Periodica Mathematica Hungarica manuscript No.
(will be inserted by the editor)

Extension of Mathieu series and alternating Mathieu series involving Neumann function Y_ν

Rakesh K. Parmar · Gradimir V. Milovanović · Tibor K. Pogány

Received: date / Accepted: date

Abstract The main objective of this paper is to present a new extension of the familiar Mathieu series and the alternating Mathieu series $S(r)$ and $\widetilde{S}(r)$ which are denoted, respectively, by $\mathbb{S}_{\mu,\nu}(r)$ and $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$. The computable series expansions of their related integral representations are obtained in terms of exponential integral E_1, and convergence rate discussion is provided for the associated series expansions. Further, for the series $\mathbb{S}_{\mu,\nu}(r)$ and $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$, related expansions are presented in terms of the Riemann Zeta function and Dirichlet Eta function, also their series built in Gauss' ${}_2F_1$ functions and associated Legendre function of the second kind Q_μ^ν are given. Discussion also includes the extended versions of the complete Butzer–Flocke–Hauss Omega functions. Finally, functional bounding inequalities are derived for the investigated extensions of Mathieu–type series.

Keywords Mathieu and alternating Mathieu series · Neumann function Y_ν · Euler–Abel transformation of series · Exponential integral E_1 · Gubler–Weber formula · Associated Legendre function of second kind · Riemann Zeta function · Dirichlet Eta function · Polylogarithm · Complete Butzer–Flocke–Hauss Ω function · Functional bounding inequality.

Mathematics Subject Classification (2000) Primary: 26D15, 33E20, 40A30, 41A58; Secondary: 11M35, 40C10, 44A20

Rakesh K. Parmar
Department of Mathematics, University College of Engineering and Technology
Bikaner Technical University, Bikaner-334004, Rajasthan, India
E-mail: rakeshparmar27@gmail.com

Gradimir V. Milovanović
Serbian Academy of Sciences and Arts, 11000 Beograd, Serbia
Faculty of Science and Mathematics, University of Niš, 18000 Niš, Serbia
E-mail: gvm@mi.sanu.ac.rs

Tibor K. Pogány
Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary
Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Croatia
E-mail: pogany.tibor@nik.uni-obuda.hu, tibor.poganj@uniri.hr

1 Introduction and preliminaries

During the study of elasticity of solid bodies, Émile Leonard Mathieu (1835–1890) introduced and investigated the famous infinite functional series so-called Mathieu series of the form [20]

$$S(r) = \sum_{n\geq 1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0.$$

The alternating version of Mathieu series is introduced and investigated by Pogány et al. [27, p. 72, Eq. (2.7)]

$$\widetilde{S}(r) = \sum_{n\geq 1} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0.$$

Elegant integral forms of Mathieu series $S(r)$ and alternating Mathieu series $\widetilde{S}(r)$ was established by Emersleben [13]

$$S(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{\mathrm{e}^x - 1}\, \mathrm{d}x. \tag{1}$$

and Pogány et al. [27, p. 72, Eq. (2.8)]

$$\widetilde{S}(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{\mathrm{e}^x + 1}\, \mathrm{d}x. \tag{2}$$

Milovanović and Pogány [22] discovered other integral forms for Mathieu and alternating Mathieu series; Tomovski and Pogány [29] deduced Cauchy principal value integrals for these series; moreover see [7, 9, 12] in this integral form, and [8, 25, 26] for another similarly focused study. The present authors studied and investigated a multi–parameter extension of the well–known Mathieu series and the alternating Mathieu series in a recent paper [24].

We emphasize the integral representations [22, p. 185-186, Corollary 2.2]

$$S(r) = \pi \int_0^\infty \frac{r^2 - x^2 + \frac{1}{4}}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \frac{\mathrm{d}x}{\cosh^2(\pi x)}\,, \tag{3}$$

$$\widetilde{S}(r) = \pi \int_0^\infty \frac{x}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \frac{\sinh(\pi x)\, \mathrm{d}x}{\cosh^2(\pi x)}\,, \tag{4}$$

which will have a special treatment below.

Let $\mathbb{N}, \mathbb{Z}$, and $\mathbb{C}$ be the sets of positive integers, integers, and complex numbers, respectively. The Bessel function of the first kind of the order ν is defined by

$$J_\nu(z) = \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)}, \qquad -z \notin \mathbb{N};\ \nu \in \mathbb{C}, \tag{5}$$

where the principal branch of $J_\nu(z)$ should be considered (it corresponds to the principal value of z^ν) and $J_\nu(z)$ is analytic in the z–plane cut along the interval $(-\infty, 0]$. Moreover, for $\nu \in \mathbb{Z}$, the Bessel function of the first kind is entire in z, the whole complex plane, see [15, p. 5].

The Bessel function of the second kind (Neumann function or Weber–Bessel function) of order ν is expressible in terms of the Bessel function of the first kind defined as [30, p. 64)]:

$$Y_\nu(z) = \frac{\cos(\nu\pi)\, J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} = \cot(\nu\pi)\, J_\nu(z) - \csc(\nu\pi)\, J_{-\nu}(z), \qquad \nu \notin \mathbb{Z}. \tag{6}$$

Also Bessel functions of half–integer order have connection or recurrence formula [17, p. 925, Eq.(8.465)]

$$Y_{n+\frac{1}{2}}(z) = (-1)^{n-1}\, J_{-n-\frac{1}{2}}(z)\,.$$

On the other hand [23, pp. 228, Eq. (10.16.1)]

$$J_{\frac{1}{2}}(z) = Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\sin(z)\,.$$

The extension of Mathieu series we can realize considering the related integral representation extending the integrand by a weight function. Namely, re–write (1) into the form

$$S(r) = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{3/2}}{\mathrm{e}^x - 1} \sqrt{\frac{2}{\pi r x}}\, \sin(rx)\, \mathrm{d}x = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{3/2}}{\mathrm{e}^x - 1}\, Y_{-\frac{1}{2}}(rx)\, \mathrm{d}x\,. \tag{7}$$

The same can be done for the alternating Mathieu series, thus

$$\widetilde{S}(r) = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{3/2}}{\mathrm{e}^x + 1}\, Y_{-\frac{1}{2}}(rx)\, \mathrm{d}x\,.$$

2 Polylogarithmic approach to Mathieu and alternating Mathieu series

In the exposition we use the series definition of the Riemann Zeta function [28, p. 164, Eq. (1)]

$$\zeta(s) = \sum_{n\geq 1} n^{-s}, \qquad \Re(s) > 1\,,$$

and its integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\mathrm{e}^x - 1}\, \mathrm{d}x, \qquad \Re(s) > 1. \tag{8}$$

The close relative of the Riemann Zeta function known as Dirichlet Eta function (or the alternating Riemann Zeta function) $\eta(s)$ and its integral representation are given by [28, p. 384, Eq. (35)]

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s) = \sum_{n\geq 1} (-1)^{n-1} n^{-s}, \qquad \Re(s) > 0\,,$$

that is,

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\mathrm{e}^x + 1}\, \mathrm{d}x, \qquad \Re(s) > 0, \tag{9}$$

respectively.

The polylogarithm (de Jonquière's function) is the Dirichlet type power series in complex argument z, *viz.*

$$\mathrm{Li}_s(z) = \sum_{n\ge 1} \frac{z^n}{n^s};$$

here the defining series converges for the complex order $s \in \mathbb{C}$ for all $|z| < 1$, while by analytic continuation it can be extended to $|z| \ge 1$. There is much coverage literature available for the polylogarithm and related topic, consult the standard references [1, 14, 19, 23, 31]. Obviously $\mathrm{Li}_s(1) = \zeta(s)$, $\Re(s) > 1$.

Our interest in polylogarithm is drawn by the integral representation

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\mathrm{e}^t - z}\, \mathrm{d}t, \qquad \Re(s) > 0, z \in \mathbb{C} \setminus [1, \infty)\,. \tag{10}$$

This integral is closely connected with the Bose–Einstein distribution's integral [10]

$$G_k(x) = \frac{1}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\mathrm{e}^{t-x} - 1}\, \mathrm{d}t, \qquad k > -1\,.$$

Here $x \le 0$, in turn for $x > 0$ the Cauchy principal value integral should be used, [10]. Obviously,

$$G_k(x) = \frac{\mathrm{e}^x}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\mathrm{e}^t - \mathrm{e}^x}\, \mathrm{d}t = \mathrm{Li}_{k+1}\big(\mathrm{e}^x\big)\,. \tag{11}$$

The Fermi–Dirac distribution integral (see also Clunie's note [10])

$$F_k(x) = \frac{1}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\mathrm{e}^{t-x} + 1}\, \mathrm{d}t, \qquad k > 0\,.$$

We point out that [31]

$$F_k(x) = -\mathrm{Li}_{k+1}\big(-\mathrm{e}^x\big)\,. \tag{12}$$

The similarity to the Emersleben's integral expressions for the Mathieu series and alternating Mathieu series $\widetilde{S}(r)$ is obvious, compare (1) and (2) respectively. Motivated by these 'similarities' our next goal is to establish inter–connection formulae between polylogarithm, the series built from Riemann Zeta function, Fermi–Dirac and Bose–Einstein integrals from one, and Mathieu series and alternating Mathieu series from the other side.

