Summation of Slowly Convergent Series via Quadratures

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Abstract

In this paper we consider a few methods for summation of slowly convergent series using quadrature formulas of Gaussian type. Such summation/integration procedures are based (i) on the Laplace transform method, and (ii) on an integration over some contours in the complex plane. Especially, we investigate some series with irrational terms. Numerical results are included to illustrate these methods.

1 Introduction

We consider convergent series of the type

$$T = \sum_{k=1}^{+\infty} a_k \qquad \text{and} \qquad S = \sum_{k=1}^{+\infty} (-1)^k a_k \tag{1.1}$$

and introduce the notation: $T = T^{(m-1)} + T_m^{(\infty)}$, $S = S^{(m-1)} + S_m^{(\infty)}$,

$$T_m^{(n)} = \sum_{k=m}^{n} a_k, \qquad S_m^{(n)} = \sum_{k=m}^{n} (-1)^k a_k,$$

where $T^{(m-1)}$ and $S^{(m-1)}$ are the corresponding partial sums of (1.1).

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Some methods of summation these series can be found, for example, in the books of Henrici [5], Lindelöf [6], and Mitrinović and Kečkić [8].

Recently, a few new summation/integration procedures for slowly convergent series are developed (see [3], [1], [2], [7]). In this paper we will give a short account of these methods as well as some new approaches to this subject, including numerical examples in order to illustrate and compare these methods. Especially, we investigate some series with irrational terms.

2 Laplace Transform Method

Suppose that the general term of T (and S) is expressible in terms of the derivative of a Laplace transform, or in terms of the Laplace transform itself. Namely, let $a_k = F'(k)$, where

$$F(p) = \int_0^{+\infty} e^{-pt} f(t)\, dt, \qquad \operatorname{Re} p \geq 1.$$

Then

$$\sum_{k=1}^{+\infty} F'(k) = -\sum_{k=1}^{+\infty} \int_0^{+\infty} t e^{-kt} f(t)\, dt = -\int_0^{+\infty} \frac{t}{e^t - 1} f(t)\, dt.$$

Similarly, for "alternating" series, one obtains

$$\sum_{k=1}^{+\infty} (-1)^k F'(k) = \int_0^{+\infty} \frac{t}{e^t + 1} f(t)\, dt$$

and

$$\sum_{k=1}^{+\infty} (-1)^k F(k) = -\int_0^{+\infty} \frac{1}{e^t + 1} f(t)\, dt.$$

In a joint paper with Gautschi [3] we considered the construction of Gaussian quadrature formulas on $(0, +\infty)$,

$$\int_0^{+\infty} g(t) w(t)\, dt = \sum_{\nu=1}^{n} \lambda_\nu g(\tau_\nu) + R_n(g), \tag{2.1}$$

with respect to the weight functions

$$w(t) = \varepsilon(t) = \frac{t}{e^t - 1} \qquad \text{(Einstein's function)}$$

and

$$w(t) = \varphi(t) = \frac{1}{e^t + 1} \qquad \text{(Fermi's function)}.$$

We obtained the first $n = 40$ coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials accurately to 25 decimal digits (see [3, Tables 1 and 3]). If the series T and S are slowly convergent and the respective function f on the right of the equalities above is smooth, then low-order Gaussian quadrature (2.1) applied to the integrals on the right, provides a possible summation procedure. Numerical examples show fast convergence of this procedure (see [3, §4]). In the sequel we refer to this procedure as the *Laplace transform method.* A problem which arises with this procedure is the determination of the original function f for a given series. For some other applications see [1] and [2].

