Generating Functions for Special Polynomials and Numbers Including Apostol-type and Humbert-type Polynomials

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Abstract. The aim of this paper is to give generating functions and to prove various properties for some new families of special polynomials and numbers. Several interesting properties of such families and their connections with other polynomials and numbers of the Bernoulli, Euler, Apostol-Bernoulli, Apostol-Euler, Genocchi and Fibonacci type are presented. Furthermore, the Fibonacci type polynomials of higher order in two variables and a new family of special polynomials $(x,y) \mapsto \mathbb{G}_d(x,y;k,m,n)$, including several paricular cases, are introduced and studied. Finally, a class of polynomials and corresponding numbers, obtained by a modification of the generating function of Humbert's polynomials, is also considered.

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1. Introduction and preliminaries

The special polynomials and numbers play an important role in many branches of mathematics and their development is always actual. Many papers and books were published in this very wide area. We mention only a few books connected with our results in this work (cf. [4], [7], [27], [28]).

In this paper we consider some new families of numbers and polynomials, including their generating functions, several interesting properties, as well as their

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connections with other polynomials and numbers of the Bernoulli, Euler, Apostol-Bernoulli, Apostol-Euler, Genocchi, Fibonacci and Lucas type. In order to give our results, we need to mention several special classes of polynomials and numbers with their generating functions.

1° *The Bernoulli polynomials* of higher order $B_d^{(h)}(x)$ are defined by means of the following generating function

$$F_{Bh}(x,t;h)=\left(\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t-1}\right)^h=\sum_{d=0}^{\infty}B_d^{(h)}(x)\frac{t^d}{d!}. \tag{1.1}$$

For $h=1$, (1.1) reduces to the generating function of the classical Bernoulli polynomials, $B_d^{(1)}(x)=B_d(x)$. Furthermore, for $x=0$, this gives the well known Bernoulli numbers $B_d=B_d(0)$. For details see [1]–[7], [13]–[22], [29].

2° *The Apostol-Bernoulli polynomials* were introduced in 1951 by Apostol [1] by means of the following generating function

$$F_{AB}(x,t;\lambda)=\frac{t\mathrm{e}^{xt}}{\lambda\mathrm{e}^t-1}=\sum_{d=0}^{\infty}\mathscr{B}_d(x,\lambda)\frac{t^d}{d!}, \tag{1.2}$$

where $|t+\log\lambda|<2\pi$ (for details see [1]–[7], [13]–[22], [29]). Several their interesting properties, formulas and extensions have been obtained by Srivastava [26] (see also the recent book [27]). Using the suitable generating functions several authors have obtained different generalizations and unifications of these numbers and polynomials (cf. [2], [5], [13], [14], [16], [17], [22], [29]).

Substituting $x=0$ in (1.2), for $\lambda\neq 1$, we get the Apostol-Bernoulli numbers $\mathscr{B}_d(\lambda)$,

$$\mathscr{B}_d(\lambda)=\mathscr{B}_d(0,\lambda), \tag{1.3}$$

and they can be expressed it terms of Stirling numbers of the second kind [1, Eq. (3.7)]. Setting $\lambda=1$ in (1.2), we get the classical Bernoulli polynomials $B_d(x)=\mathscr{B}_d(x,1)$.

Alternatively, the Apostol-Bernoulli numbers can be expressed in the form

$$\mathscr{B}_0(\lambda)=0,\quad \mathscr{B}_d(\lambda)=(-1)^{d-1}d\,\frac{\lambda\varphi_{d-2}(\lambda)}{(\lambda-1)^d},\quad d\geq 1, \tag{1.4}$$

where $\varphi_k(\lambda)$ are monic polynomials in λ and of degree k and $\varphi_k(0)=1$. Using the generating function (1.2) for $x=0$ and (1.4), it is easy to prove that the polynomials $\varphi_k(\lambda)$ are self-inversive (cf. [20, pp. 16–18]), i.e., $\lambda^k\varphi_k(1/\lambda)\equiv\varphi_k(\lambda)$. Also, we can prove that

$$\varphi_k(\lambda)=(1-\lambda)^k+\lambda\sum_{\nu=1}^{k}\binom{k+1}{\nu}(1-\lambda)^{\nu-1}\varphi_{k-\nu}(\lambda),\quad k\geq 0, \tag{1.5}$$

as well as the following determinant form

$$\varphi_k(\lambda)=(-1)^k\lambda^k\begin{vmatrix} -1/\lambda & 0 & 0 & \cdots & 0 & 1\\ \binom{2}{1} & -1/\lambda & 0 & \cdots & 0 & \xi\\ \binom{3}{1}\xi & \binom{3}{2} & -1/\lambda & \cdots & 0 & \xi^2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \binom{k}{1}\xi^{k-2} & \binom{k}{2}\xi^{k-3} & \binom{k}{3}\xi^{k-4} & \cdots & -1/\lambda & \xi^{k-1}\\ \binom{k+1}{1}\xi^{k-1} & \binom{k+1}{2}\xi^{k-2} & \binom{k+1}{3}\xi^{k-3} & \cdots & \binom{k+1}{k} & \xi^k \end{vmatrix},$$

where $\xi=1-\lambda$. For example, we have

$$\begin{aligned}
&\varphi_0(\lambda)=1,\quad \varphi_1(\lambda)=\lambda+1,\quad \varphi_2(\lambda)=\lambda^2+4\lambda+1,\\
&\varphi_3(\lambda)=\lambda^3+11\lambda^2+11\lambda+1,\quad \varphi_4(\lambda)=\lambda^4+26\lambda^3+66\lambda^2+26\lambda+1,\\
&\varphi_5(\lambda)=\lambda^5+57\lambda^4+302\lambda^3+302\lambda^2+57\lambda+1,\\
&\varphi_6(\lambda)=\lambda^6+120\lambda^5+1191\lambda^4+2416\lambda^3+1191\lambda^2+120\lambda+1,
\end{aligned}$$

etc. Using (1.5) we can conclude that $\varphi_k(1)=(k+1)!$.

