}\delta_{0,\ell} + \frac{\pi}{2}\delta_{1,\ell},\\
&\int_0^{2\pi} \cos x \sin(k+1/2)x \sin(\ell+1/2)x w_m(x)\,\mathrm{d}x = \frac{\pi}{2}\delta_{k,\ell\pm1},\ k \ge 1,\\
&\int_0^{2\pi} \cos x \sin(x/2) \sin(\ell+1/2)x w_m(x)\,\mathrm{d}x = -\frac{\pi}{2}\delta_{0,\ell} + \frac{\pi}{2}\delta_{1,\ell},\\
&\int_0^{2\pi} \cos x \cos(k+1/2)x \sin(\ell+1/2)x w_1(x)\,\mathrm{d}x = \frac{\pi}{4}(\delta_{k,1-\ell} + \delta_{k,\ell-2} - \delta_{k,\ell+2}),\\
&\int_0^{2\pi} \cos x \cos(k+1/2)x \sin(\ell+1/2)x w_m(x)\,\mathrm{d}x\\
&\qquad = \frac{\pi}{4}(\delta_{k,m-\ell} + \delta_{k,m-\ell-2} + \delta_{k,\ell-m\pm1} - \delta_{k,\ell+m\pm1}), \quad m > 1.
\end{aligned}
$$

Knowing these integrals it is easy to see that cases $m=1$, an odd $m>1$, and an even m must be separately considered.

Theorem 2 *For the weight function $w_1(x)=1+\sin x$ we have the following explicit formulas for coefficients in recurrence relations* (5) *and* (6) $(k\in\mathbb{N})$*:*

$$\begin{aligned} &\alpha_k^{(1)}=-\delta_k^{(1)}=(-1)^{k+1}\frac{4k}{(2k-1)(2k+1)},\quad \alpha_k^{(2)}=\delta_k^{(2)}=1\quad (k>1), \qquad (10)\\ &\beta_k^{(1)}=-\gamma_k^{(1)}=\frac{-2}{(2k-1)(2k+1)},\quad \beta_k^{(2)}=\gamma_k^{(2)}=(-1)^{k+1}\frac{2}{2k-1}\quad (k>1),\\ &\alpha_1^{(2)}=\beta_1^{(2)}=\gamma_1^{(2)}=\delta_1^{(2)}=0. \end{aligned}$$

Proof. In order to prove this theorem, we will prove the following explicit formulas for integrals $(\nu\in\mathbb{N}_0)$:

$$I_\nu^C=I_\nu^S=\frac{(\nu+1)\pi}{2\nu+1},\ J_\nu^C=-J_\nu^S=\frac{(-1)^\nu\pi}{2\nu+1},\ I_\nu=\frac{(-1)^\nu\pi}{2(2\nu+1)},\ J_\nu=0; \qquad (11)$$

and, also, explicit formulas for coefficients of representations (3) and (4):

for odd k

$$\begin{aligned} &c_{2i}^{(k)}=-g_{2i}^{(k)}=(-1)^{[k/2]+1+i}\frac{k-2i}{2k+1},\quad i=0,1,\ldots,[k/2],\\ &c_{2i-1}^{(k)}=g_{2i-1}^{(k)}=(-1)^{[k/2]+1+i}\frac{k+2i}{2k+1},\quad i=1,\ldots,[k/2],\\ &d_{2i}^{(k)}=-f_{2i}^{(k)}=(-1)^{[k/2]+i}\frac{k+1+2i}{2k+1},\quad i=0,1,\ldots,[k/2],\\ &d_{2i-1}^{(k)}=f_{2i-1}^{(k)}=(-1)^{[k/2]+1+i}\frac{k+1-2i}{2k+1},\quad i=1,\ldots,[k/2], \end{aligned}$$

and for even k

$$\begin{aligned} &c_{2i}^{(k)}=g_{2i}^{(k)}=(-1)^{k/2+i}\frac{k+1+2i}{2k+1},\quad i=0,1,\ldots,k/2,\\ &c_{2i-1}^{(k)}=-g_{2i-1}^{(k)}=(-1)^{k/2+i}\frac{k+1-2i}{2k+1},\quad i=1,\ldots,k/2,\\ &d_{2i}^{(k)}=f_{2i}^{(k)}=(-1)^{k/2+1+i}\frac{k-2i}{2k+1},\quad i=0,1,\ldots,k/2,\\ &d_{2i-1}^{(k)}=-f_{2i-1}^{(k)}=(-1)^{k/2+i}\frac{k+2i}{2k+1},\quad i=1,\ldots,k/2. \end{aligned}$$

By direct calculations, using (7), (8), and formulas in Theorem 1, we verify formulas (11) for $\nu = 0$; formulas for recurrence coefficients $\alpha_1^{(1)}$, $\beta_1^{(1)}$, $\gamma_1^{(1)}$ and $\delta_1^{(1)}$; formulas for coefficients $c_0^{(1)}$, $d_0^{(1)}$, $f_0^{(1)}$ and $g_0^{(1)}$; and (11) for $\nu = 1$.

Suppose that given explicit formulas are exact for two successive nonnegative integers $k-2$ and $k-1$. Starting with formulas (11) for $\nu = k-2, k-1$, $k \geq 2$ by direct calculation (using formulas (7)) we obtain recurrence coefficients (10). We have

$$D_{k-1} = \left(\frac{k\pi}{2k-1}\right)^2 - \frac{\pi^2}{4(2k-1)^2} = \frac{\pi^2(2k+1)}{4(2k-1)}, \quad D_{k-2} = \frac{\pi^2(2k-1)}{4(2k-3)},$$

and

$$\begin{aligned}
\alpha_k^{(1)} &= \frac{I_{k-1}^S J_{k-1}^C - I_{k-1} J_{k-1}}{D_{k-1}} = \frac{I_{k-1}^S J_{k-1}^C}{D_{k-1}} \\
&= \frac{k\pi}{2k-1} \cdot \frac{(-1)^{k-1}\pi}{2k-1} \cdot \frac{4(2k-1)}{\pi^2(2k+1)} = \frac{(-1)^{k-1} 4k}{(2k-1)(2k+1)}, \\
\alpha_k^{(2)} &= \frac{I_{k-1}^C I_{k-2}^S - I_{k-1} I_{k-2}}{D_{k-2}} \\
&= \left(\frac{k\pi}{2k-1} \cdot \frac{(k-1)\pi}{2k-3} - \frac{(-1)^{k-1}\pi}{2(2k-1)} \cdot \frac{(-1)^{k-2}\pi}{2(2k-3)}\right) \frac{4(2k-3)}{\pi^2(2k-1)} = 1, \\
\beta_k^{(1)} &= \frac{I_{k-1}^C J_{k-1} - I_{k-1} J_{k-1}^C}{D_{k-1}} = -\frac{I_{k-1} J_{k-1}^C}{D_{k-1}} \\
&= -\frac{(-1)^{k-1}\pi}{2(2k-1)} \cdot \frac{(-1)^{k-1}\pi}{2k-1} \cdot \frac{4(2k-1)}{\pi^2(2k+1)} = \frac{-2}{(2k-1)(2k+1)}, \\
\beta_k^{(2)} &= \frac{I_{k-1} I_{k-2}^C - I_{k-1}^C I_{k-2}}{D_{k-2}} \\
&= \left(\frac{(-1)^{k-1}\pi}{2(2k-1)} \cdot \frac{(k-1)\pi}{2k-3} - \frac{k\pi}{2k-1} \cdot \frac{(-1)^{k-2}\pi}{2(2k-3)}\right) \cdot \frac{4(2k-3)}{\pi^2(2k-1)} \\
&= (-1)^{k-1} \left(\frac{k-1}{2k-1} - \frac{-k}{2k-1}\right) \frac{2}{2k-1} = (-1)^{k-1} \frac{2}{2k-1}, \\
\gamma_k^{(1)} &= \frac{I_{k-1}^S J_{k-1} - I_{k-1} J_{k-1}^S}{D_{k-1}} = -\frac{I_{k-1} J_{k-1}^S}{D_{k-1}} = -\beta_k^{(1)}, \\
\delta_k^{(1)} &= \frac{I_{k-1}^C J_{k-1}^S - I_{k-1} J_{k-1}}{D_{k-1}} = \frac{I_{k-1}^C J_{k-1}^S}{D_{k-1}} = -\alpha_k^{(1)}, \\
\gamma_k^{(2)} &= \frac{I_{k-1} I_{k-2}^S - I_{k-1}^S I_{k-2}}{D_{k-2}} = \beta_k^{(2)}, \quad \delta_k^{(2)} = \frac{I_{k-1}^S I_{k-2}^C - I_{k-1} I_{k-2}}{D_{k-2}} = \alpha_k^{(2)}.
\end{aligned}$$