3 Contour Integration Over a Rectangle

Suppose that $a_k = f(k)$, where $z \mapsto f(z)$ is a holomorphic function in the region

$$G_m = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha, \ m - 1 < \alpha < m\}, \quad m \in \mathbb{N}.$$

In [7] we derived an alternative summation/integration method for the series (1.1) which requires the indefinite integral F of f chosen so as to satisfy the following decay conditions:

(C1) *F is a holomorphic function in the region G_m;*

(C2) $\lim\limits_{|t|\to+\infty} e^{-c|t|} F\,(x + it/\pi) = 0$*, uniformly for* $x \geq \alpha$;

(C3) $\lim\limits_{x\to+\infty} \int_{-\infty}^{+\infty} e^{-c|t|}\, |F\,(x + it/\pi)|\, dt = 0$,

where $c = 2$ or $c = 1$, when we consider $T_m^{(n)}$ or $S_m^{(n)}$, respectively.

Namely, taking $\Gamma = \partial G$ and

$$G = \Big\{z \in \mathbb{C} \mid \alpha \leq \operatorname{Re} z \leq \beta, \, |\operatorname{Im} z| \leq \frac{\delta}{\pi}\Big\},$$

where $m - 1 < \alpha < m$, $n < \beta < n + 1$ $(m, n \in \mathbb{Z}, m \leq n)$, we obtain that

$$T_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma f(z) \frac{\pi}{\tan \pi z}\, dz \quad \text{and} \quad S_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma f(z) \frac{\pi}{\sin \pi z}\, dz.$$

After integration by parts, these formulas reduce to

$$T_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 F(z)\, dz \tag{3.1}$$

and

$$S_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 \cos \pi z\, F(z)\, dz, \tag{3.2}$$

where $z \mapsto F(z)$ is an integral of $z \mapsto f(z)$.

Taking $\alpha = \alpha_m = m - 1/2$, $\beta = \beta_n = n + 1/2$, and letting $\delta \to +\infty$ and $n \to +\infty$, under conditions (C1) – (C3), the integrals in (3.1) and (3.2) over Γ reduce to integrals along the line $z = \alpha_m + iy$ $(-\infty < y < +\infty)$.

After some calculations, we reduce T and S to a problem of quadrature on $(0, +\infty)$ with respect to the weight functions

$$w_1(t) = \frac{1}{\cosh^2 t} \qquad \text{and} \qquad w_2(t) = \frac{\sinh t}{\cosh^2 t}, \tag{3.3}$$

respectively. Thus,

$$T = T^{(m-1)} + \int_0^{+\infty} \Phi\left(\alpha_m, t/\pi\right) w_1(t)\, dt$$

and

$$S = S^{(m-1)} + \int_0^{+\infty} \Psi\left(\alpha_m, t/\pi\right) w_2(t)\, dt,$$

where w_1 and w_2 are defined in (3.3) and

$$\Phi(x, y) = -\frac{1}{2}\left[F(x+iy) + F(x-iy)\right],$$

$$\Psi(x, y) = \frac{(-1)^m}{2i}\left[F(x+iy) - F(x-iy)\right].$$

Numerical quadratures of Gaussian type with respect to the weights w_1 and w_2 were constructed in [7]. The first $n = 40$ coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials were obtained accurately to 30 decimal digits.

Numerical experiments shows that is enough to use only the quadrature with respect to the first weight $w_1(t) = 1/\cosh^2 t$. Namely, in the series S we can include the hyperbolic sine as a factor in the corresponding integrand so that

$$S = S^{(m-1)} + \int_0^{+\infty} \Psi\left(\alpha_m, t/\pi\right) \sinh(t) w_1(t)\, dt. \tag{3.4}$$

4 Series With Irrational Terms

In this section we consider some series of the form

$$U_{\pm}(a,\nu) = \sum_{k=1}^{+\infty} \frac{(\pm 1)^{k-1}}{(k^2+a^2)^{\nu+1/2}}\,.$$

