Applied Mathematics and Computation 248 (2014) 631–651

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some results on the extended beta and extended hypergeometric functions

Min-Jie Luo [a], Gradimir V. Milovanovic [b,*], Praveen Agarwal [c]

[a] Department of Mathematics, East China Normal University, Shanghai 200241, People's Republic of China
[b] Mathematical Institute of Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11000 Beograd, Serbia
[c] Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India

ARTICLE INFO

Keywords:
Extended beta function
Extended Gauss hypergeometric function
Extended generalized hypergeometric function
Fractional integral
H-function
Laguerre polynomial

ABSTRACT

The purpose of this paper is to present a systematic study of some extended special functions like $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$, ${}_2F_1^{(\alpha,\beta;\rho,\lambda)}[z;b]$ and ${}_pF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$. We obtain various properties of these extended functions and establish their some connections with the Laguerre polynomial and Fox's H-function. Furthermore, we also establish the extended Riemann–Liouville type fractional integral operator and extended Kober type fractional integral operators.

1. Introduction

A fairly wide range of important functions in applied sciences (which are popularly known as special functions) are defined via improper integrals or infinite series (or infinite products). During last four decades or so, several special functions (such as the gamma and beta functions, the Gauss hypergeometric function, and so on) becomes essential tools for scientists and engineers due to their applications in mathematical physics, probability theory and other areas. The above-mentioned applications have largely motivated their extensions and generalizations.

In the present paper, we consider the following special function and its related functions.

Definition 1.1. The extended beta function $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$ with $\mathrm{Re}(b)>0$ is defined by

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{b}{t^{\rho}(1-t)^{\lambda}}\right)\mathrm{d}t, \tag{1}$$

where

$$\rho\geqslant 0,\lambda\geqslant 0,\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0,\quad \mathrm{Re}(x)>-\mathrm{Re}(\rho\alpha),\quad \mathrm{Re}(y)>-\mathrm{Re}(\lambda\alpha).$$

When $b=0$, (1) reduces to the ordinary beta function $B(x,y)$ $(\min\{\mathrm{Re}(x),\mathrm{Re}(y)\}>0)$.

Special cases of (1) are given below.

- When $\rho=\lambda=1$, the function (1) becomes the one that has been discussed in [30].
- For $\alpha=\beta$, we get

$$B_{b;\rho,\lambda}(x,y):=\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\alpha)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{b}{t^{\rho}(1-t)^{\lambda}}\right)\mathrm{d}t. \tag{2}$$

* Corresponding author.
E-mail addresses: mathwinnie@gmail.com (M.-J. Luo), gvm@mi.sanu.ac.rs (G.V. Milovanovic), goyal.praveen2011@gmail.com (P. Agarwal).

http://dx.doi.org/10.1016/j.amc.2014.09.110
0096-3003/

The case when $\rho = \lambda = m > 0$ has been studied in [20].

- For $\rho = \lambda = 1$ and $\alpha = \beta$, we get

$$B_b(x,y) := \mathcal{B}_{b;1,1}^{(\alpha,\alpha)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{b}{t(1-t)}\right)\mathrm{d}t, \tag{3}$$

which has been considered in [9,23]. A complete description of (3) can be found in [8].

With the help of these new beta functions many extended hypergeometric functions (including multivariate cases) have been created and considered (For univariate cases, see [10,24,26,30]; for multivariate cases, see [29,33]). Their applications on fractional calculus and statistics are also given in [1,5,24]. In this paper, we focus on the following two definitions. For convenience, we shall always use the notation:

$$\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n,y) := \frac{\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n,y)}{\mathcal{B}_{0;\rho,\lambda}^{(\alpha,\beta)}(x,y)} = \frac{\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n,y)}{B(x,y)}. \tag{4}$$

Definition 1.2. The extended Gauss hypergeometric function is defined by

$${}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1, & x_2 \\ & y_1 \end{matrix};z;b\right] = \sum_{n=0}^{\infty}(x_1)_n\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_2+n,y_1-x_2)\frac{z^n}{n!} \tag{5}$$

$$(\rho \geqslant 0, \lambda \geqslant 0; \quad \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\} > 0; \quad \mathrm{Re}(y_1) > \mathrm{Re}(x_2) > 0, \quad |z| < 1).$$

Generally, we can define the following extended generalized hypergeometric function. In fact, function of this type has been introduced and studied in [26].

Definition 1.3. The extended generalized hypergeometric function is defined by

$${}_pF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_1,\ldots,y_q \end{matrix};z;b\right] = \sum_{n=0}^{\infty}\mathfrak{A}(n|p,q)\frac{z^n}{n!}, \tag{6}$$

where $\rho \geqslant 0, \lambda \geqslant 0, \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\} > 0$, and its coefficient is determined by

$$\mathfrak{A}(n|p,q) = \begin{cases} (x_1)_n\prod_{j=1}^{q}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_{j+1}+n,y_j-x_{j+1}), & (p=q+1;\mathrm{Re}(y_j) > \mathrm{Re}(x_{j+1}) > 0;\ |z|<1), \\ \prod_{j=1}^{q}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_j+n,y_j-x_j), & (p=q;\mathrm{Re}(y_j) > \mathrm{Re}(x_j) > 0;\ z\in\mathbb{C}), \\ \prod_{i=1}^{r}\frac{1}{(y_i)_n}\prod_{j=1}^{p}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_j+n,y_{r+j}-x_j), & (r=q-p, p<q;\mathrm{Re}(y_{r+j}) > \mathrm{Re}(x_j) > 0;\ z\in\mathbb{C}). \end{cases}$$

When the variable b vanishes, function (6) reduces to the ordinary generalized hypergeometric function defined by

$${}_pF_q(x_1,\ldots,x_p;y_1,\ldots,y_q;z) \equiv {}_pF_q\left[\begin{matrix} x_1,\ldots,x_p \\ y_1,\ldots,y_q \end{matrix};z\right] = \sum_{n=0}^{\infty}\frac{(x_1)_n\cdots(x_p)_n}{(y_1)_n\cdots(y_q)_n}\frac{z^n}{n!},$$

where $(x)_n$ denotes the Pochhammer symbol defined, in terms of the familiar gamma function, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n=0; x\in\mathbb{C}\setminus\{0\}) \\ x(x+1)\cdots(x+n-1) & (n\in\mathbb{N}; x\in\mathbb{C}). \end{cases}$$

For conditions of convergence and other related details of this function, see [4,17,27].

We also need some other special functions such as Meijer's G-function, Fox's H-function and the Fox–Wright function, etc. Their definitions will be given in the corresponding sections.

Definition 1.4 [31]. Let $f(z) := \sum_{n=0}^{\infty}a_nz^n$ and $g(z) := \sum_{n=0}^{\infty}b_nz^n$ be two power series whose radii of convergence are denoted by R_f and R_g, respectively. Then their Hadamard product is the power series defined by

$$(f*g)(z) := \sum_{n=0}^{\infty}a_nb_nz^n.$$

The radius of convergence R of the Hadamard product series $(f*g)(z)$ satisfies $R_f \cdot R_g \leqslant R$.

If, in particular, one of the power series defines an entire function, then the Hadamard product series defines an entire function, too.

The concept of the Hadamard product is very useful in our investigation. It can help us decompose a newly-emerged function into two known functions. Let us consider the function ${}_pF_{p+r}^{(\alpha,\beta;\rho,\lambda)}[z;b]$. Its decomposition is illustrative. That is

$$ {}_pF_{p+r}^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_1,\ldots,y_{p+r}\end{matrix};z;b\right] = {}_1F_r\left[\begin{matrix} 1 \\ y_1,\ldots,y_r\end{matrix};z\right] * {}_pF_p^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_{1+r},\ldots,y_{p+r}\end{matrix};z;b\right] \quad (|z|<\infty). $$

This paper is organized as follows. Section 2 establishes several new properties of the extended beta functions with different parameters, in particular some connections with the Laguerre polynomial and Fox's H-function are found. In Section 3, we first prove a very general Mellin–Barnes type contour integral representation for the extended generalized hypergeometric function, and then derive a few important results such as the extended Gauss summation formula, extended Kummer's first transformation and extended Nørlund's expansion etc. In Section 4, results depending on some new extended fractional integral operators are obtained. With the help of the extended Riemann–Liouville fractional integral, a decompositional formula of the extended generalized hypergeometric function is formulated. Further, several Kober type fractional integral operators are also considered.

2. New results of the extended beta functions

In this section, we present various properties for the extended beta function $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$ defined by (1) and discuses their (some) connections with the hypergeometric functions, Laguerre polynomials and Fox's H-function.

Theorem 2.1. *The function defined by integral (1) exists under the conditions*

$$ \rho \geqslant 0, \lambda \geqslant 0, \quad \min\{\mathrm{Re}(\beta),\mathrm{Re}(\alpha)\} > 0, \quad \mathrm{Re}(b) > 0 $$

and

$$ \mathrm{Re}(x) > -\mathrm{Re}(\rho\alpha), \quad \mathrm{Re}(y) > -\mathrm{Re}(\lambda\alpha). $$

Further, if $\alpha=\beta$, the integral also exists when $x,y\in\mathbb{C}$. If $b=0$, the conditions of existence becomes $\min\{\mathrm{Re}(x),\mathrm{Re}(y)\}>0$.

Proof. For convenience, we write $\Xi(t) := \frac{b}{t^\rho(1-t)^\lambda}$ $(\mathrm{Re}(b)>0)$. Observe that

- if $\rho>0$ and $\lambda>0$, then $|\Xi(t)|\to\infty$ as $t\to 0$ (or $t\to 1$);
- if $\rho>0$ and $\lambda=0$, then $|\Xi(t)|\to\infty$ as $t\to 0$, and $\Xi(t)\to b$ as $t\to 1$;
- if $\lambda>0$ and $\rho=0$, then $|\Xi(t)|\to\infty$ as $t\to 1$, and $\Xi(t)\to b$ as $t\to 0$.

We only need to consider the first case, and the other cases can be easily included. Let s_1 $(s_1>0)$ and s_2 $(s_2<1)$ be the numbers such that, for any $t\in[s_1,s_2]$ and b fixed, the following inequality

$$ |\Xi(t)| \leqslant T < \infty $$

holds, where T is a positive number. Then, we rewrite integral (1) as

$$ \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y) = \left(\int_0^{s_1} + \int_{s_1}^{s_2} + \int_{s_2}^1\right) t^{x-1}(1-t)^{y-1}{}_1F_1(\alpha;\beta;-\Xi(t))\mathrm{d}t = I_1+I_2+I_3. $$

Since the asymptotic behavior of confluent hypergeometric function ${}_1F_1(x)$ as $\mathrm{Re}(x)\to-\infty$ is given by (see [12, p. 278])

$$ {}_1F_1(\alpha;\beta;x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(-x)^{-\alpha}\left[1+\mathcal{O}\left(|x|^{-1}\right)\right], \quad \mathrm{Re}(x)\to-\infty, $$

it is easy to find that

$$ \begin{aligned} |I_1| &\leqslant \int_0^{s_1} t^{\mathrm{Re}(x)-1}(1-t)^{\mathrm{Re}(y)-1}|{}_1F_1(\alpha;\beta;-\Xi(t))|\mathrm{d}t \\ &\leqslant K\int_0^{s_1} t^{\mathrm{Re}(x)-1}(1-t)^{\mathrm{Re}(y)-1}|\Xi(t)|^{-\mathrm{Re}(\alpha)}e^{\mathrm{Im}(\alpha)\arg(\Xi(t))}\mathrm{d}t \\ &\leqslant K_1\int_0^{s_1} t^{\mathrm{Re}(x)-1}(1-t)^{\mathrm{Re}(y)-1}\left|\frac{b}{t^\rho(1-t)^\lambda}\right|^{-\mathrm{Re}(\alpha)}\mathrm{d}t \\ &= \frac{K_1}{|b|^{\mathrm{Re}(\alpha)}}\int_0^{s_1} t^{\mathrm{Re}(x)+\mathrm{Re}(\rho\alpha)-1}(1-t)^{\mathrm{Re}(y)+\mathrm{Re}(\lambda\alpha)-1}\mathrm{d}t \\ &\leqslant \frac{K_1}{|b|^{\mathrm{Re}(\alpha)}}B(\mathrm{Re}(x)+\mathrm{Re}(\rho\alpha),\mathrm{Re}(y)+\mathrm{Re}(\lambda\alpha)) \end{aligned} $$

$$(K, K_1 > 0, \quad \mathrm{Re}(x) + \mathrm{Re}(\rho\alpha) > 0, \quad \mathrm{Re}(y) + \mathrm{Re}(\lambda\alpha) > 0).$$

The boundedness of integral I_3 can be obtained in the same manner.

Now, we estimate the integral I_2. By the continuity of ${}_1F_1(\alpha;\beta;z)$ we have

$$|{}_1F_1(\alpha;\beta;-\Xi(t))| \leqslant M, \quad t \in [s_1, s_2].$$

Then, by writing $N := \max_{t\in[s_1,s_2]} t^{\mathrm{Re}(x)-1}(1-t)^{\mathrm{Re}(y)-1}$, we get

$$|I_2| \leqslant M \int_{s_1}^{s_2} t^{\mathrm{Re}(x)-1}(1-t)^{\mathrm{Re}(y)-1}\mathrm{d}t \leqslant MN(s_2 - s_1).$$

This completes the proof. □

Remark 2.2. From the proof of Theorem 2.1, it is clear that there always exists a positive number $\mathfrak{C}$ such that

$$\left|\mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x,y)\right| \leqslant \mathfrak{C}B(x,y),$$

if $\min\{\mathrm{Re}(x), \mathrm{Re}(y)\} > 0$. Thus in most cases our extended definitions share most of the properties of the classical ones.

