On zeros of the regular power series of a quaternionic variable

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Abstract

Using tools from the newly developed theory of regular functions and polynomials with quaternionic coefficients located on only one side of the variable, we derive zero-free regions for the related subclass of regular power series and obtain discs that are not centered at the origin, containing all the zeros of these polynomials. The results obtained for this particular subclass of regular functions lead to generalizations of several results that are known from the relevant literature.

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1 Introduction

A classical study in geometric function theory is to locate the zeros of a polynomial in the plane using various approaches and techniques. This kind of study is considered to be very significant and has deeply influenced the development of mathematics and its application areas, such as physical systems. This study, in addition to having multiple applications, has inspired much more research, both theoretically and practically. The first result in this direction is the well-known Cauchy method [2], giving the upper bound for the moduli of the zeros of a polynomial, with complex coefficients, in the complex plane, but this bound can be very crude. Therefore, in order to attain better and sharp zero bounds, it is desirable to put some restrictions on the coefficients of the polynomial. In this connection, we state the following elegant result concerning the distribution of zeros of a polynomial when its coefficients are restricted is known in the literature as Eneström-Kakeya theorem.

Theorem 1.1 ([15]) *If $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ $(z \in \mathbb{C})$ is a polynomial of degree n with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $T(z)$ lie in $|z| \leq 1$.

We refer the reader to the comprehensive books of Marden [15] and Milovanović et al. [19] for an exhaustive survey of extensions and refinements of this well-known result. We get the following equivalent form of Theorem 1.1 by applying it to the polynomial $z^n T(1/z)$.

Theorem 1.2 *If $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ $(z \in \mathbb{C})$ is a polynomial of degree n with real coefficients and satisfying*

$$a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0,$$

then $T(z)$ does not vanish in $|z| \leq 1$.

The extension of Theorem 1.2 to a class of related analytic functions was established by Aziz and Mohammad [1] in the form of the following result.

Theorem 1.3 *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \not\equiv 0$ be analytic in $|z| \leq t$, $t > 0$. If*

$$a_\nu > 0 \quad \text{and} \quad a_{\nu-1} - t a_\nu \geq 0, \quad \nu = 1, 2, 3, \ldots,$$

then $f(z)$ does not vanish in $|z| < t$.

Numerous applications and extensions of the above results form an essential part of the classical content of geometric function theory and are equally important in modern papers dealing with the regional location of zeros of regular functions of quaternionic variables. Given the richness of the complex setting, a natural question is: what kind of results in the quaternionic setting can be obtained? It is then natural to study this class of functions with emphasis on the distribution of their zeros.

This paper is organized as follows. Section 2 contains some known concepts and results useful in the next sections. In section 3, we derive zero-free regions of a quaternionic power series with coefficients located on only one side of the variable. Section 4 yields discs that are not centred at the origin and include all of the zeros of a quaternionic polynomial with coefficients whose real and imaginary components satisfy appropriate conditions. We end this paper with a brief conclusion in Section 5.

2 A brief overview of quaternions and quaternionic functions

The noncommutative skew field $\mathbb{H}$ of quaternions consists of elements of the form $q = \alpha + \beta i + \gamma j + \delta k$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, with the following properties of the imaginary units i, j, k,

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Each element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is given by the real part $\mathrm{Re}(q) = \alpha$ and the imaginary part $\mathrm{Im}(q) = \beta i + \gamma j + \delta k$. The conjugate of q is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$, so that the norm of q is given by $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of an arbitrary non zero element $q \in \mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$.

Now, we define the ball $B(0, r) = \{q \in \mathbb{H};\ |q| < r\}$ for $r > 0$, and then by $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centred at the origin, i.e.,

$$\mathbb{B} = \big\{q = \alpha + \beta i + \gamma j + \delta k : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 1\big\}.$$

Since the multiplication in $\mathbb{H}$ is not commutative, one can consider unilateral quaternionic polynomials of the form

$$T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$$

and power series of the form

$$f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$$

of the quaternionic variable q on the left and with quaternionic coefficients a_ν on the right.

Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule): given $T_1(q) =$

$\sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ and $T_2(q) = \sum_{\mu=0}^{m} q^{\mu} b_{\mu}$, we define

$$(T_1 * T_2)(q) := \sum_{\substack{\nu=0,1,\ldots,n \\ \mu=0,1,\ldots,m}} q^{\nu+\mu} a_{\nu} b_{\mu}.$$

Given two quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ and $g(q) = \sum_{\nu=0}^{\infty} q^{\nu} b_{\nu}$ with radii of convergence greater than R, we define the regular product of f and g as the series

$$(f * g)(q) = \sum_{\nu=0}^{\infty} q^{\nu} c_{\nu},$$

where $c_{\nu} = \sum_{k=0}^{\nu} a_k b_{\nu-k}$ for each ν. As observed in ([5], [8]) for each quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$, there exists a ball $B(0,R) = \{q \in \mathbb{H};\ |q| < R\}$ such that f converges absolutely and uniformly on each compact subset of $B(0,R)$ and that function f is regular. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [29] for the basic features of quaternionic functions (see also [11] and [28]).

By using some useful tools from the theory on slice regular functions, Gentili and Stoppato [9] (see also [7]) gave a necessary and sufficient condition for a regular quaternionic power series to have a zero at a point in the form of the following result.

Theorem 2.1 *Let $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ be a given quaternionic power series with radius of convergence R, and let $p \in B(0,R)$. Then $f(p) = 0$ if and only if there exists a quaternionic power series $g(q)$ with radius of convergence R such that*

$$f(q) = (q - p) * g(q).$$

This extends to quaterniomic power series the theory presented in [13] for polynomials. The following result which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors is from [13] (see also [7] and [9]).

Theorem 2.2 *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\left(f(q_0)^{-1} q_0 f(q_0)\right) = 0$.*

Gentili and Struppa [8] established a maximum modulus theorem for regular functions, which includes convergent power series and polynomials in the form of the following result.