3° *The Apostol-Euler polynomials of the first kind* $\mathcal{E}_d(x,\lambda)$ are defined by means of the generating function

$$F_{AE}(x,t;\lambda)=\frac{2\mathrm{e}^{xt}}{\lambda\mathrm{e}^t+1}=\sum_{d=0}^{\infty}\mathcal{E}_d(x,\lambda)\frac{t^d}{d!},\tag{1.6}$$

where $|2t+\log\lambda|<\pi$ (cf. [1]–[7], [22], [29]). For $\lambda\neq 1$, substituting $x=1/2$ in (1.6) and making some arrangement, we obtain the Apostol-Euler numbers. Setting $\lambda=1$ in (1.6), we get the first kind Euler polynomials $E_d(x)=\mathcal{E}_d(x,1)$.

4° *The Apostol-Euler polynomials of the second kind* are defined by means of the generating function

$$\frac{2}{\lambda\mathrm{e}^t+\lambda^{-1}\mathrm{e}^{-t}}\mathrm{e}^{tx}=\sum_{d=0}^{\infty}\mathcal{E}_d^*(x,\lambda)\frac{t^d}{d!}\tag{1.7}$$

(cf. [25]). A special kind of these polynomials for $\lambda=1$ are denoted by $\mathcal{E}_d^*(x)=\mathcal{E}_d^*(x,1)$, and the corresponding numbers by $\mathcal{E}_d^*=\mathcal{E}_d^*(0)$. By using (1.6) and (1.7), for $x=0$, we have the following relation

$$\mathcal{E}_d^*(0,\lambda)=2^d\lambda\mathcal{E}_d\left(\frac{1}{2},\lambda^2\right).$$

The second kind Euler numbers E_d^* are defined by the special case of the first kind Euler polynomials, $E_d^*=2^dE_d(1/2)$.

5° *The Euler polynomials of higher order* $E_d^{(h)}(x)$ are defined by means of the following generating function

$$F_{Eh}(x,t;h)=\left(\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t+1}\right)^h=\sum_{d=0}^{\infty}E_d^{(h)}(x)\frac{t^d}{d!},\tag{1.8}$$

so that, obviously, $E_d^{(1)}(x) = E_d(x)$.

6° *The Genocchi numbers and polynomials and their generalizations.* The Genocchi numbers G_d are defined by the generating function

$$F_g(t) = \frac{2t}{e^t + 1} = \sum_{d=0}^{\infty} G_d \frac{t^d}{d!}, \tag{1.9}$$

where $|t| < \pi$ (cf. [13], [16], [22], [29]).

In general, for these numbers we have $G_0 = 0$, $G_1 = 1$, and $G_{2d+1} = 0$ for $d \in \mathbb{N}$. Some relations between the Genocchi, Bernoulli and Euler numbers are given by $G_{2d} = 2\left(1 - 2^{2d}\right)B_{2d}$ and $G_{2d} = 2dE_{2d-1}$. The sequence of Genocchi numbers is

$$\{g_d\}_{d \geq 0} = \{0, 1, -1, 0, 1, 0, -3, 0, 17, 0, -155, 0, \ldots\}.$$

The Genocchi polynomials $G_d(x)$ are defined by the following generating function

$$F_g(x;t) = F_g(t)\, e^{xt} = \sum_{d=0}^{\infty} G_d(x) \frac{t^d}{d!}, \tag{1.10}$$

where $|t| < \pi$. Using (1.10), it is easy to see that

$$G_d(x) = \sum_{k=0}^{d} \binom{d}{k} G_k x^{d-k}$$

The first seven Genocchi polynomials are

$$\begin{aligned}
&G_0(x) = 0, \quad G_1(x) = 1, \quad G_2(x) = 2x - 1, \quad G_3(x) = 3x^2 - 3x, \\
&G_4(x) = 4x^3 - 6x^2 + 1, \quad G_5(x) = 5x^4 - 10x^3 + 5x, \\
&G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3.
\end{aligned}$$

The Apostol-Genocchi polynomials $g_d(x, \lambda)$ are defined by the generating function

$$\frac{2t}{\lambda e^t + 1} e^{xt} = \sum_{d=0}^{\infty} G_d(x, \lambda) \frac{t^d}{d!}, \tag{1.11}$$

where $|2t + \log \lambda| < \pi$. Setting $\lambda = 1$ in (1.11), we get the classical Genocchi polynomials $G_d(x) = G_d(x, 1)$, which reduce to the classical Genocchi numbers $G_d = G_d(0)$ for $x = 0$.

Substituting $x = 0$ in (1.11), for $\lambda \neq 1$, we obtain the Apostol-Genocchi numbers $G_d(\lambda) = G_d(0, \lambda)$. For some details, properties and other generalizations see [11], [13], [16], [22], [26], [27], [29].

7° *The Stirling numbers of the second kind* $S_2(n, k; \lambda)$ are defined by means of the following generating function (cf. [3], [24], [26]):

$$F_S(t, k; \lambda) = \frac{\left(\lambda e^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k; \lambda) \frac{t^n}{n!}, \tag{1.12}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$.

The generalized Stirling numbers and polynomials have been defined by means of the following generating function (cf. [3]):

$$\frac{(\mathrm{e}^t-1)^k}{k!}\mathrm{e}^{t\alpha}=\sum_{n=0}^{\infty}S^{(\alpha)}(n,k)\frac{t^n}{n!}. \tag{1.13}$$

Several combinatorial properties of these polynomials have been proved in [3].

Simsek [24] has modified the generating function (1.13), defining the so-called λ-array polynomials $S_k^n(x;\lambda)$ by means of the following generating function

$$F_A(t,x,k;\lambda)=\frac{(\lambda\mathrm{e}^t-1)^k}{k!}\mathrm{e}^{tx}=\sum_{n=0}^{\infty}S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{1.14}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. Substituting $\lambda=1$, the λ-array polynomials reduce to the array polynomials, $S^{(\alpha)}(n,k)=S_k^n(\alpha;1)$ (cf. [3], [24]).