2.1. Relations between the extended beta functions and hypergeometric functions

We begin with the following theorem, which establishes a very useful integral representation of $\mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x,y)$.

Theorem 2.3. *The following integral expression holds true:*

$$\mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1} B_{bs;\rho,\lambda}(x,y)\mathrm{d}s, \tag{7}$$

$$(\rho \geqslant 0, \lambda \geqslant 0; \quad \mathrm{Re}(\beta) > \mathrm{Re}(\alpha) > 0; \quad \mathrm{Re}(b) > 0),$$

where the function $B_{bs;\rho,\lambda}(x,y)$ is defined by (2).

Proof. By using the integral representation of (1) and the Euler integral representation of confluent hypergeometric function [13, p. 126, Theorem 7.1], we have

$$\begin{aligned}
\mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\alpha;\beta;-\Xi(t))\mathrm{d}t \\
&= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 t^{x-1}(1-t)^{y-1}\left[\int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1} e^{-\frac{bs}{t^\rho(1-t)^\lambda}}\mathrm{d}s\right]\mathrm{d}t \\
&= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1}\left[\int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{bs}{t^\rho(1-t)^\lambda}}\mathrm{d}t\right]\mathrm{d}s \\
&= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1} B_{bs;\rho,\lambda}(x,y)\mathrm{d}s.
\end{aligned}$$

This completes the proof. □

Setting $\rho = \lambda = 1$ in (7) gets

$$\mathcal{B}^{(\alpha,\beta)}_{b;1,1}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1} B_{bs}(x,y)\mathrm{d}s, \tag{8}$$

which contains a simpler integrand. Since the generalized beta function $B_{\frac{1}{4}b}(x,y)$ have the following expansion (see [23, pp. 24–25], see also [8, pp. 234–235]):

$$\begin{aligned}
B_{\frac{1}{4}b}(x,y) &= \sqrt{\pi}2^{1-x-y}\frac{\Gamma(x)\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)\Gamma\left(\frac{1+x+y}{2}\right)}\,{}_2F_2\left[\begin{matrix}\frac{2-x-y}{2}, \frac{1-x-y}{2} \\ 1-x, 1-y\end{matrix}; -b\right] \\
&+ \sqrt{\pi}2^{1-x-y}\frac{\Gamma(-x)\Gamma(y-x)}{\Gamma\left(\frac{y-x}{2}\right)\Gamma\left(\frac{1+y-x}{2}\right)} b^x{}_2F_2\left[\begin{matrix}\frac{2+x-y}{2}, \frac{1+x-y}{2} \\ 1+x, 1+x-y\end{matrix}; -b\right] \\
&+ \sqrt{\pi}2^{1-x-y}\frac{\Gamma(-y)\Gamma(x-y)}{\Gamma\left(\frac{x-y}{2}\right)\Gamma\left(\frac{1+x-y}{2}\right)} b^y{}_2F_2\left[\begin{matrix}\frac{2+y-x}{2}, \frac{1+y-x}{2} \\ 1+y, 1+y-x\end{matrix}; -b\right],
\end{aligned}$$

we can substitute this into (8) and integrate out s to get the following result.

Corollary 2.4. *The following hypergeometric series expression holds true:*

$$\begin{aligned}\mathcal{B}_{b;1,1}^{(\alpha,\beta)}(x,y) &= \sqrt{\pi}2^{1-x-y}\frac{\Gamma(x)\Gamma(y)}{\Gamma(\frac{x+y}{2})\Gamma(\frac{1+x+y}{2})}{}_3F_3\left[\begin{matrix}\frac{2-x-y}{2},\frac{1-x-y}{2},\alpha\\ 1-x,1-y,\beta\end{matrix};-4b\right]\\ &+\sqrt{\pi}2^{1+x-y}\frac{\Gamma(\beta)\Gamma(\alpha+x)}{\Gamma(\alpha)\Gamma(\beta+x)}\frac{\Gamma(-x)\Gamma(y-x)b^x}{\Gamma(\frac{y-x}{2})\Gamma(\frac{1+y-x}{2})}{}_3F_3\left[\begin{matrix}\frac{2+x-y}{2},\frac{1+x-y}{2},\alpha+x\\ 1+x,1+x-y,\beta+x\end{matrix};-4b\right]\\ &+\sqrt{\pi}2^{1-x+y}\frac{\Gamma(\beta)\Gamma(\alpha+x)}{\Gamma(\alpha)\Gamma(\beta+x)}\frac{\Gamma(-y)\Gamma(x-y)b^y}{\Gamma(\frac{x-y}{2})\Gamma(\frac{1+x-y}{2})}{}_3F_3\left[\begin{matrix}\frac{2+y-x}{2},\frac{1+y-x}{2},\alpha+y\\ 1+y,1+y-x,\beta+y\end{matrix};-4b\right],\end{aligned}$$

where $\min\{\mathrm{Re}(x),\mathrm{Re}(y),\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(b)\}>0$, *and no two members of* $\{0,x,y\}$ *differ by an integer.*

Corollary 2.5. *If* $b,\rho,\lambda>0,\min\{\mathrm{Re}(x),\mathrm{Re}(y)\}>0,\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)>0$, *then we have the following inequality*

$$\left|\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)\right|\leqslant\Omega_{\rho,\lambda}^{\alpha,\beta}(b)B(\mathrm{Re}(x),\mathrm{Re}(y)),\tag{9}$$

where, and in what follows, $\Upsilon(\rho,\lambda):=\frac{(\rho+\lambda)^{\rho+\lambda}}{\rho^\rho\lambda^\lambda}$ *and*

$$\Omega_{\rho,\lambda}^{\alpha,\beta}(b):={}_1F_1\left[\begin{matrix}\mathrm{Re}(\alpha)\\ \mathrm{Re}(\beta)\end{matrix};-\Upsilon(\rho,\lambda)b\right]\frac{\Gamma(\mathrm{Re}(\alpha))\Gamma(\mathrm{Re}(\beta)-\mathrm{Re}(\alpha))}{|\Gamma(\alpha)\Gamma(\beta-\alpha)|}.$$

Proof. Since the function $\Xi(t)$ attains its minimum at $t=\rho/(\rho+\lambda)$, we get

$$|B_{b;\rho;\lambda}(x,y)|\leqslant\int_0^1 t^{\mathrm{Re}(x)-1}(1-t)^{\mathrm{Re}(y)-1}\exp(-\Xi(t))\mathrm{d}t\leqslant\exp(-\Upsilon(\rho,\lambda)b)B(\mathrm{Re}(x),\mathrm{Re}(y)).\tag{10}$$

By using (7) and (10) together we find that

$$\begin{aligned}\left|\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)\right| &\leqslant\frac{|\Gamma(\beta)|}{|\Gamma(\alpha)\Gamma(\beta-\alpha)|}\int_0^1 s^{\mathrm{Re}(\alpha)-1}(1-s)^{\mathrm{Re}(\beta-\alpha)-1}|B_{bs;\rho,\lambda}(x,y)|\mathrm{d}s\\ &\leqslant\frac{B(\mathrm{Re}(x),\mathrm{Re}(y))\Gamma(\mathrm{Re}(\beta))}{|\Gamma(\alpha)\Gamma(\beta-\alpha)|}\int_0^1 s^{\mathrm{Re}(\alpha)-1}(1-s)^{\mathrm{Re}(\beta-\alpha)-1}\exp(-\Upsilon(\rho,\lambda)bs)\mathrm{d}s\\ &={}_1F_1\left[\begin{matrix}\mathrm{Re}(\alpha)\\ \mathrm{Re}(\beta)\end{matrix};-\Upsilon(\rho,\lambda)b\right]\frac{\Gamma(\mathrm{Re}(\alpha))\Gamma(\mathrm{Re}(\beta)-\mathrm{Re}(\alpha))}{|\Gamma(\alpha)\Gamma(\beta-\alpha)|}B(\mathrm{Re}(x),\mathrm{Re}(y)).\qquad\square\end{aligned}$$

Remark 2.6. If we set $\rho=\lambda=\alpha=\beta=1$ and $x,y,b>0$, inequality (10) reduces to

$$B_b(x,y)\leqslant e^{-4b}B(x,y),$$

which has been proved in [8, p. 224, Theorem 5.5].

When all parameters in inequality (9) become real, it gives

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)\leqslant\Omega_{\rho,\lambda}^{\alpha,\beta}(b)B(x,y)\Rightarrow\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n,y)\leqslant\Omega_{\rho,\lambda}^{\alpha,\beta}(b)\frac{(x)_n}{(x+y)_n},$$

where $\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n,y)$ is defined by (4) and

$$\Omega_{\rho,\lambda}^{\alpha,\beta}(b)={}_1F_1\left[\begin{matrix}\alpha\\ \beta\end{matrix};-\Upsilon(\rho,\lambda)b\right].$$

Applying this inequality to (6) we have the following.

Corollary 2.7. *If* $b,\rho,\lambda>0,\alpha,\beta>0,x_i,y_j>0$ $(i=1,\dots,p;j=1,\dots,q)$ *and* x_i,y_j *also satisfy the conditions given in Definition 1.3, we have*

$${}_pF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\dots,x_p\\ y_1,\dots,y_q\end{matrix};z;b\right]\leqslant\begin{cases}\left[\Omega_{\rho,\lambda}^{\alpha,\beta}(b)\right]^q{}_pF_q\left[\begin{matrix}x_1,\dots,x_p\\ y_1,\dots,y_q\end{matrix};z\right], & p=q;\ p=q+1;\\ \left[\Omega_{\rho,\lambda}^{\alpha,\beta}(b)\right]^p{}_pF_q\left[\begin{matrix}x_1,\dots,x_p\\ y_1,\dots,y_q\end{matrix};z\right], & p<q.\end{cases}$$

2.2. Relations between the extended beta functions and Laguerre polynomials

Some nice representations of the extended beta function $B_b(x,y)$ in terms of the Laguerre polynomials have been established in [8,23]. Recently, the same method has also been used to find some useful series representation of the extended Mittag–Leffler function [28]. The following theorems focus on the connections between the extended beta function $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$ and the Laguerre polynomials $L_n(x)$ defined by the generating function [3, p. 176, Eq. (5.33)]

$$(1-u)^{-1}\exp\left(-\frac{xu}{1-u}\right)=\sum_{n=0}^{\infty}L_n(x)u^n, \qquad |u|<1,\ 0\leqslant x<\infty. \tag{11}$$

Theorem 2.8. *The extended beta function defined by (2) possesses the following series expression*

$$B_{b;\rho,\lambda}(x,y)=e^{-b}\sum_{n=0}^{\infty}L_n(b)\mathbf{S}_n(1) \quad (\rho\geqslant 0,\ \lambda\geqslant 0,\ b\geqslant 0;\quad \mathrm{Re}(x)>-\rho,\ \mathrm{Re}(y)>-\lambda), \tag{12}$$

where $\mathbf{S}_n(z)$ is a polynomial defined by

$$\mathbf{S}_n(z)=\sum_{m=0}^{n}\frac{(-n)_m}{m!}\frac{\Gamma(x+(m+1)\rho)\Gamma(y+(m+1)\lambda)}{\Gamma(x+y+(m+1)(\rho+\lambda))}z^k. \tag{13}$$

For $\rho=M,\lambda=N$ $(M,N\in\mathbb{N})$, it reduces to

$$\begin{aligned}B_{b;M,N}(x,y) &= B(x,y)\frac{(x)_M(y)_N}{(x+y)_{M+N}}e^{-b}\\ &\quad\times\sum_{n=0}^{\infty}L_n(b)_{M+N+1}F_{M+N}\left[\begin{matrix}-n,\Delta(M;x+M),\Delta(N;y+N)\\ \Delta(M+N;x+y+M+N)\end{matrix};\frac{M^MN^N}{(M+N)^{M+N}}\right],\end{aligned} \tag{14}$$

where, and in what follows, $\Delta(M;x)$ abbreviates the array of M parameters

$$\frac{x}{M},\ \frac{x+1}{M},\ \ldots,\ \frac{x+M-1}{M},\ M=1,2,\ldots.$$

Proof. In order to prove the theorem we need a useful variant of (11), that is,

$$e^{-\frac{b}{u}}=e^{-b}u\sum_{n=0}^{\infty}L_n(b)(1-u)^n,\quad |u|<1, 0\leqslant b<\infty. \tag{15}$$

Observe that $0\leqslant|t^{\rho}(1-t)^{\lambda}|\leqslant\Upsilon^{-1}(\rho,\lambda)<1,\forall t\in[0,1]$. We can set $u=t^{\rho}(1-t)^{\lambda}$ in (15) to get

$$\begin{aligned}e^{-\frac{b}{t^{\rho}(1-t)^{\lambda}}} &= e^{-b}t^{\rho}(1-t)^{\lambda}\sum_{n=0}^{\infty}L_n(b)\left[1-t^{\rho}(1-t)^{\lambda}\right]^n\\ &= e^{-b}t^{\rho}(1-t)^{\lambda}\sum_{n=0}^{\infty}L_n(b)\sum_{m=0}^{n}\frac{(-n)_m}{m!}t^{\rho m}(1-t)^{\lambda m}.\end{aligned}$$

Then, by using definition (2) and interchanging the order of integration and summation, we may find that

$$\begin{aligned}B_{b;\rho,\lambda}(x,y) &= e^{-b}\sum_{n=0}^{\infty}L_n(b)\sum_{m=0}^{n}\frac{(-n)_m}{m!}\int_0^1 t^{x+\rho+\rho m-1}(1-t)^{y+\lambda+\lambda m-1}\mathrm{d}t\\ &= e^{-b}\sum_{n=0}^{\infty}L_n(b)\sum_{m=0}^{n}\frac{(-n)_m}{m!}\frac{\Gamma(x+(m+1)\rho)\Gamma(y+(m+1)\lambda)}{\Gamma(x+y+(m+1)(\rho+\lambda))},\end{aligned}$$

which completes the proof of the first assertion.