Theorem 2.3 (Maximum Modulus Theorem) *Let $B = B(0, r)$ be a ball in $\mathbb{H}$ with centre 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is a constant on B.*

It is worth noting that the proof of the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ from an algebraic point of view was given by Niven (for reference, see [25], [26]). This led to the complete identification of the zeros of polynomials in terms of their factorization, for reference see [27]. Thus it became an interesting perspective to think about the regions containing all the zeros of a regular polynomial of quaternionic variable. The earliest attempts to find the zeros of regular functions of a quaternionic variable were made by Niven [25], and there has been a lot of activity in this area of study recently. Most of these recent works deal with the generalisations and extensions of the zero bounds of polynomials with restricted quaternionic coefficients.

Slice regular functions of a quaternionic variable have been intensively studied in the past decade, and this study is extremely useful in replicating many useful properties of holomorphic functions of complex variables. The Eneström-Kakeya theorem and its various generalizations, as mentioned in Section 1, has recently been extended to polynomials of a quaternionic variable by Carney et al. [3] as follows:

Theorem 2.4 *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n, where q is a quaternionic variable with real coefficients and satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(q)$ lie in $|q| \leq 1$.

In the same paper, Carney et al. [3] also established an extension of Theorem 2.4 to quaternionic coefficients in the form of the following result.

Theorem 2.5 *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a quaternionic polynomial of degree n, where $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 \geq 0, \quad \alpha_n \neq 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leq 1 + \frac{2}{\alpha_n} \sum_{\nu=0}^{n} \big(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu|\big).$$

Concurrently, Tripathi [30, Theorem 3.1] established a generalization of Theorem 2.4 in the form of the following result.

Theorem 2.6 *Let $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ be a polynomial of degree n, where q is a quaternionic variable with quaternionic coefficients, where $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$ for $\nu = 0, 1, 2, \dots, n$, satisfying*

$$\begin{aligned} \alpha_n &\geq \alpha_{n-1} \geq \cdots \geq \alpha_{\ell}, \\ \beta_n &\geq \beta_{n-1} \geq \cdots \geq \beta_{\ell}, \\ \gamma_n &\geq \gamma_{n-1} \geq \cdots \geq \gamma_{\ell}, \\ \delta_n &\geq \delta_{n-1} \geq \ldots \geq \delta_{\ell}, \end{aligned}$$

for $0 \leq l \leq n$. Then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{1}{|a_n|} \Big[|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_{\ell}) + (\beta_n - \beta_{\ell}) + (\gamma_n - \gamma_{\ell}) + (\delta_n - \delta_{\ell}) + M_{\ell} \Big],$$

where

$$M_{\ell} = \sum_{\nu=1}^{\ell} \Big[|\alpha_{\nu} - \alpha_{\nu-1}| + |\beta_{\nu} - \beta_{\nu-1}| + |\gamma_{\nu} - \gamma_{\nu-1}| + |\delta_{\nu} - \delta_{\nu-1}| \Big].$$

The need for estimation of the bounds for the zeros of regular functions arises frequently in geometric function theory, and this study finds numerous applications in quantum physics, functional calculus, and operator theory. These estimates are also frequently employed in a wide range of applications in numerical mathematics and engineering domains, as they provide a simple and efficient way to express the relationship between the variables of the system. Recently, several works appeared in the literature, including generalizations and refinements of the above results; see, e.g., [6], [16–18], [20–22]. In addition, we also mention here some recent results on the Eneström-Kakeya theorem for quaternionic polynomials derived under certain complicated restrictions for polynomial coefficients (see [10], [14], [23], [24]).

Existing results in the literature also show that there is a need to find explicit bounds for polynomials and regular functions, e.g., those having restrictions on the coefficients. For this reason, it is desirable to limit the coefficients of the aforementioned regular functions to obtain their zero inclusion regions and unify the derivation of various existing and new Eneström-Kakeya type bounds. There is now a very ample literature on the location of zeros in quaternionic polynomials, but not on the zeros in quaternionic power series.

3 Zero-free regions of quaternionic power series

In this section, we establish several new results that pertain to the zero-free regions of quaternionic power series with coefficients located on only one side of the variable. In proofs, we apply methods that are far different from the known ones. The obtained results for this subclass of regular functions produce generalisations of a number of results known in the literature on this subject. We start with the following extension of Theorem 1.3 to slice regular functions, regular in the ball $B(0, R)$ with centre at the origin and radius $R > 0$.

Theorem 3.1 *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ for all $q \in B(0,R)$. If a_{ν}, $\nu = 0,1,2,\ldots,$ are real and positive satisfying*

$$a_{\nu-1} - ta_{\nu} \geq 0, \ \ \nu = 1,2,\ldots$$

where $0 < t < R$. Then $f(q)$ does not vanish in $|q| < t$.

Proof Consider the power series

$$\begin{aligned} F(q) &= (t-q) * f(q) \\ &= (t-q) * (a_0 + qa_1 + q^2 a_2 + \cdots) \\ &= ta_0 - \left[q(a_0 - ta_1) + q^2(a_1 - ta_2) + \cdots\right] \\ &= ta_0 - q\psi(q), \end{aligned}$$

where

$$\psi(q) = \sum_{\nu=1}^{\infty} q^{\nu-1}(a_{\nu-1} - ta_{\nu}).$$

For $|q| = t$, we have

$$\begin{aligned} |\psi(q)| &\leq \sum_{\nu=1}^{\infty} |q|^{\nu-1} |a_{\nu-1} - ta_{\nu}| \\ &= \sum_{\nu=1}^{\infty} t^{\nu-1}(a_{\nu-1} - ta_{\nu}) \\ &= a_0. \end{aligned}$$

Since $\psi(q)$ is regular in $|q| \leq t$, it follows by Theorem 2.3, that

$$|\psi(q)| \leq a_0 \quad \text{for} \quad |q| \leq t. \tag{1}$$

For $|q| \leq t$, we have

$$\begin{aligned} |F(q)| &= |ta_0 - q\psi(q)| \\ &\geq |ta_0| - |q|\,|\psi(q)| \\ &\geq a_0(t - |q|) \quad \text{by} \quad (1). \end{aligned}$$

Thus in $|q| \leq t$, $|F(q)| > 0$ if $|q| < t$, i.e., $F(q) \neq 0$ for $|q| < t$. Since by Theorem 2.1, the only zeros of $(t-q) * f(q)$ are $q = t$ and the zeros of $f(q)$, therefore, $f(q) \neq 0$ for $|q| < t$. This proves Theorem 3.1. □

Now, we present an extension of Theorem 3.1 to quaternionic coefficients.