Using (10), and formulas given in Theorem 1, we directly verify given formulas for $c_i^{(k)}$, $d_i^{(k)}$, $f_i^{(k)}$, $g_i^{(k)}$, $i = 0, 1, \ldots, k$, ($c_k^{(k)} = g_k^{(k)} = 1$ and $d_k^{(k)} = f_k^{(k)} = 0$). For example, if k is even, then $[(k-1)/2] = k/2 - 1$ and for $i = 1, \ldots, [(k-3)/2]$ we obtain

$$
\begin{aligned}
c_{2i-1}^{(k)} &= c_{2i-2}^{(k-1)} + c_{2i}^{(k-1)} - \alpha_k^{(1)} c_{2i-1}^{(k-1)} - \beta_k^{(1)} f_{2i-1}^{(k-1)} - \alpha_k^{(2)} c_{2i-1}^{(k-2)} - \beta_k^{(2)} f_{2i-1}^{(k-2)} \\
&= (-1)^{k/2+i-1} \frac{k-2i+1}{2k-1} + (-1)^{k/2+i} \frac{k-1-2i}{2k-1} \\
&\quad + (-1)^{k/2+i} \frac{4k(k-1+2i)}{(2k-1)^2(2k+1)} + (-1)^{k/2+i} \frac{2(k-2i)}{(2k-1)^2(2k+1)} \\
&\quad - (-1)^{k/2-1+i} \frac{k-1-2i}{2k-3} + (-1)^{k/2+i} \frac{2(k-2+2i)}{(2k-1)(2k-3)} \\
&= (-1)^{k/2+i} \frac{k+1-2i}{2k+1}, \\
c_{2i}^{(k)} &= c_{2i-1}^{(k-1)} + c_{2i+1}^{(k-1)} - \alpha_k^{(1)} c_{2i}^{(k-1)} - \beta_k^{(1)} f_{2i}^{(k-1)} - \alpha_k^{(2)} c_{2i}^{(k-2)} - \beta_k^{(2)} f_{2i}^{(k-2)} \\
&= (-1)^{k/2+i} \frac{k-1+2i}{2k-1} + (-1)^{k/2+i+1} \frac{k+1+2i}{2k-1} \\
&\quad + (-1)^{k/2+i} \frac{4k(k-1-2i)}{(2k-1)^2(2k+1)} + (-1)^{k/2+i} \frac{2(k+2i)}{(2k-1)^2(2k+1)} \\
&\quad - (-1)^{k/2-1+i} \frac{k-1+2i}{2k-3} + (-1)^{k/2+i} \frac{2(k-2-2i)}{(2k-1)(2k-3)} \\
&= (-1)^{k/2+i} \frac{k+1+2i}{2k+1}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
c_{k-1}^{(k)} &= c_{k-2}^{(k-1)} - \alpha_k^{(1)} \\
&= (-1)^{k/2-1+1+k/2-1} \frac{k-1-2(k/2-1)}{2(k-1)+1} - \frac{(-1)^{k+1} 4k}{(2k-1)(2k+1)} = \frac{1}{2k+1}, \\
c_{k-2}^{(k)} &= 1 + c_{k-3}^{(k-1)} - \alpha_k^{(1)} c_{k-2}^{(k-1)} - \beta_k^{(1)} f_{k-2}^{(k-1)} - \alpha_k^{(2)} \\
&= 1 - \frac{2k-3}{2k-1} - \frac{4k}{(2k-1)^2(2k+1)} - \frac{2(k+k-2)}{(2k-1)^2(2k+1)} - 1 = -\frac{2k-1}{2k+1}, \\
c_0^{(k)} &= c_0^{(k-1)} + c_1^{(k-1)} - \alpha_k^{(1)} c_0^{(k-1)} - \beta_k^{(1)} f_0^{(k-1)} - \alpha_k^{(2)} c_0^{(k-2)} - \beta_k^{(2)} f_0^{(k-2)} \\
&= (-1)^{k/2} \frac{k-1}{2k-1} + (-1)^{k/2+1} \frac{k+1}{2k-1} + (-1)^{k/2} \frac{4k(k-1)}{(2k-1)^2(2k+1)} \\
&\quad + (-1)^{k/2} \frac{2k}{(2k-1)^2(2k+1)} - (-1)^{k/2-1} \frac{k-1}{2k-3} \\
&\quad + (-1)^{k/2} \frac{2(k-2)}{(2k-1)(2k-3)} = (-1)^{k/2} \frac{k+1}{2k+1}.
\end{aligned}
$$

Analogously we can verify formulas for $c_i^{(k)}$, $i = 0, 1, \ldots, k-1$, in case when k is odd, and also formulas for $d_i^{(k)}, f_i^{(k)}, g_i^{(k)}$, $i = 0, 1, \ldots, k-1$, $k \in \mathbb{N}$.

Now, only we need to do is to verify formulas (11) for $\nu = k$, using given explicit formulas for coefficients $c_i^{(k)}$, $d_i^{(k)}$, $f_i^{(k)}$, $g_i^{(k)}$, $i = 0, 1, \ldots, k$. Using (9) we have