The second assertion follows directly from the identity

$$(a)_{kn}=\left(\frac{a}{k}\right)_n\left(\frac{a+1}{k}\right)_n\cdots\left(\frac{a+k-1}{k}\right)_n k^{kn}. \qquad \square$$

Since the Laguerre polynomial $L_n(x)$ possesses the orthogonality property [14, p. 809]

$$\int_0^{\infty}e^{-x}L_m(x)L_n(x)\mathrm{d}x=\delta_{mn}, \tag{16}$$

we easily deduce the following integral identity.

Corollary 2.9. *If* $m, M, N \in \mathbb{N}$ *and* $\min\{\mathrm{Re}(x), \mathrm{Re}(y)\} > 0$, *then the following integral identity*

$$\int_0^\infty L_m(b) B_{b;M,N}(x,y) \mathrm{d}b = B(x,y) \frac{(x)_M (y)_N}{(x+y)_{M+N}} {}_{M+N+1}F_{M+N}\left[\begin{matrix} -m, \Delta(M; x+M), \Delta(N; y+N) \\ \Delta(M+N; x+y+M+N) \end{matrix}; \frac{M^M N^N}{(M+N)^{M+N}}\right] \tag{17}$$

holds true.

Remark 2.10. There is another way to derive this nice formula. Since the equation [14, p. 809]

$$\int_0^\infty e^{-\beta x} L_m(x) \mathrm{d}x = \frac{(\beta-1)^m}{\beta^{m+1}}, \quad \mathrm{Re}(\beta) > 0$$

holds, the integral on the left-hand side of (17) can be evaluated as

$$\begin{aligned} \int_0^\infty L_m(b) B_{b;M,N}(x,y) \mathrm{d}b &= \int_0^\infty L_m(b) \left[\int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{b}{t^M(1-t)^N}\right) \mathrm{d}t\right] \mathrm{d}b \\ &= \int_0^1 t^{x-1}(1-t)^{y-1} \left[\int_0^\infty L_m(b) \exp\left(-\frac{b}{t^M(1-t)^N}\right) \mathrm{d}b\right] \mathrm{d}t \\ &= \int_0^1 t^{x+M-1}(1-t)^{y+N-1} \left(1 - t^M(1-t)^N\right)^m \mathrm{d}t, \end{aligned}$$

which, after a brief computation, is in fact the result (17). The justification of the interchange of the order of integration follows easily from the inequality [27, p. 450, Eq. (18.14.8)]:

$$|L_m(b)| < e^{b/2} \quad (b \geqslant 0).$$

It is also worth mentioning that we can consider (14) and (17) from another point of view. By a Laguerre series, we mean a series of the form (see [3, p. 179, Theorem 5.2])

$$f(b) = \sum_{m=0}^\infty c_m L_m(b), \quad 0 < b < \infty, \tag{18}$$

where the expansion coefficients c_m is determined by

$$c_m = \int_0^\infty e^{-b} f(b) L_m(b) \mathrm{d}b, \quad m = 0, 1, 2, \ldots \tag{19}$$

If f is piecewise smooth in every finite interval $b_1 \leqslant b \leqslant b_2, 0 < b_1 < b_2 < \infty$, and

$$\int_0^\infty e^{-b} f^2(b) \mathrm{d}b < \infty,$$

then the Laguerre series (18) with coefficients determined by (19) converges pointwise to $f(b)$ at every continuity point of f. At points of discontinuity, the series converges to the average value $\frac{1}{2}[f(x^+) + f(x^-)]$.

Now, we can set $f(b) = e^b B_{b;M,N}(x,y)$. Then (17) gives the expression of c_m. Substituting this c_m into the Laguerre series (18) we can find (14).

Applying (16) to equation ([8, p. 238, Theorem 5.13])

$$B_b(x,y) = e^{-2b} \sum_{m,n=0}^\infty B(x+m+1, y+n+1) L_m(b) L_n(b), \quad \mathrm{Re}(x) > -1, \quad \mathrm{Re}(y) > -1, \tag{20}$$

we can deduce the following theorem.

Theorem 2.11. *Let* $\min\{\mathrm{Re}(x), \mathrm{Re}(y)\} > -1$. *Then there holds the formula*

$$\int_0^\infty e^b B_b(x,y) \mathrm{d}b = B(x+1, y+1) {}_3F_2\left[\begin{matrix} 1, x+1, y+1 \\ \frac{x+y}{2}+1, \frac{x+y+1}{2}+1 \end{matrix}; \frac{1}{4}\right], \tag{21}$$

where function $B_b(x,y)$ *is defined by* (3).

Proof. We multiply both sides of (20) by e^b and integrate out b, then the orthogonality property (16) help us reduce the double series involving in the right-hand side of (20) to a single series.

There is also another way to prove (21) without using orthogonality. Express the extended beta function $B_b(x,y)$ in its integral representation, then we have

$$\begin{aligned}\int_0^\infty e^b B_b(x,y)\mathrm{d}b &= \int_0^\infty e^b\left[\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{b}{t(1-t)}\right)\mathrm{d}t\right]\mathrm{d}b\\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\left[\int_0^\infty e^b\exp\left(-\frac{b}{t(1-t)}\right)\mathrm{d}b\right]\mathrm{d}t.\end{aligned}$$

It is straightforward to justify the interchange of the orders of integration. Write $f(t) := \frac{1}{t(1-t)} - 1$. Then $f(t)$ remains positive for all $t \in (0,1)$. Exactly, $f(t) \geqslant 3, \forall t \in [0,1]$. Thus, the inner integral is evaluated as

$$\int_0^\infty e^{-f(t)b}\mathrm{d}b = \frac{1}{f(t)} = \frac{t(1-t)}{1-t(1-t)}.$$

Now,

$$\begin{aligned}\int_0^\infty e^b B_b(x,y)\mathrm{d}b &= \int_0^1 \frac{t^x(1-t)^y}{1-t(1-t)}\mathrm{d}t\\ &= \sum_{k=0}^\infty \frac{(1)_k}{k!}\frac{\Gamma(x+1+k)\Gamma(y+1+k)}{\Gamma(x+y+2+2k)}\\ &= B(x+1,y+1)\sum_{k=0}^\infty \frac{(1)_k(x+1)_k(y+1)_k}{(x+y+2)_{2k}k!}\\ &= B(x+1,y+1)\sum_{k=0}^\infty \frac{(1)_k(x+1)_k(y+1)_k}{\left(\frac{x+y}{2}+1\right)_k\left(\frac{x+y+1}{2}+1\right)_k}\frac{\left(\frac{1}{4}\right)^k}{k!}.\end{aligned}$$

The result follows from interpreting the final series as a generalized hypergeometric function. □

Employing the same method we give the following statement.

Theorem 2.12. *Let $\rho, \lambda > 0, \min\{\mathrm{Re}(x), \mathrm{Re}(y)\} > 0$. Then there holds the formula*

$$\int_0^\infty e^b B_{b;\rho,\lambda}(x,y)\mathrm{d}b = {}_3\Psi_1\left[1\left|\begin{matrix}(1,1),(x+\rho,\rho),(y+\lambda,\lambda)\\ (x+y+\rho+\lambda,\rho+\lambda)\end{matrix}\right.\right], \tag{22}$$

where ${}_p\Psi_q(z)$ is the Fox–Wright function defined by (see [17, pp. 56–58])

$${}_p\Psi_q(z) = {}_p\Psi_q\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right] := \sum_{k=0}^\infty \frac{\prod_{i=1}^p\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^q\Gamma(b_j+\beta_j k)}\frac{z^k}{k!} \tag{23}$$

$$\left(z, a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j \in \mathbb{R}(i=1,\ldots,p; j=1,\ldots,q);\quad \Delta := \sum_{j=1}^q\beta_j - \sum_{i=1}^p\alpha_i \geqslant -1\right).$$

Proof. The derivation of (22) is direct. However, the convergence of ${}_3\Psi_1(1)$ needs more consideration. According to the conditions given in [17, Theorem 1.5], the series in (23) is absolutely convergent for

$$|z| < \delta := \prod_{i=1}^p|\alpha_i|^{-\alpha_i}\prod_{j=1}^q|\beta_j|^{\beta_j},$$

provided that $\Delta = -1$. In our case, we have

$$\Delta = -1 \quad \text{and} \quad \delta = \left(1+\frac{\lambda}{\rho}\right)^\rho\left(1+\frac{\rho}{\lambda}\right)^\lambda > 1 \quad (\rho > 0, \lambda > 0).$$

Thus, ${}_3\Psi_1(1)$ is convergent. □

Theorem 2.13. *The extended beta function defined by (1) possesses the following series expression*

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y) = \sum_{n=0}^\infty \mathbf{S}_n(1)\,{}_2F_2\left[\begin{matrix}n+1,\alpha\\ 1,\beta\end{matrix};-b\right],$$

where the polynomial $\mathbf{S}_n(z)$ is defined by (13).

Proof. We start by recalling the following useful identity [14, p. 810, Eq. (7.415)]

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}e^{-\beta x}L_n(\beta x)\mathrm{d}x = B(\lambda,\mu)\,{}_2F_2\left[\begin{matrix}n+1,\lambda\\1,\lambda+\mu\end{matrix};-\beta\right] \quad (\mathrm{Re}(\lambda)>0,\ \mathrm{Re}(\mu)>0). \tag{24}$$

Then, using (12) in integral representation (7), we get

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1}e^{-bs}\sum_{n=0}^{\infty}L_n(bs)\mathbf{S}_n(1)\mathrm{d}s.$$

Interchanging the order of summation and integration shows that

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y) = \sum_{n=0}^{\infty}\frac{\mathbf{S}_n(1)}{B(\alpha,\beta-\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\beta-\alpha-1}e^{-bs}L_n(bs)\mathrm{d}s.$$

Using (24) with suitable parameters, we get the result. $\square$

2.3. H-function and G-function representations of $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$

For integers m,n,p,q such that $0\leqslant m\leqslant q, 0\leqslant n\leqslant p$, and for parameters $a_i,b_i\in\mathbb{C}$ and for parameters $\alpha_i,\beta_j\in\mathbb{R}_+=(0,\infty)$ $(i=1,\ldots,p; j=1,\ldots,q)$, the H-function is defined in terms of a Mellin–Barnes type integral in the following manner ([16, pp. 1–2]; see also [18, p. 343, Definition E.1.] and [22, p. 2, Definition 1.1]):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}}\Theta(s)z^{-s}\mathrm{d}s,$$

where

$$\Theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}$$

and the contour $\mathfrak{L}$ is suitably chosen, and an empty product, if it occurs, is taken to be unity.

When $\alpha_i=\beta_j=1$ $(i=1,\ldots,p; j=1,\ldots,q)$, it reduces to the Meijer's G-function, i.e.,

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,1)_{1,p}\\(b_j,1)_{1,q}\end{matrix}\right.\right].$$

More detailed information about Meijer's G-function may be found in [3,21].

Theorem 2.14. *Let*

$$\rho>0,\ \lambda>0;\quad \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(b)\}>0;\quad \mathrm{Re}(x)>-\mathrm{Re}(\rho\alpha),\quad \mathrm{Re}(y)>-\mathrm{Re}(\lambda\alpha).$$

Then the extended beta function (1) can be expressed as

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}H_{2,4}^{3,1}\left[b\left|\begin{matrix}(1-\alpha,1),(x+y,\rho+\lambda)\\(0,1),(x,\rho),(y,\lambda),(1-\beta,1)\end{matrix}\right.\right].$$

Let

$$\rho=0,\lambda>0;\quad \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(b),\mathrm{Re}(x)\}>0;\quad \mathrm{Re}(y)>-\mathrm{Re}(\lambda\alpha).$$

Then we have

$$\mathcal{B}_{\beta;0,\lambda}^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)\Gamma(x)}{\Gamma(\alpha)}H_{2,3}^{2,1}\left[b\left|\begin{matrix}(1-\alpha,1),(x+y,\lambda)\\(0,1),(y,\lambda),(1-\beta,1)\end{matrix}\right.\right]. \tag{25}$$

Similarly, we have

$$\mathcal{B}_{\beta;\rho,0}^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)\Gamma(y)}{\Gamma(\alpha)}H_{2,3}^{2,1}\left[b\left|\begin{matrix}(1-\alpha,1),(x+y,\rho)\\(0,1),(x,\rho),(1-\beta,1)\end{matrix}\right.\right], \tag{26}$$

where

$$\rho>0,\lambda=0;\quad \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(b),\mathrm{Re}(y)\}>0;\quad \mathrm{Re}(x)>-\mathrm{Re}(\rho\alpha).$$

Proof. For Kummer's confluent hypergeometric function ${}_1F_1(\alpha;\beta;z)$, the following contour integral representation holds true (see [13, p. 127, Theorem 7.2]):

$$ {}_1F_1(\alpha;\beta;z)=\frac{1}{2\pi \mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{-I\infty}^{+I\infty}\frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\beta+s)}(-z)^s\mathrm{d}s,|\arg(-z)|<\frac{\pi}{2}. $$

The path of integration separates all the poles $s=-\alpha-n(n\in\mathbb{N}_0)$ to the left and all poles $s=n\in\mathbb{N}_0$ to the right.