In 1916 Kapteyn (see [9, p. 386]) proved the formula

$$U_{+}(a,\nu) = \sum_{k=1}^{+\infty} \frac{1}{(k^2+a^2)^{\nu+1/2}} = \frac{\sqrt{\pi}}{(2a)^{\nu}\Gamma(\nu+1/2)} \int_0^{+\infty} \frac{t^{\nu}}{e^t-1}\, J_{\nu}(at)\,dt$$

which is valid when $\operatorname{Re}\nu > 0$ and $|\operatorname{Im} a| < 1$. Here, J_ν is the Bessel function of the order ν. Since for $F(p) = 1/(p^2+a^2)^{\nu+1/2}$ $(\operatorname{Re}\nu > -1/2,\ \operatorname{Re} p > |\operatorname{Im} a|)$ the original function is

$$f(t) = \frac{\sqrt{\pi}}{(2a)^{\nu}\Gamma(\nu+1/2)}\, t^{\nu} J_{\nu}(at),$$

using the Laplace transform method we find

$$U_{-}(a,\nu) = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{(k^2+a^2)^{\nu+1/2}} = \frac{\sqrt{\pi}}{(2a)^{\nu}\Gamma(\nu+1/2)} \int_0^{+\infty} \frac{t^{\nu}}{e^t+1}\, J_{\nu}(at)\,dt.$$

Thus, this method leads to an integration of the Bessel function $t \mapsto J_\nu(at)$ with Einstein's weight $\varepsilon(t)$ or Fermi's weight $\varphi(t)$. For some special values of ν, we can use also quadratures with respect to the weights $t^{\pm 1/2}\varepsilon(t)$ and $t^{\pm 1/2}\varphi(t)$ (see [3] and [1]).

In order to sum the series $U_{-}(a,0)$, $a > 0$, we can integrate the function $z \mapsto F(z) = g(z)/\sqrt{z^2+a^2}$, with $g(z) = \pi/\sin \pi z$, over the circle

$$C_n = \Big\{ z \in \mathbb{C} \;\Big|\; |z| = n + \frac{1}{2} \Big\}, \qquad n > a,$$

with cuts along the imaginary axis, so that the critical singularities ia and $-ia$ are eliminated (cf. [8, p. 217]). Precisely, the contour of integration Γ is given by $\Gamma = C_n^1 \cup l_1 \cup \gamma_1 \cup l_2 \cup C_n^2 \cup l_3 \cup \gamma_2 \cup l_4$, where C_n^1 and C_n^2 are parts of the circle C_n, γ_1 and γ_2 are small circular parts of radius ε and centres at $\pm ia$, and l_k $(k = 1,2,3,4)$ are the corresponding line segments.

Let $F^*(z)$ be the branch of $F(z)$ which corresponds to the value of the square root which is positive for $z = 1$. Since

$$\oint_\Gamma F^*(z)\,dz = 2\pi i \sum_{k=-n}^{n} \frac{(-1)^k}{\sqrt{k^2+a^2}},$$

and $\int_{\gamma_1} \to 0$, $\int_{\gamma_2} \to 0$, when $\varepsilon \to +0$, and $\int_{C_n^1 \cup C_n^2} \to 0$, when $n \to +\infty$, we obtain

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt{k^2+a^2}} = -\frac{1}{2a} + \int_a^{+\infty} \frac{du}{\sinh \pi u \sqrt{u^2-a^2}},$$

i.e.,

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{\sqrt{k^2+a^2}} = \frac{1}{2a} - \frac{1}{2}\int_{-1}^{+1} \left(t \sinh \frac{\pi a}{t} \right)^{-1} \frac{dt}{\sqrt{1-t^2}} \,.$$

Thus, we reduced $U_-(a,0)$ to a problem of Gauss-Chebyshev quadrature. Since $t \mapsto (t \sinh(\pi a/t))^{-1}$ is an even function we can apply the $(2n)$-point Gaussian approximations with only n functional evaluations, so that we have

$$U_-(a,0) \approx GC(n) = \frac{1}{2a} - \frac{\pi}{2n} \sum_{k=1}^{n} \Big(\tau_k \sinh \frac{\pi a}{\tau_k} \Big)^{-1}, \tag{4.1}$$

where $\tau_k = \cos((2k-1)\pi/(4n))$, $k = 1, \ldots, n$.