8° *The Humbert polynomials* $\left\{\Pi_{n,m}^{\lambda}\right\}_{n=0}^{\infty}$ were defined in 1921 by Humbert [12]. Their generating function is

$$\left(1-mxt+t^m\right)^{-\lambda}=\sum_{n=0}^{\infty}\Pi_{n,m}^{\lambda}(x)\,t^n. \tag{1.15}$$

This function satisfies the following recurrence relation (cf. [7], [18], [19] and references therein):

$$(n+1)\,\Pi_{n+1,m}^{\lambda}(x)-mx\,(n+\lambda)\,\Pi_{n,m}^{\lambda}(x)-(n+m\lambda-m+1)\,\Pi_{n-m+1,m}^{\lambda}(x)=0.$$

A special case of these polynomials are the Gegenbauer polynomials given as follows [8]:

$$C_n^{\lambda}(x)=\Pi_{n,2}^{\lambda}(x)$$

and also the Pincherle polynomials given as follows (see [23], [12]):

$$\mathscr{P}_n(x)=\Pi_{n,3}^{-1/2}(x)\,.$$

Later, Gould [9] studied a class of generalized Humbert polynomials, $P_n(m,x,y,p,C)$, defined by

$$\left(C-mxt+yt^m\right)^p=\sum_{n=0}^{\infty}P_n(m,x,y,p,C)\,t^n,$$

where $m\geq 1$ is an integer and the other parameters are unrestricted in general (cf. [7], [10]).

Some special cases of the generalized Humbert polynomials, $P_n(m,x,y,p,C)$, can be given as follows (cf. [12]):

$$\begin{array}{lcll}
P_n\left(2,x,1,-\frac{1}{2},1\right) & = & \mathrm{P}_n(x) & \text{Legendre (1784),}\\
P_n\left(2,x,1,-\nu,1\right) & = & C_n^{\nu}(x) & \text{Geganbauer (1874),}\\
P_n\left(3,x,1,-\frac{1}{2},1\right) & = & \mathscr{P}_n(x) & \text{Pincherle (1890),}\\
P_n\left(m,x,1,-\nu,1\right) & = & h_{n,m}^{\nu}(x) & \text{Humbert (1921).}
\end{array}$$

9° *The Fibonacci type polynomials in two variables* $(x,y) \mapsto \mathscr{G}_j(x,y;k,m,n)$ has been recently defined by Ozdemir and Simsek [21] by the following generating function

$$H(t;x,y;k,m,n) = \sum_{j=0}^{\infty} \mathscr{G}_j(x,y;k,m,n)\, t^j = \frac{1}{1 - x^k t - y^m t^{m+n}}, \tag{1.16}$$

where $k,m,n \in \mathbb{N}_0$. An explicit formula for the polynomials $\mathscr{G}_j(x,y;k,m,n)$, $j = 0,1,\ldots$, can be done in the following form [21]

$$\mathscr{G}_j(x,y;k,m,n) = \sum_{c=0}^{\left[\frac{j}{m+n}\right]} \binom{j - c(m+n-1)}{c} y^{mc} x^{jk - mck - nck},$$

where $[a]$ is the largest integer $\leq a$.

In this paper we give some new identities for the previous classes of polynomials and investigate some new properties of these polynomials. Moreover, by using their generating functions, we give some applications which are associated with the Fibonacci type polynomials of higher order in two variables.

The paper is organized as follows. Fibonacci type polynomials of higher order in two variables and a new family of special polynomials $(x,y) \mapsto \mathbb{G}_d(x,y;k,m,n)$ are introduced and studied in Sections 2 and 3, respectively. Special cases of polynomials $\mathbb{G}_d(x,y;k,m,n)$ are investigated in Section 4. Finally, Section 5 is devoted to a class of polynomials and corresponding numbers, obtained by a modification of the generating function of Humbert's polynomials.

2. Fibonacci type polynomials of higher order in two variables

In this section we give a new generalization of the Fibonacci type polynomials in two variables.

Definition 2.1. *Two variable Fibonacci type polynomials of higher order* $(x,y) \mapsto \mathscr{G}_j^{(h)}(x,y;k,m,n)$ *are defined by the following generating function*

$$\sum_{j=0}^{\infty} \mathscr{G}_j^{(h)}(x,y;k,m,n)\, t^j = \frac{1}{\left(1 - x^k t - y^m t^{n+m}\right)^h} \tag{2.1}$$

where h is a positive integer.

Observe that

$$\mathscr{G}_j^{(1)}(x,y;k,m,n) = \mathscr{G}_j(x,y;k,m,n).$$

We give now a computation formula of two variable Fibonacci type polynomials of higher order h in the following statement.

Theorem 2.2. *We have*

$$\mathscr{G}_j^{(h_1+h_2)}(x,y;k,m,n) = \sum_{\ell=0}^{j} \mathscr{G}_\ell^{(h_1)}(x,y;k,m,n)\, \mathscr{G}_{j-\ell}^{(h_2)}(x,y;k,m,n). \tag{2.2}$$

Proof. Setting $h = h_1 + h_2$ into (2.1), we start with

$$\sum_{j=0}^{\infty} \mathscr{G}_j^{(h_1+h_2)}(x,y;k,m,n)\,t^j = \frac{1}{\left(1-x^k t - y^m t^{n+m}\right)^{h_1}} \cdot \frac{1}{\left(1-x^k t - y^m t^{n+m}\right)^{h_2}},$$

and then, using again (2.1), we get

$$\sum_{j=0}^{\infty} \mathscr{G}_j^{(h_1+h_2)}(x,y;k,m,n)\,t^j = \sum_{j=0}^{\infty} \mathscr{G}_j^{(h_1)}(x,y;k,m,n)\,t^j \sum_{j=0}^{\infty} \mathscr{G}_j^{(h_2)}(x,y;k,m,n)\,t^j.$$