Then, from integral representation (1),

$$ \begin{aligned} \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\alpha,\beta;-\Xi(t))\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_0^1 t^{x-1}(1-t)^{y-1}\left[\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\beta+s)}\left(\frac{b}{t^\rho(1-t)^\lambda}\right)^s\mathrm{d}s\right]\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\beta+s)}b^s\left[\int_0^1 t^{x-\rho s-1}(1-t)^{y-\lambda s-1}\mathrm{d}t\right]\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\Gamma(\alpha+s)\Gamma(-s)\Gamma(x-\rho s)\Gamma(y-\lambda s)}{\Gamma(\beta+s)\Gamma(x+y-(\rho+\lambda)s)}b^s\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\Gamma(s)\Gamma(x+\rho s)\Gamma(y+\lambda s)\Gamma(\alpha-s)}{\Gamma(\beta-s)\Gamma(x+y+(\rho+\lambda)s)}b^{-s}\mathrm{d}s\ (s\to -s) \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha)}H_{2,4}^{3,1}\left[b\,\middle|\,\begin{matrix}(1-\alpha,1),(x+y,\rho+\lambda)\\(0,1),(x,\rho),(y,\lambda),(1-\beta,1)\end{matrix}\right]. \end{aligned} $$

Eqs. (25) and (26) can be derived in the same way. □

By using the famous Gauss–Legendre multiplication formula [27, p. 138. Eq. (5.5.6)]:

$$ \Gamma(mz)=(2\pi)^{\frac{1-m}{2}}m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right),\quad z\neq 0,-\frac{1}{m},-\frac{2}{m},\ldots;\ m\in\mathbb{N}, $$

it is straightforward to find the following results.

Corollary 2.15. *If $m,n\in\mathbb{N},\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0$, we have*

$$ \mathcal{B}_{b;m,n}^{(\alpha,\beta)}(x,y)=\sqrt{2\pi}\frac{m^{x-\frac{1}{2}}n^{y-\frac{1}{2}}}{(m+n)^{x+y-\frac{1}{2}}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}G_{m+n+1,m+n+2}^{m+n+1,0}\left[\Upsilon(m,n)b\,\middle|\,\begin{matrix}1-\alpha,\Delta(m+n,x+y)\\0,\Delta(m;x),\Delta(n;y),1-\beta\end{matrix}\right]. $$

Further, the extended beta function $\mathcal{B}_{b;m,n}^{(\alpha,\beta)}(x,y)$ (as a function of b) satisfies the following ordinary differential equation:

$$ \left[-\Upsilon(m,n)b(\vartheta_b+\alpha)\mathfrak{D}_b^{m+n;x+y}-\vartheta_b\mathfrak{D}_b^{m;x}\mathfrak{D}_b^{n;y}(\vartheta_b-1+\beta)\right]w(b)=0, $$

where $w(b):=\mathcal{B}_{b;m,n}^{(\alpha,\beta)}(x,y)$, the differential operators $\mathfrak{D}_b^{m;x}$ and ϑ_b are defined by

$$ \mathfrak{D}_b^{m;x}:=\prod_{j=1}^{m}\left(\vartheta_b-\frac{x+j-1}{m}\right)\quad\text{and}\quad\vartheta_b:=b\frac{\mathrm{d}}{\mathrm{d}b}. $$

Remark 2.16. When $\alpha=\beta$ and $\rho=\lambda=1$ we can easily find:

$$ B_b(x,y)=\sqrt{\pi}2^{1-x-y}G_{2,3}^{3,0}\left[4b\,\middle|\,\begin{matrix}\frac{x+y}{2},\frac{x+y+1}{2}\\0,x,y\end{matrix}\right]\quad(\mathrm{Re}(b)>0), $$

which has been proved in [8, p. 232, Theorem 5.12].

2.4. A finite sum of $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$

Theorem 2.17. *The extended beta function defined by (1) have the following finite sum*

$$ \sum_{k=0}^{N}(-1)^N\binom{N}{k}\frac{(\beta-\alpha)_k}{(\beta)_k}\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta+k)}(x,y)=\frac{(\alpha)_N}{(\beta)_N}\mathcal{B}_{b;\rho,\lambda}^{(\alpha+N,\beta+N)}(x,y), $$

where

$$ N\in\mathbb{N},\ \mathrm{Re}(b)>0,\quad\min\{\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)\}>0;\quad\rho\geqslant 0,\ \lambda\geqslant 0 $$

and

$$ \mathrm{Re}(x)>-\mathrm{Re}(\rho\alpha),\quad\mathrm{Re}(y)>-\mathrm{Re}(\lambda\alpha). $$

Proof. For Kummer's confluent hypergeometric function ${}_1F_1(z)$ we have the following finite sum [6, p. 411, 5.14.1 (1)]

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{(\beta-\alpha)_k}{(\beta)_k}\,{}_1F_1\left[\begin{matrix}\alpha\\ \beta+k\end{matrix};z\right]=\frac{(\alpha)_N}{(\beta)_N}\,{}_1F_1\left[\begin{matrix}\alpha+N\\ \beta+N\end{matrix};z\right]. \tag{27}$$

Replacing z with $-\Xi(t)$, and multiplying by $t^{x-1}(1-t)^{y-1}$ on both sides of (27) we get

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{(\beta-\alpha)_k}{(\beta)_k}t^{x-1}(1-t)^{y-1}\,{}_1F_1\left[\begin{matrix}\alpha\\ \beta+k\end{matrix};-\Xi(t)\right]=\frac{(\alpha)_N}{(\beta)_N}t^{x-1}(1-t)^{y-1}\,{}_1F_1\left[\begin{matrix}\alpha+N\\ \beta+N\end{matrix};-\Xi(t)\right]. \tag{28}$$

Finally, we obtain Theorem 2.17 by integrating (28) with respect to t over $(0,1)$. □

An application of Theorem 2.17 gives the following result.

Corollary 2.18. *The extended Gauss hypergeometric function defined by (5) have the following finite sum with respect to parameters α and β, that is,*

$${}_2F_1^{(\alpha+N,\beta+N;\rho,\lambda)}\left[\begin{matrix}x_1, & x_2\\ & y_1\end{matrix};z;b\right]=\frac{(\beta)_N}{(\alpha)_N}\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{(\beta-\alpha)_k}{(\beta)_k}\,{}_2F_1^{(\alpha,\beta+k;\rho,\lambda)}\left[\begin{matrix}x_1, & x_2\\ & y_1\end{matrix};z;b\right],$$

where

$$N\in\mathbb{N},\ \min\{\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)\}>0;\quad \rho\geqslant 0,\lambda\geqslant 0;\quad \mathrm{Re}(y_1)>\mathrm{Re}(x_2)>0,\ |z|<1.$$

Remark 2.19. In fact, many finite sum formulas of Kummer's confluent hypergeometric function listed in [6] can be used to establish some useful finite sum formulas for the extended beta function, and further the finite sum formulas of the extended Gauss hypergeometric function.

3. Contour integral representation of ${}_pF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$

The Mellin transform of a function $f(t)$, denoted by $F(s)$, is defined by (see [22, p. 46]; see also [17])

$$F(s):=\mathfrak{M}\{f(t)\}(s)=\int_0^\infty t^{s-1}f(t)\mathrm{d}t, \tag{29}$$

provided that the integral converges. The inverse Mellin transform is given by the contour integral

$$f(x)=\mathfrak{M}^{-1}\{F(s)\}(x)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}F(s)x^{-s}\mathrm{d}s\quad(\gamma=\mathrm{Re}(s)). \tag{30}$$

If $F(s)$ is analytic in the relevant strip then $f(x)$ is uniquely determined by $F(s)$ by using the formula (30). For a general theory of the Mellin transform, we refer to [7].

In order to establish the main result of this section, we need the following well-known theorem which is widely used to evaluate definite integrals and infinite series.

Theorem 3.1 (*Ramanujan's Master Theorem [2]*). *Assume f admits an expansion of the form*

$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(n)}{n!}(-x)^n\quad(\varphi(0)\neq 0).$$

Then the Mellin transform of f is given by

$$\int_0^\infty x^{s-1}f(x)\mathrm{d}x=\Gamma(s)\varphi(-s).$$

We are now ready to present a Mellin–Barnes type contour integral representation of the extended generalized hypergeometric function.

Theorem 3.2. *For $p=q+1$, the extended generalized hypergeometric functions ${}_pF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$ possesses the following integral representation:*

$$\begin{aligned}{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};z;b\right]&=\frac{1}{(2\pi i)^q}\frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_q-i\infty}^{\gamma_q+i\infty}H_q(\mathbf{x},\mathbf{y};\mathbf{s})\prod_{j=1}^{q}\frac{\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(\beta-s_j)}\\&\quad\times{}_{q+1}F_q\left[\begin{matrix}x_1,x_2+\rho s_1,\ldots,x_{q+1}+\rho s_q\\ y_1+(\rho+\lambda)s_1,\ldots,y_q+(\rho+\lambda)s_q\end{matrix};z\right]b^{-\sum_{j=1}^{q}s_j}\mathrm{d}s_1\cdots\mathrm{d}s_q,\end{aligned} \tag{31}$$

where

$$\rho,\lambda>0,\ \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(b)\}>0,\quad \mathrm{Re}(y_j)>\mathrm{Re}(x_{j+1})>0,\quad \gamma_j:=\mathrm{Re}(s_j)>0,$$

$$H_q(\mathbf{x},\mathbf{y};\mathbf{s}):=\frac{C_q(\mathbf{x},\mathbf{y};\mathbf{s})}{C_q(\mathbf{x},\mathbf{y};\mathbf{0})}\quad\text{and}\quad C_q(\mathbf{x},\mathbf{y};\mathbf{s}):=\prod_{j=1}^{q}\frac{\Gamma(x_{j+1}+\rho s_j)\Gamma(y_j-x_{j+1}+\lambda s_j)}{\Gamma(y_j+(\rho+\lambda)s_j)}.$$

In the above contour integral, we require that each contour in the complex s_j-plane starting at $-i\infty$ and ending at $+i\infty$ separates all the poles of $\Gamma(x_{j+1}+\rho s_j)\Gamma(y_j-x_{j+1}+\lambda s_j)\Gamma(s_j)$(i.e., $s_j=-n; s_j=\frac{-x_{j+1}-n}{\rho}; s_j=\frac{x_{j+1}-y_j-n}{\lambda}$ $(n\in\mathbb{N}_0)$) to the left and all the poles of $\Gamma(\alpha-s_j)$ (i.e., $s_j=\alpha+n$ $(n\in\mathbb{N}_0)$) to the right.

For $p=q$, we have

$$ {}_qF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_q\\ y_1,\ldots,y_q\end{matrix};z;b\right]=\frac{1}{(2\pi i)^q}\frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_q-i\infty}^{\gamma_q+i\infty}\widetilde{H}_q(\mathbf{x},\mathbf{y};\mathbf{s})\prod_{j=1}^{q}\frac{\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(\beta-s_j)}\times{}_qF_q\left[\begin{matrix}x_1+\rho s_1,\ldots,x_q+\rho s_q\\ y_1+(\rho+\lambda)s_1,\ldots,y_q+(\rho+\lambda)s_q\end{matrix};z\right]b^{-\sum_{j=1}^{q}s_j}\mathrm{d}s_1\cdots\mathrm{d}s_q, \tag{32}$$

where

$$\rho,\lambda>0,\ \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\ \mathrm{Re}(b)\}>0,\quad \mathrm{Re}(y_j)>\mathrm{Re}(x_j)>0,\quad \gamma_j:=\mathrm{Re}(s_j)>0,$$

$$\widetilde{H}_q(\mathbf{x},\mathbf{y};\mathbf{s}):=\frac{\widetilde{C}_q(\mathbf{x},\mathbf{y};\mathbf{s})}{\widetilde{C}_q(\mathbf{x},\mathbf{y};\mathbf{0})}\quad\text{and}\quad \widetilde{C}_q(\mathbf{x},\mathbf{y};\mathbf{s}):=\prod_{j=1}^{q}\frac{\Gamma(x_j+\rho s_j)\Gamma(y_j-x_j+\lambda s_j)}{\Gamma(y_j+(\rho+\lambda)s_j)}.$$

For $p<q (r=q-p)$ we have

$$ {}_pF_{p+r}^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_p\\ y_1,\ldots,y_{p+r}\end{matrix};z;b\right]=\frac{1}{(2\pi i)^p}\frac{\Gamma^p(\beta)}{\Gamma^p(\alpha)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_p-i\infty}^{\gamma_p+i\infty}K_p(\mathbf{x},\mathbf{y};\mathbf{s})\prod_{j=1}^{p}\frac{\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(\beta-s_j)}\times{}_pF_{p+r}\left[\begin{matrix}x_1+\rho s_1,\ldots,x_p+\rho s_p\\ y_1,\ldots,y_r,y_{r+1}+(\rho+\lambda)s_1,\ldots,y_{p+r}+(\rho+\lambda)s_p\end{matrix};z\right]b^{-\sum_{j=1}^{p}s_j}\mathrm{d}s_1\cdots\mathrm{d}s_p, \tag{33}$$

where

$$\rho,\lambda>0,\ \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(b)\}>0,\quad \mathrm{Re}(y_{r+j})>\mathrm{Re}(x_{j+1})>0,\quad \gamma_j:=\mathrm{Re}(s_j)>0,$$

$$K_p(\mathbf{x},\mathbf{y};\mathbf{s}):=\frac{D_p(\mathbf{x},\mathbf{y};\mathbf{s})}{D_p(\mathbf{x},\mathbf{y};\mathbf{0})}\quad\text{and}\quad D_p(\mathbf{x},\mathbf{y};\mathbf{s}):=\prod_{j=1}^{p}\frac{\Gamma(x_j+\rho s_j)\Gamma(y_{j+r}-x_j+\lambda s_j)}{\Gamma(y_{j+r}+(\rho+\lambda)s_j)}.$$

Note that the description of the contours in (32) and (33) is similar to that of (31).