Theorem 3.2 *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ for all $q \in B(0,R)$. If $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$, $\nu = 0,1,2,\ldots,$ are quaternionic coefficients satisfying*

$$0 < |a_0| \leq t|a_1| \leq \cdots \leq t^{\lambda}|a_{\lambda}| \geq t^{\lambda+1}|a_{\lambda+1}| \geq \ldots,$$

where λ is some finite non negative integer and $0 < t < R$. Then $f(q)$ does not vanish in

$$|q| < \frac{t}{\left(2t^{\lambda}\left|\frac{a_{\lambda}}{a_0}\right| - 1\right) + \frac{2}{|a_0|}\sum_{\nu=0}^{\infty}\big|a_{\nu} - |a_{\nu}|\big|t^{\nu}}.$$

Proof Again we consider the power series

$$\begin{aligned}(t-q)*f(q) &= (t-q)*(a_0+qa_1+q^2a_2+\cdots)\\ &= ta_0-\big[q(a_0-ta_1)+q^2(a_1-ta_2)+\cdots\big]\\ &= ta_0-q\psi(q)\quad \text{(say)}.\end{aligned}$$

For $|q|=t$, we have

$$\begin{aligned}|\psi(q)| &\le \sum_{\nu=1}^{\infty} t^{\nu-1}|a_{\nu-1}-ta_\nu|\\ &\le \sum_{\nu=1}^{\infty} t^{\nu-1}\big|t|a_\nu|-|a_{\nu-1}|\big| + \sum_{\nu=1}^{\infty} t^{\nu-1}\big|t(a_\nu-|a_\nu|)-(a_{\nu-1}-|a_{\nu-1}|)\big|\\ &= \sum_{\nu=1}^{\lambda} t^{\nu-1}\big(t|a_\nu|-|a_{\nu-1}|\big) + \sum_{v=\lambda+1}^{\infty} t^{\nu-1}\big(|a_{\nu-1}|-t|a_\nu|\big)\\ &\quad + \sum_{\nu=1}^{\infty} t^{\nu-1}\big|t(a_\nu-|a_\nu|)-(a_{\nu-1}-|a_{\nu-1}|)\big|\\ &= 2t^\lambda|a_\lambda|-|a_0|+\sum_{\nu=1}^{\infty} t^{\nu-1}\big|t(a_\nu-|a_\nu|)-(a_{\nu-1}-|a_{\nu-1}|)\big|\\ &\le 2t^\lambda|a_\lambda|-|a_0|+2\sum_{\nu=0}^{\infty} t^{\nu}\big|a_\nu-|a_\nu|\big|. \end{aligned}\tag{2}$$

Now proceeding as in the proof of Theorem 3.1, it follows that for $|q|\le t$, by (2), we have

$$\begin{aligned}\big|(t-q)*f(q)\big| &\ge t|a_0|-|q||\psi(q)|\\ &\ge |a_0|\left[t-|q|\left(2t^\lambda\left|\frac{a_\lambda}{a_0}\right|-1+\frac{2}{|a_0|}\sum_{\nu=0}^{\infty}t^\nu\big|a_\nu-|a_\nu|\big|\right)\right].\end{aligned}$$

Thus in $|q|\le t,\ |(t-q)*f(q)|>0$ if

$$|q|<\frac{t}{2t^\lambda\left|\frac{a_\lambda}{a_0}\right|-1+\frac{2}{|a_0|}\sum_{\nu=0}^{\infty}t^\nu\big|a_\nu-|a_\nu|\big|}.$$

Since by Theorem 2.1, the only zeros of $(t-q)*f(q)$ are $q=t$ and the zeros of $f(q)$, therefore, $f(q)\neq 0$ if

$$|q|<\frac{t}{2t^\lambda\left|\frac{a_\lambda}{a_0}\right|-1+\frac{2}{|a_0|}\sum_{\nu=0}^{\infty}t^\nu\big|a_\nu-|a_\nu|\big|}.$$

This completes the proof of Theorem 3.2. □

It we take $\beta_\nu=\gamma_\nu=\delta_\nu=0$ for $\nu=0,1,2,\ldots$, in Theorem 3.2, we get the following generalization of Theorem 3.1.

Corollary 3.3 *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ in the quaternionic variable q. If a_{ν}, $\nu = 0,1,2,\ldots$, are real and positive, satisfying*

$$a_0 \leq ta_1 \leq \cdots \leq t^{\lambda} a_{\lambda} \geq t^{\lambda+1} a_{\lambda+1} \geq \ldots,$$

where λ is some finite non negative integer and $0 < t < R$. Then $f(q)$ does not vanish in

$$|q| < \frac{t}{2\left(\frac{a_{\lambda}}{a_0}\right) t^{\lambda} - 1}.$$

Next, we obtain a zero free region for a regular quaternionic power series with quaternionic coefficients in which the real component satisfying some suitable inequalities.