Theorem 1 *For all $|r| < 1$ we have*

$$S(r) = 2 \sum_{n\ge 0} \frac{(-1)^n\, (2)_n\, r^{2n}}{n!}\, \zeta(2n+3), \tag{13}$$

$$\widetilde{S}(r) = 2 \sum_{n\ge 0} \frac{(-1)^n\, (2)_n\, r^{2n}}{n!}\, \eta(2n+3)\,. \tag{14}$$

Proof Consider the integral representation (1). By the Taylor expansion of the sine function in the integrand we conclude

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x}{\mathrm{e}^x - 1}\sum_{n\ge 0}\frac{(-1)^n (rx)^{2n+1}}{(2n+1)!}\,\mathrm{d}x = \sum_{n\ge 0}\frac{(-1)^n r^{2n}}{(2n+1)!}\int_0^\infty \frac{x^{2n+2}}{\mathrm{e}^x - 1}\,\mathrm{d}x\,.$$

In turn, by (10) and (11) we confirm that

$$\int_0^\infty \frac{x^{2n+2}}{\mathrm{e}^x - 1}\,\mathrm{d}x = \Gamma(2n+3)\,G_{2n+2}(0) = \Gamma(2n+3)\,\mathrm{Li}_{2n+3}(1) = (2n+2)!\,\zeta(2n+3)\,, \qquad (15)$$

which results in

$$S(r) = 2\sum_{n\ge 0}(-1)^n r^{2n}(n+1)\,\zeta(2n+3)\,,$$

getting (13). Next, starting now from (2) we infer by similar proving procedure the second asserted formula which holds for the alternating Mathieu series $\widetilde{S}(r)$. Indeed, applying (12), we conclude

$$\begin{aligned}\widetilde{S}(r) &= \frac{1}{r}\int_0^\infty \frac{x}{\mathrm{e}^x + 1}\sum_{n\ge 0}\frac{(-1)^n (rx)^{2n+1}}{(2n+1)!}\,\mathrm{d}x = 2\sum_{n\ge 0}(-1)^n r^{2n}(n+1)\int_0^\infty \frac{x^{2n+2}}{\mathrm{e}^x + 1}\,\mathrm{d}x \\ &= 2\sum_{n\ge 0}(-1)^n r^{2n}(n+1)F_{2n+2}(0) = -2\sum_{n\ge 0}(-1)^n r^{2n}(n+1)\,\mathrm{Li}_{2n+3}(-1) \\ &= 2\sum_{n\ge 0}(-1)^n r^{2n}(n+1)\,\eta(2n+3)\,,\end{aligned}$$

which completes the proof. □

Remark 1 We point out that (13) and (14) are not new; in fact these relations coincide with the series representations [24, Eqs. (1.7-8)], also see [27, p. 72, Proposition 1.] for (14). We also point out that there are no reasons to consider the series $S(r)$ and $\widetilde{S}(r)$ exclusively for $r > 0$; the exception can be Mathieu's original mathematical model in which he described the vibration of clamped rectangular plates and membranes, see the discussion in the memoir [24, §8.3]. So the importance of the previously presented results.

3 Series expansions of integrals (3) and (4)

The derivation of integral expressions (3) and (4) associated to $S(r)$ and $\widetilde{S}(r)$ is realized by complex analytical and integral transformation methods, see [22]. Then, since their integrands include reciprocals of hyperbolic functions, we explore other series expansions of these integrals.

First, we introduce the exponential integral of the first order [1, p. 228, Eq. 5.1.1]

$$E_1(z) = ⨍_z^\infty x^{-1}\mathrm{e}^{-x}\,\mathrm{d}x\,, \qquad |\arg(z)| < \pi,$$

which mirror symmetry property reads $E_1(\overline{z}) = \overline{E_1(z)}$, see [1, p. 229, Eq. 5.1.13]. Obivously, we consider here the principal value of the integral when $z \neq 0$, consult [23, p. 150, Eq. 6.2.1].

Moreover in the `Mathematica` package the exponential integral is defined also as the principal value of the integral [23, p. 150, Eq. 6.2.5]

$$\mathrm{Ei}(x) = -\fint_{-x}^{\infty} t^{-1}\, \mathrm{e}^{-t}\, \mathrm{d}t\,, \qquad x > 0.$$

However, the inter–connection $E_1(x) = -\mathrm{Ei}(-x)$ holds true, see [23, p. 150, Eq. 6.2.6].

Theorem 2 *For all $r > 0$ we have the following series expansions*

$$S(r) = \frac{1}{r} \sum_{n\geq 0} s \left\{ \mathrm{e}^{-rs} \Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{rs} \Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] \right\}\Big|_{s=2\pi(n+1)}\,, \tag{16}$$

$$\widetilde{S}(r) = \frac{1}{r} \sum_{n\geq 0} s \left\{ \mathrm{e}^{rs} \Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs} \Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] \right\}\Big|_{s=\pi(2n+1)}, \tag{17}$$

where $\Re[z]$ denotes the real part of $z \in \mathbb{C}$.

Proof Expanding the secant hyperbolic kernel in the integrand of (3), for all $x > 0$ we have

$$\frac{1}{\cosh^2(\pi x)} = \frac{4\,\mathrm{e}^{-2\pi x}}{\big(1+\mathrm{e}^{-2\pi x}\big)^2} = 4\sum_{n\geq 0}(-1)^n(n+1)\,\mathrm{e}^{-2\pi(n+1)x} = 4\,\mathrm{e}^{-2\pi x}\,{}_1F_0\big[2;-;-\mathrm{e}^{-2\pi x}\big]\,. \tag{18}$$

Let us denote $\mathscr{L}_x[f](s)$ the Laplace transform of a suitable function f with respect to the input variable x of the output variable s. By the expansion (18), the integral (3) becomes a series of Laplace transforms which reads

$$S(r) = 4\pi \sum_{n\geq 0} \frac{(-1)^n (2)_n}{n!} \mathscr{L}_x\Big[\frac{r^2 - x^2 + \frac14}{\big(x^2 - r^2 + \frac14\big)^2 + r^2}\Big]\big(2\pi(n+1)\big)\,. \tag{19}$$

Next, we need the related Laplace integral property [1, p. 230, Eq. 5.1.28]

$$\int_0^\infty \frac{\mathrm{e}^{-sx}}{x+a}\,\mathrm{d}x = \mathscr{L}_x[(x+a)^{-1}](s) = \mathrm{e}^{as}\,E_1(as)\,, \qquad s>0,\ a>0\,.$$

The partial fraction decomposition of the integrand is

$$\frac{r^2 - x^2 + \frac14}{\big(x^2 - r^2 + \frac14\big)^2 + r^2} = \frac{1}{4r}\left\{ \frac{1}{x+r+\frac{\mathrm{i}}{2}} + \frac{1}{x+r-\frac{\mathrm{i}}{2}} - \frac{1}{x-r+\frac{\mathrm{i}}{2}} - \frac{1}{x-r-\frac{\mathrm{i}}{2}} \right\}.$$

Hence, applying the previously listed results, we have

$$\begin{aligned}\mathscr{L}_x\left[\frac{r^2 - x^2 + \frac14}{\big(x^2 - r^2 + \frac14\big)^2 + r^2}\right](s) = \frac{1}{4r}\Big[&\mathrm{e}^{(r+\frac{\mathrm{i}}{2})s}\,E_1\big((r+\tfrac{\mathrm{i}}{2})s\big) + \mathrm{e}^{(r-\frac{\mathrm{i}}{2})s}\,E_1\big((r-\tfrac{\mathrm{i}}{2})s\big) \\ &- \mathrm{e}^{-(r-\frac{\mathrm{i}}{2})s}\,E_1\big(-(r-\tfrac{\mathrm{i}}{2})s\big) - \mathrm{e}^{-(r+\frac{\mathrm{i}}{2})s}\,E_1\big((-(r+\tfrac{\mathrm{i}}{2})s\big)\Big]\,.\end{aligned}$$

By the mirror symmetry of the exponential integral we readily conclude

$$\mathscr{L}_x\left[\frac{r^2 - x^2 + \frac14}{\big(x^2 - r^2 + \frac14\big)^2 + r^2}\right](s) = \frac{1}{2r}\Big\{\mathrm{e}^{rs}\,\Re\big[\mathrm{e}^{\frac{\mathrm{i}}{2}s}\,E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs}\,\Re\big[\mathrm{e}^{\frac{\mathrm{i}}{2}s}\,E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big]\Big\}\,,$$

which right hand side for $s = 2\pi(n+1)$ reduces to

$$\frac{(-1)^{n+1}}{2r}\Big\{\mathrm{e}^{2r\pi(n+1)}\Re\big[E_1\big(2\pi(r+\tfrac{\mathrm{i}}{2})(n+1)\big)\big] - \mathrm{e}^{-2r\pi(n+1)}\Re\big[E_1\big(2\pi(-r+\tfrac{\mathrm{i}}{2})(n+1)\big)\big]\Big\}.$$

Inserting the last expression into (19) we arrive at the asserted series expansion (16).