$$
\begin{aligned}
I_k^C =& \pi\left(1 + c_0^{(k)} d_0^{(k)}\right) + \pi \sum_{\nu=0}^{k-1} \left({c_\nu^{(k)}}^2 + {d_\nu^{(k)}}^2\right) + \pi \sum_{\nu=0}^{k-1} \left(c_\nu^{(k)} d_{\nu+1}^{(k)} - c_{\nu+1}^{(k)} d_\nu^{(k)}\right), \\
I_k^S =& \pi\left(1 + f_0^{(k)} g_0^{(k)}\right) + \pi \sum_{\nu=0}^{k-1} \left({f_\nu^{(k)}}^2 + {g_\nu^{(k)}}^2\right) + \pi \sum_{\nu=0}^{k-1} \left(f_\nu^{(k)} g_{\nu+1}^{(k)} - f_{\nu+1}^{(k)} g_\nu^{(k)}\right), \\
I_k =& \frac{\pi}{2}\left(c_0^{(k)} g_0^{(k)} + f_0^{(k)} d_0^{(k)}\right) + \pi \sum_{\nu=0}^{k-1} \left(c_\nu^{(k)} f_\nu^{(k)} + d_\nu^{(k)} g_\nu^{(k)}\right) \\
&+ \frac{\pi}{2} \sum_{\nu=0}^{k-1} \left(c_\nu^{(k)} g_{\nu+1}^{(k)} + f_\nu^{(k)} d_{\nu+1}^{(k)}\right) - \frac{\pi}{2} \sum_{\nu=0}^{k-1} \left(c_{\nu+1}^{(k)} g_\nu^{(k)} + f_{\nu+1}^{(k)} d_\nu^{(k)}\right), \\
J_k^C =& \pi\left({c_0^{(k)}}^2 - {d_0^{(k)}}^2 + c_0^{(k)} d_1^{(k)} + d_0^{(k)} c_1^{(k)}\right) \\
&+ 2\pi \sum_{\nu=0}^{k-1} \left(c_\nu^{(k)} c_{\nu+1}^{(k)} + d_\nu^{(k)} d_{\nu+1}^{(k)}\right) + \pi \sum_{\nu=0}^{k-2} \left(c_\nu^{(k)} d_{\nu+2}^{(k)} - c_{\nu+2}^{(k)} d_\nu^{(k)}\right), \\
J_k^S =& \pi\left({f_0^{(k)}}^2 - {g_0^{(k)}}^2 + f_0^{(k)} g_1^{(k)} + g_0^{(k)} f_1^{(k)}\right) \\
&+ 2\pi \sum_{\nu=0}^{k-1} \left(f_\nu^{(k)} f_{\nu+1}^{(k)} + g_\nu^{(k)} g_{\nu+1}^{(k)}\right) + \pi \sum_{\nu=0}^{k-2} \left(f_\nu^{(k)} g_{\nu+2}^{(k)} - f_{\nu+2}^{(k)} g_\nu^{(k)}\right), \\
J_k =& \pi\left(c_0^{(k)} f_0^{(k)} - d_0^{(k)} g_0^{(k)}\right) + \frac{\pi}{2}\left(c_0^{(k)} g_1^{(k)} + c_1^{(k)} g_0^{(k)} + f_0^{(k)} d_1^{(k)} + f_1^{(k)} d_0^{(k)}\right) \\
&+ \pi \sum_{\nu=0}^{k-1} \left(c_\nu^{(k)} f_{\nu+1}^{(k)} + g_\nu^{(k)} d_{\nu+1}^{(k)} + g_{\nu+1}^{(k)} d_\nu^{(k)} + c_{\nu+1}^{(k)} f_\nu^{(k)}\right) \\
&+ \frac{\pi}{2} \sum_{\nu=0}^{k-2} \left(c_\nu^{(k)} g_{\nu+2}^{(k)} - c_{\nu+2}^{(k)} g_\nu^{(k)} + f_\nu^{(k)} d_{\nu+2}^{(k)} - f_{\nu+2}^{(k)} d_\nu^{(k)}\right);
\end{aligned}
$$

Substituting here explicit formulas for coefficients $c_i^{(k)}$, $d_i^{(k)}$, $f_i^{(k)}$, $g_i^{(k)}$, $i = 0, 1, \ldots, k$ we obtain formulas (11):

$$
\begin{aligned}
I_k =& \frac{(-1)^{k+1}\pi}{(2k+1)^2} 2 \sum_{i=0}^{k-1} (k-i)(k+1+i) + \frac{(-1)^k \pi}{2(2k+1)^2} \left(2 \sum_{i=0}^{k-1} (k-i)(k+i) - k^2\right) \\
&+ \frac{(-1)^k \pi}{2(2k+1)^2} \left(2 \sum_{i=0}^{k-1} (k-i)(k+2+i) + (k+1)^2\right) \\
=& \frac{(-1)^{k+1}\pi}{(2k+1)^2} \frac{2}{3} k(k+1)(2k+1) + \frac{(-1)^k \pi}{2(2k+1)^2} \left(\frac{2}{6} k(k+1)(4k-1) - k^2\right) \\
&+ \frac{(-1)^k \pi}{2(2k+1)^2} \left(\frac{2}{6} k(k+1)(4k+5) + (k+1)^2\right) = (-1)^k \frac{\pi}{2(2k+1)},
\end{aligned}
$$

$$
\begin{aligned}
I_k^C = I_k^S &= \frac{\pi}{(2k+1)^2}\sum_{i=0}^{2k+1} i^2 - \frac{\pi}{(2k+1)^2}\sum_{i=0}^{2k} i(i+1)\\
&= \frac{\pi}{(2k+1)^2}\frac{1}{6}(2k+1)(2k+2)(4k+3)\\
&\quad -\frac{\pi}{(2k+1)^2}\frac{1}{3}2k(2k+1)(2k+2) = \frac{\pi(k+1)}{2k+1}.
\end{aligned}
$$

On similar way we obtain formulas for integrals J_k^C, J_k^S and J_k. $\square$

Theorem 3 *For the weight function $w_m(x) = 1 + \sin mx$, where $m \geq 3$ is an odd integer, we have the following explicit formulas for coefficients in five-term recurrence relations* (5) *and* (6) ($\ell \in \mathbb{N}_0$)*:*

$\alpha_1^{(1)} = -\delta_1^{(1)} = 1$, $\alpha_1^{(2)} = \delta_1^{(2)} = 0$,
for $k = \ell m$, $\ell \geq 1$*:* $\alpha_k^{(2)} = \delta_k^{(2)} = 1$, $\alpha_k^{(1)} = -\delta_k^{(1)} = \dfrac{(-1)^{\ell+1}}{2\ell+1}$,
for $k = \ell m + 1$, $\ell \geq 1$*:*

$$
\alpha_k^{(2)} = \delta_k^{(2)} = \frac{(2\ell+1)^2 - 1}{(2\ell+1)^2}, \quad \alpha_k^{(1)} = -\delta_k^{(1)} = \frac{(-1)^\ell}{2\ell+1},
$$

for all other values of integers $k > 1$ *hold* $\alpha_k^{(2)} = \delta_k^{(2)} = 1$, $\alpha_k^{(1)} = \delta_k^{(1)} = 0$,
for $k = [m/2] + \ell m + 1$*:* $\beta_k^{(2)} = \gamma_k^{(2)} = \dfrac{(-1)^\ell}{2(\ell+1)}$,
for $k = [m/2] + \ell m + 2$*:* $\beta_k^{(2)} = \gamma_k^{(2)} = \dfrac{(-1)^{\ell+1}}{2(\ell+1)}$,
for all other values of positive integers k *hold* $\beta_k^{(2)} = \gamma_k^{(2)} = 0$, *and for all positive integers* k *hold* $\beta_k^{(1)} = \gamma_k^{(1)} = 0$.

Proof. The steps in proof is the same as in proof of Theorem 2.