Remark 4.1. The same method can be applied to the summation of the series

$$\sum_{k=-\infty}^{+\infty} f(k, \sqrt{k^2+a^2}) \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} (-1)^k f(k, \sqrt{k^2+a^2}) \quad (a > 0),$$

where f is a rational function. Then we integrate the function $z \mapsto F(z) = f(z, \sqrt{z^2+a^2})g(z)$, with $g(z) = \pi/\tan \pi z$ and $g(z) = \pi/\sin \pi z$, respectively, over the circle C_n with the cuts.

5 Numerical Results

In this section we illustrate the previous methods taking the series $U_-(a,0)$, with $a = 0.25$, 0.5, 1, 2, 4. All computations were done in Q-arithmetic on the MICROVAX 3400 computer (machine precision $\approx 1.93 \times 10^{-34}$).

Table 5.1 shows the relative errors in Gaussian approximations $GC(n)$ (cf. (4.1)) and $GF(n)$ (the Laplace transform method with Fermi's weight), as well as

$$S_{n,m} = S^{(m-1)} + \sum_{\nu=1}^{n} \lambda_\nu \Psi(\alpha_m, \tau_\nu/\pi) \sinh(\tau_\nu), \quad m = 1, 2, 3, 4,$$

(cf. (3.4)) for $n = 5(5)40$ and $a = 0.25$. (Numbers in parentheses indicate decimal exponents.)

TABLE 5.1
Relative errors in Gaussian approximation of the sum $U_-(a, 0)$ for $a = 0.25$

n	$GC(n)$	$GF(n)$	$S_{n,1}$	$S_{n,2}$	$S_{n,3}$	$S_{n,4}$
5	1.8(−2)	2.7(−10)	2.4(−4)	1.6(−4)	1.4(−4)	1.2(−4)
10	7.2(−4)	8.8(−19)	9.2(−7)	4.2(−9)	3.9(−9)	3.6(−9)
15	1.9(−4)	1.8(−24)	1.8(−8)	9.2(−14)	8.6(−14)	8.2(−14)
20	2.9(−5)		5.6(−10)	1.3(−16)	1.7(−18)	1.7(−18)
25	1.2(−5)		1.2(−10)	1.5(−18)	2.6(−23)	3.2(−23)
30	9.0(−8)		1.1(−11)	3.0(−20)	8.7(−27)	6.1(−28)
35	1.0(−6)		6.1(−13)	2.5(−23)	3.1(−28)	1.1(−32)
40	3.5(−7)		7.0(−13)	1.8(−23)	2.0(−30)	1.6(−33)

The Bessel function J_0 was evaluated by means of the rational approximations in [4] indexed 5852, 6553 and 6953, with precision 23.22, 23.37 and 23.46, respectively. Because of that, some entries in third column are empty.

TABLE 5.2
Relative errors in Gaussian approximation of the sum $U_-(a, 0)$ for $a = 1$

n	$GC(n)$	$GF(n)$	$S_{n,1}$	$S_{n,2}$	$S_{n,3}$	$S_{n,4}$
5	3.1(−5)	2.0(−4)	2.0(−4)	2.4(−4)	2.1(−4)	1.8(−4)
10	5.1(−7)	2.1(−7)	6.2(−6)	6.3(−9)	5.9(−9)	5.4(−9)
15	9.4(−9)	9.0(−12)	2.2(−6)	1.6(−12)	1.3(−13)	1.2(−13)
20	5.0(−10)	2.3(−14)	3.8(−7)	5.9(−15)	2.4(−18)	2.5(−18)
25	4.5(−11)	2.5(−19)	3.9(−8)	2.1(−16)	1.1(−22)	4.9(−23)
30	3.6(−12)	2.3(−21)	1.8(−8)	2.5(−17)	2.0(−24)	1.0(−27)
35	6.6(−14)	7.0(−26)	3.3(−9)	1.4(−18)	4.8(−26)	6.4(−31)
40	4.4(−14)		2.3(−9)	3.4(−20)	3.1(−27)	6.3(−33)