Next, as to (17), since

$$\begin{aligned}\frac{\sinh(\pi x)}{\cosh^2(\pi x)} &= -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{\cosh(\pi x)}\Big) = 2\sum_{n\geq 0}(-1)^n(2n+1)\mathrm{e}^{-(2n+1)\pi x}\\ &= 2\mathrm{e}^{-\pi x}\sum_{n\geq 0}\frac{(1)_n(\frac{3}{2})_n}{(\frac{1}{2})_n\, n!}\big(-\mathrm{e}^{-2\pi x}\big)^n = 2\mathrm{e}^{-\pi x}\,{}_2F_1\Big[\begin{matrix}1,\frac{3}{2}\\ \frac{1}{2}\end{matrix}\Big| -\mathrm{e}^{-2\pi x}\Big],\end{aligned}$$

the integral expression (4) becomes the following series of Laplace transforms

$$\widetilde{S}(r) = 2\pi\sum_{n\geq 0}(-1)^n\frac{(1)_n(\frac{3}{2})_n}{(\frac{1}{2})_n\, n!}\mathscr{L}_x\Big[\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2}\Big]\big((2n+1)\pi\big).$$

The partial fraction decomposition of the Laplace transform input function reads

$$\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2} = -\frac{\mathrm{i}}{4r}\Big\{\frac{1}{x+r+\frac{\mathrm{i}}{2}} - \frac{1}{x+r-\frac{\mathrm{i}}{2}} - \frac{1}{x-r+\frac{\mathrm{i}}{2}} + \frac{1}{x-r-\frac{\mathrm{i}}{2}}\Big\},$$

therefore

$$\begin{aligned}\mathscr{L}_x\Big[\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2}\Big](s) = -\frac{\mathrm{i}}{4r}\Big[&\mathrm{e}^{(r+\frac{\mathrm{i}}{2})s}\,E_1\big((r+\tfrac{\mathrm{i}}{2})s\big) - \mathrm{e}^{(r-\frac{\mathrm{i}}{2})s}\,E_1\big((r-\tfrac{\mathrm{i}}{2})s\big)\\ &- \mathrm{e}^{-(r-\frac{\mathrm{i}}{2})s}\,E_1\big(-(r-\tfrac{\mathrm{i}}{2})s\big) + \mathrm{e}^{-(r+\frac{\mathrm{i}}{2})s}\,E_1\big((-(r+\tfrac{\mathrm{i}}{2})s\big)\Big].\end{aligned}$$

Again by the mirror symmetry of the exponential integral $E_1(z)$, inserting $s = \pi(2n+1)$, we conclude that

$$\begin{aligned}\mathscr{L}_x\Big[\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2}\Big]\big(\pi(2n+1)\big) = \frac{(-1)^n}{2r}\Big\{&\mathrm{e}^{r\pi(2n+1)}\Re\big[E_1\big(\pi(r+\tfrac{\mathrm{i}}{2})(2n+1)\big)\big]\\ &- \mathrm{e}^{-r\pi(2n+1)}\Re\big[E_1\big(\pi(-r+\tfrac{\mathrm{i}}{2})(2n+1)\big)\big]\Big\}.\end{aligned}$$

The rest is obvious. □

Unfortunately, the series (16) for the sum $S(r)$ is slowly convergent. Denote its general term by $u_n(r)$, i.e.,

$$u_n(r) = \frac{s}{r}\Big\{\mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big]\Big\}\Big|_{s=2\pi(n+1)},$$

and consider another auxiliary series

$$T(r) = \frac{1}{2}u_0(r) + \frac{1}{2}\sum_{n\geq 0}\big(u_n(r)+u_{n+1}(r)\big), \tag{20}$$

where in both $n \geq 0$ and $r > 0$. Let $S_n(r)$ and $T_n(r)$ be nth partial sums of the series $S(r)$ and $T(r)$, respectively. Since the series (16) is convergent, it must be $\lim_{n\to+\infty} u_n(r) = 0$, and according to $T_n(r) - S_n(r) = \frac{1}{2}u_{n+1}(r)$, we conclude that $T(r)$ is also a convergent series, with the same sum $S(r)$.

Remark 2 Numerical calculations show that for fixed values of r, $u_n(r) > 0$ for even n, and negative for odd n, so that the transformation of the series (16), given by (20), is, in fact, the well-known *Euler–Abel transformation*. The series (16) is extremely slowly convergent and practically is not usable for numerical calculations. On the other hand, the transformed series (20) shows a relatively fast convergence, so that a reasonable number of initial terms is enough to approximate the sum $S(r)$ with the required accuracy. The following examples illustrate these properties.

Example 1 In Figure 1 (left) we present the errors

$$E_{S,n}(r) := T_n(r) - S(r) = \frac{1}{2}u_0(r) + \frac{1}{2}\sum_{k=0}^{n}\big(u_k(r) + u_{k+1}(r)\big), \tag{21}$$

with only $n = 0, 1, 2,$ and 5. As the exact value $S(r)$ we take a very precise approximation obtained by using the Gaussian quadrature formula with respect to the hyperbolic weight function (see [21, 22]), applied directly to the integral (3). As we can see, only for small values of r the errors $E_{S,n}(r)$ are significant if $n \leq 5$. In the same figure (right) we present the corresponding relative errors $R_{S,n}(r) = |E_{S,n}(r)/S(r)|$, taking the partial sums in (21) for $n = 5, 10, 50$ and 100 terms. For example, with $n = 100$, the relative error for $r \in [0, 1]$ is less than 10^{-6}, and for larger $r > 1$ this error is less than 10^{-8}, which means that we obtain the values of $S(r)$ with at least 6 and 8 exact decimal digits, respectively.

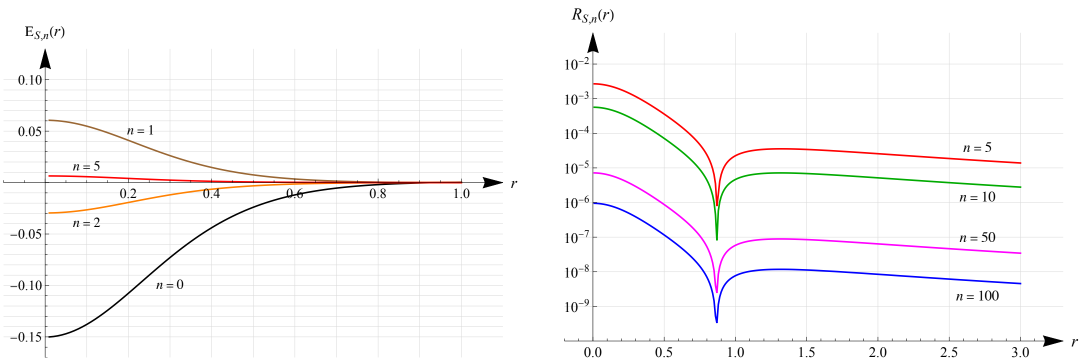

Fig. 1 Errors $E_{S,n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $R_{S,n}(r)$ for $0 \leq r \leq 3$, for $n = 5, 10, 50,$ and 100 (right)

A series with more faster convergence can be obtained by repeating the previous transformation to the series (20). Then we get

$$\frac{1}{4}\Big(3u_0(r) + u_1(r) + \sum_{n\geq 0}\big(u_n(r) + 2u_{n+1}(r) + u_{n+2}(r)\big)\Big). \tag{22}$$

The corresponding errors in the partial sums are denoted by $\mathcal{E}_{S,n}(r)$ and presented in Figure 2 (left), as well as the relative errors $\mathcal{R}_{S,n}(r)$ in the same figure (right).