Proof. By applying Ramanujan's Master Theorem to the series representation of the extended generalized hypergeometric function ${}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}[-z;b]$, we have

$$\int_0^\infty z^{s-1}{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\quad x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};-z;b\right]\mathrm{d}z=\Gamma(s)\mathfrak{A}(-s|q+1,q),$$

where

$$\mathfrak{A}(-s|q+1,q)=\frac{\Gamma(x_1-s)}{\Gamma(x_1)}\prod_{j=1}^{q}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_{j+1}-s,y_j-x_{j+1}).$$

The use of the inverse Mellin transform (30) gives the following contour integral:

$$ {}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\quad x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};z;b\right]=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\Gamma(s)\mathfrak{A}(-s|q+1,q)(-z)^{-s}\mathrm{d}s$$
$$=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\prod_{j=1}^{q}\frac{\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_{j+1}-s,y_j-x_{j+1})}{B(x_{j+1},y_j-x_{j+1})}\frac{\Gamma(s)\Gamma(x_1-s)}{\Gamma(x_1)}(-z)^{-s}\mathrm{d}s. \tag{34}$$

Here $\gamma := \mathrm{Re}(s) > 0, |\arg(-z)| < \pi$ and the path of integration separates all the poles of $\Gamma(s)$ (i.e., $s = -n \in \mathbb{N}_0$) to the left and all the poles of $\mathfrak{A}(-s|q+1, q)$ to the right. The poles of $\Gamma(x_1 - s)$ are $s = x_1 + n (n \in \mathbb{N}_0)$. However, the poles of function $\mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x_{j+1} - s, y_j - x_{j+1})$ do not always exist. If we choose $\alpha = \beta$, the poles of $B_{b;\rho,\lambda}\,(x_{j+1} - s, y_j - x_{j+1})$ will disappear.

If we set $b = 0$ in (34), the contour integral gives the standard expression of the classical generalized hypergeometric function ${}_{q+1}F_q(z)$ (see [17, p. 30, Eq. (1.6.29)]):

$$ {}_{q+1}F_q\left[\begin{matrix} x_1, \ldots, x_{q+1} \\ y_1, \ldots, y_q \end{matrix}; z\right] = \frac{1}{2\pi \mathrm{i}} \frac{\prod_{j=1}^{q} \Gamma(y_j)}{\prod_{l=1}^{q+1} \Gamma(x_l)} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\prod_{l=1}^{q+1} \Gamma(x_l - s)}{\prod_{j=1}^{q} \Gamma(y_j - s)} (-z)^{-s} \Gamma(s) \mathrm{d}s. \tag{35} $$

The derivation of (35) by applying Ramanujan's Master Theorem is given in [2].

In Theorem 2.14, we have proved that the extended beta function $\mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x,y)$ can be regarded as a H-function. Thus, we can write

$$ \begin{aligned} \mathcal{B}^{(\alpha,\beta)}_{b;\rho,\lambda}(x_{j+1} - s, y_j - x_{j+1}) &= \frac{\Gamma(\beta)}{\Gamma(\alpha)} H^{3,1}_{2,4}\left[b \left| \begin{matrix} (1-\alpha, 1), (y_j - s, \rho + \lambda) \\ (0,1), (x_{j+1} - s, \rho), (y_j - x_{j+1}, \lambda), (1 - \beta, 1) \end{matrix} \right. \right] \\ &= \frac{1}{2\pi \mathrm{i}} \frac{\Gamma(\beta)}{\Gamma(\alpha)} \int_{\gamma_j - \mathrm{i}\infty}^{\gamma_j + \mathrm{i}\infty} \frac{\Gamma(s_j)\Gamma(\alpha - s_j)\Gamma(y_j - x_{j+1} + \lambda s_j)}{\Gamma(\beta - s_j)} \frac{\Gamma(x_{j+1} - s + \rho s_j)}{\Gamma(y_j - s + (\rho+\lambda)s_j)} b^{-s_j} \mathrm{d}s_j, \end{aligned} $$

where $\gamma_j := \mathrm{Re}(s_j) > 0$, the path of integration starts at $\gamma_j - i\infty$ and terminates at $\gamma_j + i\infty$.

Substituting this into (34) and interchanging the orders of integration, we have

$$ \begin{aligned} & {}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1, & x_2, \ldots, x_{q+1} \\ & y_1, \ldots, y_q \end{matrix}; z; b\right] \\ &= \frac{1}{(2\pi i)^q} \frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} \cdots \int_{\gamma_q - \mathrm{i}\infty}^{\gamma_q + \mathrm{i}\infty} \prod_{j=1}^{q} \frac{\Gamma(s_j)\Gamma(\alpha - s_j)\Gamma(y_j - x_{j+1} + \lambda s_j)}{\Gamma(\beta - s_j) B(x_{j+1}, y_j - x_{j+1})} \\ &\times \left[\frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \prod_{j=1}^{q} \frac{\Gamma(x_{j+1} - s + \rho s_j)}{\Gamma(y_j - s + (\rho+\lambda)s_j)} \frac{\Gamma(s)\Gamma(x_1 - s)}{\Gamma(x_1)} (-z)^{-s} \mathrm{d}s \right] b^{-\sum_{j=1}^{q} s_j} \mathrm{d}s_1 \cdots \mathrm{d}s_q \\ &= \frac{1}{(2\pi i)^q} \frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} \cdots \int_{\gamma_q - \mathrm{i}\infty}^{\gamma_q + \mathrm{i}\infty} \prod_{j=1}^{q} \frac{\Gamma(s_j)\Gamma(\alpha - s_j)\Gamma(y_j - x_{j+1} + \lambda s_j)}{\Gamma(\beta - s_j) B(x_{j+1}, y_j - x_{j+1})} \prod_{j=1}^{q} \frac{\Gamma(x_{j+1} + \rho s_j)}{\Gamma(y_j + (\rho+\lambda)s_j)} \\ &\times {}_{q+1}F_q\left[\begin{matrix} x_1, x_2 + \rho s_1, \ldots, x_{q+1} + \rho s_q \\ y_1 + (\rho+\lambda)s_1, \ldots, y_q + (\rho+\lambda)s_q \end{matrix}; z\right] b^{-\sum_{j=1}^{q} s_j} \mathrm{d}s_1 \cdots \mathrm{d}s_q. \end{aligned} \tag{36} $$

In the above contour integral, each contour starting at $\gamma_j - i\infty$ and ending at $\gamma_j + i\infty$ separates all the poles of $\Gamma(x_{j+1} + \rho s_j)\Gamma(y_j - x_{j+1} + \lambda s_j)\Gamma(s_j)$ to the left and all the poles of $\Gamma(\alpha - s_j)$ to the right. Since the validity of (35) requires that $y_j \notin \mathbb{Z}_0^-$, we also need $y_j + (\rho+\lambda)s_j \notin \mathbb{Z}_0^-$ such that the evaluation of (36) is allowed. Actually, for $\rho, \lambda > 0$, and $\mathrm{Re}(y_j) > 0$ we always have

$$ y_j + (\rho+\lambda)s_j^* \notin \mathbb{Z}_0^-, \quad j = 1, \ldots, q, $$

where $s_j^* = \alpha + n (n \in \mathbb{N}_0)$ are the poles of $\Gamma(\alpha - s_j)$, and for other cases, this requirement can be easily satisfied.

The proof of (31) is now complete. The integral representations (32) and (33) can be proved in a similar manner. $\square$

Remark 3.3. The Mellin transform of the extended Gauss hypergeometric function (with different parameters) have been studied by many authors. For instance, Özergin et al. [30, p. 4608, Corollary 3.6] show that

$$ {}_2F_1^{(\alpha,\beta;1,1)}\left[\begin{matrix} x_1, & x_2 \\ & y_1 \end{matrix}; z; b\right] = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} \frac{\Gamma^{(\alpha,\beta)}(s) B(x_2 + s, y_1 + s - x_2)}{B(x_2, y_1 - x_2)} {}_2F_1\left[\begin{matrix} x_1, & x_2 + s \\ & y_1 + 2s \end{matrix}; z\right] b^{-s} \mathrm{d}s, \tag{37} $$

where $\Gamma^{(\alpha,\beta)}(s) \equiv \Gamma_0^{(\alpha,\beta)}(s)$ is a specific case of the extended gamma function defined by

$$ \Gamma_b^{(\alpha,\beta)}(s) := \int_0^\infty t^{s-1} {}_1F_1\left(\alpha; \beta; -t - \frac{b}{t}\right) \mathrm{d}t \quad (\min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(b)\} > 0). $$

After simplification, (37) can be expressed in our notation.

Theorem 3.2 is very illuminating. To some degree, it provides an easy criteria for us to see which properties holden by a classical generalized hypergeometric function ${}_pF_q(z)$ can be inherited by its extension ${}_pF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$. Several instructive applications of this theorem are given below.

Corollary 3.4 (*Extended Gauss' summation formula*). *For* $\rho, \lambda > 0, \min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta)\} > 0, \mathrm{Re}(y_1) > \mathrm{Re}(x_2) > 0, \mathrm{Re}(b) > 0,$ *we have the following extended Gauss summation formula*

$$ {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1, & x_2 \\ & y_1 \end{matrix};1;b\right] = \frac{\Gamma(\beta)\Gamma(y_1)}{\Gamma(\alpha)\Gamma(x_2)\Gamma(y_1-x_2)} \cdot H_{2,4}^{3,1}\left[b\left| \begin{matrix} (1-\alpha),(y_1-x_2,\rho+\lambda) \\ (0,1),(x_2,\rho),(y_1-x_1-x_2,\lambda),(1-\beta,1) \end{matrix}\right.\right] \tag{38} $$

$$ = \frac{\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_2,y_1-x_1-x_2)}{B(x_2,y_1-x_2)}, \mathrm{Re}(y_1-x_1-x_2) > 0. \tag{39} $$

Proof. Let $q=1$ in (31). By using Gauss' summation formula, ${}_2F_1(1)$ can be summed as

$$ {}_2F_1\left[\begin{matrix} x_1, x_2+\rho s_1 \\ y_1+(\rho+\lambda)s_1 \end{matrix};1\right] = \frac{\Gamma(y_1+(\rho+\lambda)s_1)\Gamma(y_1-x_1-x_2+\lambda s_1)}{\Gamma(y_1-x_2+(\rho+\lambda)s_1)\Gamma(y_1-x_2+\lambda s_1)}, $$

where $\mathrm{Re}(y_1-x_1-x_2)>0$, and $\mathrm{Re}(s_1)=\gamma_1>0$.

Thus, we have

$$ {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1, & x_2 \\ & y_1 \end{matrix};1;b\right] = \frac{1}{2\pi \mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} H_1(\mathbf{x},\mathbf{y};s)\frac{\Gamma(s_1)\Gamma(\alpha-s_1)}{\Gamma(\beta-s_1)}{}_2F_1\left[\begin{matrix} x_1, x_2+\rho s_1 \\ y_1+(\rho+\lambda)s_1 \end{matrix};1\right]b^{-s_1}\mathrm{d}s_1 $$

$$ = \frac{1}{2\pi \mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} \frac{\Gamma(s_1)\Gamma(x_2+\rho s_1)\Gamma(y_1-x_1-x_2+\lambda s_1)\Gamma(\alpha-s_1)}{\Gamma(\beta-s_1)\Gamma(y_1-x_2+(\rho+\lambda)s_1)C_1(\mathbf{x},\mathbf{y};0)}b^{-s_1}\mathrm{d}s_1. $$

From the definition of H-function (38) follows. By interpreting this special H-function as the extended beta function (see Theorem 2.14) we obtain (39). □

Remark 3.5. Some special cases of (39) can be found in [10,20,30].