Theorem 3.4 *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_{\nu}$ in the quaternionic variable q. If $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$, $\nu = 0,1,2,\ldots$, are quaternionic coefficients with real components satisfying*

$$0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^{\lambda} \alpha_{\lambda} \geq t^{\lambda+1} \alpha_{\lambda+1} \geq \cdots,$$

where λ is some finite non negative integer and $0 < t < R$. Then $f(q)$ does not vanish in

$$|q| < \frac{t}{2\left(\frac{\alpha_{\lambda}}{\alpha_0}\right) t^{\lambda} - 1 + \left(\frac{2}{\alpha_0}\right) \sum\limits_{\nu=0}^{\infty} t^{\nu} \left(|\beta_{\nu}| + |\gamma_{\nu}| + |\delta_{\nu}|\right)}.$$

Proof As in the proof of Theorem 3.2, we have

$$(t-q) * f(q) = ta_0 - q\psi(q).$$

Now using the fact that

$$|a_{\nu-1} - ta_{\nu}| \leq |\alpha_{\nu-1} - t\alpha_{\nu}| + |\beta_{\nu-1}| + t|\beta_{\nu}| + |\gamma_{\nu-1}| + t|\gamma_{\nu}| + |\delta_{\nu-1}| + t|\delta_{\nu}|,$$

we get for $|q| = t$,

$$\begin{aligned}
|\psi(q)| &\leq \sum_{\nu=1}^{\infty} t^{\nu-1} |a_{\nu-1} - ta_{\nu}| \\
&\leq \sum_{\nu=1}^{\infty} t^{\nu-1} |\alpha_{\nu-1} - t\alpha_{\nu}| \\
&\quad + \sum_{\nu=1}^{\infty} t^{\nu-1} \left[|\beta_{\nu-1}| + t|\beta_{\nu}| + |\gamma_{\nu-1}| + t|\gamma_{\nu}| + |\delta_{\nu-1}| + t|\delta_{\nu}|\right] \\
&\leq 2t^{\lambda}\alpha_{\lambda} - \alpha_0 + 2\sum_{\nu=0}^{\infty} t^{\nu} \left(|\beta_{\nu}| + |\gamma_{\nu}| + |\delta_{\nu}|\right) \\
&= \alpha_0 \left[2t^{\lambda} \frac{\alpha_{\lambda}}{\alpha_0} - 1 + \frac{2}{\alpha_0} \sum_{\nu=0}^{\infty} t^{\nu} \left(|\beta_{\nu}| + |\gamma_{\nu}| + |\delta_{\nu}|\right)\right]
\end{aligned}$$

$$\leq |a_0| \left[2t^{\lambda} \frac{\alpha_\lambda}{\alpha_0} - 1 + \frac{2}{\alpha_0} \sum_{\nu=0}^{\infty} t^{\nu} \left(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu| \right) \right]. \tag{3}$$

Now proceeding as in the proof of Theorem 3.2, it follows that for $|q| \leq t$, by (3), we have

$$\begin{aligned} |(t-q) * f(q)| &\geq t|a_0| - |q||\psi(q)| \\ &\geq |a_0| \left[t - |q| \left(2t^{\lambda} \frac{\alpha_\lambda}{\alpha_0} - 1 + \frac{2}{\alpha_0} \sum_{\nu=0}^{\infty} t^{\nu} \left(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu| \right) \right) \right]. \end{aligned}$$

Thus in $|q| \leq t$, $|(t-q) * f(q)| > 0$, if

$$|q| < \frac{t}{2t^{\lambda} \frac{\alpha_\lambda}{\alpha_0} - 1 + \frac{2}{\alpha_0} \sum\limits_{\nu=0}^{\infty} t^{\nu} \left(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu| \right)}.$$

Again, since by Theorem 2.1, the only zeros of $(t-q) * f(q)$ are $q = t$ and the zeros of $f(q)$, therefore, $f(q) \neq 0$ if

$$|q| < \frac{t}{2t^{\lambda} \frac{\alpha_\lambda}{\alpha_0} - 1 + \frac{2}{\alpha_0} \sum\limits_{\nu=0}^{\infty} t^{\nu} \left(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu| \right)}.$$

This completes the proof of Theorem 3.4. □

Taking $\lambda = 0$ in Theorem 3.4, we get the following result.

Corollary 3.5 *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_\nu$ in the quaternionic variable q. If $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$, $\nu = 0, 1, 2, \ldots,$ are quaternionic coefficients with real components satisfying*

$$0 < \alpha_0 \geq t\alpha_1 \geq t^2 \alpha_2 \geq \ldots,$$

where $0 < t < R$. Then $f(q)$ does not vanish in

$$|q| < \frac{t}{1 + \frac{2}{\alpha_0} \sum\limits_{\nu=0}^{\infty} t^{\nu} \left(|\beta_\nu| + |\gamma_\nu| + |\delta_\nu| \right)}.$$

Finally, in this section, we establish a zero-free region for a regular quaternionic power series with restricted coefficients, namely coefficients whose real and imaginary components satisfy suitable inequalities.

Theorem 3.6 *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^{\nu} a_\nu$, for all $q \in B(0,R)$. If $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$, $\nu = 0, 1, 2, \ldots,$ are quaternionic coefficients satisfying*

$$0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^{\lambda} \alpha_\lambda \geq t^{\lambda+1} \alpha_{\lambda+1} \geq \ldots,$$

$$\beta_0 \leq t\beta_1 \leq \cdots \leq t^{r} \beta_r \geq t^{r+1} \beta_{r+1} \geq \ldots,$$

$$\gamma_0 \leq t\gamma_1 \leq \cdots \leq t^{s} \gamma_s \geq t^{s+1} \gamma_{s+1} \geq \ldots,$$

$$\delta_0 \le t\delta_1 \le \cdots \le t^{\mu}\delta_{\mu} \ge t^{\mu+1}\delta_{\mu+1} \ge \ldots,$$

where λ, r, s, μ *are finite non-negative integers and* $0 < t < R$. *Then* $f(q)$ *does not vanish in*

$$|q| < \frac{t|a_0|}{2(\alpha_\lambda t^\lambda + \beta_r t^r + \gamma_s t^s + \delta_\mu t^\mu) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0)}.$$