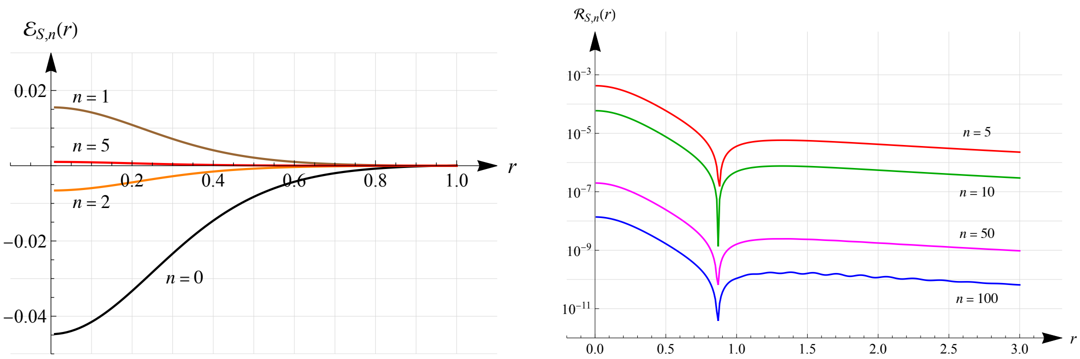

Fig. 2 Errors $\mathcal{E}_{S,n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $\mathcal{R}_{S,n}(r)$ for $0 \le r \le 3$, for $n = 5, 10, 50$, and 100 (right)

Example 2 In the case of the alternating Mathieu series $\widetilde{S}(r)$ we study the auxiliary series

$$\widetilde{T}(r) = \frac{1}{2}v_0(r) + \frac{1}{2}\sum_{n\ge 0}\big(v_n(r) + v_{n+1}(r)\big)\,, \tag{23}$$

with the general term

$$v_n(r) = \frac{s}{r}\Big\{\mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big]\Big\}\Big|_{s=\pi(2n+1)}.$$

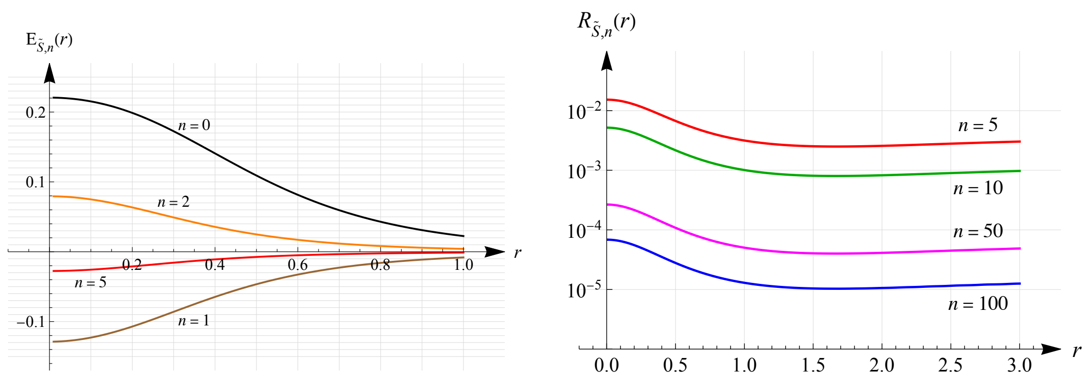

Fig. 3 Errors $E_{\tilde{S},n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $R_{\tilde{S},n}(r)$ for $0 \le r \le 3$, for $n = 5, 10, 50$, and 100 (right)

The repeated Euler–Abel transformation, in this case, gives the accelerated series in the following form:

$$\frac{1}{4}\Big(3v_0(r) + v_1(r) + \sum_{n\geq 0}(v_n(r) + 2v_{n+1}(r) + v_{n+2}(r))\Big). \tag{24}$$

The corresponding diagrams are presented in Figures 3 and 4, with the same notations as ones in the previous case for the sum $S(r)$ (Example 1).

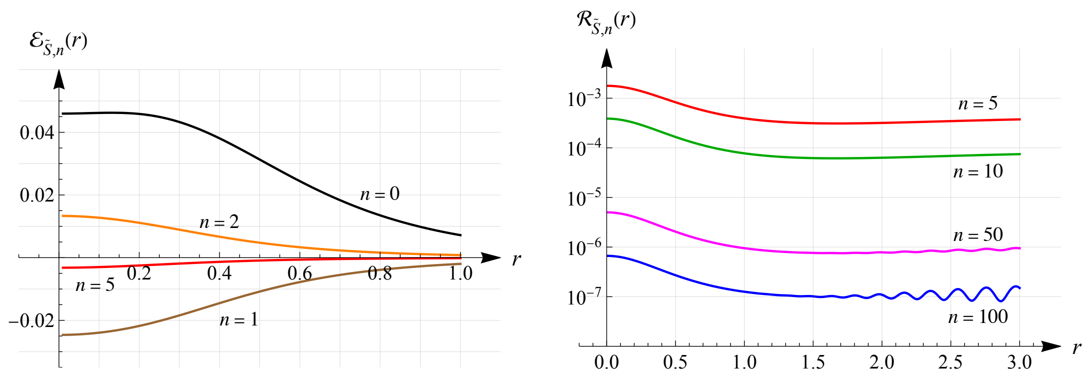

Fig. 4 Errors $\mathcal{E}_{\tilde{S},n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $\mathcal{R}_{\tilde{S},n}(r)$ for $0 \leq r \leq 3$, for $n = 5, 10, 50$, and 100 (right)

Remark 3 As we can see there exist certain oscillations in graphics for the relative errors $\mathcal{R}_{S,n}(r)$ (Figure 2 (right)) and $\mathcal{R}_{\tilde{S},n}(r)$ (Figure 4 (right)) for larger r and sufficiently large n ($n = 100$), because of unstable calculations in such cases. Namely, the values $S(r)$ and $\tilde{S}(r)$, as well as their approximations, i.e., the partial sums of series (22) and (24), respectively, are close to zero in such cases.

4 Extended Mathieu series $\mathbb{S}_{\mu,\nu}(r)$ and $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$

Motivated by (7), replacing there the kernel function $Y_{-\frac{1}{2}}$ with the general Bessel function of the second kind of order ν, we introduce the extended Mathieu series $S(r)$ and its alternating variant $\widetilde{S}(r)$ in the following forms:

$$\mathbb{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1}\, Y_\nu(rx)\,\mathrm{d}x, \qquad \mu + \nu \geq 1, \tag{25}$$

$$\widetilde{\mathbb{S}}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x + 1}\, Y_\nu(rx)\,\mathrm{d}x, \qquad \mu + \nu \geq 0, \tag{26}$$

where in both cases $r > 0$, $\mu > 0$. Clearly $\mathbb{S}_{\frac{5}{2},-\frac{1}{2}}(r) = S(r)$ and $\widetilde{\mathbb{S}}_{\frac{5}{2},-\frac{1}{2}}(r) = \widetilde{S}(r)$.

Using the recurrence formula [30, p. 66, Eq. (1)]

$$Y_{\nu-1}(z) - Y_{\nu+1}(z) = \frac{2\,\nu}{z} Y_\nu(z),$$

we obtain the following recurrence formulae

$$\frac{2\,\nu}{r}\,\mathbb{S}_{\mu,\nu}(r) = \mathbb{S}_{\mu+1,\nu-1}(r) + \mathbb{S}_{\mu+1,\nu+1}(r),$$
$$\frac{2\,\nu}{r}\,\widetilde{\mathbb{S}}_{\mu,\nu}(r) = \widetilde{\mathbb{S}}_{\mu+1,\nu-1}(r) + \widetilde{\mathbb{S}}_{\mu+1,\nu+1}(r)\,.$$

Theorem 3 *For all μ, $\nu+1>0$, $n\in\mathbb{N}$ and $\mu>|\nu|>0$, we have*

$$\begin{aligned}\mathbb{S}_{\mu,\nu}(r) &= \kappa_1(\mu,\nu)\sum_{n\ge1}\frac{1}{(n^2+r^2)^{\frac{\mu+\nu}{2}}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(1-\mu+\nu)\\ \nu+1\end{matrix}\Big|\frac{r^2}{n^2+r^2}\Big]\\ &\quad-\kappa_2(\mu,\nu)\sum_{n\ge1}\frac{1}{(n^2+r^2)^{\frac{\mu-\nu}{2}}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu-\nu),\frac12(1-\mu-\nu)\\ 1-\nu\end{matrix}\Big|\frac{r^2}{n^2+r^2}\Big]\,.\end{aligned}$$