Simultaneously, with formulas for recurrence coefficients, we prove the following formulas for integrals:

for $\nu = \ell m$

$$
I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = J_\nu = 0, \quad J_\nu^C = -J_\nu^S = (-1)^\ell \frac{(\ell+1)\pi}{(2\ell+1)^2},
$$

for $\nu = \ell m + [m/2]$

$$
I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = (-1)^\ell \frac{\pi}{2(2\ell+1)}, \quad J_\nu = J_\nu^C = J_\nu^S = 0,
$$

for $\nu = \ell m - 1$, $\ell \geq 1$

$$I_\nu^C = I_\nu^S = \frac{(2\ell+1)\pi}{4\ell}, \quad I_\nu = J_\nu = 0, \quad J_\nu^C = -J_\nu^S = (-1)^{\ell+1}\frac{\pi}{4\ell},$$

in the case $m > 3$, for $\nu = \ell m + 1, \ldots, \ell m + [m/2] - 1$

$$I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = J_\nu = J_\nu^C = J_\nu^S = 0,$$

and for $\nu = \ell m + [m/2] + 1, \ldots, (\ell+1)m - 2$

$$I_\nu^C = I_\nu^S = \frac{(2\ell+3)\pi}{4(\ell+1)}, \quad I_\nu = J_\nu = J_\nu^C = J_\nu^S = 0;$$

and for coefficients of representations (3) and (4):

for $k = \ell m + p$, $p = 0, 1, \ldots, [m/2]$

$$c_{k-2jm}^{(k)} = g_{k-2jm}^{(k)} = (-1)^j \frac{2\ell+1-2j}{2\ell+1}, \quad j = 0, 1, \ldots, [\ell/2],$$

$$c_{k-2jm-(2p+1)}^{(k)} = -g_{k-2jm-(2p+1)}^{(k)} = (-1)^{\ell+j}\frac{2j+1}{2\ell+1}, \quad j = 0, 1, \ldots, [(\ell-1)/2],$$

$$d_{k-(2j+1)m}^{(k)} = -f_{k-(2j+1)m}^{(k)} = (-1)^j \frac{2\ell-2j}{2\ell+1}, \quad j = 0, 1, \ldots, [(\ell-1)/2],$$

$$d_{k-(2j+1)m-(2p+1)}^{(k)} = f_{k-(2j+1)m-(2p+1)}^{(k)} = (-1)^{\ell+j}\frac{2(j+1)}{2\ell+1}, \quad j = 0, 1, \ldots, [\ell/2]-1;$$

for $k = \ell m - p$, $\ell \geq 1$, $p = 1, \ldots, [m/2]$

$$c_{k-2jm}^{(k)} = g_{k-2jm}^{(k)} = (-1)^j \frac{\ell-j}{\ell}, \quad j = 0, 1, \ldots, [(\ell-1)/2],$$

$$c_{k-2jm+(2p-1)}^{(k)} = -g_{k-2jm+(2p-1)}^{(k)} = (-1)^{\ell+j}\frac{j}{\ell}, \quad j = 1, \ldots, [\ell/2],$$

$$d_{k-(2j+1)m}^{(k)} = -f_{k-(2j+1)m}^{(k)} = (-1)^j \frac{2\ell-(2j+1)}{2\ell}, \quad j = 0, 1, \ldots, [\ell/2]-1,$$

$$d_{k-(2j+1)m+(2p-1)}^{(k)} = f_{k-(2j+1)m+(2p-1)}^{(k)} = (-1)^{\ell+j}\frac{2j+1}{2\ell}, \quad j = 0, 1, \ldots, [(\ell-1)/2];$$

all other coefficients are equal to 0.

By using (9), (7), and Theorem 1, it is easy to see that for all nonnegative integers $k \leq [m/2] - 1$ given formulas are correct, i.e., we have $I_0^C = I_0^S = J_0^C = -J_0^S = \pi$, $I_0 = J_0 = 0$, $\alpha_1^{(1)} = -\delta_1^{(1)} = 1, \beta_1^{(1)} = \gamma_1^{(1)} = 0$; and for $1 \leq k \leq [m/2] - 1$, $I_k^C = I_k^S = \pi$, $I_k = J_k = J_k^C = J_k^S = 0$, $c_\nu^{(k)} = d_\nu^{(k)} = f_\nu^{(k)} = g_\nu^{(k)} = 0$, $\nu = 0, 1, \ldots, k-1$ (cf. [1, §3, Example 4] [1]), and

[1] Notice that in [1] orthogonal trigonometric polynomials with leading coefficients $c_n^{(n)} = d_n^{(n)} = 1$ were considered.

$\alpha_k^{(1)} = \delta_k^{(1)} = \beta_k^{(j)} = \gamma_k^{(j)} = 0$, $\alpha_k^{(2)} = \delta_k^{(2)} = 1$ for $1 < k \le [m/2] - 1$. For $k = [m/2]$ we have also $\alpha_k^{(1)} = \delta_k^{(1)} = \beta_k^{(j)} = \gamma_k^{(j)} = 0$, $\alpha_k^{(2)} = \delta_k^{(2)} = 1$, $c_\nu^{(k)} = d_\nu^{(k)} = f_\nu^{(k)} = g_\nu^{(k)} = 0$, $\nu = 0, 1, \ldots, k-1$, and according to (9) we obtain $I_k^C = I_k^S = \pi$, $J_k^C = J_k^S = J_k = 0$, but, now, $I_k = \pi/2$, and, because of that, for $k = [m/2] + 1 = m - [m/2]$, we have $\beta_k^{(2)} = \gamma_k^{(2)} = 1/2$, as well as $d_{k-2}^{(k)} = f_{k-2}^{(k)} = -1/2$. We set apart these two values of k, in order to compare this case with case of an even m (see Theorem 4).

Starting with given formulas for $\nu = k-2, k-1$, $k \ge [m/2]+2$, by direct calculation (using (7)) we verify formulas for recurrence coefficients $\alpha_k^{(j)}, \beta_k^{(j)}, \gamma_k^{(j)}, \delta_k^{(j)}$, $j = 1, 2$. Using these formulas and Theorem 1 we obtain formulas for coefficients $c_\nu^{(k)}, d_\nu^{(k)}, f_\nu^{(k)}, g_\nu^{(k)}$, $\nu = 0, 1, \ldots, k$.

Finally, using given explicit formulas for coefficients $c_i^{(k)}$, $d_i^{(k)}$, $f_i^{(k)}$, $g_i^{(k)}$, $i = 0, 1, \ldots, k$, according to (9), similar as in proof of Theorem 2, we verify given explicit formulas for integrals I_k^C, I_k^S, I_k, J_k^C, J_k^S, J_k. $\square$

Theorem 4 *For the weight function $w_m(x) = 1 + \sin mx$, where m is an even integer, coefficients of five-term recurrence relations* (5) *and* (6) *are given by the following formulas ($\ell \in \mathbb{N}_0$):*

$\alpha_1^{(1)} = -\delta_1^{(1)} = 1$, $\alpha_1^{(2)} = \delta_1^{(2)} = 0$,
for $m = 2$: $\beta_1^{(1)} = \gamma_1^{(1)} = 1/2$,
for $m = 2$, $k = 2\ell$, $\ell \ge 1$

$$\alpha_k^{(1)} = -\delta_k^{(1)} = \frac{(-1)^{\ell+1}}{2\ell+1}, \quad \alpha_k^{(2)} = \delta_k^{(2)} = \frac{4\ell^2 - 1}{4\ell^2}, \quad \beta_k^{(1)} = \gamma_k^{(1)} = \frac{(-1)^\ell}{2\ell},$$

for $m = 2$, $k = 2\ell + 1$, $\ell \ge 1$

$$\alpha_k^{(1)} = -\delta_k^{(1)} = \frac{(-1)^\ell}{2\ell+1}, \quad \alpha_k^{(2)} = \delta_k^{(2)} = \frac{(2\ell+1)^2 - 1}{(2\ell+1)^2}, \quad \beta_k^{(1)} = \gamma_k^{(1)} = \frac{(-1)^\ell}{2(\ell+1)},$$

for $k = \ell m/2 + 1$, $m \ge 4$, where ℓ is an odd positive integer

$$\alpha_k^{(2)} = \delta_k^{(2)} = \frac{(\ell+1)^2 - 1}{(\ell+1)^2}, \quad \alpha_k^{(1)} = \delta_k^{(1)} = 0,$$

for $k = \ell m/2$, $m \ge 4$, where ℓ is an even positive integer

$$\alpha_k^{(2)} = \delta_k^{(2)} = 1, \quad \alpha_k^{(1)} = -\delta_k^{(1)} = \frac{(-1)^{\ell/2+1}}{\ell+1},$$