TABLE 5.3
Relative errors in Gaussian approximation of the sum $U_-(a,0)$ for $a=4$

n	$GC(n)$	$GF(n)$	$S_{n,1}$	$S_{n,2}$	$S_{n,3}$	$S_{n,4}$
5	8.2(−10)	1.2	9.4(−4)	7.6(−4)	6.3(−4)	5.4(−4)
10	9.0(−14)	6.5(−2)	7.0(−8)	1.3(−8)	2.0(−8)	1.9(−8)
15	4.9(−17)	1.5(−1)	3.6(−8)	1.6(−10)	6.5(−13)	4.3(−13)
20	1.5(−19)	3.3(−3)	2.5(−9)	7.0(−12)	1.5(−15)	8.9(−18)
25	9.2(−22)	1.7(−2)	2.4(−8)	3.1(−13)	2.3(−17)	4.2(−21)
30	1.1(−24)	2.8(−4)	8.6(−9)	6.4(−14)	1.1(−18)	1.4(−24)
35	8.9(−26)	1.9(−3)	4.8(−9)	7.9(−15)	5.0(−20)	1.2(−25)
40	1.4(−27)	1.1(−4)	1.9(−9)	3.4(−17)	2.2(−21)	5.7(−27)

The corresponding relative errors for $a = 1$ and $a = 4$ are presented in Tables 5.2 and 5.3, respectively. Also, we mention here the exact sums $U_-(a,0)$ (determined as $S_{40,5}$ to 30 significant digits):

$$
\begin{aligned}
U_-(0.25,0) &= 0.666326189064665806052832629421,\\
U_-(0.50,0) &= 0.599262331208773941764013859373,\\
U_-(1.00,0) &= 0.440917473865185397183787033140,\\
U_-(2.00,0) &= 0.248166827542466167909044168541,\\
U_-(4.00,0) &= 0.124997557589011259481281086361,
\end{aligned}
$$

As we can see, the Laplace transform method ($GF(n)$) is very efficient for a small parameter a. However, when a increases, the integrand $J_0(at)$ becomes a highly oscillatory function and the convergence of the process slows down considerably. On the other hand, the convergence of Gauss-Chebyshev approximations $GC(n)$ is slightly faster if the parameter a is larger. Also, we can see a rapidly increasing of convergence of the summation process $S_{n,m}$ as m increases.

References

[1] W. Gautschi, *A class of slowly convergent series and their summation by Gaussian quadrature*, Math. Comp. **57** (1991), 309–324.

[2] W. Gautschi, *On certain slowly convergent series occurring in plate contact problems*, Math. Comp. **57** (1991), 325–338.

[3] W. Gautschi and G.V. Milovanović, *Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series*, Math. Comp. **44** (1985), 177–190.

[4] J.F. Hart et al., *Computer Approximations*, Wiley, New York-London-Sydney, 1968.

[5] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1, Wiley, New York, 1984.

[6] E. Lindelöf, *Le calcul des résidus*, Gauthier–Villars, Paris, 1905.

[7] G.V. Milovanović, *Summation of series and Gaussian quadratures*, In: Approximation and Computation (R. V. M. Zahar, ed.), ISNM, Birkhäuser, Boston, 1994 (to appear).

[8] D.S. Mitrinović and J.D. Kečkić, *The Cauchy Method of Residues – Theory and Applications*, Reidel, Dordrecht, 1984.

[9] G.N. Watson, *Theory of Bessel Functions*, Cambridge, 1944.

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