Corollary 3.6 (*Extended Kummer's first transformation*). *For* $\rho,\lambda>0, \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0, \mathrm{Re}(y_1)>\mathrm{Re}(x_2)>0, \mathrm{Re}(b)>0$, *we have the following transformation*

$$ {}_1F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1 \\ y_1 \end{matrix};z;b\right] = e^z{}_1F_1^{(\alpha,\beta;\lambda,\rho)}\left[\begin{matrix} y_1-x_1 \\ y_1 \end{matrix};-z;b\right]. $$

Proof. Letting $q=1$ in (32) gives

$$ {}_1F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1 \\ y_1 \end{matrix};z;b\right] = \frac{1}{2\pi \mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} \widetilde{H}_1(\mathbf{x},\mathbf{y};\mathbf{s})\frac{\Gamma(s_1)\Gamma(\alpha-s_1)}{\Gamma(\beta-s_1)}{}_1F_1\left[\begin{matrix} x_1+\rho s_1 \\ y_1+(\rho+\lambda)s_1 \end{matrix};z\right]b^{-s_1}\mathrm{d}s_1. $$

By using Kummer's first transformation [4, p. 191]

$$ {}_1F_1\left[\begin{matrix} a \\ c \end{matrix};x\right] = e^x{}_1F_1\left[\begin{matrix} c-a \\ c \end{matrix};-x\right], $$

we have

$$ {}_1F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1 \\ y_1 \end{matrix};z;b\right] = \frac{1}{2\pi \mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} \widetilde{H}_1(\mathbf{x},\mathbf{y};\mathbf{s})\frac{\Gamma(s_1)\Gamma(\alpha-s_1)}{\Gamma(\beta-s_1)}{}_1F_1\left[\begin{matrix} x_1+\rho s_1 \\ y_1+(\rho+\lambda)s_1 \end{matrix};z\right]b^{-s_1}\mathrm{d}s_1 $$

$$ = \frac{e^z}{2\pi \mathrm{i}}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} \widetilde{H}_1(\mathbf{x},\mathbf{y};\mathbf{s})\frac{\Gamma(s_1)\Gamma(\alpha-s_1)}{\Gamma(\beta-s_1)}{}_1F_1\left[\begin{matrix} y_1-x_1+\lambda s_1 \\ y_1+(\lambda+\rho)s_1 \end{matrix};-z\right]b^{-s_1}\mathrm{d}s_1 $$

$$ = e^z{}_1F_1^{(\alpha,\beta;\lambda,\rho)}\left[\begin{matrix} y_1-x_1 \\ y_1 \end{matrix};-z;b\right]. \qquad \square $$

Remark 3.7. Some special cases of this transformation can be found in [10,20,30].

Corollary 3.8 (*Differentiation formula*). *If* $\rho,\lambda>0, \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0, \mathrm{Re}(b)>0$, *and* x_i, y_j *satisfies the conditions stated in Definition 1.3, then we have the following formula*

$$ \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{{}_pF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_1,\ldots,y_q \end{matrix};z;b\right]\right\} = \frac{(x_1)_n\cdots(x_p)_n}{(y_1)_n\cdots(y_q)_n}{}_pF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1+n,\ldots,x_p+n \\ y_1+n,\ldots,y_q+n \end{matrix};z;b\right]. $$

Proof. For a classical hypergeometric function ${}_pF_q(z)$, we have [27, p.405, Eq. (16.3.1)]

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{{}_pF_q\left[\begin{matrix}x_1,\ldots,x_p\\ y_1,\ldots,y_q\end{matrix};z\right]\right\}=\frac{(x_1)_n\cdots(x_p)_n}{(y_1)_n\cdots(y_q)_n}\,{}_pF_q\left[\begin{matrix}x_1+n,\ldots,x_p+n\\ y_1+n,\ldots,y_q+n\end{matrix};z\right].$$

To apply Theorem 3.2, write

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{{}_pF_q\left[\begin{matrix}x_1,\ldots,x_p\\ y_1,\ldots,y_q\end{matrix};z\right]\right\}=\frac{\Gamma(x_1+n)\cdots\Gamma(x_p+n)}{\Gamma(y_1+n)\cdots\Gamma(y_q+n)}\frac{\Gamma(y_1)\cdots\Gamma(y_q)}{\Gamma(x_1)\cdots\Gamma(x_p)}\,{}_pF_q\left[\begin{matrix}x_1+n,\ldots,x_p+n\\ y_1+n,\ldots,y_q+n\end{matrix};z\right].$$

For $p=q+1$, we have

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}z^n}&\left\{{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};z;b\right]\right\}=\frac{1}{(2\pi i)^q}\frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_q-i\infty}^{\gamma_q+i\infty}H_q(\mathbf{x},\mathbf{y};\mathbf{s})\prod_{j=1}^{q}\frac{\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(\beta-s_j)}\\
&\times\frac{\Gamma(x_1+n)\Gamma(x_2+n+\rho s_1)\cdots\Gamma(x_{q+1}+n+\rho s_q)}{\Gamma(y_1+n+(\rho+\lambda)s_1)\cdots\Gamma(y_q+n+(\rho+\lambda)s_q)}\frac{\Gamma(y_1+(\rho+\lambda)s_1)\cdots\Gamma(y_q+(\rho+\lambda)s_q)}{\Gamma(x_1)\Gamma(x_2+\rho s_1)\cdots\Gamma(x_{q+1}+\rho s_q)}\\
&\times{}_{q+1}F_q\left[\begin{matrix}x_1+n,x_2+n+\rho s_1,\ldots,x_{q+1}+n+\rho s_q\\ y_1+n+(\rho+\lambda)s_1,\ldots,y_q+n+(\rho+\lambda)s_q\end{matrix};z\right]b^{-\sum_{j=1}^{q}s_j}\mathrm{d}s_1\cdots\mathrm{d}s_q\\
&=\frac{(x_1)_n}{(2\pi i)^q}\frac{C_q(\mathbf{x}+\mathbf{n},\mathbf{y}+\mathbf{n};0)}{C_q(\mathbf{x},\mathbf{y};0)}\frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_q-i\infty}^{\gamma_q+i\infty}H_q(\mathbf{x}+\mathbf{n},\mathbf{y}+\mathbf{n};\mathbf{s})\prod_{j=1}^{q}\frac{\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(\beta-s_j)}\\
&\times{}_{q+1}F_q\left[\begin{matrix}x_1+n,x_2+n+\rho s_1,\ldots,x_{q+1}+n+\rho s_q\\ y_1+n+(\rho+\lambda)s_1,\ldots,y_q+n+(\rho+\lambda)s_q\end{matrix};z\right]b^{-\sum_{j=1}^{q}s_j}\mathrm{d}s_1\cdots\mathrm{d}s_q\\
&=(x_1)_n\frac{C_q(\mathbf{x}+\mathbf{n},\mathbf{y}+\mathbf{n};0)}{C_q(\mathbf{x},\mathbf{y};0)}\,{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1+n,\ldots,x_{q+1}+n\\ y_1+n,\ldots,y_q+n\end{matrix};z;b\right].
\end{aligned}\tag{40}$$

From the definition of $C_q(\mathbf{x},\mathbf{y};0)$, we have

$$\frac{C_q(\mathbf{x}+\mathbf{n},\mathbf{y}+\mathbf{n};0)}{C_q(\mathbf{x},\mathbf{y};0)}=\prod_{j=1}^{q}\frac{\Gamma(y_j)\Gamma(x_{j+1}+n)}{\Gamma(y_j+n)\Gamma(x_{j+1})}=\prod_{j=1}^{q}\frac{(x_{j+1})_n}{(y_j)_n}.\tag{41}$$

Combining (40) with (41) gives the differentiation formula for ${}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$.

Similarly, we can prove this result for ${}_qF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$ and ${}_pF_{p+r}^{(\alpha,\beta;\rho,\lambda)}[z;b]$ $(r=q-p)$. □

Remark 3.9. A similar result has been proved (by using series manipulation technique) in [26, Theorem 3.3].

Corollary 3.10. *For $\rho,\lambda>0,\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0,\mathrm{Re}(y_j)>\mathrm{Re}(x_{j+1})>0(j=1,\ldots,q),\mathrm{Re}(b)>0$, we have the following finite sum*

$$\begin{aligned}
\sum_{k=0}^{N}(-1)^k\binom{N}{k}(ka+1)^{N-1}{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}&\left[\begin{matrix}-k,x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};\frac{z}{ka+1};b\right]\\
&=\frac{z^N}{Na+1}\prod_{j=1}^{q}\frac{\Gamma(\beta)\Gamma(y_j)}{\Gamma(\alpha)\Gamma(x_{j+1})\Gamma(y_j-x_{j+1})}H_{2,4}^{3,1}\left[b\left|\begin{matrix}(1-\alpha,1),(y_j+N,\rho+\lambda)\\ (0,1),(x_{j+1}+N,\rho),(y_j-x_{j+1},\lambda),(1-\beta,1)\end{matrix}\right.\right]
\end{aligned}\tag{42}$$

$$=\frac{z^N}{Na+1}\prod_{j=1}^{q}\frac{\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_{j+1}+N,y_j-x_{j+1})}{B(x_{j+1},y_j-x_{j+1})}.\tag{43}$$

Proof. We first state the following identity [6, p. 423, Eq. (10)]

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}(ka+1)^{N-1}{}_{q+1}F_q\left[\begin{matrix}-k,x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};\frac{z}{ka+1}\right]=\frac{z^N}{Na+1}\frac{\prod_{j=1}^{q}(x_{j+1})_N}{\prod_{j=1}^{q}(y_j)_N}.$$

For convenience, we rewrite it as

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}(ka+1)^{N-1}{}_{q+1}F_q\left[\begin{matrix}-k,x_2+\rho s_1,\ldots,x_{q+1}+\rho s_q\\ y_1+(\rho+\lambda)s_1,\ldots,y_q+(\rho+\lambda)s_q\end{matrix};\frac{z}{ka+1}\right]$$
$$=\frac{z^N}{Na+1}\prod_{j=1}^{q}\frac{\Gamma(x_{j+1}+\rho s_j+N)\Gamma(y_j+(\rho+\lambda)s_j)}{\Gamma(x_{j+1}+\rho s_j)\Gamma(y_j+(\rho+\lambda)s_j+N)}.$$

Now, we use (31) to get

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}(ka+1)^{N-1}{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}-k,x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};\frac{z}{ka+1};b\right]$$
$$=\frac{1}{(2\pi i)^q}\frac{\Gamma^q(\beta)}{\Gamma^q(\alpha)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_q-i\infty}^{\gamma_q+i\infty}H_q(\mathbf{x},\mathbf{y};\mathbf{s})\prod_{j=1}^{q}\frac{\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(\beta-s_j)}\sum_{k=0}^{N}(-1)^k\binom{N}{k}(ka+1)^{N-1}$$
$$\times\,{}_{q+1}F_q\left[\begin{matrix}-k,x_2+\rho s_1,\ldots,x_{q+1}+\rho s_q\\ y_1+(\rho+\lambda)s_1,\ldots,y_q+(\rho+\lambda)s_q\end{matrix};\frac{z}{ka+1}\right]b^{-\sum_{j=1}^{q}s_j}ds_1\cdots ds_q$$
$$=\frac{\left(\frac{z^N}{Na+1}\right)}{C_q(\mathbf{x},\mathbf{y};0)}\prod_{j=1}^{q}\frac{1}{2\pi i}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_{\gamma_j-i\infty}^{\gamma_j+i\infty}\frac{\Gamma(x_{j+1}+\rho s_j+N)\Gamma(y_j-x_{j+1}+\lambda s_j)\Gamma(s_j)\Gamma(\alpha-s_j)}{\Gamma(y_j+(\rho+\lambda)s_j+N)\Gamma(\beta-s_j)}b^{-s_j}ds_j. \tag{44}$$

Interpreting the last member of (44) in terms of H-function, we arrive at the result (42). A use of Theorem 2.14 may gives (43). □

Corollary 3.11 (*Extended Nørlund's expansion*). *For* $\rho,\lambda>0,\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\}>0,\mathrm{Re}(y_j)>\mathrm{Re}(x_{j+1})>0\ (j=1,\ldots,q),\mathrm{Re}(b)>0$, *we have the following expansion*

$${}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};z\zeta;b\right]=(1-z)^{-x_1}\sum_{k=0}^{\infty}\frac{(x_1)_k}{k!}{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}-k,x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};\zeta;b\right]\left(\frac{z}{z-1}\right)^k. \tag{45}$$

Proof. The expansion (45) follows directly from Nørlund's formula ([25, p. 294, Eq. (1.21)]; see also [27, p. 411, Eq. (16.10.2)]):

$${}_{q+1}F_q\left[\begin{matrix}x_1,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};z\zeta\right]=(1-z)^{-x_1}\sum_{k=0}^{\infty}\frac{(x_1)_k}{k!}{}_{q+1}F_q\left[\begin{matrix}-k,x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};\zeta\right]\left(\frac{z}{z-1}\right)^k. \qquad \square \tag{46}$$

Remark 3.12. The derivation of (45) from (3) is obvious. So, it is worth mentioning a proof due to N.E. Nørlund. Following Nørlund [25, Section 5] we begin by recalling Euler's transformation:

$$\sum_{k=0}^{\infty}\frac{(x_1)_k}{k!}a_k z^k=(1-z)^{-x_1}\sum_{k=0}^{\infty}\frac{(x_1)_k}{k!}\left(\Delta^k a_0\right)\left(\frac{z}{1-z}\right)^k,$$

where

$$\Delta^0 a_0=a_0,\quad \Delta^1 a_0=a_1-a_0,\quad \Delta^k a_0=\Delta^{k-1}a_1-\Delta^{k-1}a_0=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}a_i,\quad k\geqslant 2.$$

If we put

$$a_k=\prod_{j=1}^{q}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_{j+1}+k,y_j-x_{j+1})\zeta^k,$$

we get for the difference of order k the hypergeometric polynomial

$$\Delta^k a_0=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\prod_{j=1}^{q}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_{j+1}+i,y_j-x_{j+1})\zeta^i=(-1)^k{}_{q+1}F_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}-k,x_2,\ldots,x_{q+1}\\ y_1,\ldots,y_q\end{matrix};\zeta;b\right].$$

Euler's transformation then gives the relation (45).