for $k = \ell m/2 + 1$, $m \ge 4$, where ℓ is an even positive integer

$$\alpha_k^{(2)} = \delta_k^{(2)} = \frac{(\ell+1)^2 - 1}{(\ell+1)^2}, \quad \alpha_k^{(1)} = -\delta_k^{(1)} = \frac{(-1)^{\ell/2}}{\ell+1},$$

for $m \geq 4$*, for all other values of integers* $k > 1$ *hold* $\alpha_k^{(2)} = \delta_k^{(2)} = 1$ *and* $\alpha_k^{(1)} = \delta_k^{(1)} = 0$*,*
for $k = m/2 + \ell m$*,* $m \geq 4$*:* $\beta_k^{(1)} = \gamma_k^{(1)} = \dfrac{(-1)^\ell}{2(\ell+1)}$*,*
for $k = m/2 + \ell m + 1$*,* $m \geq 4$*:* $\beta_k^{(1)} = \gamma_k^{(1)} = \dfrac{(-1)^{\ell+1}}{2(\ell+1)}$*,*
for $m \geq 4$*, for all other values of positive integers* k *hold* $\beta_k^{(1)} = \gamma_k^{(1)} = 0$*,*
and for all even m*, for all positive integers* k *hold* $\beta_k^{(2)} = \gamma_k^{(2)} = 0$*.*

Proof. As in previous theorems, we prove simultaneously the following formulas:

for $m = 2$, $\nu = 2\ell$

$$I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = 0,$$

$$J_\nu^C = -J_\nu^S = (-1)^\ell \frac{(\ell+1)\pi}{(2\ell+1)^2}, \quad J_\nu = (-1)^\ell \frac{\pi}{2(2\ell+1)},$$

for $m = 2$, $\nu = 2\ell + 1$

$$I_\nu^C = I_\nu^S = \frac{(2\ell+3)\pi}{4(\ell+1)}, \quad I_\nu = 0,$$

$$J_\nu^C = -J_\nu^S = (-1)^\ell \frac{\pi}{4(\ell+1)}, \quad J_\nu = (-1)^{\ell+1} \frac{(2\ell+3)\pi}{8(\ell+1)^2},$$

for $\nu = \ell m$, $m \geq 4$

$$I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = J_\nu = 0, \quad J_\nu^C = -J_\nu^S = (-1)^\ell \frac{(\ell+1)\pi}{(2\ell+1)^2},$$

for $\nu = \ell m + m/2$, $m \geq 4$

$$I_\nu^C = I_\nu^S = \frac{(2\ell+3)\pi}{4(\ell+1)}, \quad I_\nu = J_\nu^C = J_\nu^S = 0, \quad J_\nu = (-1)^{\ell+1} \frac{(2\ell+3)\pi}{8(\ell+1)^2},$$

for $\nu = \ell m + m/2 - 1$, $m \geq 4$

$$I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = J_\nu^C = J_\nu^S = 0, \quad J_\nu = (-1)^\ell \frac{\pi}{2(2\ell+1)},$$

for $\nu = \ell m - 1$, $\ell \geq 1$, $m \geq 4$

$$I_\nu^C = I_\nu^S = \frac{(2\ell+1)\pi}{4\ell}, \quad I_\nu = J_\nu = 0, \quad J_\nu^C = -J_\nu^S = (-1)^{\ell+1} \frac{\pi}{4\ell},$$

in the case $m > 4$, for $\nu = \ell m + 1, \ldots, \ell m + m/2 - 2$

$$I_\nu^C = I_\nu^S = \frac{(\ell+1)\pi}{2\ell+1}, \quad I_\nu = J_\nu = J_\nu^C = J_\nu^S = 0,$$

and for $\nu = \ell m + m/2 + 1, \ldots, (\ell+1)m - 2$

$$I_\nu^C = I_\nu^S = \frac{(2\ell+3)\pi}{4(\ell+1)}, \quad I_\nu = J_\nu = J_\nu^C = J_\nu^S = 0;$$

for $k = \ell m + p$, $p = 0, 1, \ldots, m/2 - 1$

$$c_{k-2jm}^{(k)} = g_{k-2jm}^{(k)} = (-1)^j \frac{2\ell + 1 - 2j}{2\ell+1}, \quad j = 0, 1, \ldots, [\ell/2],$$

$$c_{k-2jm-(2p+1)}^{(k)} = -g_{k-2jm-(2p+1)}^{(k)} = (-1)^{\ell+j} \frac{2j+1}{2\ell+1}, \quad j = 0, 1, \ldots, [(\ell-1)/2],$$

$$d_{k-(2j+1)m}^{(k)} = -f_{k-(2j+1)m}^{(k)} = (-1)^j \frac{2\ell - 2j}{2\ell+1}, \quad j = 0, 1, \ldots, [(\ell-1)/2],$$

$$d_{k-(2j+1)m-(2p+1)}^{(k)} = f_{k-(2j+1)m-(2p+1)}^{(k)} = (-1)^{\ell+j} \frac{2(j+1)}{2\ell+1}, \quad j = 0, 1, \ldots, [\ell/2]-1;$$

for $k = \ell m - p$, $\ell \geq 1$, $p = 1, \ldots, m/2$

$$c_{k-2jm}^{(k)} = g_{k-2jm}^{(k)} = (-1)^j \frac{\ell - j}{\ell}, \quad j = 0, 1, \ldots, [(\ell-1)/2],$$

$$c_{k-2jm+(2p-1)}^{(k)} = -g_{k-2jm+(2p-1)}^{(k)} = (-1)^{\ell+j} \frac{j}{\ell}, \quad j = 1, \ldots, [\ell/2],$$

$$d_{k-(2j+1)m}^{(k)} = -f_{k-(2j+1)m}^{(k)} = (-1)^j \frac{2\ell - (2j+1)}{2\ell}, \quad j = 0, 1, \ldots, [\ell/2] - 1,$$

$$d_{k-(2j+1)m+(2p-1)}^{(k)} = f_{k-(2j+1)m+(2p-1)}^{(k)} = (-1)^{\ell+j} \frac{2j+1}{2\ell}, \quad j = 0, 1, \ldots, [(\ell-1)/2];$$

all other coefficients are equal to 0.

By direct computation as in proof of Theorem 3 we see that for $m \geq 4$ for all $k < m/2 - 1$ given formulas are correct, i.e., $I_0^C = I_0^S = J_0^C = -J_0^S = \pi$, $I_0 = J_0 = 0$, $\alpha_1^{(1)} = -\delta_1^{(1)} = 1, \beta_1^{(1)} = \gamma_1^{(1)} = 0$; and for $1 \leq k \leq m/2 - 2$, $I_k^C = I_k^S = \pi$, $I_k = J_k = J_k^C = J_k^S = 0$, $c_\nu^{(k)} = d_\nu^{(k)} = f_\nu^{(k)} = g_\nu^{(k)} = 0$, $\nu = 0, 1, \ldots, k-1$ and $\alpha_k^{(1)} = \delta_k^{(1)} = \beta_k^{(j)} = \gamma_k^{(j)} = 0$, $\alpha_k^{(2)} = \delta_k^{(2)} = 1$ for $1 < k \leq m/2 - 2$. Then for $k = m/2 - 1$ we get also $\alpha_k^{(1)} = \delta_k^{(1)} = \beta_k^{(j)} = \gamma_k^{(j)} = 0$, $\alpha_k^{(2)} = \delta_k^{(2)} = 1$, $c_\nu^{(k)} = d_\nu^{(k)} = f_\nu^{(k)} = g_\nu^{(k)} = 0$, $\nu = 0, 1, \ldots, k-1$ and $I_k^C = I_k^S = \pi$, $I_k = J_k^C = J_k^S = 0$, but, now, $J_k = \pi/2$, and, because of that, for $k = m/2$, we have $\beta_k^{(1)} = \gamma_k^{(1)} = 1/2$, as well as $d_{k-1}^{(k)} = f_{k-1}^{(k)} = -1/2$. We put forward these values of k to see the difference to case of an odd m.