Moreover, when $|\nu|<1$ and $\mu+\nu+1>0$ there holds

$$\begin{aligned}\widetilde{\mathbb{S}}_{\mu,\nu}(r) &= \kappa_1(\mu,\nu)\sum_{n\ge1}\frac{(-1)^{n-1}}{(n^2+r^2)^{\frac{\mu+\nu}{2}}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(1-\mu+\nu)\\ \nu+1\end{matrix}\Big|\frac{r^2}{n^2+r^2}\Big]\\ &\quad-\kappa_2(\mu,\nu)\sum_{n\ge1}\frac{(-1)^{n-1}}{(n^2+r^2)^{\frac{\mu-\nu}{2}}}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu-\nu),\frac12(1-\mu-\nu)\\ 1-\nu\end{matrix}\Big|\frac{r^2}{n^2+r^2}\Big]\,,\end{aligned}$$

where

$$\kappa_1(\mu,\nu) = \cot(\nu\pi)\frac{\sqrt{\pi}\,r^{\nu-\frac12}\,\Gamma(\mu+\nu)}{2^{\nu+\frac12}\,\Gamma(\nu+1)};\quad \kappa_2(\mu,\nu) = \csc(\nu\pi)\frac{\sqrt{\pi}\,r^{-\nu-\frac12}\,\Gamma(\mu-\nu)}{2^{\frac12-\nu}\,\Gamma(1-\nu)}.$$

Proof Insert the binomial series expansion $(\mathrm{e}^x-1)^{-1} = \sum_{n\ge1}\mathrm{e}^{-nx}$, $x>0$ into (25). The legitimate integral–sum interchange, which can be proved e.g. by the dominated convergence theorem, results in

$$\mathbb{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}}\sum_{n\ge1}\int_0^\infty x^{\mu-1}\,\mathrm{e}^{-nx}\,Y_\nu(r\,x)\,\mathrm{d}x\,.$$

Making use of the integral representation [30, p. 385, Eq. (4)] or, in other words, the Laplace-Mellin transform of the Bessel function Y_ν, i.e., $\mathcal{L}_a[x^{\mu-1}\,Y_\nu(bx)]$ and $\mathcal{M}_\mu[\mathrm{e}^{-ax}\,Y_\nu(bx)]$, respectively, we infer that

$$\begin{aligned}\int_0^\infty x^{\mu-1}\mathrm{e}^{-ax}Y_\nu(bx)\,\mathrm{d}x &= \frac{\cot(\nu\pi)\cdot\left(\frac{b}{2}\right)^\nu\Gamma(\mu+\nu)}{(a^2+b^2)^{\frac12(\mu+\nu)}\,\Gamma(\nu+1)}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\frac12(1-\mu+\nu)\\ \nu+1\end{matrix}\Big|\frac{b^2}{a^2+b^2}\Big]\\ &\quad-\frac{\csc(\nu\pi)\cdot\left(\frac{b}{2}\right)^{-\nu}\Gamma(\mu-\nu)}{(a^2+b^2)^{\frac12(\mu-\nu)}\,\Gamma(1-\nu)}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu-\nu),\frac12(1-\mu-\nu)\\ 1-\nu\end{matrix}\Big|\frac{b^2}{a^2+b^2}\Big],\end{aligned}$$

which parameter space consists from $\Re(\mu)>|\Re(\nu)|$ and $\Re(a\pm ib)>0$, specifying above $a=n$ and $b=r$, we conclude the first asserted formula. $\square$

In the sequel we need the associated Legendre function of second kind of a real argument [23, Eq. 14.3.7]

$$Q_q^p(x) = \mathrm{e}^{\pi \mathrm{i} p}\, \frac{\sqrt{\pi}\, \Gamma(p+q+1)\, (x^2-1)^{\frac{q}{2}}}{2^{p+1}\, \Gamma(p+\frac{3}{2})\, x^{p+q+1}}\; {}_2F_1\Big[\begin{matrix} \frac12(p+q)+1,\, \frac12(p+q+1) \\ p+\frac32 \end{matrix} \Big| \frac{1}{x^2} \Big], \qquad x>1\,,$$

provided the parameter range consists of $p, q \in \mathbb{C}$ and $-(p+q) \notin \mathbb{N}$.

Theorem 4 *For all μ, $\nu+1>0$, $n \in \mathbb{N}$, and $\mu > |\nu| > 0$, we have*

$$\mathbb{S}_{\mu,\nu}(r) = -\sqrt{\frac{2}{\pi r}}\, \Gamma(\mu+\nu) \sum_{n \geq 1} \frac{1}{(n^2+r^2)^{\frac12 \mu}}\, Q_{\mu-1}^{-\nu}\Big[\frac{n}{\sqrt{n^2+r^2}} \Big],$$

$$\widetilde{\mathbb{S}}_{\mu,\nu}(r) = \sqrt{\frac{2}{\pi r}}\, \Gamma(\mu+\nu) \sum_{n \geq 1} \frac{(-1)^n}{(n^2+r^2)^{\frac12 \mu}}\, Q_{\mu-1}^{-\nu}\Big[\frac{n}{\sqrt{n^2+r^2}} \Big].$$

Proof The same binomial expansion as in the previous proof and a change of the order of integration and summation gives

$$\mathbb{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}} \sum_{n\geq 1} \int_0^\infty x^{\mu-1}\, \mathrm{e}^{-nx}\, Y_\nu(r\,x)\, \mathrm{d}x\,.$$

By virtue of the integral [17, p. 700, Eq. 6.621. 2]

$$\int_0^\infty x^{\mu-1} \mathrm{e}^{-ax}\, Y_\nu(bx)\, \mathrm{d}x = -\frac{2}{\pi}\, \frac{\Gamma(\mu+\nu)}{(a^2+b^2)^{\frac12\mu}}\, Q_{\mu-1}^{-\nu}\Big[\frac{a}{\sqrt{a^2+b^2}} \Big]\,,$$

which parameter space consists from $a>0$, $b>0$, $\Re(\mu) > |\Re(\nu)|$, for $a=n$ and $b=r$ we conclude the first asserted formula.

The derivation of the series expansion for $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$ applies

$$(1+\mathrm{e}^x)^{-1} = \sum_{n\geq 1} (-1)^{n-1} \mathrm{e}^{-nx}, \quad x>0.$$

Now, the path to the final formula is obvious. □

5 Functional bounding inequalities

Recall the Gubler-Weber formula [30, p. 165, Eq. (5)]

$$Y_\nu(z) = \frac{2\, \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\, \Gamma(\nu+\frac12)} \left\{ \int_0^1 (1-t^2)^{\nu-\frac12}\, \sin(zt)\, \mathrm{d}t - \int_0^\infty (1+t^2)^{\nu-\frac12}\, \mathrm{e}^{-zt}\, \mathrm{d}t \right\},$$

which holds for $\Re(z) > 0$ and $\nu > -1/2$. Splitting the ν-domain into three disjoint intervals

$$(-1/2,\, \infty) = (-1/2,\, 1/2] \cup (1/2,\, 3/2) \cup (3/2,\, \infty) = U_1 \cup U_2 \cup U_3\,,$$

Baricz et al. [3, pp. 957–958] obtained the functional bounding inequality for the real argument Neumann function $Y_\nu(x)$ (see also [18, pp. 7–8], [11, p. 76]):

$$\big|Y_\nu(x)\big| + \frac{x^\nu}{2^\nu\,\Gamma(\nu+1)} \le \begin{cases} \dfrac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}, & -\frac12<\nu\le\frac12, \\[2ex] \dfrac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} + \dfrac{2^\nu\,\Gamma(\nu)}{\pi x^\nu}, & \frac12<\nu<\frac32, \\[2ex] \dfrac{x^{\nu-1}}{\sqrt{2\pi}\,\Gamma(\nu+\frac12)} + \dfrac{2^{2\nu-\frac32}\,\Gamma(\nu)}{\pi x^\nu}, & \nu>\frac32\,. \end{cases} \tag{27}$$

Theorem 5 *For all $\mu,\ \nu+\frac12>0$, $n\in\mathbb{N}$, and $\mu>|\nu|>0$, we have*

$$\big|\mathbb{S}_{\mu,\nu}(r)\big| \le \begin{cases} c_1(\mu,\nu)\,\zeta(\mu+\nu)+c_2(\mu,\nu)\,\zeta(\mu+\nu-1), & -\frac12<\nu\le\frac12, \\[1ex] c_1(\mu,\nu)\,\zeta(\mu+\nu)+c_2(\mu,\nu)\,\zeta(\mu+\nu-1)+c_3(\mu,\nu)\,\zeta(\mu-\nu), & \frac12<\nu<\frac32, \\[1ex] c_1(\mu,\nu)\,\zeta(\mu+\nu)+c_4(\mu,\nu)\,\zeta(\mu+\nu-1)+c_5(\mu,\nu)\,\zeta(\mu-\nu), & \nu>\frac32\,. \end{cases}$$