Proof As in the proof of Theorem 3.2, we have

$$(t - q) * f(q) = ta_0 - q\psi(q).$$

Since

$$|a_{\nu-1} - ta_\nu| \le |\alpha_{\nu-1} - t\alpha_\nu| + |\beta_{\nu-1} - t\beta_\nu| + |\gamma_{\nu-1} - t\gamma_\nu| + |\delta_{\nu-1} - t\delta_\nu|, \quad \nu = 1, 2, \ldots,$$

we have for $|q| = t$,

$$\begin{aligned}|\psi(q)| &\le \sum_{\nu=1}^{\infty} t^{\nu-1}\Big[|\alpha_{\nu-1} - t\alpha_\nu| + |\beta_{\nu-1} - t\beta_\nu| + |\gamma_{\nu-1} - t\gamma_\nu| + |\delta_{\nu-1} - t\delta_\nu|\Big]\\ &= 2(\alpha_\lambda t^\lambda + \beta_r t^r + \gamma_s t^s + \delta_\mu t^\mu) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0).\end{aligned}$$

Since $\psi(q)$ is regular in $|q| \le t$, it follows by the Maximum Modulus Theorem that

$$|\psi(q)| \le 2(\alpha_\lambda t^\lambda + \beta_r t^r + \gamma_s t^s + \delta_\mu t^\mu) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0) \quad \text{for} \quad |q| \le t. \tag{4}$$

For $|q| \le t$, by (4), we have

$$\begin{aligned}|(t - q) * f(q)| &\ge t|a_0| - |q||\psi(q)|\\ &\ge t|a_0| - |q|\Big(2(\alpha_\lambda t^\lambda + \beta_r t^r + \gamma_s t^s + \delta_\mu t^\mu) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0)\Big)\\ &> 0,\end{aligned}$$

if

$$|q| \le \frac{t|a_0|}{2(\alpha_\lambda t^\lambda + \beta_r t^r + \gamma_s t^s + \delta_\mu t^\mu) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0)}.$$

By Theorem 2.1, the only zeros of $(t - q) * f(q)$ are $q = t$ and the zeros of $f(q)$, therefore, it follows that $f(q) \ne 0$ for

$$|q| < \frac{t|a_0|}{2(\alpha_\lambda t^\lambda + \beta_r t^r + \gamma_s t^s + \delta_\mu t^\mu) - (\alpha_0 + \beta_0 + \gamma_0 + \delta_0)}.$$

This completes the proof of Theorem 3.6. □

By the Cauchy-Schwarz inequality, we have $\alpha_0 + \beta_0 + \gamma_0 + \delta_0 \le 2|a_0|$. Using this fact and taking $\lambda = r = s = \mu = 0$ in Theorem 3.6, we get the following result.

Corollary 3.7 *Let* $f : B(0, R) \to \mathbb{H}$ *be a regular power series in the quaternionic variable* q, *i.e.,* $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$, *for all* $q \in B(0, R)$. *If* $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$, $\nu = 0, 1, 2, \ldots,$ *are quaternionic coefficients satisfying*

$$0 < \alpha_0 \ge t\alpha_1 \ge t^2\alpha_2 \ge \ldots,$$

$$\beta_0 \geq t\beta_1 \geq t^2\beta_2 \geq \dots,$$

$$\gamma_0 \geq t\gamma_1 \geq t^2\gamma_2 \geq \dots,$$

$$\delta_0 \geq t\delta_1 \geq t^2\delta_2 \geq \dots$$

where $0 < t < R$. *Then* $f(q)$ *does not vanish in* $|q| < t/2$.

4 Location of all zeros of a quaternionic polynomial in a non-central disc

In this section, we obtain regions consisting of discs that are not centred at the origin and include all of the zeros of a quaternionic polynomial with coefficients whose real and imaginary components satisfy suitable inequalities. We establish a generalisation of Theorem 2.6. It will be shown that this result in particular gives the quaternionic analogue of a result due to Joyal et al. [12] and from which we can recover Theorem 2.4 as well.

Theorem 4.1 *If* $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients, where* $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$ *for* $\nu = 0, 1, 2, \dots, n$, *satisfying*

$$\lambda_1 \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_{\ell}, \quad \lambda_2 \beta_n \geq \beta_{n-1} \geq \dots \geq \beta_{\ell},$$

$$\lambda_3 \gamma_n \geq \gamma_{n-1} \geq \dots \geq \gamma_{\ell}, \quad \lambda_4 \delta_n \geq \delta_{n-1} \geq \dots \geq \delta_{\ell},$$

where $\lambda_s \geq 1$ *for* $s = 1, 2, 3, 4$, *and* $0 \leq \ell \leq n-1$. *Then all the zeros of* $T(q)$ *lie in*

$$\left| q + \frac{\lambda_1 \alpha_n + \lambda_2 \beta_n i + \lambda_3 \gamma_n j + \lambda_4 \delta_n k}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Big[(\lambda_1 \alpha_n + |\alpha_0| - \alpha_{\ell})$$

$$+ (\lambda_2 \beta_n + |\beta_0| - \beta_{\ell}) + (\lambda_3 \gamma_n + |\gamma_0| - \gamma_{\ell}) + (\lambda_4 \delta_n + |\delta_0| - \delta_{\ell}) + M_{\ell} \Big],$$

where

$$M_{\ell} = \sum_{\nu=1}^{\ell} \Big[|\alpha_{\nu} - \alpha_{\nu-1}| + |\beta_{\nu} - \beta_{\nu-1}| + |\gamma_{\nu} - \gamma_{\nu-1}| + |\delta_{\nu} - \delta_{\nu-1}| \Big].$$