Now, by using the Cauchy product in the right-hand side of the above equation, we obtain

$$\sum_{j=0}^{\infty} \mathscr{G}_j^{(h_1+h_2)}(x,y;k,m,n)\,t^j = \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \mathscr{G}_\ell^{(h_1)}(x,y;k,m,n)\,\mathscr{G}_{j-\ell}^{(h_2)}(x,y;k,m,n)\,t^j.$$

Finally, comparing the coefficients of t^j on both sides in the previous equality, we arrive at the desired result (2.2). $\square$

Remark 2.3. *Setting $h_1 = h_2 = 1$ in* (2.2)*, we obtain the following formula for computing two variable Fibonacci type polynomials of the second order,*

$$\mathscr{G}_j^{(2)}(x,y;k,m,n) = \sum_{\ell=0}^{j} \mathscr{G}_\ell(x,y;k,m,n)\,\mathscr{G}_{j-\ell}(x,y;k,m,n).$$

If we take $x := ax$, $y = -1$, $k = 1$, $m = 1$, $n = a-1$, (2.1) reduces to

$$\begin{aligned}\sum_{j=0}^{\infty} \mathscr{G}_j^{(h)}(ax,-1;1,1,a-1)\,t^j &= \frac{1}{\left(1-axt+t^a\right)^h} \\ &= \sum_{j=0}^{\infty} \Pi_{j,a}^h(x)\,t^j.\end{aligned}$$

Comparing the coefficients of t^j on both sides of the above equality, we obtain the following result:

Corollary 2.4. *A relation between two variable Fibonacci type polynomials of higher order $\mathscr{G}_j^{(h)}(x,y;k,m,n)$ and Humbert polynomials $\Pi_{n,m}^h(x)$ is given by*

$$\mathscr{G}_j^{(h)}(ax,-1;1,1,a-1) = \Pi_{j,a}^h(x).$$

3. Special polynomials including two variable Fibonacci type polynomials and Bernoulli and Euler type polynomials

In this section, in order to introduce a new family of polynomials, we modify and unify the generating functions of the Fibonacci type polynomials in two variables. By using these generating functions, we derive some relations and identities including the Apostol-Bernoulli numbers, the Bernoulli type polynomials, the Humbert polynomials and the Genocchi polynomials. These relations and identities also include the Fibonacci type polynomials in two variables.

Now, we introduce the generating function for these new special polynomials in two variables $(x,y) \mapsto \mathbb{G}_d(x,y;k,m,n)$, $d \geq 0$, with the three free parameters k,m,n.

Definition 3.1. *The polynomials $\mathbb{G}_d(x,y;k,m,n)$ are defined by means of the following generating function*

$$\begin{aligned}\mathbb{F}(z;x,y;k,m,n) &= \frac{1-x^k-y^m}{1-x^k\mathrm{e}^z-y^m\mathrm{e}^{z(m+n)}} \\ &= \sum_{d=0}^{\infty} \frac{\mathbb{G}_d(x,y;k,m,n)}{d!}\left(\frac{z}{1-x^k-y^m}\right)^d. \end{aligned} \tag{3.1}$$

A recurrence relation for the polynomials $\mathbb{G}_d(x,y;k,m,n)$ can be proved.

Theorem 3.2. *Let $\mathbb{G}_0(x,y;k,m,n)=1$ and d be a positive integer. Then we have*

$$\begin{aligned}\mathbb{G}_d(x,y;k,m,n) &= x^k\sum_{j=0}^{d}\binom{d}{j}\mathbb{G}_j(x,y;k,m,n)\left(1-x^k-y^m\right)^{d-j} \\ &\quad + y^m\sum_{j=0}^{d}\binom{d}{j}\mathbb{G}_j(x,y;k,m,n)(m+n)^{d-j}\left(1-x^k-y^m\right)^{d-j}.\end{aligned}$$

Proof. By applying the umbral calculus methods to (3.1), we get

$$\begin{aligned}1-x^k-y^m = \sum_{d=0}^{\infty}\mathbb{G}_d(x,y;k,m,n)\frac{z^d}{(1-x^k-y^m)^d\,d!} \\ -x^k\sum_{d=0}^{\infty}\left(\mathbb{G}(x,y;k,m,n)+1-x^k-y^m\right)^d\frac{z^d}{(1-x^k-y^m)^d\,d!} \\ -y^m\sum_{d=0}^{\infty}\left(\mathbb{G}(x,y;k,m,n)+(m+n)\left(1-x^k-y^m\right)\right)^d\frac{z^d}{(1-x^k-y^m)^d\,d!},\end{aligned}$$

with the usual convention of replacing $\mathbb{G}^d(x,y;k,m,n)$ by $\mathbb{G}_d(x,y;k,m,n)$. Comparing the coefficients of z^d on the both sides of the previous equality, we arrive at the desired result. □

A few first polynomials are

$$\begin{aligned}&\mathbb{G}_0(x,y;k,m,n)=1, \quad \mathbb{G}_1(x,y;k,m,n)=x^k+(m+n)y^m, \\ &\mathbb{G}_2(x,y;k,m,n)=[x^k+(m+n)y^m]^2-(m+n-1)^2x^ky^m+x^k+(m+n)^2y^m,\end{aligned}$$

etc.

3.1. Relations between the polynomials and numbers

Here, we consider some special cases of the polynomials $\mathbb{G}_d(x,y;k,m,n)$. By using the generating function from (3.1), we derive some new identities and relations, which include the polynomials $\mathbb{G}_d(x,y;k,m,n)$, the Apostol-Bernoulli and the Apostol-Euler polynomials and numbers, as well as the classical Bernoulli, Euler and Genocchi polynomials and numbers.