Moreover, when $\mu+\nu+1>0$ there holds

$$\big|\widetilde{\mathbb{S}}_{\mu,\nu}(r)\big| \le \begin{cases} c_1(\mu,\nu)\,\eta(\mu+\nu)+c_2(\mu,\nu)\,\eta(\mu+\nu-1), & -\frac12<\nu\le\frac12, \\[1ex] c_1(\mu,\nu)\,\eta(\mu+\nu)+c_2(\mu,\nu)\,\eta(\mu+\nu-1)+c_3(\mu,\nu)\,\eta(\mu-\nu), & \frac12<\nu<\frac32, \\[1ex] c_1(\mu,\nu)\,\eta(\mu+\nu)+c_4(\mu,\nu)\,\eta(\mu+\nu-1)+c_5(\mu,\nu)\,\eta(\mu-\nu), & \nu>\frac32, \end{cases}$$

where

$$c_1(\mu,\nu)=\frac{\sqrt{\pi}\,r^{\nu-\frac12}\,\Gamma(\mu+\nu)}{2^{\nu+\frac12}\,\Gamma(\nu+1)},\ c_2(\mu,\nu)=\frac{\sqrt{\pi}\,r^{\nu-\frac32}\,\Gamma(\mu+\nu-1)}{2^{\nu+\frac12}\,\Gamma(\nu+\frac12)},\ c_3(\mu,\nu)=\frac{2^{\nu-\frac12}\,\Gamma(\nu)\Gamma(\mu-\nu)}{\sqrt{\pi}\,r^{\nu+\frac12}\,\Gamma(\nu+1)},$$

$$c_4(\mu,\nu)=\frac{\sqrt{\pi}\,r^{\nu-\frac32}\,\Gamma(\mu+\nu-1)}{2\,\Gamma(\nu+\frac12)},\ c_5(\mu,\nu)=\frac{2^{2\nu-2}\,\Gamma(\nu)\Gamma(\mu-\nu)}{\sqrt{\pi}\,r^{\nu+\frac12}}.$$

Proof Starting with (25) and splitting the range of ν into three disjoint intervals

$$(-1/2,\,\infty)=(-1/2,\,1/2]\cup(1/2,\,3/2)\cup(3/2,\,\infty)=U_1\cup U_2\cup U_3,$$

and using the estimates (27), we conclude

$$|\mathbb{S}_{\mu,\nu}(r)| \le \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\,|Y_\nu(rx)|\,\mathrm{d}x \le \begin{cases} \mathbb{S}_{\mu,U_1}(r), & -\frac12<\nu\le\frac12, \\[1ex] \mathbb{S}_{\mu,U_2}(r), & \frac12<\nu<\frac32, \\[1ex] \mathbb{S}_{\mu,U_3}(r), & \nu>\frac32, \end{cases}$$

where

$$\mathbb{S}_{\mu,U_1}(r) \leq \sqrt{\frac{\pi}{2r}} \left\{ \frac{\left(\frac{r}{2}\right)^{\nu}}{\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x + \frac{\left(\frac{r}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{x^{\mu+\nu-2}}{\mathrm{e}^x - 1}\,\mathrm{d}x \right\},$$

$$\begin{aligned}\mathbb{S}_{\mu,U_2}(r) \leq \sqrt{\frac{\pi}{2r}} \Bigg\{ & \frac{\left(\frac{r}{2}\right)^{\nu}}{\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x + \frac{\left(\frac{r}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{x^{\mu+\nu-2}}{\mathrm{e}^x - 1}\,\mathrm{d}x \\ & + \frac{2^{\nu}\,\Gamma(\nu)}{\pi\, r^{\nu}\,\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x \Bigg\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}_{\mu,U_3}(r) \leq \sqrt{\frac{\pi}{2r}} \Bigg\{ & \frac{\left(\frac{r}{2}\right)^{\nu}}{\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x + \frac{r^{\nu-1}}{\sqrt{2\,\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{x^{\mu+\nu-2}}{\mathrm{e}^x - 1}\,\mathrm{d}x \\ & + \frac{2^{2\nu-\frac{3}{2}}\,\Gamma(\nu)}{\pi\, r^{\nu}\,\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x \Bigg\},\end{aligned}$$

which is equivalent to the first statement of this theorem. In the derivation procedure we apply the integral representation (8) of the Riemann Zeta function.

Similarly, if we start with the expression (26) we obtain the second formula with the aid of Dirichlet Eta function's integral form (9). In both cases the parameter constraints are controlled by the convergence conditions (8) and (9), respectively. □

6 Extended Mathieu series in terms of Riemann Zeta and Dirichlet Eta

The Bessel function of the second kind Y_ν has two fashion power series expansions depending on the nature of the order parameter. Firstly, when $\nu = n \in \mathbb{Z}$, we have [1, p. 360, Eq. 9.1.11]

$$\begin{aligned}Y_n(z) = \frac{2}{\pi}\, J_n(z)\, \log\frac{z}{2} - \frac{1}{\pi}\Big(\frac{2}{z}\Big)^n \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\Big(\frac{z^2}{4}\Big)^k \\ - \frac{1}{\pi}\Big(\frac{z}{2}\Big)^n \sum_{k\geq 0} \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!}\Big(-\frac{z^2}{4}\Big)^k,\end{aligned} \tag{28}$$

which immediately follows from (5) and (6). Here ψ is the digamma function defined by

$$\psi(x) = (\log\Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For a non-integer order $\nu \notin \mathbb{Z}$ there exist several equivalent series representations; we work with the re-formulated (6), *viz.*

$$Y_\nu(z) = \cot(\nu\pi) \sum_{n\geq 0} \frac{(-1)^n(\frac{z}{2})^{2n+\nu}}{\Gamma(n+\nu+1)\, n!} - \csc(\nu\pi) \sum_{n\geq 0} \frac{(-1)^n(\frac{z}{2})^{2n-\nu}}{\Gamma(n-\nu+1)\, n!}. \tag{29}$$

Theorem 6 *Assume that* $\mu, r > 0$ *and* $n \in \mathbb{N}$. *Then*

$$\begin{aligned}
\mathbb{S}_{\mu,n}(r) &= \sqrt{\frac{2}{\pi r}}\, \Gamma(\mu+n) \sum_{k\geq 0} \frac{(-1)^k(\mu+n)_{2k}}{(k+n)!\,k!} \left[\log\frac{r}{2} + \psi(\mu+2k+n)\right] \left(\frac{r}{2}\right)^{2k+n} \zeta(\mu+2k+n) \\
&\quad + \sqrt{\frac{2}{\pi r}}\, \Gamma(\mu+n) \sum_{k\geq 0} \frac{(-1)^k(\mu+n)_{2k}}{(k+n)!\,k!} \left(\frac{r}{2}\right)^{2k+n} \zeta'(\mu+2k+n) \\
&\quad - \frac{\Gamma(\mu-n)}{\sqrt{2\pi r}} \sum_{k=0}^{n-1} \frac{(n-k-1)!(\mu-n)_{2k}}{k!} \left(\frac{r}{2}\right)^{2k-n} \zeta(\mu+2k-n) \\
&\quad - \frac{\Gamma(\mu+n)}{\sqrt{2\pi r}} \sum_{k\geq 0} (-1)^k \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!} (\mu+n)_{2k} \left(\frac{r}{2}\right)^{2k+n} \zeta(\mu+2k+n)\,.
\end{aligned}$$

Proof Consider (25) for $\nu = n \in \mathbb{N}$. By the series (28) and by legitimate transformations we get

$$\begin{aligned}
\mathbb{S}_{\mu,n}(r) &= \sqrt{\frac{2}{\pi r}} \log\frac{r}{2} \sum_{k\geq 0} \frac{(-1)^k}{\Gamma(k+n+1)\,k!} \left(\frac{r}{2}\right)^{2k+n} \int_0^\infty \frac{x^{\mu+2k+n-1}}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&\quad + \sqrt{\frac{2}{\pi r}} \sum_{k\geq 0} \frac{(-1)^k}{\Gamma(k+n+1)\,k!} \left(\frac{r}{2}\right)^{2k+n} \int_0^\infty \frac{x^{\mu+2k+n-1}\log x}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&\quad - \frac{1}{\sqrt{2\pi r}} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{r}{2}\right)^{2k-n} \int_0^\infty \frac{x^{\mu+2k-n-1}}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&\quad - \frac{1}{\sqrt{2\pi r}} \sum_{k\geq 0} (-1)^k \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!} \left(\frac{r}{2}\right)^{2k+n} \int_0^\infty \frac{x^{\mu+2k+n-1}}{\mathrm{e}^x-1}\,\mathrm{d}x\,. \qquad (30)
\end{aligned}$$