Proof Consider the polynomial

$$\begin{aligned} T(q) * (1 - q) &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \dots + q^n(a_n - a_{n-1}) - q^{n+1} a_n \\ &= a_0 + \sum_{\nu=1}^{n-1} q^{\nu}(a_{\nu} - a_{\nu-1}) + q^n \big[(\lambda_1 \alpha_n - \alpha_{n-1}) + (\lambda_2 \beta_n - \beta_{n-1}) i \\ &\qquad + (\lambda_3 \gamma_n - \gamma_{n-1}) j + (\lambda_4 \delta_n - \delta_{n-1}) k \big] \\ &\qquad - q^n \big[(q-1) a_n + \lambda_1 \alpha_n + \lambda_2 \beta_n i + \lambda_3 \gamma_n j + \lambda_4 \delta_n k \big] \\ &= \phi(q) - q^n \big[(q-1) a_n + \lambda_1 \alpha_n + \lambda_2 \beta_n i + \lambda_3 \gamma_n j + \lambda_4 \delta_n k \big], \end{aligned}$$

where

$$\phi(q) = a_0 + \sum_{\nu=1}^{n-1} q^{\nu}(a_{\nu} - a_{\nu-1}) + q^n\big[(\lambda_1\alpha_n - \alpha_{n-1}) + (\lambda_2\beta_n - \beta_{n-1})i + (\lambda_3\gamma_n - \gamma_{n-1})j + (\lambda_4\delta_n - \delta_{n-1})k\big].$$

For $|q| = 1$, we have

$$\begin{aligned}
|\phi(q)| &\leq |a_0| + \sum_{\nu=1}^{\ell} |q|^{\nu}|a_{\nu} - a_{\nu-1}| + \sum_{\nu=\ell+1}^{n-1} |q|^{\nu}|a_{\nu} - a_{\nu-1}| + |q|^n\big|(\lambda_1\alpha_n - \alpha_{n-1}) \\
&\quad + (\lambda_2\beta_n - \beta_{n-1})i + (\lambda_3\gamma_n - \gamma_{n-1})j + (\lambda_4\delta_n - \delta_{n-1})k\big| \\
&= |\alpha_0 + \beta_0 i + \gamma_0 j + \delta_0 k| \\
&\quad + \sum_{\nu=1}^{\ell} \big|(\alpha_{\nu} - \alpha_{\nu-1}) + (\beta_{\nu} - \beta_{\nu-1})i + (\gamma_{\nu} - \gamma_{\nu-1})j + (\delta_{\nu} - \delta_{\nu-1})k\big| \\
&\quad + \sum_{\nu=\ell+1}^{n-1} \big|(\alpha_{\nu} - \alpha_{\nu-1}) + (\beta_{\nu} - \beta_{\nu-1})i + (\gamma_{\nu} - \gamma_{\nu-1})j + (\delta_{\nu} - \delta_{\nu-1})k\big| \\
&\quad + \big|(\lambda_1\alpha_n - \alpha_{n-1}) + (\lambda_2\beta_n - \beta_{n-1})i + (\lambda_3\gamma_n - \gamma_{n-1})j + (\lambda_4\delta_n - \delta_{n-1})k\big| \\
&\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| \\
&\quad + \sum_{\nu=1}^{\ell} \big[|\alpha_{\nu} - \alpha_{\nu-1}| + |\beta_{\nu} - \beta_{\nu-1}| + |\gamma_{\nu} - \gamma_{\nu-1}| + |\delta_{\nu} - \delta_{\nu-1}|\big] \\
&\quad + \sum_{\nu=\ell+1}^{n-1} \big[|\alpha_{\nu} - \alpha_{\nu-1}| + |\beta_{\nu} - \beta_{\nu-1}| + |\gamma_{\nu} - \gamma_{\nu-1}| + |\delta_{\nu} - \delta_{\nu-1}|\big] \\
&\quad + |\lambda_1\alpha_n - \alpha_{n-1}| + |\lambda_2\beta_n - \beta_{n-1}| + |\lambda_3\gamma_n - \gamma_{n-1}| + |\lambda_4\delta_n - \delta_{n-1}| \\
&= (\lambda_1\alpha_n + |\alpha_0| - \alpha_{\ell}) + (\lambda_2\beta_n + |\beta_0| - \beta_{\ell}) + (\lambda_3\gamma_n + |\gamma_0| - \gamma_{\ell}) \\
&\quad + (\lambda_4\delta_n + |\delta_0| - \delta_{\ell}) + M_{\ell},
\end{aligned}$$

where

$$M_{\ell} = \sum_{\nu=1}^{\ell} \big[|\alpha_{\nu} - \alpha_{\nu-1}| + |\beta_{\nu} - \beta_{\nu-1}| + |\gamma_{\nu} - \gamma_{\nu-1}| + |\delta_{\nu} - \delta_{\nu-1}|\big].$$

Note that, we have

$$\max_{|q|=1} \left|q^n * \phi\left(\frac{1}{q}\right)\right| = \max_{|q|=1} \left|q^n \phi\left(\frac{1}{q}\right)\right| = \max_{|q|=1} \left|\phi\left(\frac{1}{q}\right)\right| = \max_{|q|=1} |\phi(q)|,$$

it is clear that $q^n * \phi\left(1/q\right)$ has the same bound on $|q| = 1$ as ϕ, that is

$$\left|q^n * \phi\left(\frac{1}{q}\right)\right| \leq (\lambda_1\alpha_n + |\alpha_0| - \alpha_{\ell}) + (\lambda_2\beta_n + |\beta_0| - \beta_{\ell}) + (\lambda_3\gamma_n + |\gamma_0| - \gamma_{\ell}) + (\lambda_4\delta_n + |\delta_0| - \delta_{\ell}) + M_{\ell} \quad \text{for} \quad |q| = 1.$$