The proof is similar as the proof of Theorems 2 and 3. $\square$

Theorem 5 *For the weight function $w(x) = \sqrt{2} + \sin x + \cos x$ five-term recurrence coefficients are given by the following formulas:*

$$\alpha_1^{(1)} = \frac{1}{3}(3+\sqrt{2}), \quad \beta_1^{(1)} = \frac{1}{3}(-1-\sqrt{2}),$$

$$\gamma_1^{(1)} = \frac{1}{3}(-1+\sqrt{2}), \quad \delta_1^{(1)} = \frac{1}{3}(-3+\sqrt{2}),$$

and for any integer $k \geq 2$

$$\alpha_k^{(1)} = \begin{cases} \frac{1}{(2k-1)(2k+1)}\left((-1)^{[(k-1)/2]}(2k-1)+\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)(2k+1)}\left((-1)^{(k-1)/2}(2k+1)+\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\alpha_k^{(2)} = \begin{cases} \frac{1}{(2k-1)^2}\left((2k-1)^2-1+(-1)^{k/2}(2k-1)\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)^2}\left((2k-1)^2-1+(-1)^{(k+1)/2}\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\beta_k^{(1)} = \begin{cases} \frac{1}{(2k-1)(2k+1)}\left((-1)^{[(k-1)/2]}(2k+1)-\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)(2k+1)}\left((-1)^{(k+1)/2}(2k-1)-\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\beta_k^{(2)} = \begin{cases} \frac{1}{(2k-1)^2}\left(1+(-1)^{k/2}\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)^2}\left(1+(-1)^{(k-1)/2}(2k-1)\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\gamma_k^{(1)} = \begin{cases} \frac{1}{(2k-1)(2k+1)}\left((-1)^{[(k-1)/2]}(2k+1)+\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)(2k+1)}\left((-1)^{(k+1)/2}(2k-1)+\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\gamma_k^{(2)} = \begin{cases} \frac{1}{(2k-1)^2}\left(-1+(-1)^{k/2}\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)^2}\left(-1+(-1)^{(k-1)/2}(2k-1)\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\delta_k^{(1)} = \begin{cases} \frac{1}{(2k-1)(2k+1)}\left((-1)^{[(k-1)/2]+1}(2k-1)+\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)(2k+1)}\left((-1)^{(k+1)/2}(2k+1)+\sqrt{2}\right), & k-\textit{odd}, \end{cases}$$

$$\delta_k^{(2)} = \begin{cases} \frac{1}{(2k-1)^2}\left((2k-1)^2-1+(-1)^{k/2+1}(2k-1)\sqrt{2}\right), & k-\textit{even}, \\ \frac{1}{(2k-1)^2}\left((2k-1)^2-1+(-1)^{(k-1)/2}\sqrt{2}\right), & k-\textit{odd}. \end{cases}$$

Proof. The steps in proof are the same as in proof of the Theorem 2. In this case the following formulas hold (k, ℓ are nonnegative integers):

$$\int_0^{2\pi} \cos(k+1/2)x\cos(\ell+1/2)xw(x)\,\mathrm{d}x = \pi\sqrt{2}\delta_{k,\ell} + \frac{\pi}{2}\delta_{k,\ell\pm1},\ k \geq 1,$$
$$\int_0^{2\pi} \cos(x/2)\cos(\ell+1/2)xw(x)\,\mathrm{d}x = \pi(\sqrt{2}+1/2)\delta_{0,\ell} + \frac{\pi}{2}\delta_{1,\ell},$$
$$\int_0^{2\pi} \sin(k+1/2)x\sin(\ell+1/2)xw(x)\,\mathrm{d}x = \pi\sqrt{2}\delta_{k,\ell} + \frac{\pi}{2}\delta_{k,\ell\pm1},\ k \geq 1,$$
$$\int_0^{2\pi} \sin(x/2)\sin(\ell+1/2)xw(x)\,\mathrm{d}x = \pi(\sqrt{2}-1/2)\delta_{0,\ell} + \frac{\pi}{2}\delta_{1,\ell},$$
$$\int_0^{2\pi} \cos(k+1/2)x\sin(\ell+1/2)xw(x)\,\mathrm{d}x = \frac{\pi}{2}(\delta_{k,\ell-1} - \delta_{k,\ell+1}),\ k \geq 1,$$
$$\int_0^{2\pi} \cos(x/2)\sin(\ell+1/2)xw(x)\,\mathrm{d}x = \frac{\pi}{2}(\delta_{0,\ell} + \delta_{1,\ell}),$$
$$\int_0^{2\pi} \cos x\cos(k+1/2)x\cos(\ell+1/2)xw(x)\,\mathrm{d}x$$
$$= \frac{\pi}{2}\delta_{k,\ell} + \frac{\pi\sqrt{2}}{2}\delta_{k,\ell\pm1} + \frac{\pi}{4}\delta_{k,\ell\pm2},\ k > 1,$$
$$\int_0^{2\pi} \cos x\cos(x/2)\cos(\ell+1/2)xw(x)\,\mathrm{d}x$$
$$= \frac{(\sqrt{2}+1)\pi}{2}\delta_{0,\ell} + \frac{(2\sqrt{2}+1)\pi}{4}\delta_{1,\ell} + \frac{\pi}{4}\delta_{2,\ell},$$
$$\int_0^{2\pi} \cos x\cos(3x/2)\cos(\ell+1/2)xw(x)\,\mathrm{d}x$$
$$= \frac{(2\sqrt{2}+1)\pi}{4}\delta_{0,\ell} + \frac{\pi}{2}\delta_{1,\ell} + \frac{\pi\sqrt{2}}{2}\delta_{2,\ell} + \frac{\pi}{4}\delta_{3,\ell},$$
$$\int_0^{2\pi} \cos x\sin(k+1/2)x\sin(\ell+1/2)xw(x)\,\mathrm{d}x =$$
$$= \frac{\pi}{2}\delta_{k,\ell} + \frac{\pi\sqrt{2}}{2}\delta_{k,\ell\pm1} + \frac{\pi}{4}\delta_{k,\ell\pm2},\ k > 1,$$
$$\int_0^{2\pi} \cos x\sin(x/2)\sin(\ell+1/2)xw(x)\,\mathrm{d}x$$
$$= \frac{(-\sqrt{2}+1)\pi}{2}\delta_{0,\ell} + \frac{(2\sqrt{2}-1)\pi}{4}\delta_{1,\ell} + \frac{\pi}{4}\delta_{2,\ell},$$
$$\int_0^{2\pi} \cos x\sin(3x/2)\sin(\ell+1/2)xw(x)\,\mathrm{d}x$$
$$= \frac{(2\sqrt{2}-1)\pi}{4}\delta_{0,\ell} + \frac{\pi}{2}\delta_{1,\ell} + \frac{\pi\sqrt{2}}{2}\delta_{2,\ell} + \frac{\pi}{4}\delta_{3,\ell},$$
$$\int_0^{2\pi} \cos x\cos(k+1/2)x\sin(\ell+1/2)xw(x)\,\mathrm{d}x = \frac{\pi}{4}(\delta_{k,1-\ell} + \delta_{k,\ell-2} - \delta_{k,\ell+2}).$$