The first, third and fourth integrals are already known by virtue of (15), however, the second one is more challenging. Since

$$I_p = \int_0^\infty \frac{x^{p-1}\log x}{\mathrm{e}^x-1}\,\mathrm{d}x = \sum_{m\geq 0} \int_0^\infty x^{p-1}\mathrm{e}^{-(m+1)x}\log x\,\mathrm{d}x =: \sum_{m\geq 0} \mathscr{I}_m\,,$$

by the Mellin transform [16, p. 315, Eq. (9)]

$$\int_0^\infty x^{p-1}\mathrm{e}^{-qx}\log x\,\mathrm{d}x = \frac{\Gamma(p)}{q^p}\big[\psi(p) - \log q\big]\,, \qquad \Re(q) > 0,\, \Re(p) > 0\,,$$

and having in mind that

$$\sum_{n\geq 1} \frac{\log n}{n^p} = -\zeta'(p), \qquad \Re(p) > 1,$$

setting $p = \mu + 2k + n$ and $q = m + 1$, we conclude that

$$\begin{aligned}
I_{\mu+2k+n} &= \Gamma(\mu+2k+n)\,\psi(\mu+2k+n)\,\zeta(\mu+2k+n) - \Gamma(\mu+2k+n) \sum_{m\geq 0} \frac{\log(m+1)}{(m+1)^{\mu+2k+n}} \\
&= \Gamma(\mu+2k+n)\big[\psi(\mu+2k+n)\,\zeta(\mu+2k+n) + \zeta'(\mu+2k+n)\big]\,. \qquad (31)
\end{aligned}$$

Finally, applying (15) and (31) to the expression (30), after certain transformations and reduction, we arrive at the statement. □

Theorem 7 *Assume that $\mu, r > 0$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned}\widetilde{\mathbb{S}}_{\mu,n}(r) &= \sqrt{\frac{2}{\pi r}}\,\Gamma(\mu+n)\sum_{k\geq 0}\frac{(-1)^k(\mu+n)_{2k}}{(k+n)!\,k!}\left[\log\frac{r}{2}+\psi(\mu+2k+n)\right]\left(\frac{r}{2}\right)^{2k+n}\eta(\mu+2k+n)\\ &+\sqrt{\frac{2}{\pi r}}\,\Gamma(\mu+n)\sum_{k\geq 0}\frac{(-1)^k(\mu+n)_{2k}}{(k+n)!\,k!}\left(\frac{r}{2}\right)^{2k+n}\eta'(\mu+2k+n)\\ &-\frac{\Gamma(\mu-n)}{\sqrt{2\pi r}}\sum_{k=0}^{n-1}\frac{(n-k-1)!(\mu-n)_{2k}}{k!}\left(\frac{r}{2}\right)^{2k-n}\eta(\mu+2k-n)\\ &-\frac{\Gamma(\mu+n)}{\sqrt{2\pi r}}\sum_{k\geq 0}(-1)^k\frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!}(\mu+n)_{2k}\left(\frac{r}{2}\right)^{2k+n}\eta(\mu+2k+n)\,.\end{aligned}$$

Proof Applying the Mellin–transform

$$\int_0^\infty \frac{x^{p-1}}{\mathrm{e}^x+1}\,\mathrm{d}x = \Gamma(p)\,\eta(p), \qquad \Re(p)>0,$$

for all integrals which we derive by the lines of the previous proving procedure, we clearly deduce the claimed result. □

Now, we precise the Riemann Zeta building blocks series presentation of the extended Mathieu $\mathbb{S}_{\mu,\nu}(r)$ and Dirichlet Eta function terms for extended alternating Mathieu series $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$ by using the non–integer ν parameter case.

Theorem 8 *For all $\mu, r > 0$ and for $|\nu| < 1$, when $\mu \pm \nu > 1$, we have*

$$\begin{aligned}\mathbb{S}_{\mu,n}(r) &= \cot(\nu\pi)\,\Gamma(\mu+\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k(\mu+\nu)_{2k}}{\Gamma(k+\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k+\nu}\zeta(\mu+2k+\nu)\\ &\quad-\csc(\nu\pi)\,\Gamma(\mu-\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k(\mu-\nu)_{2k}}{\Gamma(k-\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k-\nu}\zeta(\mu+2k-\nu)\,.\end{aligned}$$

Moreover, for $\mu, r > 0$ and for $|\nu| < 1$, when $\mu \pm \nu > 0$ there holds

$$\begin{aligned}\widetilde{\mathbb{S}}_{\mu,n}(r) &= \cot(\nu\pi)\,\Gamma(\mu+\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k(\mu+\nu)_{2k}}{\Gamma(k+\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k+\nu}\eta(\mu+2k+\nu)\\ &\quad-\csc(\nu\pi)\,\Gamma(\mu-\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k(\mu-\nu)_{2k}}{\Gamma(k-\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k-\nu}\eta(\mu+2k-\nu)\,. \qquad (32)\end{aligned}$$

Proof We start again with the integral (25) when $\nu \in (-1,1)$. The series representation (29) implies

$$\mathbb{S}_{\mu,n}(r) = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\left\{\cot(\nu\pi)\sum_{k\geq 0}\frac{(-1)^k(\frac{rx}{2})^{2k+\nu}}{\Gamma(k+\nu+1)\,k!}-\csc(\nu\pi)\sum_{k\geq 0}\frac{(-1)^k(\frac{rx}{2})^{2k-\nu}}{\Gamma(k-\nu+1)\,k!}\right\}\mathrm{d}x$$

$$\begin{aligned}
&= \cot(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k+\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k+\nu}\int_0^\infty \frac{x^{\mu+2k+\nu-1}}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&\quad - \csc(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k-\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k-\nu}\int_0^\infty \frac{x^{\mu+2k-\nu-1}}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&= \cot(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k+\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k+\nu}\Gamma(\mu+2k+\nu)\,\zeta(\mu+2k+\nu) \\
&\quad - \csc(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k-\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k-\nu}\Gamma(\mu+2k-\nu)\,\zeta(\mu+2k-\nu)\,,
\end{aligned}$$

which is equivalent to the stated formula. The proof of (32) is now straightforward. □

7 Extending Butzer–Flocke–Hauss (complete) Omega function $\Omega(z)$ *via* Neumann functions

The notation $\Omega(z), z\in\mathbb{C}$, stands for the so–called complete Butzer–Flocke–Hauss (BHF) Omega function introduced in [4, Definition 7.1], [5] in the form

$$\Omega(z) := 2\int_{0+}^{\frac12}\sinh(zu)\cot(\pi u)\,\mathrm{d}u, \qquad z\in\mathbb{C}\,.$$

It is the Hilbert transform $\mathscr{H}_1[\mathrm{e}^{-zx}](0)$ at zero of the 1–periodic function $\left(\mathrm{e}^{-zx}\right)_1$, defined by the periodic extension of the exponential function e^{-zx}, $|x|<\frac12$, $z\in\mathbb{C}$, thus

$$\Omega(z) = \mathscr{H}_1\left[\mathrm{e}^{-zx}\right](0) = ⨍_{-\frac12}^{\frac12}\mathrm{e}^{zu}\,\cot(\pi u)\,\mathrm{d}u\,.$$

Another expressions for the complete BHF Omega function $\Omega(x)$ are given by Butzer et al. [6]:

$$\Omega(x) = \frac{2}{\pi}\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{1}{\mathrm{e}^t+1}\cos\left(\frac{xt}{2\pi}\right)\mathrm{d}t\,, \qquad x\in\mathbb{R}\,, \tag{33}$$

while the real argument complete BHF Ω function's integral form by Tomovski and Pogány reads [29, p. 10, Theorem 3.3]

$$\Omega(x) = 2\sqrt{\frac{2}{\pi}}\sinh\left(\frac{x}{2}\right)⨍_0^\infty \sinh\left(\frac{xt}{\pi}\right)\tan t\,\mathrm{d}t\,.$$

By extensions in the integrand of the Butzer–Flocke–Hauss Omega function which is intimately connected to the generalized Mathieu series (consult the extensive study by Butzer and Pogány [5]) we are faced with a new territory of ideas and series/integral conclusion upon the structure of these kind generalizations.