Since $q^n * \phi(1/q)$ is a polynomial and so is regular in $|q| \leq 1$, it follows by Theorem 2.3, that

$$\left|q^n * \phi\left(\frac{1}{q}\right)\right| = \left|q^n \phi\left(\frac{1}{q}\right)\right| \leq (\lambda_1\alpha_n + |\alpha_0| - \alpha_\ell) + (\lambda_2\beta_n + |\beta_0| - \beta_\ell) + (\lambda_3\gamma_n + |\gamma_0| - \gamma_\ell) + (\lambda_4\delta_n + |\delta_0| - \delta_\ell) + M_\ell$$

for $|q| \leq 1$. Hence

$$\left|\phi\left(\frac{1}{q}\right)\right| \leq \frac{1}{|q^n|}\Big((\lambda_1\alpha_n + |\alpha_0| - \alpha_\ell) + (\lambda_2\beta_n + |\beta_0| - \beta_\ell) + (\lambda_3\gamma_n + |\gamma_0| - \gamma_\ell) + (\lambda_4\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big) \quad \text{for} \quad |q| \leq 1.$$

Replacing q by $1/q$, we see that

$$|\phi(q)| \leq \Big((\lambda_1\alpha_n + |\alpha_0| - \alpha_\ell) + (\lambda_2\beta_n + |\beta_0| - \beta_\ell) + (\lambda_3\gamma_n + |\gamma_0| - \gamma_\ell) + (\lambda_4\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big)|q|^n \quad \text{for} \quad |q| \geq 1. \tag{5}$$

For $|q| \geq 1$, we have

$$\begin{aligned}|T(q) * (1-q)| &= \left|\phi(q) - q^n\{(q-1)a_n + \lambda_1\alpha_n + \lambda_2\beta_n i + \lambda_3\gamma_n j + \lambda_4\delta_n k\}\right| \\ &\geq |q^n||a_n|\left|q + \frac{\lambda_1\alpha_n + \lambda_2\beta_n i + \lambda_3\gamma_n j + \lambda_4\delta_n k}{a_n} - 1\right| - |\phi(q)| \\ &\geq |q|^n\Bigg[|a_n|\left|q + \frac{\lambda_1\alpha_n + \lambda_2\beta_n i + \lambda_3\gamma_n j + \lambda_4\delta_n k}{a_n}\right| \\ &\quad - \big[(\lambda_1\alpha_n + |\alpha_0| - \alpha_\ell) + (\lambda_2\beta_n + |\beta_0| - \beta_\ell) \\ &\quad + (\lambda_3\gamma_n + |\gamma_0| - \gamma_\ell) + (\lambda_4\delta_n + |\delta_0| - \delta_\ell) + M_\ell\big]\Bigg],\end{aligned}$$

where we used (5).

Thus, if

$$\left|q + \frac{\lambda_1\alpha_n + \lambda_2\beta_n i + \lambda_3\gamma_n j + \lambda_4\delta_n k}{a_n} - 1\right| > r, \tag{6}$$

where

$$r = \frac{1}{|a_n|}\big[(\lambda_1\alpha_n + |\alpha_0| - \alpha_\ell) + (\lambda_2\beta_n + |\beta_0| - \beta_\ell) + (\lambda_3\gamma_n + |\gamma_0| - \gamma_\ell) + (\lambda_4\delta_n + |\delta_0| - \delta_\ell) + M_\ell\big],$$

then $|T(q) * (1-q)| > 0$, that is $T(q) * (1-q) \neq 0$. Since by Theorem 2.2 the only zeros of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$, therefore, $T(q) \neq 0$ for all q satisfying (6). In other words, all the zeros of $T(q)$ lie in

$$\left|q + \frac{\lambda_1\alpha_n + \lambda_2\beta_n i + \lambda_3\gamma_n j + \lambda_4\delta_n k}{a_n} - 1\right| \leq r.$$

This completes the proof of Theorem 4.1. □

Taking $\lambda_s = R \geq 1$ for $s = 1, 2, 3, 4$, in Theorem 4.1 we get the following generalization of Theorem 2.6.

Corollary 4.2 *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ is a quaternionic polynomial of degree n with quaternionic coefficients, where $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$R\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_{\ell}, \quad R\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_{\ell},$$

$$R\gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_{\ell}, \quad R\delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_{\ell},$$

for some $R \geq 1$ and $0 \leq \ell \leq n-1$. Then all the zeros of $T(q)$ lie in

$$\begin{aligned} |q + R - 1| \leq \frac{1}{|a_n|} \Big[&(R\alpha_n + |\alpha_0| - \alpha_{\ell}) + (R\beta_n + |\beta_0| - \beta_{\ell}) \\ &+ (R\gamma_n + |\gamma_0| - \gamma_{\ell}) + (R\delta_n + |\delta_0| - \delta_{\ell}) + M_{\ell} \Big], \end{aligned}$$

where

$$M_{\ell} = \sum_{\nu=1}^{\ell} \Big[|\alpha_{\nu} - \alpha_{\nu-1}| + |\beta_{\nu} - \beta_{\nu-1}| + |\gamma_{\nu} - \gamma_{\nu-1}| + |\delta_{\nu} - \delta_{\nu-1}| \Big].$$

Remark 4.1 The above corollary was recently established in [16]. Taking $R = 1$ in Corollary 4.2, we recover Theorem 2.6.

Taking $\beta_{\nu} = \gamma_{\nu} = \delta_{\nu} = 0$ for $\nu = 0, 1, 2, \ldots, n$, in Corollary 4.2, we get the following result for $\ell = 0$.

Corollary 4.3 *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$, is a polynomial of degree n (where q is a quaternionic variable), with real coefficients and satisfying*

$$Ra_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

for some $R \geq 1$. Then all the zeros of $T(q)$ lie in

$$|q + R - 1| \leq \frac{Ra_n + |a_0| - a_0}{|a_n|}.$$

Remark 4.2 For $R = 1$, the above Corollary 4.3 gives the quaternionic analogue of a result due to Joyal et al. [12] (see [30] and [18]). If we take $R = 1$ and suppose $a_0 > 0$ in Corollary 4.3, we recover Theorem 2.4.