Theorem 3.3. *Let $d \geq 1$. The polynomials $\mathbb{G}_d(x,y;k,m,n)$, $d \geq 1$, are connected with the Apostol-Bernoulli numbers $\mathscr{B}_d(\lambda)$ in the following way*

$$\mathbb{G}_{d-1}(x,y;k,1,0) = -\frac{\left(1-x^k-y\right)^d}{d}\mathscr{B}_d\left(x^k+y\right), \quad d \geq 1. \tag{3.2}$$

Proof. First, according to (3.1) and (1.2), we have the following relation

$$\mathbb{F}(z;x,y;k,1,0) = -\frac{1-x^k-y}{z}F_{AB}(0,z;x^k+y),$$

i.e.,

$$\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(x,y;k,1,0)}{\left(1-x^k-y\right)^d}\frac{z^d}{d!} = -\frac{1-x^k-y}{z}\sum_{d=0}^{\infty}\mathscr{B}_d\left(x^k+y\right)\frac{z^d}{d!},$$

where we also used (1.3). However, since

$$\begin{aligned}\frac{z}{1-x^k-y}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(x,y;k,1,0)}{\left(1-x^k-y\right)^d}\cdot\frac{z^d}{d!} &= \sum_{d=0}^{\infty}\frac{(d+1)\mathbb{G}_d(x,y;k,1,0)}{\left(1-x^k-y\right)^{d+1}}\cdot\frac{z^{d+1}}{(d+1)!} \\ &= \sum_{d=1}^{\infty}\frac{d\mathbb{G}_{d-1}(x,y;k,1,0)}{\left(1-x^k-y\right)^d}\cdot\frac{z^d}{d!},\end{aligned}$$

we have

$$\sum_{d=1}^{\infty}\frac{d\mathbb{G}_{d-1}(x,y;k,1,0)}{\left(1-x^k-y\right)^d}\cdot\frac{z^d}{d!} = -\sum_{d=1}^{\infty}\mathscr{B}_d\left(x^k+y\right)\frac{z^d}{d!}, \tag{3.3}$$

because $\mathscr{B}_0(\lambda) = 0$.

Comparing the coefficients of $z^d/d!$ on both sides in (3.3), we obtain (3.2). □

By using the Apostol-Bernoulli numbers and the equality (3.2) we get another computation formula for the polynomials $\mathbb{G}_d(x,y;k,m,n)$. Thus,

$$\begin{aligned}\mathbb{G}_0(x,y;k,1,0) &= -\left(1-x^k-y\right)\mathscr{B}_1\left(x^k+y\right) = 1, \\ \mathbb{G}_1(x,y;k,1,0) &= -\frac{\left(1-x^k-y\right)^2}{2}\mathscr{B}_2\left(x^k+y\right) = x^k+y, \\ \mathbb{G}_2(x,y;k,1,0) &= -\frac{\left(1-x^k-y\right)^3}{3}\mathscr{B}_3\left(x^k+y\right) = x^{2k}+2x^ky+x^k+y^2+y, \\ \mathbb{G}_3(x,y;k,1,0) &= -\frac{\left(1-x^k-y\right)^4}{4}\mathscr{B}_4\left(x^k+y\right) = \left(x^k+y\right)\left[\left(x^k+y\right)^2+4\left(x^k+y\right)+1\right], \\ \mathbb{G}_4(x,y;k,1,0) &= -\frac{\left(1-x^k-y\right)^5}{5}\mathscr{B}_5\left(x^k+y\right) \\ &= \left(x^k+y\right)\left[\left(x^k+y\right)^3+11\left(x^k+y\right)^2+11\left(x^k+y\right)+1\right], \quad \text{etc.}\end{aligned}$$

Theorem 3.4. *Let $d \geq 0$. The relation between the polynomials $\mathbb{G}_d(x,y;k,m,n)$ and the Apostol-Bernoulli polynomials $\mathscr{B}_d(x,\lambda)$ is given by*

$$\mathscr{B}_d\left(x,x^k\right) = -d\sum_{j=0}^{d-1}\binom{d-1}{j}\frac{x^{d-1-j}}{\left(1-x^k\right)^{j+1}}\mathbb{G}_j(x,0;k,m,n). \tag{3.4}$$

Proof. Starting with (1.6) and (3.1) for $y=0$ $m\neq 0$, we conclude that

$$ze^{xz}\mathbb{F}(z;x,0;k,m,n)=ze^{xz}\frac{1-x^k}{1-x^k e^z}=\left(x^k-1\right)F_{AE}(x,z;x^k), \tag{3.5}$$

i.e.,

$$z\sum_{d=0}^{\infty}\frac{(xz)^d}{d!}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(x,0;k,m,n)}{(1-x^k)^{d+1}}\frac{z^d}{d!}=-\sum_{d=0}^{\infty}\mathscr{B}_d\left(x,x^k\right)\frac{z^d}{d!},$$

after replacing by the corresponding series representations. Now, using the Cauchy product on the left-hand side of the above equality, we obtain

$$\sum_{d=0}^{\infty}\sum_{j=0}^{d}\binom{d}{j}x^{d-j}\frac{\mathbb{G}_j(x,0;k,m,n)}{(1-x^k)^{j+1}}\frac{z^{d+1}}{d!}=-\sum_{d=0}^{\infty}\mathscr{B}_d\left(x,x^k\right)\frac{z^d}{d!},$$

i.e., (3.4). □

Remark 3.5. *By using* (3.5)*, the equality* (3.4) *can be also given in the following form*

$$\mathscr{B}_d\left(x,x^k\right)=-\sum_{j=1}^{d}\binom{d}{j}x^{d-j}\frac{j\,\mathbb{G}_{j-1}(x,0;k,m,n)}{\left(1-x^k\right)^j}.$$

Theorem 3.6. *The Euler polynomials $E_d(x)$ can be expressed in terms of the polynomials $\mathbb{G}_d(x,y;k,m,n)$ as*

$$E_d(x)=\sum_{j=0}^{d}\binom{d}{j}x^{d-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}. \tag{3.6}$$