Using these formulas, we obtain

$$
\begin{aligned}
I_n^C =& \pi(\sqrt{2}+1/2){c_0^{(n)}}^2 + \pi(\sqrt{2}-1/2){d_0^{(n)}}^2 + \pi c_0^{(n)} d_0^{(n)} \\
&+\pi\sqrt{2}\sum_{\nu=1}^{n}\left({c_\nu^{(n)}}^2 + {d_\nu^{(n)}}^2\right) + \pi\sum_{\nu=0}^{n-1}\left(c_\nu^{(n)}c_{\nu+1}^{(n)} + d_\nu^{(n)}d_{\nu+1}^{(n)}\right) \\
&+\pi\sum_{\nu=0}^{n-1}\left(c_\nu^{(n)}d_{\nu+1}^{(n)} - d_\nu^{(n)}c_{\nu+1}^{(n)}\right), \\
I_n^S =& \pi(\sqrt{2}+1/2){f_0^{(n)}}^2 + \pi(\sqrt{2}-1/2){g_0^{(n)}}^2 + \pi f_0^{(n)} g_0^{(n)} \\
&+\pi\sqrt{2}\sum_{\nu=1}^{n}\left({f_\nu^{(n)}}^2 + {g_\nu^{(n)}}^2\right) + \pi\sum_{\nu=0}^{n-1}\left(f_\nu^{(n)}f_{\nu+1}^{(n)} + g_\nu^{(n)}g_{\nu+1}^{(n)}\right) \\
&+\pi\sum_{\nu=0}^{n-1}\left(f_\nu^{(n)}g_{\nu+1}^{(n)} - g_\nu^{(n)}f_{\nu+1}^{(n)}\right), \\
I_n =& \pi(\sqrt{2}+1/2)c_0^{(n)}f_0^{(n)} + \pi(\sqrt{2}-1/2)d_0^{(n)}g_0^{(n)} + \frac{\pi}{2}\left(c_0^{(n)}g_0^{(n)} + d_0^{(n)}f_0^{(n)}\right) \\
&+\pi\sqrt{2}\sum_{\nu=1}^{n-1}\left(c_\nu^{(n)}f_\nu^{(n)} + d_\nu^{(n)}g_\nu^{(n)}\right) \\
&+\frac{\pi}{2}\sum_{\nu=0}^{n-1}\left(c_\nu^{(n)}f_{\nu+1}^{(n)} + f_\nu^{(n)}c_{\nu+1}^{(n)} + d_\nu^{(n)}g_{\nu+1}^{(n)} + g_\nu^{(n)}d_{\nu+1}^{(n)}\right) \\
&+\frac{\pi}{2}\sum_{\nu=0}^{n-1}\left(c_\nu^{(n)}g_{\nu+1}^{(n)} - g_\nu^{(n)}c_{\nu+1}^{(n)} + f_\nu^{(n)}d_{\nu+1}^{(n)} - d_\nu^{(n)}f_{\nu+1}^{(n)}\right), \\
J_n^C =& \pi(1+\sqrt{2}){c_0^{(n)}}^2 + \pi(1-\sqrt{2}){d_0^{(n)}}^2 + \pi\left(c_0^{(n)}d_1^{(n)} + c_1^{(n)}d_0^{(n)}\right) \\
&+\pi(2\sqrt{2}+1)c_0^{(n)}c_1^{(n)} + \pi(2\sqrt{2}-1)d_0^{(n)}d_1^{(n)} + \pi\sum_{\nu=1}^{n}\left({c_\nu^{(n)}}^2 + {d_\nu^{(n)}}^2\right) \\
&+2\pi\sqrt{2}\sum_{\nu=1}^{n-1}\left(c_\nu^{(n)}c_{\nu+1}^{(n)} + d_\nu^{(n)}d_{\nu+1}^{(n)}\right) \\
&+\pi\sum_{\nu=0}^{n-2}\left(c_\nu^{(n)}c_{\nu+2}^{(n)} + d_\nu^{(n)}d_{\nu+2}^{(n)} + c_\nu^{(n)}d_{\nu+2}^{(n)} - d_\nu^{(n)}c_{\nu+2}^{(n)}\right), \\
J_n^S =& \pi(1+\sqrt{2}){f_0^{(n)}}^2 + \pi(1-\sqrt{2}){g_0^{(n)}}^2 + \pi\left(f_0^{(n)}g_1^{(n)} + f_1^{(n)}g_0^{(n)}\right) \\
&+\pi(2\sqrt{2}+1)f_0^{(n)}f_1^{(n)} + \pi(2\sqrt{2}-1)g_0^{(n)}g_1^{(n)} + \pi\sum_{\nu=1}^{n}\left({f_\nu^{(n)}}^2 + {g_\nu^{(n)}}^2\right) \\
&+2\pi\sqrt{2}\sum_{\nu=1}^{n-1}\left(f_\nu^{(n)}f_{\nu+1}^{(n)} + g_\nu^{(n)}g_{\nu+1}^{(n)}\right) \\
&+\pi\sum_{\nu=0}^{n-2}\left(f_\nu^{(n)}f_{\nu+2}^{(n)} + g_\nu^{(n)}g_{\nu+2}^{(n)} + f_\nu^{(n)}g_{\nu+2}^{(n)} - g_\nu^{(n)}f_{\nu+2}^{(n)}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_n = {} & \pi(1+\sqrt{2})c_0^{(n)}f_0^{(n)} + \pi(1-\sqrt{2})d_0^{(n)}g_0^{(n)} + \pi\sum_{\nu=1}^{n}\left(c_\nu^{(n)}f_\nu^{(n)} + d_\nu^{(n)}g_\nu^{(n)}\right) \\
& + \frac{\pi}{2}\left(c_0^{(n)}g_1^{(n)} + c_1^{(n)}g_0^{(n)} + f_0^{(n)}d_1^{(n)} + f_1^{(n)}d_0^{(n)}\right) \\
& + \frac{\pi}{2}(2\sqrt{2}+1)\left(c_0^{(n)}f_1^{(n)} + c_1^{(n)}f_0^{(n)}\right) + \frac{\pi}{2}(2\sqrt{2}-1)\left(d_0^{(n)}g_1^{(n)} + d_1^{(n)}g_0^{(n)}\right) \\
& + \pi\sqrt{2}\sum_{\nu=1}^{n-1}\left(c_\nu^{(n)}f_{\nu+1}^{(n)} + f_\nu^{(n)}c_{\nu+1}^{(n)} + d_\nu^{(n)}g_{\nu+1}^{(n)} + g_\nu^{(n)}d_{\nu+1}^{(n)}\right) \\
& + \frac{\pi}{2}\sum_{\nu=0}^{n-2}\Big(c_\nu^{(n)}f_{\nu+2}^{(n)} + f_\nu^{(n)}c_{\nu+2}^{(n)} + d_\nu^{(n)}g_{\nu+2}^{(n)} + g_\nu^{(n)}d_{\nu+2}^{(n)} \\
& \qquad + c_\nu^{(n)}g_{\nu+2}^{(n)} + f_\nu^{(n)}d_{\nu+2}^{(n)} - g_\nu^{(n)}c_{\nu+2}^{(n)} - d_\nu^{(n)}f_{\nu+2}^{(n)}\Big).
\end{aligned}
$$