4. Extended fractional integral operators

First, we establish the extended Riemann–Liouville type fractional integral operator in this section. Then with the help of the extended Riemann–Liouville fractional integral operator, a decompositional formula of the extended generalized hypergeometric function ${}_pF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$ is formulated. Next, we introduce four extended Kober type fractional integral operators involving the extended Gauss hypergeometric functions, and then we establish their Mellin transform in the form of Theorems.

4.1. Extended Riemann–Liouville fractional integral operator

The classical Riemann–Liouville (left-sided) fractional integral of order $\mu \in \mathbb{C}$ $(\mathrm{Re}(\mu) > 0)$ is defined by [17]

$$I_x^\mu\{f(x)\} := \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t \quad (x > a).$$

The corresponding Riemann–Liouville fractional derivatives of order $\mu \in \mathbb{C}$ $(\mathrm{Re}(\alpha) \geqslant 0)$ is defined by

$$D_x^\mu\{y(x)\} := \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_x^{n-\mu}\{y(x)\} = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\mu-1} y(t)\mathrm{d}t \quad (n = [\mathrm{Re}(\mu)] + 1; x > a),$$

where $[\mathrm{Re}(\mu)]$ means the integral part of $\mathrm{Re}(\mu)$.

Now, by introducing new parameters, we consider the following extension of the classical Riemann–Liouville fractional integral:

$$I_z^{\mu,b}\{f(z)\} := \frac{1}{\Gamma(\mu)} \int_0^z f(t)(z-t)^{\mu-1} {}_1F_1\left(\alpha; \beta; -\frac{z^{\rho+\lambda} b}{t^\rho (z-t)^\lambda}\right)\mathrm{d}t,$$

where $\rho \geqslant 0, \lambda \geqslant 0, \min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\mu), \mathrm{Re}(b)\} > 0$. It is clear that $I_z^{\mu,b}$ may become I_z^μ when $b = 0$. The case when $\rho = \lambda = 1$ and $\alpha = \beta$ have been considered in [29]. Similar constructions are also used in [26,33].

From the definition of extended beta function (1), we have

$$I_z^{\mu,b}\{z^{\eta-1}\} = \frac{z^{\eta+\mu-1}}{\Gamma(\mu)} \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(\eta,\mu), \quad \mathrm{Re}(\eta) > 0. \tag{47}$$

Repeated application of property (47) yields the following result.

Theorem 4.1 (*Decompositional structure*). *For $\rho, \lambda > 0, \min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta)\} > 0, \mathrm{Re}(b) > 0$, we have*

$${}_pF_q^{(\alpha,\beta;\rho,\tau)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_1,\ldots,y_q \end{matrix}; z; b\right] = \begin{cases} \mathbb{I}_z^{y_q - x_{q+1}}\{\cdots \mathbb{I}_z^{y_1-x_2}\{(1-z)^{-x_1}\}\}, & p = q+1, \qquad (48) \\ \mathbb{I}_z^{y_q - x_q}\{\cdots \mathbb{I}_z^{y_1-x_1}\{e^z\}\}, & p = q, \qquad (49) \\ \mathbb{I}_z^{y_{r+p}-x_p}\left\{\cdots \mathbb{I}_z^{y_{r+1}-x_1}\left\{{}_1F_r\left[\begin{matrix} 1 \\ y_1,\ldots,y_r \end{matrix}; z\right]\right\}\right\}, & \begin{matrix} r = q-p, \\ p < q. \end{matrix} \qquad (50) \end{cases}$$

where

$$\mathbb{I}_z^{y-x}\{f(z)\} := \frac{\Gamma(y)}{\Gamma(x)} z^{1-y} I_z^{y-x,b}\{z^{x-1} f(z)\}, \quad \mathrm{Re}(y) > \mathrm{Re}(x) > 0,$$

and $I_z^{y-x,b}$ is the extended Riemann–Liouville fractional integral operator.

Proof. In order to derive (48) we proceed as follows,

$$\begin{aligned} I_z^{y_1-x_2,b}\{z^{x_2-1}(1-z)^{-x_1}\} &= \sum_{n=0}^\infty \frac{(x_1)_n}{n!} I_z^{y_1-x_2,b}\{z^{x_2+n-1}\} \\ &= \frac{z^{y_1-1}}{\Gamma(y_1-x_2)} \sum_{n=0}^\infty (x_1)_n \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_2+n, y_1-x_2) \frac{z^n}{n!} \end{aligned}$$

$$(\mathrm{Re}(y_1) > \mathrm{Re}(x_2) > 0).$$

Rearranging the equation we have

$$\frac{\Gamma(y_1)}{\Gamma(x_2)} z^{1-y_1} I_z^{y_1-x_2,b}\{z^{x_2-1}(1-z)^{-x_1}\} = {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1, & x_2 \\ & y_1 \end{matrix}; z; b\right].$$

Performing $I_z^{\mu,b}$ again in a similar manner, we have

$$\frac{\Gamma(y_2)}{\Gamma(x_3)}z^{1-y_2}I_z^{y_2-x_3,b}\left\{z^{x_3-1}{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\begin{bmatrix}x_1, & x_2 & \\ & y_1 & ;z;b\end{bmatrix}\right\}$$
$$=\frac{\Gamma(y_2)}{\Gamma(x_3)}z^{1-y_2}\sum_{n=0}^{\infty}\frac{(x_1)_n}{n!}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x_2+n,y_1-x_2)I_z^{y_2-x_3}\{z^{x_3+n-1}\}$$
$$={}_3F_2^{(\alpha,\beta;\rho,\lambda)}\begin{bmatrix}x_1,x_2,x_3 & \\ y_1,y_2 & ;z;b\end{bmatrix},\quad \mathrm{Re}(y_j)>\mathrm{Re}(x_{j+1}),\ j=1,2.$$

Continuing this process gives (48). Eqs. (49) and (50) can be similarly proved. □

4.2. Extended Kober type fractional integral operators

Definition 4.2. Define

$$\mathcal{I}[f(z)]:=\frac{\mu z^{-\eta-1}}{\Gamma(1-A)}\int_0^z {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\begin{bmatrix}A, & B \\ & C\end{bmatrix};\frac{at^\mu}{z^\mu};b\Bigg]t^\eta f(t)\mathrm{d}t \tag{51}$$

and

$$\mathcal{J}[f(z)]:=\frac{\mu z^{\delta}}{\Gamma(1-A)}\int_z^\infty {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\begin{bmatrix}A, & B \\ & C\end{bmatrix};\frac{az^\mu}{t^\mu};b\Bigg]t^{-\delta-1} f(t)\mathrm{d}t, \tag{52}$$

where $A,B,C,\eta,\mu,\delta,a\in\mathbb{C}(\mathrm{Re}(C)>\mathrm{Re}(B)>0)$ and ${}_2F_1^{(\alpha,\beta;\rho,\lambda)}[z;b]$ denotes the extended Gauss hypergeometric function.

The condition of the validity of the operators (51) and (52) are given below:

- $1\leqslant p,q<\infty,\ \frac{1}{p}+\frac{1}{q}=1,\ |\arg(1-a)|<\pi,\ \mu>0;$
- $\mathrm{Re}(\eta)>-\frac{1}{q},\ \mathrm{Re}(\delta)>-\frac{1}{p},\ \mathrm{Re}(C-A-B)>-1;$
- $f\in L_p(0,\infty)$, where $L_p(0,\infty)$ is a Lebesgue space of measurable real or complex valued functions.

Remark 4.3. It may be noted that, if we set, $b=0$ in (51) and (52), we obtain the known operators due to Kalla and Saxena [15, p. 231, Eqs. (1) and (2)]:

$$I[f(z)]=I\begin{bmatrix}A,B,C \\ \eta,\mu,a\end{bmatrix};f(z)\Bigg]=\frac{\mu z^{-\eta-1}}{\Gamma(1-A)}\int_0^z {}_2F_1\begin{bmatrix}A, & B \\ & C\end{bmatrix};\frac{at^\mu}{z^\mu}\Bigg]t^\eta f(t)\mathrm{d}t, \tag{53}$$
$$J[f(z)]=J\begin{bmatrix}A,B,C \\ \delta,\mu,a\end{bmatrix};f(z)\Bigg]=\frac{\mu z^{\delta}}{\Gamma(1-A)}\int_z^\infty {}_2F_1\begin{bmatrix}A, & B \\ & C\end{bmatrix};\frac{az^\mu}{t^\mu}\Bigg]t^{-\delta-1} f(t)\mathrm{d}t.$$

Similarly, if we set $C=B, a=1, b=0$ in (51) and (52), we get operators given by Erdélyi [11, p. 217]. In the sequel, if we set $C=B, \mu=1, a=1, b=0$ in (51) and (52), we have results due to Kober [19, p. 193].

Remark 4.4. It is necessary to point out that our new definitions are very close to their original definitions.

Let us consider the operator (53). We have, for $f(z)\in L_p(0,\infty)$,

$$|\mathcal{I}[f(z)]|\leqslant\frac{\mu}{|\Gamma(1-A)|}\int_0^1\left|{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\begin{bmatrix}A, & B \\ & C\end{bmatrix};au^\mu;b\Bigg]\right|u^{\mathrm{Re}(\eta)}|f(zu)|\mathrm{d}u.$$

From the results of Section 2, it will be not very difficult to see that

$$\begin{aligned}\left|{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\begin{bmatrix}A, & B \\ & C\end{bmatrix};au^\mu;b\Bigg]\right| &\leqslant\left|\frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)}\right|\sum_{n=0}^{\infty}|(A)_n|\left|\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(B+n,C-B)\right|\frac{|a|u^{\mu n}}{n!}\\ &\leqslant\frac{\Omega_{\rho,\lambda}^{\alpha,\beta}(b)|\Gamma(C)|}{|\Gamma(B)\Gamma(C-B)|}\sum_{n=0}^{\infty}(|A|)_n\frac{\Gamma(\mathrm{Re}(B)+n)\Gamma(\mathrm{Re}(C-B))}{\Gamma(\mathrm{Re}(C)+n)}\frac{|a|u^{\mu n}}{n!}\\ &\leqslant L\cdot{}_2F_1\begin{bmatrix}|A|, & \mathrm{Re}(B) \\ & \mathrm{Re}(C)\end{bmatrix};|a|u^\mu\Bigg],\end{aligned}$$

where $\Omega_{\rho,\lambda}^{\alpha,\beta}(b)$ is defined in Corollary 2.5 and

$$L=\Omega_{\rho,\lambda}^{\alpha,\beta}(b)\frac{|\Gamma(C)|}{\Gamma(\mathrm{Re}(C))}\frac{\Gamma(\mathrm{Re}(B))\Gamma(\mathrm{Re}(C-B))}{|\Gamma(B)\Gamma(C-B)|}.$$

If all parameters are real, the expression will be simple. Thus,

$$\begin{aligned}|\mathcal{I}[f(z)]| &\leqslant \frac{L\mu}{|\Gamma(1-A)|}\int_0^1 {}_2F_1\left[\begin{matrix}|A|, & \mathrm{Re}(B) \\ & \mathrm{Re}(C)\end{matrix}; |a|u^{\mu}\right]u^{\mathrm{Re}(\eta)}|f(zu)|\mathrm{d}u \\ &= L_1\cdot I\left[\begin{matrix}|A|, \mathrm{Re}(B), \mathrm{Re}(C) \\ \mathrm{Re}(\eta), \mu, |a|\end{matrix}; |f(z)|\right], \quad |f(z)|\in L_p(0,\infty),\end{aligned}$$

where

$$L_1 = L\cdot\frac{\Gamma(1-|A|)}{|\Gamma(1-A)|}, \quad (0<|A|<1).$$

The above inequality illustrates that for some suitable parameters, our new integral operator can be well majorized by its original form.

Theorem 4.5. *Let $f(z)\in L_p(0,\infty), 1\leqslant p\le 2$ [or $f(z)\in M_p(0,\infty)$ and $p>2$], $\mathrm{Re}(C-A-B)>0, \mu>0,$*

$$\mathrm{Re}(\eta)>\max\left\{\frac{1}{p},\frac{1}{q}\right\}, \ \frac{1}{p}+\frac{1}{q}=1, \quad \text{and} \quad |\arg(1-a)|<\pi.$$

Then the following Mellin transform formula exist:

$$\mathfrak{M}\{\mathcal{I}[f(z)]\}(s) = \frac{\mu}{\Gamma(1-A)(\eta-s+1)}\,{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; a; b\right] * {}_2F_1\left[\begin{matrix}1, & \frac{1}{\mu}(\eta-s+1) \\ & \frac{1}{\mu}(\eta-s+1)+1\end{matrix}; a\right]\mathfrak{M}\{f(z)\}(s),$$

where $\mathfrak{M}\{f(z)\}$ denotes the Mellin transform of $f(z)$ defined by (29), and $M_p(0,\infty)$ denotes the class of all functions $f(z)$ of $L_p(0,\infty)$ with $p>2$ which are inverse Mellin transforms of functions of $L_q(-\infty,\infty)$ $(q=p/(p-1))$.