Proof. As in the proof of Theorem 3.4, we assume that $m\neq 0$ and start with a special case of the generating function in (3.1), with $x=-1$, $y=0$ and $k=1$, i.e.,

$$\mathbb{F}(z;-1,0;1,m,n)=F_{AE}(0,t;1),$$

Then, by the generating function of the Euler polynomials $E_d(x)$ given by (1.8) (for $h=1$), we conclude that

$$e^{xz}\,\mathbb{F}(z;-1,0;1,m,n)=F_{Eh}(x,z;1).$$

i.e.,

$$\sum_{d=0}^{\infty}\frac{(xz)^d}{d!}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(-1,0;1,m,n)}{2^d}\frac{z^d}{d!}=\sum_{d=0}^{\infty}E_d(x)\frac{z^d}{d!}$$

or

$$\sum_{d=0}^{\infty}\sum_{j=0}^{d}\binom{d}{j}x^{d-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}\frac{z^d}{d!}=\sum_{d=0}^{\infty}E_d(x)\frac{z^d}{d!},$$

from which we obtain (3.6). □

Theorem 3.7. *The relation between the polynomials $\mathbb{G}_d(x,y;k,m,n)$ and the Genocchi polynomials $G_d(x)$ is given by*

$$G_d(x)=d\sum_{j=0}^{d-1}\binom{d-1}{j}x^{d-1-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}. \tag{3.7}$$

Proof. Assuming that $m \neq 0$ and using (3.1) and (1.10), we have

$$z\mathrm{e}^{xz}\mathbb{F}(z;-1,0;1,m,n) = F_g(x;t),$$

i.e.,

$$z\sum_{d=0}^{\infty}\frac{(xz)^d}{d!}\sum_{d=0}^{\infty}\frac{\mathbb{G}_d(-1,0;1,m,n)}{2^d}\frac{z^d}{d!} = \sum_{d=0}^{\infty}G_d(x)\frac{z^d}{d!}.$$

Since $G_0(x) = 0$, after some standard manipulations, we obtain

$$\sum_{d=1}^{\infty}\left(d\sum_{j=0}^{d-1}\binom{d-1}{j}x^{d-1-j}\frac{\mathbb{G}_j(-1,0;1,m,n)}{2^j}\right)\frac{z^d}{d!} = \sum_{d=1}^{\infty}G_d(x)\frac{z^d}{d!},$$

i.e., (3.7). □

Remark 3.8. *The relation (3.7) can be also expressed in the following form*

$$G_d(x) = \sum_{j=1}^{d}\binom{d}{j}x^{d-j}\frac{j\mathbb{G}_{j-1}(-1,0;1,m,n)}{2^{j-1}}.$$

4. Modified Humbert polynomials

In this section we modify the generating function of the Humbert polynomials in order to obtain the generating functions for some other families of special polynomials and numbers. We investigate certain properties of these generating functions and derive a few identities and relations which include the Apostol-Bernoulli and the Apostol-Euler numbers and polynomials, as well as the Bernoulli numbers of higher order, the array polynomials, and some other special numbers and polynomials.

First, we introduce a two-parameter family of the numbers $\{Y_n(\lambda;a)\}_{n\geq 0}$ by a generating function obtained from one of Humbert polynomials (1.15), by the substitution $(m,x,t,\lambda) \to (a,\lambda,\mathrm{e}^z,1)$.

Definition 4.1. *A family of the numbers $\{Y_n(\lambda;a)\}_{n\geq 0}$ is defined by*

$$F(z;\lambda,a) = \frac{1}{1-a\lambda\mathrm{e}^z+\mathrm{e}^{az}} = \sum_{n=0}^{\infty}Y_n(\lambda,a)\frac{z^n}{n!}. \tag{4.1}$$

4.1. Computing some special values of the numbers $Y_n(\lambda,a)$

Here, we consider two special cases.

CASE $a = 2$. Substituting $\lambda = 1$ and $a = 2$ into (4.1), after multiplication by z^2, we obtain

$$\sum_{n=0}^{\infty}Y_n(1,2)\frac{z^{n+2}}{n!} = \left(\frac{z}{\mathrm{e}^z-1}\right)^2 = \sum_{n=0}^{\infty}B_n^{(2)}\frac{z^n}{n!},$$

i.e.,

$$\sum_{n=2}^{\infty}Y_{n-2}(1,2)n(n-1)\frac{z^n}{n!} = \sum_{n=0}^{\infty}B_n^{(2)}\frac{z^n}{n!},$$

after using the series representation (1.1). Therefore, we have

$$B_n^{(2)} = n(n-1)Y_{n-2}(1,2), \quad n \neq 2,$$

where $B_n^{(2)}$ denotes the Bernoulli numbers of the second order.

Now, we are interested for a case when $\lambda = -\frac{1}{2}\left(\beta + \beta^{-1}\right)$, where $\beta > 1$.

Theorem 4.2. *If $\beta > 1$ we have*

$$Y_n\left(-\tfrac{1}{2}\left(\beta + \beta^{-1}\right), 2\right) = \frac{1}{4}\sum_{j=0}^{n}\binom{n}{j}\mathscr{E}_j\left(0, \beta\right)\mathscr{E}_{n-j}\left(0, \beta^{-1}\right). \tag{4.2}$$

Proof. Starting from (4.1), for $a = 2$, $\lambda = -\frac{1}{2}\left(\beta + \beta^{-1}\right)$, and $\beta > 1$, i.e.,

$$F\left(z; -\frac{1}{2}\left(\beta + \beta^{-1}\right), 2\right) = \frac{1}{4}F_{AE}(0, z; \beta)F_{AE}\left(0, z; \beta^{-1}\right),$$

and using the corresponding series representations (4.1) and (1.6), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Y_n\left(-\tfrac{1}{2}\left(\beta + \beta^{-1}\right), 2\right)\frac{z^n}{n!} &= \frac{1}{4}\left(\sum_{i=0}^{\infty}\mathscr{E}_i\left(0, \beta\right)\frac{z^i}{i!}\right)\left(\sum_{j=0}^{\infty}\mathscr{E}_j\left(0, \beta^{-1}\right)\frac{z^j}{j!}\right) \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathscr{E}_j\left(0, \beta\right)\mathscr{E}_{n-j}\left(0, \beta^{-1}\right)\right)\frac{z^n}{n!},\end{aligned}$$

i.e., (4.2). □

In a particular case for $\beta = 2$, the equality (4.2) reduces to the following identity

$$Y_n\left(-\tfrac{5}{4}, 2\right) = \frac{1}{4}\sum_{j=0}^{n}\binom{n}{j}\mathscr{E}_j\left(0, 2\right)\mathscr{E}_{n-j}\left(0, \tfrac{1}{2}\right)).$$