In order to prove this theorem we prove the following formulas for integrals ($\nu \in \mathbb{N}_0$):

$$
I_\nu^C = \begin{cases} \dfrac{\pi}{2(2\nu+1)}\left((-1)^{\nu/2} + 2(\nu+1)\sqrt{2}\right), & \nu - \text{even}, \\ \dfrac{\pi}{2(2\nu+1)}\left((-1)^{[\nu/2]+1} + 2(\nu+1)\sqrt{2}\right), & \nu - \text{odd}, \end{cases}
$$

$$
I_\nu^S = \begin{cases} \dfrac{\pi}{2(2\nu+1)}\left((-1)^{\nu/2+1} + 2(\nu+1)\sqrt{2}\right), & \nu - \text{even}, \\ \dfrac{\pi}{2(2\nu+1)}\left((-1)^{[\nu/2]} + 2(\nu+1)\sqrt{2}\right), & \nu - \text{odd}, \end{cases}
$$

$$
I_\nu = (-1)^{[\nu/2]}\frac{\pi}{2(2\nu+1)},
$$

$$
J_\nu^C = \begin{cases} \dfrac{\pi}{(2\nu+1)^2}\left(1 + (-1)^{\nu/2}(\nu+1)\sqrt{2}\right), & \nu - \text{even}, \\ \dfrac{\pi}{(2\nu+1)^2}\left(1 + (-1)^{[\nu/2]}\nu\sqrt{2}\right), & \nu - \text{odd}, \end{cases}
$$

$$
J_\nu^S = \begin{cases} \dfrac{\pi}{(2\nu+1)^2}\left(1 + (-1)^{\nu/2+1}(\nu+1)\sqrt{2}\right), & \nu - \text{even}, \\ \dfrac{\pi}{(2\nu+1)^2}\left(1 + (-1)^{[\nu/2]+1}\nu\sqrt{2}\right), & \nu - \text{odd}, \end{cases}
$$

$$
J_\nu = \begin{cases} (-1)^{\nu/2+1}\dfrac{\pi\nu\sqrt{2}}{(2\nu+1)^2}, & \nu - \text{even}, \\ (-1)^{[\nu/2]}\dfrac{\pi(\nu+1)\sqrt{2}}{(2\nu+1)^2}, & \nu - \text{odd}. \end{cases}
$$

Finally, coefficients of representations (3) and (4) are given as follows. For a positive integer n, let denote $k = [n/4]$ and $m = n - 4[n/4]$. Then for an even n, for $\ell = 0, 1, \ldots, k$ we have

$$
\begin{aligned}
c_{4\ell}^{(n)} &= \frac{(-1)^{k+\ell}}{2n+1}\left((-1)^{(2-m)/2}\left(2(k-\ell)+\frac{m}{2}\right)\sqrt{2}+\frac{2-m}{2}(4(k+\ell)+1)\right),\\
d_{4\ell}^{(n)} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left(\frac{m}{2}(4(k+\ell)+3)+\left(2(k-\ell)+\frac{m}{2}\right)\sqrt{2}\right),
\end{aligned}
$$

for $\ell = 0, 1, \ldots, k-(2-m)/2$

$$
\begin{aligned}
c_{4\ell+1}^{(n)} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left(\frac{m}{2}(4(k-\ell)+1)+\left(2(k+\ell)+1+\frac{m}{2}\right)\sqrt{2}\right),\\
c_{4\ell+2}^{(n)} &= \frac{(-1)^{k+\ell}}{2n+1}\left(\left(2(k-\ell)-1+\frac{m}{2}\right)\sqrt{2}+\frac{m}{2}(4(k+\ell)+5)\right),\\
d_{4\ell+1}^{(n)} &= \frac{(-1)^{k+\ell}}{2n+1}\left((-1)^{(2-m)/2}\left(2(k+\ell)+1+\frac{m}{2}\right)\sqrt{2}+\frac{2-m}{2}(4(k-\ell)-1)\right),\\
d_{4\ell+2}^{(n)} &= \frac{(-1)^{k+\ell}}{2n+1}\left(\frac{2-m}{2}(4(k+\ell)+3)+(-1)^{(2-m)/2}\left(2(k-\ell)-1+\frac{m}{2}\right)\sqrt{2}\right),
\end{aligned}
$$

and, for $\ell = 0, 1, \ldots, k-1$

$$
\begin{aligned}
c_{4\ell+3}^{(n)} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left(\frac{2-m}{2}(4(k-\ell)-3)+(-1)^{(2-m)/2}\left(2(k+\ell+1)+\frac{m}{2}\right)\sqrt{2}\right),\\
d_{4\ell+3}^{(n)} &== \frac{(-1)^{k+\ell+1}}{2n+1}\left(\left(2(k+\ell+1)+\frac{m}{2}\right)\sqrt{2}+\frac{m}{2}(4(k-\ell)-1)\right).
\end{aligned}
$$

For an odd n, for $\ell = 0, 1, \ldots, k$ we have

$$
\begin{aligned}
c_{4\ell}^{(n)} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left(\frac{m-1}{2}(4(k-\ell)+3)+(-1)^{(m-1)/2}\left(2(k+\ell)+1+\frac{m-1}{2}\right)\sqrt{2}\right),\\
c_{4\ell+1}^{(n)} &= \frac{(-1)^{k+\ell}}{2n+1}\left(\left(2(k-\ell)+\frac{m-1}{2}\right)\sqrt{2}+\frac{3-m}{2}(4(k+\ell)+3)\right),\\
d_{4\ell}^{(n)} &= \frac{(-1)^{k+\ell}}{2n+1}\left(\left(2(k+\ell)+1+\frac{m-1}{2}\right)\sqrt{2}+\frac{3-m}{2}(4(k-\ell)+1)\right),\\
d_{4\ell+1}^{(n)} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left(\frac{m-1}{2}(4(k+\ell)+5)+(-1)^{(m-1)/2}\left(2(k-\ell)+\frac{m-1}{2}\right)\sqrt{2}\right),
\end{aligned}
$$

and, finally, for $\ell = 0, 1, \ldots, k-(3-m)/2$ we have

$$
\begin{aligned}
c^{(n)}_{4\ell+2} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left(\frac{3-m}{2}(4(k-\ell)-1)+\left(2(k+\ell+1)+\frac{m-1}{2}\right)\sqrt{2}\right),\\
c^{(n)}_{4\ell+3} &= \frac{(-1)^{k+\ell}}{2n+1}\left((-1)^{(m-1)/2}\left(2(k-\ell)-\frac{3-m}{2}\right)\sqrt{2}+\frac{m-1}{2}(4(k+\ell)+7)\right),\\
d^{(n)}_{4\ell+2} &= \frac{(-1)^{k+\ell+1}}{2n+1}\left((-1)^{(m-1)/2}\left(2(k+\ell+1)+\frac{m-1}{2}\right)\sqrt{2}+\frac{m-1}{2}(4(k-\ell)+1)\right),\\
d^{(n)}_{4\ell+3} &= \frac{(-1)^{k+\ell}}{2n+1}\left(\frac{3-m}{2}(4(k+\ell)+5)+\left(2(k-\ell)-\frac{3-m}{2}\right)\sqrt{2}\right).
\end{aligned}
$$

Coefficients $g_\nu^{(n)}$ can be obtained from expressions for $c_\nu^{(n)}$ multiplying by -1 the first addend in the brackets on the right hand side, and coefficients $f_\nu^{(n)}$ can be obtained from expressions for $d_\nu^{(n)}$ multiplying by -1 the first addend in the brackets on the right hand side.

All explicit formulas can be obtained by direct calculation using the same steps as in Theorem 2. □

We use symbolic computations in Mathematica and software package OrthogonalPolynomials described in [3] in order to verify all given formulas.

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