Proof. From (51) we easily have by using (29)

$$\mathfrak{M}\{\mathcal{I}[f(z)]\}(s) = \frac{\mu}{\Gamma(1-A)}\int_0^\infty z^{s-\eta-2}\left\{\int_0^z {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; \frac{at^{\mu}}{z^{\mu}}; b\right]t^{\eta}f(t)\mathrm{d}t\right\}\mathrm{d}z.$$

By changing the order of integration, we obtain

$$\mathfrak{M}\{\mathcal{I}[f(z)]\}(s) = \frac{\mu}{\Gamma(1-A)}\int_0^\infty t^{\eta}f(t)\left\{\int_t^\infty z^{s-\eta-2}{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; \frac{at^{\mu}}{z^{\mu}}; b\right]\mathrm{d}z\right\}\mathrm{d}t. \tag{54}$$

For convenience, we denote the inner integral in (54) by $\mathfrak{J}$. Then

$$\begin{aligned}\mathfrak{J} &= t^{s-\eta-1}\int_0^1 u^{\eta-s}{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; au^{\mu}; b\right]\mathrm{d}u \\ &= \frac{t^{s-\eta-1}}{\mu}\frac{\Gamma\left(\frac{1}{\mu}(\eta-s+1)\right)}{\Gamma\left(\frac{1}{\mu}(\eta-s+1)+1\right)}\sum_{n=0}^{\infty}\frac{(A)_n\left(\frac{1}{\mu}(\eta-s+1)\right)_n}{\left(\frac{1}{\mu}(\eta-s+1)+1\right)_n}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(B+n, C-B)\frac{a^n}{n!} \\ &= \frac{t^{s-\eta-1}}{\mu}\frac{\Gamma\left(\frac{1}{\mu}(\eta-s+1)\right)}{\Gamma\left(\frac{1}{\mu}(\eta-s+1)+1\right)}{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; a; b\right] * {}_2F_1\left[\begin{matrix}1, & \frac{1}{\mu}(\eta-s+1) \\ & \frac{1}{\mu}(\eta-s+1)+1\end{matrix}; a\right].\end{aligned} \tag{55}$$

The final step above is obtained by applying the Hadamard product. Substituting (55) into (54), the result follows. □

By following a similar procedure we obtain the following theorem.

Theorem 4.6. *If $f(z)\in L_p(0,\infty), 1\leqslant p\le 2$ [or $f(z)\in M_p(0,\infty)$ and $p>2$], $\mathrm{Re}(C-A-B)>0, \mu>0,$*

$$\mathrm{Re}(\delta)>\max\left\{\frac{1}{p},\frac{1}{q}\right\}, \frac{1}{p}+\frac{1}{q}=1 \quad \text{and} \quad |\arg(1-a)|<\pi,$$

then we have

$$\mathfrak{M}\{\mathcal{J}[f(z)]\}(s) = \frac{\mu}{\Gamma(1-A)(s+\delta)}\,{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; a; b\right] * {}_2F_1\left[\begin{matrix}1, & \frac{1}{\mu}(\delta+s) \\ & \frac{1}{\mu}(\delta+s+\mu)\end{matrix}; a\right]\mathfrak{M}\{f(z)\}(s).$$

Definition 4.7. Define

$$\mathcal{R}[f(z)] = \frac{z^{-\sigma-\delta}}{\Gamma(\delta)}\int_0^z t^{\sigma}(z-t)^{\delta-1}{}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, & B \\ & C\end{matrix}; a\left(1-\frac{t}{z}\right); b\right]f(t)\mathrm{d}t \tag{56}$$

and

$$\mathcal{K}[f(z)] = \frac{z^{\zeta}}{\Gamma(\delta)} \int_z^{\infty} t^{-\zeta-\delta}(t-z)^{\delta-1} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; a\left(1-\frac{z}{t}\right); b\right] f(t)\mathrm{d}t, \tag{57}$$

where $A, B, C, \zeta, \sigma, \delta, a \in \mathbb{C}(\mathrm{Re}(C) > \mathrm{Re}(B) > 0)$ and ${}_2F_1^{(\alpha,\beta;\rho,\lambda)}[z; b]$ denotes the extended Gauss hypergeometric function.

The condition of the validity of the operator (56) and (57) are given below:

- $1 \leqslant p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, |\arg(1-a)| < \pi, \mu > 0.$
- $\mathrm{Re}(\sigma) > -\frac{1}{q}, \mathrm{Re}(\zeta) > -\frac{1}{p}, \mathrm{Re}(\delta) > 0, \mathrm{Re}(C-A-B) > -1.$
- $f \in L_p(0,\infty)$, where $L_p(0,\infty)$ denotes the Lebesgue space.

Remark 4.8. If we set $b = 0$ in (56) and (57), we get the known operators introduced by Saxena and Kumbhat [32].

Theorem 4.9. *If $f(z) \in L_p(0,\infty), 1 \leqslant p \leq 2$ [or $f(z) \in M_p(0,\infty)$ and $p > 2$], $\mathrm{Re}(C-A-B) > 0, \mu > 0$,*

$$\mathrm{Re}(\sigma) > \max\left\{\frac{1}{p}, \frac{1}{q}\right\}, \ \frac{1}{p} + \frac{1}{q} = 1, \quad \text{and} \quad |\arg(1-a)| < \pi,$$

then the following Mellin transform formula holds true:

$$\mathfrak{M}\{\mathcal{R}[f(z)]\}(s) = \frac{\Gamma(\sigma-s+1)}{\Gamma(\delta+\sigma-s+1)} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; a; b\right] * {}_2F_1\left[\begin{matrix} 1, & \frac{1}{\mu}(\delta) \\ & \frac{1}{\mu}(\delta+\sigma-s+1)+1 \end{matrix}; a\right] \mathfrak{M}\{f(z)\}(s),$$

where $M_p(0,\infty)$ denotes the class of all functions $f(z)$ of $L_p(0,\infty)$ with $p > 2$ which are inverse Mellin transforms of functions of $L_q(-\infty,\infty)$ $(q = p/(p-1))$.

Proof. From (56) we easily get

$$\mathfrak{M}\{\mathcal{R}[f(z)]\}(s) = \frac{1}{\Gamma(\delta)} \int_0^{\infty} z^{s-\sigma-\delta-1}\left\{\int_0^z t^{\sigma}(z-t)^{\delta-1} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; a\left(1-\frac{t}{z}\right); b\right] f(t)\mathrm{d}t\right\}\mathrm{d}z.$$

Changing the order of integration, which is permissible under the conditions stated in Theorem 4.9, we get

$$\begin{aligned}&\mathfrak{M}\{\mathcal{R}[f(z)]\}(s) \\ &= \frac{1}{\Gamma(\delta)} \int_0^{\infty} t^{\sigma} f(t)\left\{\int_t^{\infty} z^{s-\sigma-\delta-1}(z-t)^{\delta-1} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; a\left(1-\frac{t}{z}\right); b\right]\mathrm{d}z\right\}\mathrm{d}t.\end{aligned} \tag{58}$$

For convenience, we also denote the inner integral in (58) by $\mathfrak{I}'$. Then

$$\begin{aligned}\mathfrak{I}' &= t^{s-\sigma-1} \int_0^1 u^{\delta-1}(1-u)^{\sigma-s+1-1} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; au; b\right]\mathrm{d}u \\ &= t^{s-\sigma-1}\frac{\Gamma(\delta)\Gamma(\sigma-s+1)}{\Gamma(\delta+\sigma-s+1)}\sum_{n=0}^{\infty}\frac{(A)_n(\delta)_n}{(\delta+\sigma-s+1)_n}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(B+n, C-B)\frac{a^n}{n!} \\ &= t^{s-\sigma-1}\frac{\Gamma(\delta)\Gamma(\sigma-s+1)}{\Gamma(\delta+\sigma-s+1)} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; a; b\right] * {}_2F_1\left[\begin{matrix} 1, & \delta \\ & \delta+\sigma-s+1 \end{matrix}; a\right].\end{aligned} \tag{59}$$

Finally by applying the Hadamard product and substituting (59) into (58), we get the desired result. □

The same method gives the following theorem.

Theorem 4.10. *If $f(z) \in L_p(0,\infty), 1 \leqslant p \leq 2$ [or $f(z) \in M_p(0,\infty)$ and $p > 2$], $\mathrm{Re}(C-A-B) > 0, \mu > 0$,*

$$\mathrm{Re}(\zeta) > \max\left\{\frac{1}{p}, \frac{1}{q}\right\}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \mathrm{Re}(\delta) > 0, \quad \text{and} \quad |\arg(1-a)| < \pi,$$

then the following Mellin transform formula holds:

$$\mathfrak{M}\{\mathcal{K}[f(z)]\}(s) = \frac{\Gamma(s+\zeta)}{\Gamma((s+\zeta)+\delta)} {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} A, & B \\ & C \end{matrix}; a; b\right] * {}_2F_1\left[\begin{matrix} 1, & \frac{1}{\mu}(\delta) \\ & \frac{1}{\mu}(\delta+\zeta+s) \end{matrix}; a\right] \mathfrak{M}\{f(z)\}(s).$$

5. Concluding remarks and observations

As a conclusive remark we point out that the new special functions like $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y), {}_2F_1^{(\alpha,\beta;\rho,\lambda)}[z;b]$ and ${}_pF_q^{(\alpha,\beta;\rho,\lambda)}[z;b]$ introduced in this paper should be regarded as a class of continuous analogues (with respect to parameter b) of the classical beta function and hypergeometric functions.

The importance of these functions is that they inherit most of the properties of the original functions and provide new relations between different functions. In addition, these new extensions are very compatible with the fractional calculus. Therefore, results of this paper are general in character and give some contributions to the theory of the special function.

Acknowledgement

The second author was supported by the Serbian Ministry of Education, Science and Technological Development (No. #OI174015).

References

[1] P. Agarwal, Certain properties of the generalized Gauss hypergeometric functions, Appl. Math. Inf. Sci. 8 (5) (2014) 2315–2320.
[2] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V.H. Moll, A. Straub, Ramanujans master theorem, Ramanujan J. 29 (1–3) (2012) 103–120.
[3] L.C. Andrews, Special Functions for Engineers and Applied Mathematicians, Macmillan, New York, 1985.
[4] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge Univ Press, 1999.
[5] D. Baleanu, P. Agarwal, On generalized fractional integral operators and the generalized Gauss hypergeometric functions, Abstr. Appl. Anal. 2014 (2014). Article ID 630840, 5 pages. http://dx.doi.org/10.1155/2014/630840.
[6] Yu.A. Brychkov, Handbook of Special Functions: Derivatives, Integrals, Series and other Formulas, CRC Press, 2008.
[7] P.L. Butzer, S. Jansche, A direct approach to the Mellin transform, J. Fourier Anal. 3 (1997) 325–376.
[8] M.A. Chaudhry, S.M. Zubair, On a Class of Incomplete Gamma Functions with Applications, Chapman and Hall (CRC Press Company), Boca Raton, London, New York, Washington, D.C, 2001.
[9] M.A. Chaudhry, A. Qadir, M. Rafique, S.M. Zubair, Extension of Euler's beta function, J. Comput. Appl. Math. 78 (1997) 19–32.
[10] M.A. Chaudhry, A. Qadir, H.M. Srivastava, R.B. Paris, Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput. 159 (2) (2004) 589–602.
[11] A. Erdélyi, On some functional transformations, Univ. Politec. Torino. Rend. Sem. Mat. 10 (1950/51) 217–234.
[12] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. 1, McGraw-Hill, New York, Toronto, London, 1953.
[13] G. Kristensson, Second Order Differential Equation: Special Functions and Their Classification, Springer, New York, 2010.
[14] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, seventh ed., Academic Press, 2007.
[15] S.L. Kalla, R.K. Saxena, Integral operators involving hypergeometric functions, Math. Z. 108 (1969) 231–234.
[16] A.A. Kilbas, M. Saigo, H-Transforms: Theory and Applications, Chapman and Hall (CRC Press Company), Boca Raton, London, New York, Washington, D.C, 2004.
[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, in: Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[18] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman & J. Wiley, Harlow, New York, 1994.
[19] H. Kober, On fractional integrals and derivatives, Quart. J. Math. (Oxford) 11 (1940) 193–211.
[20] D.M. Lee, A.K. Rathie, R.K. Parmar, Y.S. Kim, Generalization of extended beta function, hypergeometric and confluent hypergeometric functions, Honam Math. J. 33 (2) (2011) 187–206.
[21] A.M. Mathai, R.K. Saxena, Generalized hypergeometric functions with applications in statistics and physical sciences, Lecture Notes in Mathematics, vol. 348, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
[22] A.M. Mathai, R.K. Saxena, H.J. Haubold, The H-function: Theory and Applications, Springer, New York, 2010.
[23] A.R. Miller, Remarks on a generalized beta function, J. Comput. Appl. Math. 100 (1998) 23–32.
[24] D.K. Nagar, N.A. Moran-Vasquez, A.K. Gupta, Properties and applications of extended hypergeometric functions, Ingenieria y Ciencia 10 (19) (2014) 11–31.
[25] N.E. Nrlund, Hypergeometric functions, Acta Math. 94 (1955) 289–349.
[26] M.-J. Luo, R.K. Raina, Extended generalized hypergeometric functions and their applications, Bull. Math. Anal. Appl. 5 (4) (2013) 65–77.
[27] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010.
[28] M.A. Özarslan, B. Yilmaz, The extended Mittag–Leffler function and its properties, J. Inequal. Appl. 85 (2014).
[29] M.A. Özarslan, E. Özergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, Math. Comput. Model. 52 (2010) 1825–1833.
[30] E. Özergin, M.A. Özarslan, A. Altin, Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math. 235 (2011) 4601–4610.
[31] T. Pohlen, The Hadamard product and universal power series (Dissertation), Universität Trier, 2009.
[32] R.K. Saxena, R.K. Kumbhat, A generalization of Kober operators, Vijnana Parishad Anusandhan Patrika 16 (1973) 177–186.
[33] H.M. Srivastava, R.K. Parmar, P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms I (2012) 238–258.