We give the following functional equations related to the numbers $Y_n\left(\lambda, a\right)$:

$$zF\left(z; \lambda, 1\right) = -F_{AB}(0, z; \lambda - 1)$$

and

$$2F\left(z; \lambda, 1\right) = F_{AE}(0, z; 1 - \lambda).$$

Combining the above equations with (4.1), (1.2) and also (1.6), we get

$$Y_{n-1}\left(\lambda, 1\right) = -\frac{1}{n}\mathscr{B}_n(\lambda - 1) \quad \text{and} \quad Y_n\left(\lambda, 1\right) = \frac{1}{2}\mathscr{E}_n(0, 1 - \lambda).$$

4.2. A recurrence relation for the numbers $Y_n(\lambda, a)$

By applying the Umbral calculus methods to (4.1), we find a recurrence relation for these numbers.

Theorem 4.3. *Let $2 \neq a\lambda$ and*

$$Y_0\left(\lambda, a\right) = \frac{1}{2 - a\lambda}.$$

Then, for $n \geq 1$, we have

$$Y_n(\lambda, a) = \sum_{j=0}^{n}\binom{n}{j}(a\lambda - a^{n-j})Y_j(\lambda, a). \tag{4.3}$$

Proof. Starting from (4.1), we get

$$\sum_{n=0}^{\infty} Y_n(\lambda,a)\frac{z^n}{n!} - a\lambda\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{n=0}^{\infty} Y_n(\lambda,a)\frac{z^n}{n!} + \sum_{n=0}^{\infty}\frac{(az)^n}{n!}\sum_{n=0}^{\infty} Y_n(\lambda,a)\frac{z^n}{n!} = 1.$$

Now, using the Cauchy product rule in the left-hand side of this equality, we obtain

$$\sum_{n=0}^{\infty} Y_n(\lambda,a)\frac{z^n}{n!} - a\lambda\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} Y_j(\lambda,a)\frac{z^n}{n!} + \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} a^{n-j} Y_j(\lambda,a)\frac{z^n}{n!} = 1.$$

Therefore,

$$\sum_{n=0}^{\infty} Y_n(\lambda,a)\frac{z^n}{n!} = 1 + \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\left(a\lambda - a^{n-j}\right) Y_j(\lambda,a)\right)\frac{z^n}{n!}.$$

Comparing the coefficients of $z^n/n!$ on both sides of the above equality, we arrive at the desired result. □

According to (4.3), we can recursively compute the values of the numbers $Y_n(\lambda,a)$ for $a\lambda \neq 2$,

$$Y_n(\lambda,a) = \frac{1}{2-a\lambda}\sum_{j=0}^{n-1}\binom{n}{j}(a\lambda - a^{n-j})Y_j(\lambda,a).$$

This formula gives

$$\begin{aligned}
Y_0(\lambda,a) &= \frac{1}{2-a\lambda},\\
Y_1(\lambda,a) &= \frac{a\lambda - a}{(2-a\lambda)^2},\\
Y_2(\lambda,a) &= \frac{a^2\lambda^2 + (2-4a+a^2)a\lambda}{(2-a\lambda)^3},\\
Y_3(\lambda,a) &= \frac{a^3\lambda^3 + \left(8-12a+6a^2-a^3\right)a^2\lambda^2 + \left(4-12a+6a^2-2a^3\right)a\lambda + 2a^3}{(2-a\lambda)^4},
\end{aligned}$$

etc.

Remark 4.4. *All numbers $Y_n(\lambda,a)$ are rational functions of real parameters a and λ, with a pole $\lambda = 2/a$ of order $n+1$.*

4.3. A new family of polynomials $P_n(x;\lambda,a)$

By (4.1), we can define a new family of polynomials $P_n(x;\lambda,a)$ by means of the following generating function:

$$G(z;x;\lambda,a) = \mathrm{e}^{xz}F(z;\lambda,a),$$

i.e.,

$$G(z;x;\lambda,a) = \sum_{n=0}^{\infty} P_n(x;\lambda,a)\frac{z^n}{n!} = \frac{\mathrm{e}^{xz}}{1-a\lambda\mathrm{e}^{z}+\mathrm{e}^{az}}. \tag{4.4}$$

Using (4.1), (4.4), as well as the numbers $Y_j(\lambda,a)$, we obtain the following representation of the polynomials $P_n(x;\lambda,a)$.

Theorem 4.5. *For $n \in \mathbb{N}_0$ we have*

$$P_n(x;\lambda,a) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} Y_j(\lambda,a).$$

Proof. According to (4.4), we have

$$\sum_{n=0}^{\infty} P_n(x;\lambda,a)\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty} x^n \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} Y_n(\lambda,a)\frac{z^n}{n!}\right),$$

i.e.,

$$\sum_{n=0}^{\infty} P_n(x;\lambda,a)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} x^{n-j} Y_j(\lambda,a)\right)\frac{z^n}{n!}.$$

The last equality gives the desired result. □

Theorem 4.6. *For $n \geq 1$ we have*

$$\frac{\partial}{\partial x} P_n(x;\lambda,a) = nP_{n-1}(x;\lambda,a).$$

Proof. By differentiating the generating function (4.4) with respect to x, we conclude that

$$\frac{\partial}{\partial x} G(z;x;\lambda,a) = zG(z;x;\lambda,a).$$

Then, using the corresponding series representation, we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} P_n(x;\lambda,a)\frac{z^n}{n!} = \sum_{n=1}^{\infty} nP_{n-1}(x;\lambda,a)\frac{z^n}{n!},$$

from which the desired result directly follows. □

Theorem 4.7. *The following identities*

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} P_{n-2k}(x;\lambda,a) = \frac{1}{2}\Big(P_n(x+1;\lambda,a) + P_n(x-1;\lambda,a)\Big) \tag{4.5}$$

and

$$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} P_{n-2k-1}(x;\lambda,a) = \frac{1}{2}\Big(P_n(x+1;\lambda,a) - P_n(x-1;\lambda,a)\Big) \tag{4.6}$$

hold.