5 Conclusion

The historical Cauchy's and the Eneström-Kakeya theorems form an essential part of the classical content of geometric function theory. They are equally important in modern papers dealing with the regional location of zeros of polynomials and regular functions with quaternionic coefficients located on only one side of the variable. Here, we find upper bounds for the zeros of these polynomials and deduce zero-free regions for the associated subclass of regular power series by employing tools from the recently developed theory of regular functions and polynomials with quaternionic coefficients. A number of results known in the literature on this topic are generalised by the results found for this subclass of regular functions.

Declarations

Ethical Approval

Not applicable.

Competing interests

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References

[1] Aziz, A., Mohammad, Q.G.: On the zeros of a certain class of polynomials and related analytic functions. J. Math. Anal. Appl. **75**, 495–502 (1980)

[2] Cauchy, A.: Exercises de mathématique. In: Œeuvres (2) **9**, 122 (1829)

[3] Carney, N., Gardner, R., Keaton, R., Powers, A.: The Eneström-Kakeya theorem for polynomials of a quaternionic variable. J. Approx. Theory **250**, (Art. 105325), 1–10 (2020)

[4] Coroianu, L., Gal, S.G.: On the inequalities of Turán, Bernstein and Erdős-Lax in quaternionic setting. RACSAM, **115** (Art. 187), 1–20 (2021)

[5] Gal, S.G., Sabadini, I.: On the Bernstein and Erdős-Lax inequality for quaternionic polynomials. C. R. Acad. Sci. Paris Ser. I **353**, 5–9 (2015)

[6] Gardner, R.B., Taylor, M.A.: Generalization of an Eneström-Kakeya type theorem to the quaternions. Armenian J. Math. **14**, 1–8 (2022)

[7] Gentili, G., Struppa, D.C.: On the multiplicity of zeros of polynomials with quaternionic coefficients. Milan J. Math. **76**, 15–25 (2008)

[8] Gentili, G., Struppa, D.C.: A new theory of regular function of a quaternionic variable. Adv. Math. **216**, 279–301 (2007)

[9] Gentili, G., Stoppato, C.: Zeros of regular function and polynomials of a quaternionic variable. Michigan Math. J. **56**, 655–667 (2008)

[10] Hussain, A.: A note on Enström-Kakeya theorem for quaternionic polynomials. Korean J. Math. **30** (3), 503–512 (2022)

[11] Ivanov, S., Petkov, A.: The almost Schur lemma in quaternionic contact geometry. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2), Paper No. 77, 18 pp. (2023).

[12] Joyal, A., Labelle, G., Rahman, Q.I.: On the location of zeros polynomials. Canad. Math. Bull. **10**, 53–63 (1967)

[13] Lam, T.Y.: A First Course in Noncommutative Rings. Springer, New York (1991)

[14] Liman, A., Hussain, S., Hussain, I.: A note on a generalisation of Eneström-Kakeya theorem for quaternionic polynomials. Mediterr. J. Math. **19** (4), Paper No. 192, 10 pp. (2022)

[15] Marden, M.: Geometry of Polynomials. Math. Surveys, **3** (1966).

[16] Milovanović, G.V., Mir, A.: Zeros of one class of quaternionic polynomials. Filomat **36** (19), 6655–6667 (2022)

[17] Milovanović, G.V., Mir, A.: On the zero bounds of polynomials and regular functions of a quaternionic variable. Appl. Anal. Discrete Math. **17** (1), 216–231 (2023)

[18] Milovanović, G.V., Mir, A., Ahmad, A.: On the zeros of a quaternionic polynomial with restricted coefficients. Linear Algebra Appl. **653**, 231–245 (2022)

[19] Milovanović, G.V., Mitrinović, D.S., Rassias, Th.M.: Topics in Polynomials: Extremal problems, Inequalities, Zeros. World Scientific, Singapore (1994).

[20] Mir, A.: On the zeros of a quaternionic polynomial: an extension of the Eneström-Kakeya theorem. Czechoslovak Math. J. **73**(148) (2023), no. 3, 649–662.

[21] Mir, A.: Estimation of bounds for the zeros of polynomials and regular functions with quaternionic coefficients. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **118** (1), Paper No. 1, 14 pp. (2024).

[22] Mir, A., Ahmad, A.: On the Eneström-Kakeya theorem for quaternionic polynomials. C. R. Math. Acad. Sci. Paris **361** (2023), 1051–1062.

[23] Mir, S., Liman, A.: On the Eneström-Kakeya theorem for quaternionic polynomials. Mediterr. J. Math. **19** (6), Paper No. 264, 12 pp. (2022)

[24] Mir, S., Liman, A.: Some extensions of Eneström-Kakeya theorem for quaternionic polynomials. Korean J. Math. **30** (4), 615–628 (2022)

[25] Niven, J.: Equations in quaternions. Amer. Math Mon. **48**, 654–661 (1941)

[26] Niven, J.: The roots of a quaternion. Amer. Math Mon. **49**, 386–388 (1942)

[27] Serôdio R., Siu, L.S.: Zeros of quaternion polynomials. Appl. Math. Lett. **14**, 237–239 (2001)

[28] Srivastava, H. M., Shah, F. A., Teali, A. A.: Short-time special affine Fourier transform for quaternion-valued functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (2), Paper No. 66, 20 pp. (2022).

[29] A. Sudbery, A.: Quaternion analysis. Math. Proc. Cambridge Philos. Soc. **85**, 199–224 (1979)

[30] Tripathi, D.: A note on Eneström-Kakeya theorem for a polynomial with quaternionic variable. Arab. J. Math. **9**, 707–714 (2020)