Inspired by (33), we can write

$$\Omega(x) = -\frac{\sqrt{x}}{\pi}\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{\sqrt{t}}{\mathrm{e}^t+1}\,Y_{\frac12}\left(\frac{xt}{2\pi}\right)\mathrm{d}t$$

having in mind that $\cos(z) = -\sqrt{\pi z/2}\, Y_{\frac{1}{2}}(z)$ implementing the Neumann function of the general order ν instead of $Y_{-\frac{1}{2}}$ in the kernel in the following way

$$\Omega_{\mu,\nu}(x) = -\frac{\sqrt{x}}{\pi} \sinh\Big(\frac{x}{2}\Big) \int_0^\infty \frac{t^{\mu-1}}{\mathrm{e}^t+1}\, Y_\nu\Big(\frac{xt}{2\pi}\Big)\,\mathrm{d}t\,. \tag{34}$$

The parameter range derivation will be our first goal. In turn, recognizing that the same integral consist both $\Omega_{\mu,\nu}(x)$ and $\mathbb{S}_{\mu,\nu}(r)$ in (34) and (26), respectively, we deduce the relation

$$\Omega_{\mu,\nu}(x) = -\frac{x}{\pi^2}\, \sinh\Big(\frac{x}{2}\Big)\, \widetilde{\mathbb{S}}_{\mu,\nu}\Big(\frac{x}{2\pi}\Big)\,. \tag{35}$$

Therefore the parameter spaces coincide for any $x > 0$.

Next, the power series form of the complete BHF Ω function whose coefficients are built by finite sums containing Dirichlet Eta function terms is reported in [5, p. 901, Theorem 5.4. (ii)]

$$\Omega(z) = \frac{z}{\pi} \sum_{n\geq 0} \sum_{k=0}^{n} \frac{(-1)^k\, \eta(2k+1)}{\pi^{2k}(2(n-k)+1)!} \Big(\frac{z}{2}\Big)^{2n}, \qquad |z| < 2\pi\,,$$

which shows that Ω is intimately connected with the Eta function. In [5] the authors discussed the relations of the Mathieu-, and the alternating Mathieu series and its generalized variants from one, and the $\Omega(z)$ function from other side by the Taylor expansion of the Hilbert–Eisenstein series $\mathfrak{h}_1(z)$ of the first order and the polygamma function $\psi^{(r)}$ of order r (see also [2]).

However, our recent considerations are developed in another direction, according to the series representation of the expanded complete BHF $\Omega_{\mu,\nu}(x)$ in terms of the Dirichlet Eta function. In turn, bearing in mind (35), the counterpart results valid for $\Omega_{\mu,\nu}(x)$ exposed in Eq. (14) of Theorem 1, Theorem 7 and finally in Eq. (32) of Theorem 8 turn out to be their immediate consequences. So, we leave the formulation of these functional bound results to the interested reader.

Acknowledgements

The authors are grateful to the anonymous referee for useful suggestions which certainly improve the accuracy and clarity of the exposition. The work of RKP is supported by the Mathematical Research Impact Centric Support (MATRICS), SERB, Department of Science & Technology (DST), India (File No. MTR/2019/001328). The work of the second author GVM was supported in part by the Serbian Academy of Sciences and Arts (Φ-96). The research activities of TKP have been supported in part by the University of Rijeka, Croatia under the project labeled `uniri-pr-prirod-19-16`.

References

1. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972)

2. Á. Baricz, P. L. Butzer, T. K. Pogány, Alternating Mathieu series, Hilbert – Eisenstein series and their generalized Omega functions. In G. V. Milovanović, M. Th. Rassias (eds.), *Analytic Number Theory, Approximation Theory, and Special Functions* – In Honor of Hari M. Srivastava. (Springer, New York, 2014), 775–808.
3. Á. Baricz, D. Jankov Maširević, T. K. Pogány, Integral representations for Neumann-type series of Bessel functions I_ν, Y_ν and K_ν. Proc. Amer. Math. Soc. **140**(3), 951–960, (2012)
4. P. L. Butzer, S. Flocke, M. Hauss, Euler functions $E_\alpha(z)$ with complex α and applications, In G.A. Anastassiou, S.T. Rachev (eds.), *Approximation, Probability and Related Fields.* (Plenum Press, New York, 1994), 127–150.
5. P. L. Butzer, T. K. Pogány, A fresh approach to classical Eisenstein series and the newer Hilbert-Eisenstein series. Int. J. Number Theory **13**(4), 885–911 (2017)
6. P. L. Butzer, T. K. Pogány, H. M. Srivastava, A linear ODE for the Omega function associated with the Euler function $E_\alpha(z)$ and the Bernoulli function $B_\alpha(z)$. Appl. Math. Lett. **19**, 1073–1077 (2006)
7. P. Cerone, C. T. Lenard, On integral forms of generalized Mathieu series. JIPAM J. Inequal. Pure Appl. Math. **4**(5), Article No. 100, 1–11 (2003)
8. Junesang Choi, R. K. Parmar, T. K. Pogány, Mathieu-type series built by (p, q)–extended Gaussian hypergeometric function. Bull. Korean Math. Soc. **54**(3), 789–797 (2017)
9. Junesang Choi, H. M. Srivastava, Mathieu series and associated sums involving the Zeta functions. Comput. Math. Appl. **59**(2), 861–867 (2010)
10. J. Clunie, On Bose-Einstein functions. Proc. Phys. Soc. Sect. A. **67**(7), 632–636 (1954)
11. Dj. Cvijović, T. K. Pogány, Second type Neumann series related to Nicholson's and to Dixon–Ferrar formula. In V. Kravchenko, S. Sitnik (eds.), *Transmutation Operators and Applications*, Chapter 4. (Birkhäuser Verlag, Springer Basel AC, 2020), 67–84
12. N. Elezović, H. M. Srivastava, Ž. Tomovski, Integral representations and integral transforms of some families of Mathieu type series, Integral Transforms Spec. Functions **19**(7), 481–495 (2008)
13. O. Emersleben, Über die Reihe $\sum_{k=1}^{\infty} \frac{k}{(k^2+r^2)^2}$. Math. Ann. **125**, 165–171 (1952)
14. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**. (McGraw-Hill Book Company, New York, Toronto and London, 1953)
15. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. **II**. (McGraw-Hill Book Company, New York, Toronto and London, 1953)
16. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of Integral Transforms*, Vol. **I**. (McGraw-Hill Book Company, New York, Toronto and London, 1954)
17. I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Series, and Products.* (Corrected and Enlarged edition prepared by A. Jeffrey), seventh edition. (Academic Press, New York, 2007)
18. D. Jankov Maširević, T. K. Pogány, Second type Neumann series of generalized Nicholson function. Results Mat. **75**(1), Article No. 12, 14pp (2020)
19. L. Lewin, *Polylogarithms and Associated Functions.* (North-Holland, New York, 1981)
20. E. L. Mathieu, *Traité de Physique Mathématique* VI-VII: *Théorie de l'élasticité des corps solides.* (Gauthier–Villars, Paris, 1890)
21. G. V. Milovanović, Summation of series and Gaussian quadratures. In: R. V. M. Zahar (ed.), *Approximation and Computation* ISNM Vol. 119, 459–475, (Birkhäuser, Basel–Boston– Berlin, 1994)
22. G. V. Milovanović, T. K. Pogány, New integral forms of generalized Mathieu series and related applications. Appl. Anal. Discrete Math. **7**, 180–192 (2013)
23. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (Eds.), *NIST Handbook of Mathematical Functions.* (Cambridge University Press, Cambridge, 2010)
24. R. K. Parmar, G. V. Milovanović, T. K. Pogány, Multi-parameter Mathieu, and alternating Mathieu series. Appl. Math. Comp. **400**, Article ID 126099, 27pp (2021)
25. R. K. Parmar, T. K. Pogány, On Mathieu–type series for the unified Gaussian hypergeometric functions. Appl. Anal. Discrete Math. **14**(1), 138–149 (2020)
26. T. K. Pogány, R. K. Parmar, On p–extended Mathieu series. Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **22(534)**, 107–117 (2018)
27. T. K. Pogány, H. M. Srivastava, Ž. Tomovski, Some families of Mathieu **a**-series and alternating Mathieu **a**-series. Appl. Math. Comput. **173**(1), 69–108 (2006)
28. H. M. Srivastava, Junesang Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals.* (Elsevier Science, Publishers, Amsterdam, London and New York, 2012)
29. Ž. Tomovski, T. K. Pogány, Integral expressions for Mathieu–type power series and for the Butzer-Flocke-Hauss Ω–function. Fract. Calc. Appl. Anal. **14**(4), 623–634 (2011)
30. G. N. Watson, *A Treatise on the Theory of Bessel Functions.* (Cambridge University Press, London, 1922)
31. D. C. Wood, The computation of polylogarithms. Technical Report 15-92 (PS), Canterbury, UK: University of Kent Computing Laboratory, 19pp (1992)