Proof. According to (4.4), we find that

$$G(z;x+y;\lambda,a) = \mathrm{e}^{yz} G(z;x;\lambda,a),$$

as well as the following equality

$$\sum_{n=0}^{\infty} P_n(x+y;\lambda,a)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} y^{n-j} P_j(x;\lambda,a)\right)\frac{z^n}{n!},$$

i.e.,

$$P_n(x+y;\lambda,a) = \sum_{j=0}^{n}\binom{n}{j} y^j P_{n-j}(x;\lambda,a).$$

Now, substituting $y = 1$ and $y = -1$ into this equality, we obtain

$$P_n(x+1;\lambda,a) = \sum_{j=0}^{n} \binom{n}{j} P_{n-j}(x;\lambda,a)$$

and

$$P_n(x-1;\lambda,a) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} P_{n-j}(x;\lambda,a),$$

respectively. Finally, adding and subtracting these equalities we get the identities (4.5) or (4.6). $\square$

References

[1] Apostol, T.M.: On the Lerch Zeta function. Pacific J. Math. **1** (2), 161–167 (1951)

[2] Bayad, A., Simsek, Y., Srivastava, H. M.: Some array type polynomials associated with special numbers and polynomials. Appl. Math. Comput. **244**, 149–157 (2014)

[3] Cakić, N.P., Milovanović, G.V.: On generalized Stirling numbers and polynomials. Math. Balkanica **18**, 241–248 (2004)

[4] Comtet, L.: Advanced Combinatorics: The Art of Finite and Infinite Expansions. Reidel, Dordrecht and Boston (1974)

[5] Dere, R., Simsek, Y., Srivastava, H. M.: A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. Number Theory **133** , 3245–3263 (2013)

[6] Djordjević, G.B.: Polynomials related to generalized Chebyshev polynomials. Filomat **23** (3), 279–290 (2009)

[7] Djordjević, G.B., Milovanović, G.V.: Special Classes of Polynomials. University of Niš, Faculty of Technology, Leskovac (2014)

[8] Gegenbauer, L.: Zur Theorie der Functionen $C_n^v(x)$. Osterreichische Akademie der Wissenschaften Mathematisch Naturwissen Schaftliche Klasse Denkscriften **48**, 293–316 (1884)

[9] Gould, H.W.: Inverse series relations and other expansions involving Humbert polynomials. Duke Math. J. **32** (4), 697–712 (1965)

[10] Henry, W., He, T.: Characterization of (c)-Riordan arrays, Gegenbauer-Humbert-type polynomial sequences, and (c)-Bell polynomials. J. Math. Res. Appl. **33** (5), 505–527 (2013)

[11] Horadam, A.F.: Genocchi polynomials. In: Applications of Fibonacci Numbers, Vol. 4 (Winston-Salem, NC, 1990), 145–166, Kluwer Acad. Publ., Dordrecht (1991)

[12] Humbert, P.: Some extensions of Pincherle's polynomials. Proc. Edinburgh Math. Soc. (1) **39**, 21–24 (1921)

[13] Kim, T.: On the q-Genocchi numbers and polynomials. Adv. Stud. Contemp. Math. **17**, 9–15 (2008)

[14] Lu, D.-Q., Srivastava, H. M.: Some series identities involving the generalized Apostol type and related polynomials, Comput. Math. Appl. **62**, 3591–2602 (2011)

[15] Luo, Q.-M.: On the Apostol-Bernoulli polynomials. Cent. Eur. J. Math. **2** (4), 509–515 (2004)

[16] Luo, Q.-M., Srivastava, H.M.: Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind. Appl. Math. Comput. **217**, 5702–5728 (2011)

[17] Luo, Q.-M., Srivastava, H. M.: Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Comput. Math. Appl. **51**, 631–642 (2006)

[18] Milovanović, G.V., Djordević, G.P.: On some properties of Humbert's polynomials. Fibonacci Quart. **25**, 356–360 (1987)

[19] Milovanović, G.V., Djordević, G.P.: On some properties of Humbert's polynomials, II. Facta Univ. Ser. Math. Inform. **6**, 23–30 (1991)

[20] Milovanović, G.V., Mitrinović, D.S., Rassias, Th.M.: Topics in Polynomials: Extremal Problems, Inequalities, Zeros. World Scientific Publ. Co., Singapore – New Jersey – London –Hong Kong (1994)

[21] Ozdemir, G., Simsek, Y.: Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers. Filomat **30** (4), 969–975 (2016)

[22] Ozden, H., Simsek, Y., Srivastava, H. M.: A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, Comput. Math. Appl. **60**, 2779–2787 (2010)

[23] Pincherle, S.: Una nuova extensione delle funzione spherich. Mem. R. Accad. Bologna **5**, 337–362 (1890)

[24] Simsek, Y.: Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications. Fixed Point Theory Appl. 2013, **87**, 28 pp. (2013)

[25] Simsek, Y.: Computation methods for combinatorial sums and Euler-type numbers related to new families of numbers. Math. Meth. Appl. Sci. DOI: 10.1002/mma.4143 (2016)

[26] Srivastava, H.M.: Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Inf. Sci. **5** (3), 390–444 (2011)

[27] Srivastava, H.M., Choi, J.: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, Inc., Amsterdam (2012)

[28] Srivastava, H.M., Manocha, H.L.: A Treatise on Generating Functions. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York (1984)

[29] Srivastava, H. M., Kurt, B., Simsek, Y.: Some families of Genocchi type polynomials and their interpolation functions, Integral Transforms Spec. Funct. **23**, 919–938 (2012)

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